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ALI FARDOUN

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On equivariant p-harmonic maps

by

Ali FARDOUN

Department of Mathematics, University of Brest,
6, avenue Le Gorgeu, 29275 Brest Cedex, France.
E-mail: fardoun@univ-brest.fr

ABSTRACT. – In the first section of this paper we study the Dirichlet problem for equivariant (rotationally symmetric) p-harmonic maps from the Euclidean ball B^m to the closed upper ellipsoid $E_+^m(b)$ ($p \geq 2$, $m \geq 3$): in particular, we establish a condition which is necessary and sufficient for the existence of an equivariant smooth solution with prescribed boundary values. In the last section, we obtain existence results for equivariant p-harmonic maps between spheres and ellipsoids. © Elsevier, Paris

Key words: p-harmonic maps, existence and regularity, equivariant theory.

RÉSUMÉ. – Dans la première partie de cet article nous étudions un problème de Dirichlet pour les applications p-harmoniques, équivariantes (rotationnellement symétriques), de la boule euclidienne B^m à valeurs dans la partie supérieure fermée de l'ellipsoïde $E_+^m(b)$ ($p \geq 2$, $m \geq 3$) : en particulier, nous déterminons une condition nécessaire et suffisante pour l'existence d'une solution équivariante de classe C^∞ avec données initiales au bord. Dans la dernière partie de l'article, nous obtenons des résultats d'existence pour les applications p-harmoniques équivariantes entre sphères ou ellipsoïdes. © Elsevier, Paris

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0. INTRODUCTION

In this paper we study p-harmonic maps in several equivariant contexts. If $f:(M, g) \rightarrow (N, h)$ is a smooth map between two Riemannian manifolds, its p-energy is defined by

$$E_p(f) = \frac{1}{p} \int_M |df|^p \, dv_g \qquad (p \geq 2). \tag{0.1}$$

The p-energy functional (0.1) includes as a special case ($p = 2$) the energy functional, whose critical points are the usual harmonic maps (see [5] for background). We say that f is a *p-harmonic map* if it is a critical point of the p-energy functional, that is to say, if it satisfies the Euler-Lagrange equation of the functional (0.1), that is,

$$\operatorname{div}(|df|^{p-2} \, df) = 0. \tag{0.2}$$

In particular, we note that every harmonic map with constant energy density is p-harmonic for all $p \geq 2$. By the theorem of Nash, we can suppose that the target manifold N is isometrically embedded in $\mathbb{R}^{n+k}$, where $n = \dim N$ and k is large enough. Let $F = i \circ f$, where i denotes the embedding of N into $\mathbb{R}^{n+k}$. The p-energy of F is still defined by (0.1) and equation (0.2) becomes equivalent to

$$-\operatorname{div}(|dF|^{p-2} \, dF) + |dF|^{p-2} A(F)(dF, dF) = 0, \tag{0.3}$$

where A denotes the second fundamental form of N in $\mathbb{R}^{n+k}$. We can then consider the Sobolev space

$$H^{1,p}(M, N) = \{u \in H^{1,p}(M, \mathbb{R}^{n+k}) : u(x) \in N \text{ a.e.}\}. \tag{0.4}$$

If $F \in H^{1,p}(M, N)$, we say that F is *weakly p-harmonic* if it is a weak solution of (0.3): that is, if

$$\int_M |dF|^{p-2} \{dF.d\phi + \phi.A(F)(dF, dF)\} \, dv_g = 0 \tag{0.5}$$
$$\text{for all } \phi \in C_0^\infty(M, \mathbb{R}^{n+k}).$$

When $p = 2$, a key regularity theorem [5] says that any continuous weakly harmonic map is smooth (thus harmonic). But, if $p > 2$, the regularity of weakly p-harmonic maps is more difficult to obtain because equation (0.3) is not elliptic at points where $|dF| = 0$ (for instance, see [4], [8] for more general results on the regularity of p-harmonic maps).

It is well known that, in some favorable cases, certain geometrical symmetries allow us to reduce the existence of harmonic maps to the study of an ordinary differential equation (equivariant theory, see [6]). In this paper, we apply the equivariant methods to p-harmonic maps: in general, we can say that the reduction technics are easily extended to the case that $p>2$. By contrast, the resulting ordinary differential equations are more difficult to handle.

Our first results concern the Dirichlet problem for maps of the Euclidean ball into an ellipsoid: these are stated in Section 1 and complement the analysis of Baldes [1] and Jäger-Kaul [11]. Section 2 contains the proofs of the results stated in Section 1. In Section 3, we obtain existence results for p-harmonic maps between spheres and ellipsoids. Some of the results of this paper were announced in [7].

1. DIRICHLET PROBLEM: NOTATION AND RESULTS

We assume that $m \geq 3$ unless otherwise specified. For $b > 0$, let

$$B^m = \{x \in \mathbb{R}^m : |x| \leq 1\}$$

$$\text{and} \quad E_+^m(b) = \left\{(w, y) \in \mathbb{R}^m \times \mathbb{R} : |w|^2 + \frac{y^2}{b^2} = 1,\ y \geq 0\right\}.$$

For $x \in B^m$, we set $r = |x|$ and say that $F = F_\alpha : B^m \to E_+^m(b)$ is *equivariant (or rotationally symmetric)* if

$$F_\alpha(x) = \left(\frac{x}{r}\sin\alpha(r),\ b\cos\alpha(r)\right), \quad \text{where } \alpha : [0, 1] \to [0, \pi/2]. \tag{1.1}$$

In particular, the equator map $u^*(x) = (\frac{x}{r}, 0)$ belongs to $H^{1,p}(B^m, E_+^m(b))$ and is weakly p-harmonic if $p < m$. In the quadratic case ($p = 2$), Baldes [1] obtained the following results (Theorem [1]).

(i) If $b^2 \geq 4\frac{m-1}{(m-2)^2}$ and $F_\alpha : B^m \to E_+^m(b)$ is an equivariant weakly harmonic map such that $F_\alpha \equiv u^*$ on ∂B^m, then $F_\alpha \equiv u^*$ on B^m.

(ii) If $b^2 < 4\frac{m-1}{(m-2)^2}$, then there exists an equivariant harmonic map $F_\alpha \in C^\infty(B^m, E_+^m(b))$ such that $F_\alpha \equiv u^*$ on ∂B^m.

We generalize this result to the case $p \geq 2$: indeed, our first results are

THEOREM 1. – *Suppose* $2 \leq p < m$.

(i) *If* $b^2 \geq 4 \frac{m-1}{(m-p)^2}$ *and* $F_\alpha : B^m \longrightarrow E^m_+(b)$ *is an equivariant weakly p-harmonic map such that* $F_\alpha \equiv u^*$ *on* ∂B^m, *then* $F_\alpha \equiv u^*$ *on* B^m.

(ii) *If* $b^2 < 4 \frac{m-1}{(m-p)^2}$, *then there exists a unique equivariant p-harmonic map* $F_\alpha \in C^\infty(B^m, E^m_+(b))$ *such that* $F_\alpha \equiv u^*$ *on* ∂B^m *and*

$$E_p(F_\alpha) = \operatorname{Inf}\{E_p(u) : u \in H^{1,p}(B^m, E^m_+(b)),\quad u \text{ equivariant and } u \equiv u^* \text{ on } \partial B^m\}. \tag{1.2}$$

THEOREM 2. – *Suppose* $p \geq m \geq 2$ *and* $b > 0$. *Then there exists a unique equivariant p-harmonic map* $F_\alpha \in C^\infty(B^m, E^m_+(b))$ *such that* $F_\alpha \equiv u^*$ *on* ∂B^m *and which satisfies* (1.2).

We can extend α to $[0, +\infty)$ in such a way that F_α extends to a map $\overline{F}_\alpha : \mathbb{R}^m \longrightarrow E^m(b)$. We shall study the asymptotic behavior of solutions α. In a similar spirit, let $\rho \in (0, \pi]$: we also state the following *Dirichlet problem* $\operatorname{Dir}(\rho, m)$: *Does there exists an equivariant p-harmonic map* $F_\alpha : B^m \longrightarrow E^m(b)$ *such that* $\alpha(0) = 0$ *and* $\alpha(1) = \rho$?

If we set $r = e^t$ and $A(t) = \alpha(e^t)$, our results in this context are stated in the following propositions and generalize the analysis of Jäger-Kaul for $p = 2$ and $b = 1$ [11].

PROPOSITION 1. – *Let* $2 \leq p < m$.

(i) *Suppose* $b^2 \geq 4 \frac{m-1}{(m-p)^2}$:

If $\pi/2 \leq \rho \leq \pi$, *then* $\operatorname{Dir}(\rho, m)$ *has no solution.*

If $0 < \rho < \pi/2$, *then* $\operatorname{Dir}(\rho, m)$ *admits a solution: the function* $A(t)$ *associated to this solution satisfies* $\dot{A}(t) > 0$, $\lim_{t\to+\infty} A(t) = \pi/2$ *and* $\lim_{t\to+\infty} \dot{A}(t) = 0$. *In the phase plane* $(A, \dot{A})$, *the point* $(\pi/2, 0)$ *is an improper node if* $b^2 > 4 \frac{m-1}{(m-p)^2}$, *a proper node otherwise. Then the image of the extension* $\overline{F}_\alpha : \mathbb{R}^m \longrightarrow E^m(b)$ *coincide with the interior of* $E^m_+(b)$.

(ii) *Suppose* $b^2 < 4 \frac{m-1}{(m-p)^2}$: *then there exists* $\sigma \in \mathbb{R}$ *such that* $\pi/2 < \sigma < \pi$ *and* $\operatorname{Dir}(\rho, m)$ *admits at least one solution if* $0 < \rho < \sigma$, *while* $\operatorname{Dir}(\rho, m)$ *has no solution if* $\rho > \sigma$. *Moreover, if* $A(t) = \alpha(e^t)\, t \in \mathbb{R}$, *is an extension of a solution of* $\operatorname{Dir}(\rho, m)$, *then* $0 < A(t) < \pi$ $\lim_{t\to+\infty} A(t) = \pi/2$ *and* $\lim_{t\to+\infty} \dot{A}(t) = 0$. *In the phase plane* $(A, \dot{A})$, *the point* $(\pi/2, 0)$ *is a focus and then* $A(t)$ *oscillate around* $\pi/2$ *when* t *tends to* $+\infty$.

PROPOSITION 2. – (i) *Suppose* $p = m \geq 2$. *Then* $\operatorname{Dir}(\rho, m)$ *admits a solution* $A(t) = \alpha(e^t)$ *for all* $0 < \rho < \pi$. *Moreover,* $0 < A(t) < \pi$, *the function* A *is increasing on* $\mathbb{R}$ *and* $\lim_{t \to +\infty} A(t) = \pi$. *In particular, the images of the extensions* $\overline{F}_\alpha : \mathbb{R}^m \to E^m(b)$ *coincide with* $E^m(b)$*-{South Pole}.*

(ii) *Suppose* $p > m \geq 2$. *Then, for any* $0 < \rho \leq \pi$ $\operatorname{Dir}(\rho, m)$ *admits an infinite number of solutions* $A(t) = \alpha(e^t)$. *Moreover, the function* A *is increasing on* $\mathbb{R}$ *and* $\lim_{t \to +\infty} A(t) = +\infty$. *Then the extensions* $\overline{F}_\alpha : \mathbb{R}^m \to E^m(b)$ *cover* $E^m(b)$ *an infinite number of times.*

Remark 1. – The conclusion $\lim_{t \to +\infty} A(t) = +\infty$ in Proposition 2 (ii) shows that for all $p > m \geq 2$ and $b > 0$, the Dirichlet problem $u : B^m \to E^m(b)$, $u|_{\partial B^m} \equiv$ South Pole, admits an infinite number of nonconstant solutions (whose homotopic classification can be found in [15] Lemma 4.1, Corollaries 4.1 and 4.2).

The situation is completely different if $p < m$. Indeed, we obtain the following generalization of a result of Karcher and Wood [10]:

THEOREM 3. – *Let* N *be a Riemannian manifold of dimension* ≥ 2 *and suppose* $2 \leq p < m$. *If* $u : B^m \to N$ *is a p-harmonic map such that* $u|_{\partial B^m} \equiv$ *constant, then* u *is constant on* B^m.

Remark 2. – The conclusion of Theorem 3 is still true if we suppose B^m equipped with a metric of the form $g = f^2 g_0$, where g_0 is the Euclidean metric and f a positive function which satisfies $\frac{\partial}{\partial r}(r f(x)) \geq 0$ for all $x \in B^m$.

Remark 3. – The conclusion of Theorem 3 is still true if $p = 2 = m$ (see [14]). It would be interesting to extend this result to the case $p = m \geq 3$.

2. PROOFS OF THE RESULTS STATED IN § 1

In order to study equivariant maps as in (1.1), it is convenient to introduce the following function spaces:

(2.1)

$$X = \left\{ \alpha \in H^{1,p}([0, 1]; \mathbb{R}) : \|\alpha\|^p = \int_0^1 [|\dot{\alpha}|^p + |\alpha|^p]\, r^{m-1}\, dr < \infty \right\}$$

$$X_0 = \{\alpha \in X : 0 \leq \alpha(r) \leq \pi/2;\ \alpha(1) = \pi/2\} \qquad \overset{\circ}{A} = \overset{\circ}{H}{}^{1,p}([0, 1]; \mathbb{R}).$$

The p-energy of F_α is given by

$$E_p(F_\alpha) = \operatorname{vol}(S^{m-1})\, J_p(\alpha), \tag{2.2}$$

where

$$J_p(\alpha) = \frac{1}{p} \int_0^1 [\dot{\alpha}^2(r)\, h^2(\alpha(r)) + \frac{m-1}{r^2} \sin^2(\alpha(r))]^{p/2}\, r^{m-1}\, dr,$$

$$\text{with } h^2(\alpha) = b^2 \sin^2(\alpha) + \cos^2(\alpha).$$

Proof of Theorem 1. – We denote by $\alpha_{\pi/2}$ the constant critical point $\alpha \equiv \pi/2$. Theorem 1 is obtained essentially by minimisation of the functional $J_p(\alpha)$ on X_0. More precisely, assertion (i) follows from the fact that, if $b^2 \geq 4\,\frac{m-1}{(m-p)^2}$, then $\alpha_{\pi/2}$ is the absolute minimum of J_p on X_0. In this case, *a priori* estimates allow us to exclude the existence of other equivariant solutions. As for part (ii), we first prove that the minimum α_0 is different from $\alpha_{\pi/2}$ (indeed, if $b^2 < 4\,\frac{m-1}{(m-p)^2}$ then $\alpha_{\pi/2}$ is an unstable critical point). Next, we prove that α_0 is smooth on (0, 1] and $\lim_{r\to 0} \alpha_0(r) = 0$. Finally, we prove the regularity of F_{α_0} through the origin of B^m: to this end, we need a sharp analysis of the asymptotic behavior of solutions of the Euler-Lagrange equation associated to the functional (2.2).

The proof of Theorem 1 is divided into 9 steps. The first seven steps lead us to the proof of part (ii), while the last two steps prove part (i).

STEP 1. – *There exists a map $\alpha_0 \in X_0$ which minimises J_p on X_0 and satisfies*

$$(2.3)\quad \int_0^1 \left[\dot{\alpha}_0^2\, h^2(\alpha_0) + \frac{m-1}{r^2} \sin^2 \alpha_0\right]^{\frac{p-2}{2}}$$
$$\times \left\{ h^2(\alpha_0)\, \dot{\alpha}_0\, \dot{\zeta} + \zeta \sin \alpha_0 \cos \alpha_0 \left((b^2-1)\, \dot{\alpha}_0^2 + \frac{m-1}{r^2} \right) \right\} r^{m-1}\, dr = 0,$$
$$\textit{for all } \zeta \in \overset{\circ}{A}.$$

(For $\zeta \in C_0^\infty([0, 1])$, (2.3) is the weak Euler-Lagrange equation associated to (2.2). We will need (2.3) for $\zeta \in \overset{\circ}{A}$ in Step 3 below)

Proof. – We observe that $J_p(\alpha) < \infty$ for all $\alpha \in X$. Moreover, the functional J_p is lower semi-continuous. We set $c = \operatorname{Inf}(1; b^2)$. For all $\alpha \in X_0$, we have

$$(2.4)\quad J_p(\alpha) \geq \frac{c^p}{p} \int_0^1 |\dot{\alpha}|^p\, r^{m-1}\, dr = \frac{c^p}{p} \left\{ \|\alpha\|^p - \int_0^1 |\alpha|^p\, r^{m-1}\, dr \right\}$$
$$\geq \frac{c^p}{p} \|\alpha\|^p - \frac{(\pi/2\, c)^p}{p} \frac{1}{m}.$$

It follows that the functional J_p is coercive on X_0. Then, there exists a map $\alpha_0 \in X_0$ which minimises J_p over X_0. Set $X_1 = \{\alpha \in X : \alpha(1) = \pi/2\}$: we will prove that α_0 is a critical point of J_p on X_1. For any $\alpha \in X_1$, define $\alpha^* \in X_0$ by

$$\alpha^* = \begin{cases} \pi/2 & \text{if} \quad \alpha(r) > \pi/2, \\ \alpha & \text{if} \quad 0 \leq \alpha(r) \leq \pi/2, \\ 0 & \text{if} \quad \alpha(r) < 0. \end{cases}$$

Let

$$F_p(r, \alpha, \dot\alpha) = \left[\dot\alpha^2 h^2(\alpha) + \frac{m-1}{r^2}\sin^2(\alpha)\right]^{p/2} = [\dot\alpha^2 h^2(\alpha) + F(r, \alpha)]^{p/2}.$$

Also, let

$$G_p(r, \alpha, \dot\alpha) = [\dot\alpha^2 h^2(\alpha^*) + G(r, \alpha)]^{p/2} \quad \text{for all } \alpha \in X_1,$$
$$\text{where } G(r, \alpha) = F(r, \alpha^*),$$

and

$$J_p^*(\alpha) = \frac{1}{p}\int_0^1 G_p(r, \alpha(r), \dot\alpha(r))\, r^{m-1}\, dr, \quad \text{where} \quad \alpha \in X_1.$$

For all $\alpha \in X_1$ and $r \in [0, 1]$, we get $F(r, \alpha^*) = G(r, \alpha) = G(r, \alpha^*)$. It follows that, for all $\alpha \in X_1$, $J_p(\alpha^*) = J_p^*(\alpha^*) \leq J_p^*(\alpha)$ which implies

$$\operatorname{Inf}\{J_p(\alpha) : \alpha \in X_0\} = \operatorname{Inf}\{J_p^*(\alpha) : \alpha \in X_1\} = c_0.$$

Now, let $\{\alpha_i\}$ be a minimizing sequence in X_1 for J_p^*: by passing to $\{\alpha_i^*\}$ if necessary, we can suppose that $\alpha_i \in X_0$. Since $J_p^*(\alpha_i) \to c_0$ and $\alpha_i \in X_0$, the inequality (2.4) shows that $\{\alpha_i\}$ is bounded in X_1. Therefore, there exists a subsequence which converges weakly in X_1 to some $\alpha_0 \in X_0$. The semi-continuity of J_p^* yields $J_p^*(\alpha_0) = c_0$; then α_0 is a critical point of J_p^* in X_1. To prove (2.3), a short computation leads us to the Hardy inequality

(2.5) $$\int_0^1 \left|\frac{\zeta}{r}\right|^p dr \leq \frac{p}{p-1}\int_0^1 |\dot\zeta|^p\, dr \qquad \text{for all } \zeta \in \overset{\circ}{A}.$$

It follows that

(2.6) $$\int_0^1 \left|\frac{\zeta}{r}\right|^p r^{m-1}\, dr < \infty.$$

Next, for $\lambda \in (-1, 1)$ we get

$$p \frac{J_p^* (\alpha_0 + \lambda \zeta) - J_p^* (\alpha_0)}{\lambda}$$
$$= \frac{1}{\lambda} \int_0^1 r^{m-1} \int_0^1 \frac{d}{dt} G_p (r, \alpha_0 + t\, \lambda \zeta, \dot{\alpha}_0 + t\, \lambda \dot{\zeta})\, dt\, dr$$
$$= \int_0^1 r^{m-1} \int_0^1 \frac{\partial G_p}{\partial y} (r, \alpha_0 + t\, \lambda \zeta, \dot{\alpha}_0 + t\, \lambda \dot{\zeta})\, \dot{\zeta}$$
$$+ \frac{\partial G_p}{\partial x} (r, \alpha_0 + t\, \lambda \zeta, \dot{\alpha}_0 + t\, \lambda \dot{\zeta})\, \zeta\, dt\, dr,$$

where

$$G_p (r, x, y) = \begin{cases} \left[b^2\, y^2 + \dfrac{m-1}{r^2} \right]^{p/2} & \text{if x} > \pi/2, \\ \left[h^2 (x)\, y^2 + \dfrac{m-1}{r^2} \sin^2 x \right]^{p/2} & \text{if} \quad 0 \leq x \leq \pi/2, \\ y^p & \text{if} \quad x < 0. \end{cases}$$

Finally, by using the fact that ζ satisfies (2.6) and the Hölder inequality, we can easily see that

$$\left| r^{m-1} \int_0^1 \frac{\partial G_p}{\partial y} (r, \alpha_0 + t\, \lambda \zeta, \dot{\alpha}_0 + t\, \lambda \dot{\zeta})\, \dot{\zeta} \right.$$
$$\left. + \frac{\partial G_p}{\partial x} (r, \alpha_0 + t\, \lambda \zeta, \dot{\alpha}_0 + t\, \lambda \dot{\zeta})\, \zeta\, dt \right|$$

is dominated independently of λ by a function in $L^1 ([0, 1])$. By the Lebesgues dominated convergence Theorem, we have

$$\lim_{\lambda \to 0} \frac{J_p^* (\alpha_0 + \lambda \zeta) - J_p^* (\alpha_0)}{\lambda}$$
$$= \frac{1}{p} \int_0^1 \left\{ \frac{\partial G_p}{\partial y} (r, \alpha_0, \dot{\alpha}_0)\, \dot{\zeta} + \frac{\partial G_p}{\partial x} (r, \alpha_0, \dot{\alpha}_0)\, \zeta \right\} r^{m-1}\, dr.$$

And, since $\alpha_0 \in X_0$, we conclude that α_0 satisfies (2.3). This completes the proof of Step 1.

STEP 2. – *If* $b^2 < 4\frac{m-1}{(m-p)^2}$ *then* $\alpha_0 \neq \alpha_{\pi/2}$.

Proof. – First, we calculate the second variation,

$$Q(\zeta) = \frac{d^2}{dt^2} J_p\left(\frac{\pi}{2} + t\,\zeta\right)\Big|_{t=0} \qquad \text{for} \quad \zeta \in \overset{\circ}{A}.$$

We note that

$$J_p\left(\frac{\pi}{2} + t\,\zeta\right) = \frac{1}{p}\int_0^1 F_p\left(r,\, \frac{\pi}{2} + t\,\zeta,\, t\,\dot{\zeta}\right) r^{m-1}\, dr$$

where

$$F_p\left(r,\, \frac{\pi}{2} + t\,\zeta,\, t\,\dot{\zeta}\right) = \left[t^2\,\dot{\zeta}^2\, k^2(t\,\zeta) + \frac{m-1}{r^2}\cos^2(t\,\zeta)\right]^{p/2}$$

$$\text{and} \quad k^2(t\,\zeta) = b^2\cos^2(t\,\zeta) + \sin^2(t\,\zeta).$$

If t is sufficiently close to 0, there exists $\varepsilon > 0$ such that $\cos^2(t\,\zeta) \geq \cos^2\varepsilon > 0$. Thus we get $F_p(r,\, \frac{\pi}{2} + t\,\zeta,\, t\,\dot{\zeta}) \neq 0$. Next, we can easily see that $|r^{m-1}\frac{d^2}{dt^2} F_p(r,\, \frac{\pi}{2} + t\,\zeta,\, t\,\dot{\zeta})|$ is dominated independently of t by a function in $L^1([0,\, 1])$. By the Lebesgues dominated convergence Theorem, we have

$$Q(\zeta) = (m-1)^{\frac{p}{2}}\int_0^1 \left[b^2\,\dot{\zeta}^2(r) - \frac{m-1}{r^2}\,\zeta^2(r)\right] r^{m-p+1}\, dr.$$

In order to end Step 2, it is enough to find a function $\zeta \in \overset{\circ}{A}$ such that $\zeta \leq 0$ and $Q(\zeta) < 0$ if $b^2 < 4\frac{m-1}{(m-p)^2}$. Indeed, the fact that $\alpha_{\pi/2}$ is a critical point of J_p togeher with $Q(\zeta) < 0$, imply that

$$J_p(\alpha_{\pi/2} + t\,\zeta) < J_p(\alpha_{\pi/2}),$$

provided that $t > 0$ is sufficiently small. It follows that the map $\alpha_{\pi/2}$ is not the minimum of J_p over X_0, that is $\alpha_0 \neq \alpha_{\pi/2}$. To define ζ, let

$$\nu = \frac{(m-p)^2}{4} - \frac{m-1}{b^2} + \varepsilon, \quad \text{where } \varepsilon \text{ is small enough as to have } \nu < 0.$$

Then we set

$$(2.7) \qquad \zeta(r) = r^{\frac{p-m}{2}}\sin(\sqrt{-\nu}\,\mathrm{Log}\, r) \qquad \text{if} \quad r_0 \leq r \leq 1$$
$$\text{and} \quad \zeta(r) = 0 \qquad \text{if} \quad 0 \leq r \leq r_0,\ r_0 = \exp(-\pi/\sqrt{-\nu}) < 1.$$

A direct computation shows that $Q(\zeta) < 0$, so the proof of Step 2 is complete.

STEP 3. – $\alpha_0 \in C^\infty((0, 1])$.

Proof. – On any compact set $[a, 1]$ $(a > 0)$, $\alpha_0 \in H^{1,p}([a, 1])$. It follows from the Sobolev embedding Theorem that α_0 is continuous on $[a, 1]$, then on $(0, 1]$. Next, we set

$$\Gamma = \{r > 0 : \alpha_0(r) = 0\}.$$

We first suppose that $\Gamma = \varnothing$: then $\alpha_0(r) > 0$ for all $r \in (0, 1]$.

To simplify the notations, we set

$$b_1(x; \mu; \eta) = \left[\eta^2 h^2(\mu) + \frac{m-1}{x^2} \sin^2 \mu\right]^{\frac{p-2}{2}} h^2(\mu)\, \eta\, x^{m-1}$$

and

$$\begin{aligned} b(x; \mu; \eta) = &\left[\eta^2 h^2(\mu) + \frac{m-1}{x^2} \sin^2 \mu\right]^{\frac{p-2}{2}} \\ &\times \sin \mu \cos \mu \left((b^2 - 1)\, \eta^2 + \frac{m-1}{x^2}\right) x^{m-1}. \end{aligned}$$

We know that α_0 satisfies (2.3)

$$\int_0^1 b_1(x; \alpha_0(x); \dot{\alpha}_0(x))\, \dot{\zeta}(x) + b(x; \alpha_0(x); \dot{\alpha}_0(x))\, \zeta(x)\, dx = 0$$
$$\text{for all } \zeta \in \overset{\circ}{A}.$$

Now, there exists $N_a > 0$ such that $\alpha_0(r) \geq N_a$ for all $r \in [a, 1]$. Next, for all $\eta \in \mathbb{R}$, $\mu \geq N_a$ and $x \in [a, 1]$, a short computation tell us that the functions b_1 and b satisfy the conditions (3.1), (3.2), (5.7) of Theorem 5.2 in [12]. Then $\alpha_0 \in H^{2,2}_{\mathrm{loc}}((a, 1))$ for all $a > 0$. It follows from the Sobolev embedding Theorem that $\alpha_0 \in C^1((0, 1))$. Now we will prove that $\alpha_0 \in C^1((0, 1])$. To this end, we set $r = e^t$ and $A(t) = \alpha_0(e^t)\,(t \in (-\infty, 0])$. We calculate the Euler-Lagrange equation associated to (2.2) in terms of $A(t)$ and we get

$$\begin{aligned}(2.8)\quad \ddot{A} = \dot{A} - (m-1)\,\dot{A}\,&\frac{[h^2(A)\,\dot{A}^2 + (m-1)\sin^2(A)]}{[(p-1)\,h^2(A)\,\dot{A}^2 + (m-1)\sin^2(A)]}\\ &+ \frac{(p-2)(m-1)\sin A\,\dot{A}\,[\sin A - \dot{A}\cos A]}{[(p-1)\,h^2(A)\,\dot{A}^2 + (m-1)\sin^2(A)]}\\ &+ (m-1)\,\frac{\sin A\cos A}{h^2(A)}\,\frac{[h^2(A)\,\dot{A}^2 + (m-1)\sin^2(A)]}{[(p-1)\,h^2(A)\,\dot{A}^2 + (m-1)\sin^2(A)]}\\ &- (b^2-1)\,\frac{\sin A\cos A}{h^2(A)}\,\dot{A}^2\end{aligned}$$

where

$$h^2(A) = b^2\sin^2(A) + \cos^2(A).$$

Let $H(A;\,\dot{A})$ be the right hand side of (2.8). For all $a < 0$, a short computation tells us that $H(A;\,\dot{A}) \in L^1((a,\,0))$. For all $t < 0$, we have the following equality

(2.9)

$$\dot{A}(t) = \int_{t_1}^{t} H(A(u);\,\dot{A}(u))\,du + \dot{A}(t_1),\ \text{where}\ t_1 < 0\ \text{is a fixed constant.}$$

By passing to the limit when $t \to 0$ in (2.9), we find that $\lim_{t\to 0}\dot{A}(t)$ exists and is finited. Thus the map $\alpha_0 \in C^1((0,\,1])$. Finally, since the function $A \in C^1((-\infty,\,0])$ and is a solution of (2.8), we conclude that $A \in C^\infty((-\infty,\,0])$ and so $\alpha_0 \in C^\infty((0,\,1])$. Now, suppose $\Gamma \neq \emptyset$ and let $r_0 = \sup\Gamma$ ($r_0 < 1$ because $\alpha_0(1) = \frac{\pi}{2}$). An argument similar to the previous case shows that the map $\alpha_0 \in C^\infty((r_0,\,1])$. Since α_0 is the minimum of J_p over X_0, it follows that $\alpha_0(r) = 0$ for all $r \leq r_0$. Since $\lim_{t\to \operatorname{Log} r_0}\dot{A}(t)$ exists and is finited, we conclude that $\dot{\alpha}_0(r_0^+)$ exists and is finited. Now we will prove that $\dot{\alpha}_0(r_0^+) = 0$. Because the map α_0 is solution of (2.3), we can choose a test function $\zeta \in C_0^\infty([0,\,1])$ such that

$$\operatorname{supp}\zeta \subset [r_0 - \varepsilon;\, r_0 + \varepsilon] \qquad \text{and} \qquad \zeta(r_0) \neq 0$$
$$(\text{where}\ \varepsilon > 0\ \text{is close to}\ 0).$$

After an integration by parts, we find that (2.3) is equivalent to

$$\begin{aligned}(2.10)\quad &\int_{r_0}^{r_0+\varepsilon}\left\{\frac{d}{dr}\left[r^{m-1}\,\frac{\partial F_p}{\partial y}(r,\,\alpha_0,\,\dot{\alpha}_0)\right] - \frac{\partial F_p}{\partial x}(r,\,\alpha_0,\,\dot{\alpha}_0)\,r^{m-1}\right\}\zeta\,dr\\ &\quad + \left[r^{m-1}\,\frac{\partial F_p}{\partial y}(r,\,\alpha_0,\,\dot{\alpha}_0)\,\zeta\right]_{r_0}^{r_0+\varepsilon} = 0\end{aligned}$$

where $F_p(r, x, y) = [h^2(x) y^2 + \frac{m-1}{r^2} \sin^2 x]^{p/2}$.

We observe that the integrand vanishes because $\alpha_0 \in C^\infty((r_0, 1])$ and so α_0 is a strong solution on $(r_0, 1]$. It follows that (2.10) reduces to

$$F_{p-2}(r_0, \alpha_0(r_0), \dot\alpha_0(r_0))\, \zeta(r_0)\, \dot\alpha_0(r_0^+)\, h^2(\alpha_0(r_0))\, r_0^{m-1} = 0,$$

and then $\dot\alpha_0(r_0^+) = 0$. Now, we let t tend to Log r_0 in (2.8) and we note that

$$A^{(n)}(\operatorname{Log} r_0) = 0 \qquad \text{for all} \quad n \geq 0.$$

Next, there exist two constants $C_1, C_2 > 0$ such that any solution Z of (2.8) satisfies the following inequality

$$|\ddot Z| \leq C_1 |\dot Z| + C_2 |Z|. \tag{2.11}$$

Since the function $A(t)$ is a solution of (2.8) for $t \in (\operatorname{Log} r_0, 0]$ and $A \equiv 0$ on $(-\infty, \operatorname{Log} r_0]$, we deduce that A satisfies (2.11) on $\mathbb{R}$. By the unique continuation principle, it follows that $A \equiv 0$ on $\mathbb{R}$. This fact contradicts $A(0) = \pi/2$. (Because the function $H(A; \dot A)$ is not of class C^1 in a neighborhood of (0, 0), it seemed to us preferable to use the unique continuation principle rather than the Cauchy uniqueness Theorem.)

STEP 4. – $\dot\alpha_0(r) > 0$ *for all* $r \in (0, 1]$ *and* $\lim_{r \to 0} \alpha_0(r) = 0$.

Proof. – (We develop ideas of [6].) Set

$$K(t) = e^{-2t}[h^2(A)\dot A^2 + (m-1)\sin^2(A)].$$

The differential equation (2.8) becomes equivalent to

$$\ddot A + \dot A \left\{ (m-2) + \frac{p-2}{2}\frac{\dot K}{K} + \frac{(b^2-1)}{h^2(A)} \dot A \sin A \cos A \right\} = (m-1)\frac{\sin A \cos A}{h^2(A)}. \tag{2.12}$$

According to Step 3, the solution $A(t) = \alpha_0(e^t)$ satisfies

$$0 < A(t) \leq \frac{\pi}{2} \quad \text{for all } t \in (-\infty, 0] \quad \text{and} \quad A(0) = \frac{\pi}{2}.$$

We first prove that the zeroes of $\dot A$ are isolated. We note that $\dot A(0) \neq 0$; for otherwise $A \equiv \frac{\pi}{2}$ by the Cauchy uniqueness Theorem. Let $t_0 < 0$ be such that $\dot A(t_0) = 0$. Then $A(t_0) \neq 0$ and $A(t_0) \neq \frac{\pi}{2}$ (for otherwise $A \equiv \frac{\pi}{2}$); we deduce by (2.8) that $\ddot A(t_0) \neq 0$. Then there exists a neighborhood J of t_0 which contains no more zeroes of $\dot A$.

Now we prove that $\dot{A}(t) > 0$ (which of course implies $\dot{\alpha}_0(r) > 0$). We argue by contradiction: let $\bar{t} < 0$ be the first point such that $\dot{A}(\bar{t}) = 0$; set $I = (\bar{t}, 0]$ and

$$(2.13)\quad P_A = 2\left\{\frac{\ddot{A}}{\dot{A}} + (m-2) + \frac{p-2}{2}\frac{\dot{K}}{K} + \frac{(b^2-1)}{h^2(A)}\dot{A}\sin A\cos A\right\},$$
$$Q_A = 2(m-1)\frac{\sin A\cos A}{h^2(A)\dot{A}}.$$

Since A is a solution of (2.12), we have $P_A \equiv Q_A$ on I. We consider the following first order linear differential equation

$$(2.14)\qquad y'(t) + P_A(t)\,y(t) = Q_A(t).$$

The constant function $y \equiv 1$ is a solution of (2.14) and we note that

(2.15)
$$1 \equiv y(t) = \frac{1 - \int_t^{t_0} Q_A(u)\left(\exp - \int_u^{t_0} P_A(r)\,dr\right)du}{\exp - \int_t^{t_0} P_A(u)\,du}\qquad \bar{t} < t_0,\ t < 0.$$

Let $T < \bar{t}$ be the first point (if it exists) such that $\dot{A}(T) = 0$. Thus (2.15) holds on $(T, \bar{t})$. Direct integration of (2.15) gives

$$1 \equiv \frac{N(t)}{D(t)}\qquad \text{for all}\quad t \in (T, \bar{t}),$$

where

$$N(t) = 1 - \frac{2(m-1)}{h^2(A(t_0))\,\dot{A}^2(t_0)\,K^{p-2}(t_0)\,e^{2(m-2)t_0}}$$
$$\times \int_t^{t_0} \dot{A}(r)\,e^{2(m-2)r}\,K^{p-2}(r)\sin A(r)\cos A(r)\,dr,$$

and

$$D(t) = \frac{\dot{A}^2(t)\,K^{p-2}(t)\,e^{2(m-2)t}\,h^2(A(t))}{h^2(A(t_0))\,\dot{A}^2(t_0)\,K^{p-2}(t_0)\,e^{2(m-2)t_0}}\qquad (t_0 \in (T, \bar{t})).$$

For all $t \in (T, \bar{t})$, we have

$$(2.16)\qquad N(t) > 0$$

and

(2.17) $$\dot{N}(t) \neq 0 \quad \text{because } \dot{A}(t) \neq 0.$$

We claim that $T = -\infty$. For otherwise, since $D(T) = D(\bar{t}) = 0$ we would get $N(T) = N(\bar{t}) = 0$. Then (2.16) implies that N must have a maximum on $(T, \bar{t})$, a fact which contradicts (2.17). Moreover, since $0 < A(t) \leq \frac{\pi}{2}$ there exists $t'' \in (-\infty, \bar{t})$ such that $\dot{A}(t'')$ is close to 0. So $D(t'')$ and then $N(t'')$ would be arbitrarly close to 0. If follows from $N(\bar{t}) = 0$ and (2.16) that N has a local maximum on $(-\infty, \bar{t})$, which again contradicts (2.17). Then $\dot{A}(t) \neq 0$ for all $t \in (-\infty, 0]$. We conclude then that $\dot{A}(t) > 0$ because $0 < A(t) \leq \frac{\pi}{2}$ and $A(0) = \frac{\pi}{2}$. Finally, $\lim_{t \to -\infty} A(t)$ exists and is finited. Now, there exists a sequence $t_n \to -\infty$ such that $\ddot{A}(t_n) \to 0$ and $\dot{A}(t_n) \to 0$. By passing to the limit in (2.8), we get $\lim_{t \to -\infty} A(t) = 0$. This achieves the proof of Step 4.

STEP. 5 – *If $b^2 < 4\frac{m-1}{(m-p)^2}$, then $\dot{\alpha}_0(0)$ exists and is positive.*

Proof. – The qualitative study of the autonomous equation (2.8) is simplified by the observation that the following function

$$V_p(A) = k(A)^{\frac{p-2}{2}} \{(p-1) h^2(A) \dot{A}^2 - (m-1) \sin^2(A)\}$$
$$\text{where} \quad k(A) = h^2(A) \dot{A}^2 + (m-1) \sin^2(A),$$

is a Lyapunov function (see [13]) associated to (2.8). Indeed, if A is a solution of (2.8), we have

(2.18) $$\frac{d}{dt}(V_p(A(t))) = k(A(t))^{\frac{p-2}{2}} \{p(p-m) h^2(A(t)) \dot{A}^2(t)\} \leq 0$$

for all $t \in (-\infty, 0]$.

For future use, we note that (2.18) is equivalent to

(2.19) $$\frac{d}{dt}(V_p(A(t)) e^{(m-p)t}) = (p-m) e^{(m-p)t} k(A(t))^{\frac{p}{2}} \leq 0.$$

From

$$E_p(F_{\alpha_0}) = \frac{1}{p} \operatorname{vol}(S^{m-1}) \int_{-\infty}^{0} e^{(m-p)t} k(A(t))^{\frac{p}{2}} dt < \infty,$$

it follows that $\liminf_{t\to-\infty} e^{(m-p)\,t}\, k\,(A\,(t))^{\frac{p}{2}} = 0$. Next, by using (2.19), we have

$$E_p\,(F_{\alpha_0}) = \frac{1}{(m-p)\,p}\,\mathrm{vol}\,(S^{m-1})\,[-(p-1)\,b^2\,\dot{A}^2\,(0) + (m-1)] \times [b^2\,\dot{A}^2\,(0) + (m-1)]^{\frac{p-2}{2}}. \tag{2.20}$$

and

$$V_p\,(A\,(t)) < 0 \qquad \text{for all} \quad t \in (-\infty,\, 0]. \tag{2.21}$$

From the Lyapunov functions theory (see [13]) and (2.21), we get $\dot{A}\,(t) \to 0$ when $t \to -\infty$. Next, it is convenient to set

$$\dot{G} = \frac{\dot{A}\,h\,(A)}{\sin A}.$$

Then equation (2.8) becomes equivalent to

(2.22)

$$\ddot{G} = -\,(p-1)\,\frac{[\dot{G}^2 + (m-1)]}{[(p-1)\,\dot{G}^2 + (m-1)]} \times \left\{(\dot{G}-1)\left(\dot{G} + \frac{m-1}{p-1}\right) - \left(1 - \frac{\cos A}{h\,(A)}\right)\left(\dot{G}^2 - \frac{m-1}{p-1}\right)\right\}$$

For future use, we note that (2.21) implies

$$\left(\dot{G}^2\,(t) - \frac{m-1}{p-1}\right) < 0 \qquad \text{for all} \quad t \in (-\infty,\, 0], \tag{2.23}$$

which in turn yields

$$\lim_{t\to-\infty}\left(1 - \frac{\cos\,(A\,(t))}{h\,(A\,(t))}\right)\left(\dot{G}^2\,(t) - \frac{m-1}{p-1}\right) = 0. \tag{2.24}$$

First, we prove that $\lim_{t\to-\infty} \dot{G}\,(t) = 1$. For this purpose, we compare solutions of (2.22) with solutions of the following differential equation

$$\ddot{G} = -(p-1)\,\frac{[\dot{G}^2 + (m-1)]}{[(p-1)\,\dot{G}^2 + (m-1)]}\left\{(\dot{G}-1)\left(\dot{G} + \frac{m-1}{p-1}\right)\right\}. \tag{2.25}$$

Thus we need to investigate the qualitative behavior of solutions to (2.25). This equation has two constant solutions $\dot{G}_1 \equiv 1$ and $\dot{G}_2 \equiv -\frac{m-1}{p-1}$.

Let $\dot{G}_r$ be a nonconstant solution of (2.25). As for nonconstant solutions, there are two possible cases:

1) There exists $\bar{t} \in \mathbb{R}$ such that $\dot{G}_r(\bar{t}) > 1$. Then, we get $\dot{G}_r(t) > 1$ for all $t \in \mathbb{R}$. For otherwise, there would exist t_0 such that $\dot{G}_r(t_0) = 1$ and so $\dot{G}_r \equiv 1$ (by the Cauchy uniqueness Theorem). The function $V(\dot{G}_r) = (\dot{G}_r - 1)$ is a Lyapunov function associated to (2.25) because $\frac{d}{dt} V(\dot{G}_r(t)) = \ddot{G}_r(t) < 0$ for all $t \in \mathbb{R}$. It follows that $\lim_{t \to -\infty} \dot{G}_r(t) = +\infty$ and $\lim_{t \to +\infty} \dot{G}_r(t) = 1$.

2) There exists $\bar{t} \in \mathbb{R}$ such that $\dot{G}_r(\bar{t}) < 1$. Then, we get

$$-\frac{m-1}{p-1} < \dot{G}_r(t) < 1 \text{ for all } t \in \mathbb{R} \text{ (by the Cauchy uniqueness Theorem)}$$

Next, because of (2.25), it follows that $\ddot{G}_r(t) > 0$ for all $t \in \mathbb{R}$ and so $\dot{G}_r$ is increasing; then $\lim_{t \to \pm\infty} \dot{G}_r(t)$ exists. Moreover, for any $\varepsilon > 0$ and $C < 0$, there exists $t_1 < C$ such that $\ddot{G}_r(t_1) < \varepsilon$ (that is, there exists a sequence $t_n \to -\infty$ such that $\ddot{G}_r(t_n) \to 0$). For otherwise, we have

(2.26) *there exist two constants $A_1 > 0$ and $T < 0$ such that $\ddot{G}_r(t) > A_1$ for all $t < T$.*

If $t \ll 0$, we can write

$$\dot{G}_r(t) = \int_T^t \ddot{G}_r(u)\, du + \dot{G}_r(T) \tag{2.27}$$

From (2.26) and (2.27), it follows that $\lim_{t \to -\infty} \dot{G}_r(t) = -\infty$. This contradicts the fact that the function $\dot{G}_r$ is bounded. Now, using (2.25) and since $\dot{G}_r$ is increasing we get $\lim_{t \to -\infty} \dot{G}_r(t) = -\frac{m-1}{p-1}$. Similarly, we get $\lim_{t \to +\infty} \dot{G}_r(t) = 1$. Now we use these qualitative facts about (2.25) to show that $\lim_{t \to -\infty} \dot{G}(t) = 1$. We argue by contradiction: then there exist $\varepsilon > 0$ and a sequence $t_n \to -\infty$ such that either

$$\dot{G}(t_n) \geq 1 + \varepsilon, \tag{2.28}$$

or

$$\dot{G}(t_n) \leq 1 - \varepsilon. \tag{2.29}$$

If (2.28) is satisfied, then by (2.24) we can take $t_n < 0$ large enough such that

$$\left(1 - \frac{\cos A}{h(A)}\right)\left(\dot{G}^2 - \frac{m-1}{p-1}\right) \text{ is close to } 0.$$

So $\dot{G}$ would be approximated in a compact set by a solution $\dot{G}_r$ of (2.25) which satisfies $\dot{G}_r(t_n) \geq 1 + \varepsilon$. By 1), we have $\lim\limits_{t \to -\infty} \dot{G}_r(t) = +\infty$, so there would exist t' such that $\dot{G}(t')$ is arbitrarily large. This contradicts the fact that $\dot{G}$ is bounded. Similarly, using 2) we find that (2.29) is not possible, so $\lim\limits_{t \to -\infty} \dot{G}(t) = 1$.

Now, we need to prove that $\dot{G}(t) < 1$ for all $t \in (-\infty, 0]$. To this end, we note that $\ddot{G}(0) < 0$ because $\ddot{A}(0) < 0$ (by (2.8)). And using (2.23), we get the following

$$-\left(1 - \frac{\cos(A(t))}{h(A(t))}\right)\left(\dot{G}^2(t) - \frac{m-1}{p-1}\right) > 0 \quad \text{for all} \quad t \in (-\infty, 0]. \tag{2.30}$$

Next, we have to consider the following two possibilities:

1) suppose $\dot{G}(0) < 1$: if there exists $t' < 0$ such that $\dot{G}(t') \geq 1$, using $\lim\limits_{t \to -\infty} \dot{G}(t) = 1$, we find that there exists $t'' < 0$ such that $\dot{G}(t'') \geq 1$ and $\ddot{G}(t'') = 0$. Then, simple inspection of (2.30) and (2.22) leads us to a contradiction;

2) suppose $\dot{G}(0) \geq 1$: then, since $\ddot{G}(0) < 0$ and $\lim\limits_{t \to -\infty} \dot{G}(t) = 1$, there exists $t' < 0$ such that $\dot{G}(t') > 1$ and $\ddot{G}(t') = 0$. This fact contradicts (2.22) and (2.30).

By way of summary, the function $\dot{G}$ satisfies $\dot{G}(t) < 1$ for all $t \in (-\infty, 0]$.

Next, we prove that there exist two constants $C_1, C_2 > 0$ such that

$$1 - C_1 e^{C_2 t} \leq \dot{G}(t) \leq 1 \quad \text{for all } t \in (-\infty, 0]. \tag{2.31}$$

Coming back to $\alpha_0(r)$, (2.31) takes the form

$$\frac{1 - C_1 r^{C_2}}{r} \leq \frac{\dot{\alpha}_0(r)\, h(\alpha_0(r))}{\sin(\alpha_0(r))} \leq \frac{1}{r}. \tag{2.32}$$

To prove (2.31), write $\dot{G}(t) = 1 - \gamma(t)$ with $\gamma(t) > 0$ and $\lim_{t \to -\infty} \gamma(t) = 0$. Then equation (2.22) becomes equivalent to

$$\gamma'(t) + c_1(t)\gamma(t) = c_2(t)\left(1 - \frac{\cos A(t)}{h(A(t))}\right), \tag{2.33}$$

where

$$c_1(t) = (p-1)\frac{[(1-\gamma(t))^2 + (m-1)]}{[(p-1)(1-\gamma(t))^2 + (m-1)]}\left(1 - \gamma(t) + \frac{m-1}{p-1}\right),$$

and

$$c_2(t) = -\frac{[(p-1)(1-\gamma(t))^2 - (m-1)]}{[(p-1)(1-\gamma(t))^2 + (m-1)]}[(1-\gamma(t))^2 + (m-1)].$$

We note that $\lim_{t \to -\infty} c_1(t) = m$. Moreover, there exist two constants M, $N > 0$ such that

$$M \leq c_2(t) \leq N \quad \text{for all } t \in (-\infty, 0].$$

Next, Taylor's expansion centered at $A(-\infty) = 0$ shows that there exists a function g such that

$$\left(1 - \frac{\cos(A)}{h(A)}\right) = \frac{b^2 A^2}{2} + A^2 g(A) \quad \text{and} \quad \lim_{t \to -\infty} g(A(t)) = 0.$$

Thus there exist $T \ll 0$, two constants K_1, $K_2 > 0$ and a function c_3 such that

$$K_1 \leq c_3(t) \leq K_2 \quad \text{for all } t \leq T,$$

and γ is a solution of the following differential equation

$$\gamma'(t) + c_1(t)\gamma(t) = c_3(t) A^2(t). \tag{2.34}$$

Since $\lim_{t \to -\infty} \frac{\dot{A}(t)}{A(t)} = \lim_{t \to -\infty} \dot{G}(t) = 1$, it follows that for any $\varepsilon > 0$ there exists $T_1 \leq T$ such that

$$\frac{\dot{A}(t)}{A(t)} \geq 1 - \varepsilon \quad \text{for all } t \leq T_1. \tag{2.35}$$

Integrating (2.35) between T_1 and t, we find that there exist $T_2 \ll 0$ and a constant $C > 0$ such that the function A satisfies

$$0 < A(t) \leq Ce^{(1-\varepsilon)t} \quad \text{for all } t \leq T_2. \tag{2.36}$$

From (2.34) and (2.36), it follows that there exist $T_3 \ll 0$, a constant $K_3 > 0$ and a function c_4 such that

$$0 < c_4(t) \leq K_3 \quad \text{for all } t \leq T_3, \tag{2.37}$$

and γ is a solution of the following differential equation

$$\gamma'(t) + c_1(t)\,\gamma(t) = c_4(t)\,e^t. \tag{2.38}$$

An explicit computation gives

$$\gamma(t) = \frac{\int_{-\infty}^{t} c_4(s) \exp\left(s + \int_{T_3}^{s} c_1(r)\,dr\right) ds}{\exp \int_{T_3}^{t} c_1(r)\,dr}.$$

Taking T_3 sufficiently large and using the fact that $\lim_{t \to -\infty} c_1(t) = m$ and (2.37), it follows that there exist two constants C_1, $C_2 > 0$ such that (2.31) is satisfied for all $t \leq T_3$. Next, it is easy to show that (2.31) is satisfied for all $t \in (-\infty, 0]$ provided C_1 is large enough. Finally, using (2.32) we will prove that $\dot{\alpha}_0(0)$ exists and is positive. For this purpose, it is convenient to write down the primitives $I(r)$ of the function $\frac{\dot{\alpha}_0\, h(\alpha_0)}{\sin(\alpha_0)}$. Set

$$f(r) = \frac{\operatorname{cotg} \alpha_0(r)}{b} + \sqrt{1 + \frac{\operatorname{cotg}^2 \alpha_0(r)}{b^2}}, \quad \beta = \frac{1}{b^2} - \frac{1}{2}, \quad \mu = \frac{4}{b^2}\,[b^2 - 1].$$

Then, for all $0 < r \leq 1$, we have:
If $b^2 < 1$,

$$I(r) = \operatorname{Log}\left(b \frac{\sin \alpha_0(r)}{\cos \alpha_0(r) + h(\alpha_0(r))}\right) + \sqrt{-b^2 + 1}\,\operatorname{Log}\left(\frac{f^2(r) + 2\beta - \sqrt{-\mu}}{f^2(r) + 2\beta + \sqrt{-\mu}}\right). \tag{2.39}$$

If $b^2 = 1$,

$$I(r) = \operatorname{Log}\left(\frac{\sin \alpha_0(r)}{\cos \alpha_0(r) + 1}\right) = \operatorname{Log}\left(\operatorname{tg} \frac{\alpha_0(r)}{2}\right). \tag{2.40}$$

If $b^2 > 1$,

$$I(r) = \operatorname{Log}\left(b \frac{\sin \alpha_0(r)}{\cos \alpha_0(r) + h(\alpha_0(r))}\right) - \sqrt{b^2 - 1}\,\operatorname{Arctg}\left(\frac{f^2(r) + 2\beta}{\sqrt{\mu}}\right). \tag{2.41}$$

Now, we choose $0 < \varepsilon < 1$ and we write

$e^{I(r)} = \sin \alpha_0 (r) U (r)$ where U is a positive continuous function on $[0, \varepsilon]$. We integrate (2.32) between ε and r, and by passing to the exponential, we obtain the following inequality

$$\frac{\sin \alpha_0 (\varepsilon)}{\varepsilon} \leq \frac{U (r)}{U (\varepsilon)} \frac{\sin \alpha_0 (r)}{r} \leq \frac{\sin \alpha_0 (\varepsilon)}{\varepsilon} e^{\frac{C_1}{C_2} \varepsilon^{C_2}}. \tag{2.42}$$

Let us fix $\varepsilon = \varepsilon_0 < 1$. Since U is a positive continuous function on $[0, \varepsilon_0]$, U is bounded on this compact set. Then there exist two constants $M_1, M_2 > 0$ such that

$$0 < M_1 \leq \frac{\sin \alpha_0 (r)}{r} \leq M_2 \quad \text{for all } 0 \leq r \leq \varepsilon_0. \tag{2.43}$$

Now, suppose that $\lim\limits_{r \to 0} \frac{\sin \alpha_0 (r)}{r}$ does not exist: By (2.43) it follows that there exist two sequences (R_i) and (T_i) tending to 0 as $i \to +\infty$ and two real numbers $l_1, l_2 > 0$ $(l_1 \neq l_2)$ such that $\frac{\sin \alpha_0 (R_i)}{R_i} \to l_1$ and $\frac{\sin \alpha_0 (T_i)}{T_i} \to l_2$. Next, from (2.42) we get, the following inequality

$$\begin{aligned} &\left| \frac{\sin \alpha_0 (R_i)}{R_i} U (R_i) - \frac{\sin \alpha_0 (T_i)}{T_i} U (T_i) \right| \\ &\leq U (\varepsilon) \frac{\sin \alpha_0 (\varepsilon)}{\varepsilon} |e^{\frac{C_1}{C_2} \varepsilon^{C_2}} - 1| \quad \text{for all } \varepsilon \leq \varepsilon_0. \end{aligned} \tag{2.44}$$

Since U is continuous on $[0, \varepsilon_0]$, $U (0) > 0$ and $\frac{\sin \alpha_0 (\varepsilon)}{\varepsilon}$ is bounded, we conclude that when $i \to +\infty$ the left hand side of (2.44) tends to $U (0) |l_1 - l_2|$ while the right hand side tends to 0 when $\varepsilon \to 0$. This is a contradiction and so

$$\lim_{r \to 0} \frac{\sin \alpha_0 (r)}{r} = \dot{\alpha}_0 (0) \quad \text{exists and is positive by (2.43).}$$

STEP 6. – *F_{α_0} is a weakly p-harmonic map.*

Proof. – A short computation gives the following equality

$$\begin{aligned} \operatorname{div} [(|dF_{\alpha_0}|^{p-2} dF_{\alpha_0}).\phi] =& \langle \operatorname{div} (|dF_{\alpha_0}|^{p-2} dF_{\alpha_0}), \phi \rangle \\ &+ \langle |dF_{\alpha_0}|^{p-2} dF_{\alpha_0}, d\phi \rangle \\ &\text{for all } \phi \in C_0^\infty (B^m. \mathbb{R}^{m+1}). \end{aligned} \tag{2.45}$$

Set $B_\varepsilon^m = \{x \in \mathbb{R}^m : |x| \leq \varepsilon\}$: F_{α_0} is a weakly p-harmonic map if F_{α_0} satisfies (0.5) or equivalently

(2.46)

$$\lim_{\varepsilon \to 0} \int_{B^m - B_\varepsilon^m} |dF_{\alpha_0}|^{p-2} \{dF_{\alpha_0}.d\phi + \phi.A(F_{\alpha_0})(dF_{\alpha_0}, dF_{\alpha_0})\}\, dx = 0$$
$$\text{for all } \phi \in C_0^\infty(B^m, \mathbb{R}^{m+1})$$

Now, we observe that

$$(2.47)\ -\mathrm{div}\,(|dF_{\alpha_0}|^{p-2}\, dF_{\alpha_0}) + |dF_{\alpha_0}|^{p-2} A(F_{\alpha_0})(dF_{\alpha_0}, dF_{\alpha_0}) = 0 \quad \text{on } B^m - B_\varepsilon^m,$$

because α_0 is smooth on $[\varepsilon, 1]$ and then satisfies strongly the Euler-Lagrange equation associated to (2.2), that is (2.47). From (2.45) and (2.47), it follows that (2.46) becomes equivalent to

$$(2.48) \qquad \lim_{\varepsilon \to 0} \int_{B^m - B_\varepsilon^m} \mathrm{div}\,[(|dF_{\alpha_0}|^{p-2}\, dF_{\alpha_0})\cdot\phi]\, dx = 0 \quad \text{for all } \phi \in C_0^\infty(B^m, \mathbb{R}^{m+1}).$$

Let ν be the unit normal to S^{m-1}. Using the Stokes Theorem and the fact that ϕ is compactly supported, we see that the left hand side of (2.48) is equal to

$$\lim_{\varepsilon \to 0} \int_{\partial(B^m - B_\varepsilon^m)} |dF_{\alpha_0}|^{p-2} \langle dF_{\alpha_0}\cdot\phi, \nu\rangle dx$$
$$= \lim_{\varepsilon \to 0} \int_{\partial(B_\varepsilon^m)} |dF_{\alpha_0}|^{p-2} \langle dF_{\alpha_0}\cdot\phi, \nu\rangle\, d\theta_\varepsilon,$$

Now let $M = \sup\{\phi(x) : x \in B^m\}$. We obtain the following inequality

$$\left| \int_{\partial(B_\varepsilon^m)} |dF_{\alpha_0}|^{p-2} \langle dF_{\alpha_0}\cdot\phi, \nu\rangle\, d\theta_\varepsilon \right| \leq M \int_{\partial(B_\varepsilon^m)} |dF_{\alpha_0}|^{p-1}\, d\theta_\varepsilon$$
$$= M \left[\dot{\alpha}^2(\varepsilon)\, h^2(\alpha(\varepsilon)) + \frac{m-1}{\varepsilon^2} \sin^2 \alpha(\varepsilon)\right]^{p-1} \varepsilon^{m-1}\, \mathrm{vol}\,(S_\varepsilon^{m-1}).$$

The right hand side tends to 0 when ε tends to 0, thus (2.48) is satisfied. This completes the proof of Step 6.

STEP 7. – *We are now in the right position to proceed to the proof of part* (ii) *of Theorem* 1. *The map F_{α_0} is of class C^1 on B^m because $\dot{\alpha}_0(0)$ exists. Moreover $|dF_{\alpha_0}| > 0$ on B^m because of Steps* 4 *and* 5. *Therefore, using Lemma* 2.1 *of* [3], *we conclude that F_{α_0} is of class C^∞ on B^m. Finally, suppose that there exist two maps F_{α_1} and F_{α_2} which satisfy* (1.2). *Set $A_1(t) = \alpha_1(e^t)$ and $A_2(t) = \alpha_2(e^t)$. Their p-energy calculated in* (2.20) *is an increasing function of the variable $\dot{A}^2(0)$. Then F_{α_1} and F_{α_2} satisfy* (1.2) *if $A_1(0) = A_2(0) = \frac{\pi}{2}$ and $\dot{A}_1(0) = \dot{A}_2(0) > 0$. Since A_1 and A_2 are solutions of* (2.8) *we use the Cauchy uniqueness Theorem to conclude that $A_1 \equiv A_2$ and then $F_{\alpha_1} = F_{\alpha_2}$. Therefore the proof of part* (ii) *of Theorem* 1 *is complete.*

Now we prove part (i) of Theorem 1. For this purpose, we need the following two steps.

STEP 8. – *If $F_\alpha \neq u^*$ is an equivariant weakly p-harmonic map such that $F_\alpha \equiv u^*$ on ∂B^m, then $\dot{\alpha}(r) > 0$ on [0, 1], $\alpha(0) = 0$ and $F_\alpha \in C^\infty(B^m, E_+^m(b))$.*

Proof. – Since F_α is weakly p-harmonic, then it satisfies (0.5). Now, we want to prove that α satisfies (2.3). To this end, let $x = (\theta, r)$ be the polar coordinates in B^m and consider in (0.5) the following type of test-functions

$$\begin{aligned} \phi : B^m &\to \mathbb{R}^{m+1} \\ (\theta, r) &\to (0, \varphi(r)) \quad \text{where } \varphi \in C_0^\infty([0, 1], \mathbb{R}). \end{aligned}$$

We deduce that α is a weak solution of the following equation

$$\frac{d}{dr}\left[\dot{\alpha} \sin \alpha \, F_{p-2}(r, \alpha, \dot{\alpha})\, r^{m-1}\right] = \frac{\overline{K}(r, \alpha, \dot{\alpha})}{h^2(\alpha)} \cos \alpha \, F_{p-2}(r, \alpha, \dot{\alpha})\, r^{m-1}, \tag{2.49}$$

where $\overline{K}(r, \alpha, \dot{\alpha}) = \dot{\alpha}^2 + \frac{m-1}{r^2} \sin^2(\alpha)$. Similarly, if we consider in (0.5) the following type of test-functions

$$\begin{aligned} \phi : B^m &\to \mathbb{R}^{m+1} \\ (\theta, r) &\to (\theta\varphi(r), 0) \quad \text{where } \varphi \in C_0^\infty([0, 1], \mathbb{R}), \end{aligned}$$

we find that α is a weak solution of

$$\begin{aligned} &\frac{d}{dr}\left[\dot{\alpha} \cos \alpha \, F_{p-2}(r, \alpha, \dot{\alpha})\, r^{m-1}\right] \\ &\quad = \left[\frac{m-1}{r^2} - b^2 \frac{\overline{K}(r, \alpha, \dot{\alpha})}{h^2(\alpha)}\right] \sin \alpha \, F_{p-2}(r, \alpha, \dot{\alpha})\, r^{m-1}. \end{aligned} \tag{2.50}$$

Multiply (2.49) by $b^2 \sin \alpha$ and (2.50) by $\cos \alpha$, then sum these two expressions. The result is that α is a weak solution of the Euler-Lagrange associated to (2.2). Now, by using (2.5) and by density, we prove that α satisfies equation (2.3). We now study the qualitative behavior of α. For all compact sets $[a, 1]$ $(a > 0)$, $\alpha \in H^{1, p}([a, 1])$. By the Sobolev embedding Theorem, it follows that α is a continuous function on $[a, 1]$ and then on $(0, 1]$. From equation (2.49), it follows that the function defined on $[0, 1]$ by

$$r \to \dot{\alpha}(r) \sin \alpha(r) F_{p-2}(r, \alpha(r), \dot{\alpha}(r)) r^{m-1}$$

belongs to $H^{1,1}([0, 1])$. So, by the Sobolev embedding Theorem there exists a continuous function f on $[0, 1]$ which satisfies

$$(2.51) \quad f(s) = \dot{\alpha}(s) \sin \alpha(s) F_{p-2}(s, \alpha(s), \dot{\alpha}(s)) s^{m-1} \quad \text{a.e. on } [0, 1].$$

Moreover, integrating (2.49) between s and t we get

$$(2.52) \quad f(s) - f(t) = -\int_s^t \frac{\overline{K}(r, \alpha, \dot{\alpha})}{h^2(\alpha)} \cos \alpha F_{p-2}(r, \alpha, \dot{\alpha}) r^{m-1} dr \quad \text{for all } 0 \leq s < t \leq 1.$$

Now, by means of a study of the zeroes of the function f, we analyse the behavior of the function α. Suppose that there exists $s_0 \in (0, 1]$ such that $f(s_0) = 0$. If follows by (2.51) and (2.52) that α is nonincreasing on $(0, s_0]$ and is increasing on $[s_0, 1]$. The function α being continuous on $(0, 1]$, we can consider the following cases: $\alpha(s_0) = 0$ or $\alpha(s_0) \neq 0$. If $\alpha(s_0) \neq 0$, then $\alpha(s) \neq 0$ for all $s \in (0, 1]$. Since α is a solution of (2.3), different from $\alpha_{\pi/2}$, then by Steps 3 and 4 $\dot{\alpha}(s) > 0$ for all $s \in (0, 1]$. This contradicts the fact that the function α is nonincreasing on $(0, s_0]$, so $\alpha(s_0) = 0$. In this case: we set,

$$b' = \operatorname{Sup}\{s \geq s_0 \text{ such that } \alpha \equiv 0 \text{ on } [s_0, s]\} \qquad (s_0 \leq b' < 1).$$

α is a positive function on $(b', 1]$. An argument similar to Steps 3 and 4 above shows that $\alpha \in C^\infty((b', 1])$ and is increasing on $(b', 1]$. Set

$$a' = \operatorname{Inf}\{s \leq s_0 \text{ such that } \alpha \equiv 0 \text{ on } [s, s_0]\} \qquad (0 \leq a' \leq s_0).$$

Suppose that $a' \neq 0$: according to the definition of a' and since the function α is nonincreasing on $(0, a')$, the function α has no zero on $(0, a')$.

Now, an argument analogous to Step 3 above shows that $\alpha \in C^\infty((0, a'))$. Next, it is convenient to set $A(t) = \alpha(e^t)$, $t \in (-\infty, \operatorname{Log} a')$. Since the function A is a solution of (2.8) on $(-\infty, \operatorname{Log} a')$, as in Step 4 one shows that $\dot{A}(t) < 0$ for all $t \in (-\infty, \operatorname{Log} a')$. Since A is a solution of (2.14) and has no zero on $(-\infty, \operatorname{Log} a')$ we get

$$N(t) = D(t) \quad \text{for all } t \in (-\infty, \operatorname{Log} a').$$

Now, we set

$$w(t) = \dot{A}^2(t)\, K^{p-2}(t)\, e^{2(m-2)t}\, h^2(A(t)),$$

and use the explicit expressions of N and D. Thus we find that

$$\begin{aligned} w(t) =& h^2(A(t_0))\, \dot{A}^2(t_0)\, K^{p-2}(t_0)\, e^{2(m-2)t_0} - 2(m-1) \\ &\times \int_t^{t_0} \dot{A}(r)\, e^{2(m-2)r}\, K^{p-2}(r) \sin A \cos A\, dr \\ &\text{for all } t < t_0 \quad (\text{for some } t_0 < \operatorname{Log} a'). \end{aligned} \tag{2.53}$$

It follows from (2.53) that w is a positive, nonincreasing function. So, $\lim_{t \to -\infty} w(t) = l$ exists $(0 < l \leq +\infty)$. Now, we prove the following assertion: $0 < l \leq +\infty$ is not possible, so $a' = 0$. For otherwise, from $0 < l \leq +\infty$, it follows that there exist $C < 0$ and $C' > 0$ such that $w(t) \geq C' > 0$ for all $t \leq C$. Then, since A is a nonincreasing function and $0 < A(t) \leq \pi/2$ for all $t < \operatorname{Log} a'$, it follows by (2.8) that $\lim_{t \to -\infty} A(t) = \pi/2$ so $\lim_{t \to -\infty} \sin A(t) = 1$. Then there exist $C_0 > 0$ and $T_0 < 0$ such that $\dot{A}$ satisfies the inequality

$$\dot{A}^2(t)\, [1 + \dot{A}^2(t)]^{p-2} \geq C_0\, e^{2(p-m)t} \quad \text{for all } t \leq T_0. \tag{2.54}$$

Now, we prove that there exist two constants $B < 0$ and $\beta > 0$ such that we have the inequality

$$\dot{A}(t) \geq C_0\, e^{-\beta t} \quad \text{for all } t \leq B. \tag{2.55}$$

For otherwise for all $B < 0$ and $\beta > 0$, there exists $t_1 \leq B$ such that

$$\dot{A}(t_1) < C_0\, e^{-\beta t_1}.$$

From this inequality, it follows that

$$\dot{A}^2(t_1)\, [1 + \dot{A}^2(t_1)]^{p-2} < C_0^2\, e^{-2\beta t_1}\, [C_0^2\, e^{-2\beta t_1} + 1]^{p-2}.$$

Next, take $\beta = \frac{m-p}{2(p-1)}$ and $B < 0$ large enough as to have

$$1 < C_0^2\, e^{-2\beta t_1} \qquad \text{and} \qquad C_0^{2(p-1)}\, 2^{p-2}\, e^{-2(p-1)\beta t_1} < C_0\, e^{2(p-m)t_1}.$$

Then we get

$$\dot{A}^2(t_1)\,[1+\dot{A}^2(t_1)]^{p-2} < C_0\, e^{2(p-m)t_1}.$$

This contradicts (2.54). Now, if we integrate (2.55) between B and t, we get $A(t) \leq A(B) - \frac{C_0}{\beta}[e^{-\beta t} - e^{-B\beta}]$ so $\lim_{t\to-\infty} A(t) = -\infty$ which is a contradiction. Then the function α vanishes on $[0, b']$, and is increasing and smooth on $(b', 1]$. An analogous proof to the one given at the end of Step 3 shows that $\alpha \equiv 0$. This contradicts the fact that $\alpha(1) = \pi/2$. Next, the function f has no zero on $(0, 1]$. And since f is a continuous function, then f has a constant sign on $(0, 1]$. If f is negative, then the function α is nonincreasing on $(0, 1]$. This contradicts the fact that $\alpha(1) = \pi/2$ and $F_\alpha \neq u^*$. Finally, the function f is positive so α is increasing on $(0, 1]$. The above study shows that α has no zero on $(0, 1]$. An argument similar to Steps 3, 4, 5 shows that $\dot{\alpha}(r) > 0$ on $[0, 1]$, $\alpha(0) = 0$ and $F_\alpha \in C^\infty(B^m, E_+^m(b))$, so ending Step 8.

Remark 4. – Since the function A is a solution of (2.8) on $(-\infty, 0]$, a similar proof to that of Step 8 shows that $\lim_{t\to-\infty} w(t) = 0$, so that $\lim_{r\to 0} f(r) = 0$.

STEP 9. – (A priori *estimates*)

(A) *If F_α is a weakly p-harmonic map and $F_\alpha \equiv u^*$ on ∂B^m, then $E_p(F_\alpha) \leq E_p(u^*)$.*

(B) *If $F_\alpha \equiv u^*$ on ∂B^m and if $b^2 \geq 4\frac{m-1}{(m-p)^2}$ then, we get the following inequality:*

$$(2.56)\quad E_p(F_\alpha) - E_p(u^*) \geq \frac{1}{p}(m-p)^{\frac{p-2}{2}}\operatorname{vol}(S^{m-1}) \times \left\{b^2 - 4\frac{m-1}{(m-p)^2}\right\}^{\frac{p}{2}} \left\{\int_0^1 \dot{\alpha}^2(\sin\alpha)^2\, r^{m-p+1}\, dr\right\}^{\frac{p}{2}}.$$

In particular, $E_p(F_\alpha) \geq E_p(u^*)$ and equality holds if and only if $F_\alpha \equiv u^*$.

Proof. – (A) Since α is a strong solution of the Euler-Lagrange equation associated to (2.2) (by Step 8), α satisfies the equation

$$\frac{d}{dr}\left[\dot{\alpha}\, h^2(\alpha)\, F_{p-2}(r,\,\alpha,\,\dot{\alpha})\, r^{m-1}\right] = \left[\dot{\alpha}^2 (b^2-1) + \frac{m-1}{r^2}\right] \cos\alpha\, \sin\,\alpha\, F_{p-2}(r,\,\alpha,\,\dot{\alpha})\, r^{m-1}. \tag{2.57}$$

We note that

$$\frac{d}{dr}\left[\dot{\alpha}\, h^2(\alpha)\,\mathrm{cotg}\,\alpha\, F_{p-2}(r,\,\alpha,\,\dot{\alpha})\, r^{m-1}\right] = \mathrm{cotg}\,\alpha\,\frac{d}{dr}\left[\dot{\alpha}\, h^2(\alpha)\, F_{p-2}(r,\,\alpha,\,\dot{\alpha})\, r^{m-1}\right] - \frac{\dot{\alpha}^2\, h^2(\alpha)}{\sin^2\alpha}\, F_{p-2}(r,\,\alpha,\,\dot{\alpha})\, r^{m-1}.$$

Multiplying (2.57) by cotg α, we get the following equation

$$F_p(r,\,\alpha,\,\dot{\alpha})\, r^{m-1} = \left[\dot{\alpha}^2 b^2 + \frac{m-1}{r^2} - \frac{\dot{\alpha}^2\, h^2(\alpha)}{\sin^2\alpha}\right] F_{p-2}(r,\,\alpha,\,\dot{\alpha})\, r^{m-1} - \frac{d}{dr}\left[\dot{\alpha}\, h^2(\alpha)\,\mathrm{cotg}\,\alpha\, F_{p-2}(r,\,\alpha,\,\dot{\alpha})\, r^{m-1}\right]. \tag{2.58}$$

Set

$$g(r) = \dot{\alpha}(r)\, h^2(\alpha(r))\, F_{p-2}(r,\,\alpha(r),\,\dot{\alpha}(r))\, r^{m-1}.$$

By remark 4, we have: $\lim_{r\to 0} f(r) = \lim_{r\to 0} g(r) = 0$. By (2.57), the function g satisfies the following inequality

$$\frac{d}{ds} g(s) \le \left[\dot{\alpha}^2(s)(b^2+1) + \frac{m-1}{s^2}\right] \sin\,\alpha(s)\, F_{p-2}(s,\,\alpha(s),\,\dot{\alpha}(s))\, s^{m-1} \quad \text{for all } s \in [0,\,1]. \tag{2.59}$$

Since the left hand side of (2.59) is in $L^1([0,\,1])$, integrating (2.59) between 0 and r, we find

$$0 \le g(r) \le \sin\,\alpha(r) \int_0^r \left[\dot{\alpha}^2(s)(b^2+1) + \frac{m-1}{s^2}\right] \times F_{p-2}(s,\,\alpha(s),\,\dot{\alpha}(s))\, s^{m-1}\, ds,$$

because α is an increasing function. So $\lim_{r\to 0} \frac{g(r)}{\sin\alpha(r)} = \lim_{r\to 0} g(r)\operatorname{cotg}\alpha(r) = 0$. Next, integrating (2.58) between 0 and 1 we get

(2.60)
$$E_p(F_\alpha) \le \frac{1}{p}\operatorname{vol}(S^{m-1}) \int_0^1 \frac{m-1}{r^2} F_{p-2}(r, \alpha(r), \dot\alpha(r))\, r^{m-1}\, dr.$$

Young's inequality gives

$$\frac{m-1}{r^2} F_{p-2}(r, \alpha(r), \dot\alpha(r))\, r^{m-1} \le \frac{2}{p}\frac{(m-1)^{\frac{p}{2}}}{r^p} r^{m-1} + \frac{p-2}{p} F_p(r, \alpha(r), \dot\alpha(r))\, r^{m-1}.$$

Finally, by using this inequality in (2.60), we get $E_p(F_\alpha) \le E_p(u^*)$.

As for assertion (B), set $J(\alpha) = \int_0^1 F_2(r, \alpha(r), \dot\alpha(r))\, r^{m-p+1}\, dr$. First, we get easily the inequality

$$J(\alpha) \ge b^2 \int_0^1 \dot\alpha^2(r) \sin^2(\alpha(r))\, r^{m-p+1}\, dr + \frac{m-1}{m-p} - (m-1)\int_0^1 \cos^2(\alpha(r))\, r^{m-p-1}\, dr.$$

Next, an integration by parts gives

$$\int_0^1 \cos^2(\alpha(r))\, r^{m-p-1} dr = \frac{2}{m-p}\int_0^1 \dot\alpha(r) \sin\alpha(r) \cos\alpha(r)\, r^{m-p}\, dr.$$

Using the inequality, $ab \le \frac{a^2}{2} + \frac{b^2}{2}$ it follows that

$$\int_0^1 \cos^2(\alpha(r))\, r^{m-p-1}\, dr \le \frac{4}{(m-p)^2}\int_0^1 \dot\alpha^2(r) \sin^2(\alpha(r))\, r^{m-p+1}\, dr.$$

Then the function J satisfies the following inequality

(2.61)
$$J(\alpha) \ge \left\{ b^2 - 4\frac{m-1}{(m-p)^2} \right\} \int_0^1 \dot\alpha^2(r) \sin^2(\alpha(r))\, r^{m-p+1}\, dr + \frac{m-1}{m-p}.$$

Now, by the Hölder inequality we get

$$p\, J_p(\alpha) \ge J(\alpha)^{\frac{p}{2}} (m-p)^{\frac{p-2}{2}}. \tag{2.62}$$

So, suppose that $b^2 \ge 4\frac{m-1}{(m-p)^2}$. Then (B) follows from (2.61) and (2.62). Part (i) of Theorem 1 is an immediate consequence of (A) and (B).

Remark 5. – Step 2 shows that u^* is an unstable critical point if $b^2 < 4\,\frac{m-1}{(m-p)^2}$. On the other hand, it follows from Theorem 1 (i) that if $b^2 \geq 4\,\frac{m-1}{(m-p)^2}$, u^* is *the minimum over the class of equivariant maps* F_α *such that* $F_\alpha \equiv u^*$ *on* ∂B^m. However, if $b^2 \geq 4\,\frac{m-1}{(m-p)^2}$ we proved that u^* *is a strictly stable map.*

Proof of Theorem 2. – The proof of Theorem 2 follows from the ideas of 1-7 above. The only relevant difference concerns Step 5 and moreover, Step 2 is unnecessary. Let us then show how to modify Step 5 in this context.

Suppose $p = m$. Set $r = e^t$ and $A(t) = \alpha_0(e^t)$ $(t \in (-\infty, 0])$. Thus

$$E_p(F_{\alpha_0}) = \frac{1}{p}\operatorname{vol}(S^{m-1}) \int_{-\infty}^{0} k(A(t))^{\frac{p}{2}}\, dt. \tag{2.62}$$

The Hamiltonian

$$H = \frac{1}{p}\operatorname{vol}(S^{m-1})\, k(A)^{\frac{p-2}{2}}\,(p-1)\,\{-h^2(A)\,\dot{A}^2 + \sin^2(A)\}$$

associated to the functional (2.62) is constant on each solution A because the integral (2.62) does not depend explicitly of t. This Hamiltonian is identically equal to zero, so A is different from the constant map equal to $\pi/2$. Then $F_{\alpha_0} \neq u^*$. From Steps 3 and 4, A does not vanish on $(-\infty, 0]$ and $\dot{A}(t) > 0$ on $(-\infty, 0]$, and A is a solution of the following differential equation

$$\frac{h(A)\,\dot{A}}{\sin A} = 1. \tag{2.63}$$

From (2.63), it follows that the map α_0 is a solution of the following differential equation

$$\frac{h(\alpha_0(r))\,\dot{\alpha}_0(r)}{\sin(\alpha_0(r))} = \frac{1}{r} \quad \text{for all } r \in (0, 1]. \tag{2.64}$$

In Step 5, we calculated the primitive of the left hand side of (2.64) and wrote it in the form $e^{I(r)} = \sin\alpha_0(r)\,U(r)$ where U is a positive continuous function on $[0, \varepsilon]$ where $0 < \varepsilon < 1$. We integrate (2.64) and pass to the exponential to obtain

$$\frac{\sin\alpha_0(r)}{r} = \frac{C}{U(r)} \quad \text{where } C \text{ is a positive constant.}$$

Since $\alpha_0(0) = 0$, by letting r go to 0, we find that $\dot{\alpha}_0(0)$ exists and is positive.

Now, suppose $p > m$. First, we note that $F_{\alpha_0} \neq u^*$ because $E_p(u^*) = +\infty$. As for Step 5, the Lyapunov function associated to (2.8) is

$$V_p(A) = k(A)^{\frac{p-2}{2}} \{(p-1) h^2(A) \dot{A}^2 - (m-1) \sin^2(A)\},$$

and is increasing on solutions.

Now, since $V_p(A(t)) > 0$ for all $t \in (-\infty, 0]$, it follows from the theory of Lyapunov functions that $\lim_{t \to -\infty} \dot{A}(t) = 0$ or $+\infty$. We want to prove that this limit is equal to 0. For this purpose, we use a method similar to Step 8 and find that

$$\lim_{t \to -\infty} w(t) = \lim_{t \to -\infty} \dot{A}^2(t) K^{p-2}(t) e^{2(m-2)t} h^2(A(t)) = 0$$

$$\text{so} \lim_{t \to -\infty} \dot{A}^{2(p-1)}(t) e^{2(m-p)t} = 0.$$

Since $p > m$, it follows that $\lim_{t \to -\infty} \dot{A}(t) = 0$. Next, in order to prove Step 5, we set, as in the case $p < m$,

$$\dot{G} = \frac{\dot{A} h(A)}{\sin A},$$

so $\dot{G}$ is a solution of (2.22). But the assertion (2.24) is not necessarily satisfied because $V_p(A(t)) > 0$ so $\dot{G}^2(t) > \frac{m-1}{p-1}$ for all $t \in (-\infty, 0]$. However, Taylor's expansion centered at $A(-\infty) = 0$ shows that there exists a function g such that

$$\left(1 - \frac{\cos A}{h(A)}\right) = \frac{b^2 A^2}{2} + A^2 g(A) \qquad \text{and} \qquad \lim_{t \to -\infty} g(A(t)) = 0.$$

From this equality, it follows that (2.24) is satisfied. Now, we want to prove that

$$\lim_{t \to -\infty} \dot{G}(t) = 1 \qquad \text{and} \qquad \dot{G}(t) > 1 \quad \text{for all } t \in (-\infty, 0].$$

To this end, we note that $\ddot{G}(0) > 0$ because $\ddot{A}(0) > 0$ (by (2.8)). Since $\dot{G}^2(t) > \frac{m-1}{p-1}$, for all $t \in (-\infty, 0]$ we get the inequality

$$-\left(1 - \frac{\cos(A(t))}{h(A(t))}\right)\left(\dot{G}^2(t) - \frac{m-1}{p-1}\right) < 0 \quad \text{for all } t \in (-\infty, 0]. \tag{2.65}$$

Consider the following two possibilities: $\dot{G}(0) \leq 1$ or $\dot{G}(0) > 1$.

If $\dot{G}(0) \leq 1$, we claim that,

$$\ddot{G}(t) > 0 \quad \text{for all } t \in (-\infty, 0]. \tag{2.66}$$

For otherwise, let t' be the first point where $t' < 0$ and $\ddot{G}(t') = 0$. So, $\dot{G}(t') \leq 1$ (because $\ddot{G}(0) > 0$) and a simple inspection of (2.22) and (2.65) shows that this is not possible. Finally, the function $\dot{G}$ is increasing and then $\lim\limits_{t \to -\infty} \dot{G}(t)$ exists and is finited because $\dot{G}$ is bounded below by $\sqrt{\frac{m-1}{p-1}}$. From (2.22), this limit is equal to 1 or $-\frac{m-1}{p-1}$; then $\lim\limits_{t \to -\infty} \dot{G}(t) = 1$. It follows that there exists $t'' \in (-\infty, 0)$ such that $\ddot{G}(t'') = 0$ and then we contradict (2.66). Therefore, we must have $\dot{G}(0) > 1$. In this case, we consider the following two possibilities:

1) suppose that $\ddot{G}(t) > 0$ for all $t \in (-\infty, 0)$. A study similar to the previous one gives $\lim\limits_{t \to -\infty} \dot{G}(t) = 1$ and $\dot{G}(t) > 1$ for all $t \in (-\infty, 0]$;

2) let $t' < 0$ be the first point such that $\ddot{G}(t') = 0$. Now, we study the behavior of $\dot{G}$ in a neighborhood of a point $u < 0$ such that $\ddot{G}(u) = 0$. A short computation gives

$$\begin{aligned} \dddot{G}(u) =& (p-1) \frac{b^2 \sin A(u) \dot{A}(u)}{h^3(A)(u)} \\ & \times \frac{[\dot{G}^2(u) + (m-1)]}{[(p-1)\dot{G}^2(u) + (m-1)]} \left(\dot{G}^2(u) - \frac{m-1}{p-1} \right) > 0. \end{aligned}$$

A simple study of $\dot{G}$ shows that there exists $\varepsilon > 0$ such that

(2.67) $\dot{G}$ *is increasing on* $[u, u+\varepsilon]$ *and* $\dot{G}$ *is nonincreasing on* $[u-\varepsilon, u]$.

We claim that $\dot{G}(t) > 1$ for all $t \in (-\infty, 0]$. For otherwise, let $t'' < 0$ be the first point such that $\dot{G}(t'') = 1$. If $t'' < t'$, then there would exist t_0 such that $t'' < t_0 < t'$ and $\ddot{G}(t_0) = 0$. The function $\dot{G}$ is increasing on $[t'', t_0]$ and nonincreasing on $[t_0, t']$. This contradicts (2.67). If $t'' \geq t'$, then $\dot{G}(t') \leq 1$ and a simple inspection of (2.22) shows that this is not possible because $\ddot{G}(t') = 0$. Moreover, we have $\ddot{G}(t) < 0$ for all $t < t'$ (for otherwise, we would contradict (2.67)) so $\lim\limits_{t \to -\infty} \dot{G}(t)$ exists and is not finited (for otherwise, by (2.22), this limit would be equal to 1, a fact which is not possible because there would exist $T < t'$ such that $\ddot{G}(T) = 0$). Now, we write equation (2.22) in the following form

$$\ddot{G}(t) = \varphi_1(t) \dot{G}^2(t) + \varphi_2(t). \tag{2.68}$$

where

$$\varphi_1(t) = -(p-1)\frac{[\dot{G}^2(t)+(m-1)]}{[(p-1)\dot{G}^2(t)+(m-1)]} \times \left\{\left(1-\frac{1}{\dot{G}(t)}\right)\left(1+\frac{m-1}{(p-1)\dot{G}(t)}\right)\right\}$$

$$\text{and } \lim_{t\to-\infty}\varphi_1(t) = -1,$$

and

$$\varphi_2(t) = (p-1)\frac{[\dot{G}^2(t)+(m-1)]}{[(p-1)\dot{G}^2(t)+(m-1)]} \times \left(1-\frac{\cos(A(t))}{h(A(t))}\right)\left(\dot{G}^2(t)-\frac{m-1}{p-1}\right)$$

$$\text{and } \varphi_2(t) = o(\dot{G}^2).$$

From (2.68), it follows that there exists $T \ll 0$ such that

$$\frac{\ddot{G}(u)}{\dot{G}^2(u)} \le -\frac{1}{2} \quad \text{for all } u \le T. \tag{2.69}$$

Integrating (2.69) between T and t $(t \le T)$, we get

$$\frac{1}{\dot{G}(t)} \le \frac{1}{2}[t-(T-2\dot{G}(T))].$$

Set $C = (T - 2\dot{G}(T)) < T$, so that $\lim_{t\to C^+}\dot{G}(t) = +\infty$. This leads us to a contradiction because the function $\dot{G}$ is continuous on $(-\infty, 0]$. Finally, $\lim_{t\to-\infty}\dot{G}(t) = 1$ and $\dot{G}(t) > 1$ for all $t \in (-\infty, 0]$. Next, we write $\dot{G}(t) = 1+\gamma(t)$ with $\gamma(t) > 0$ and $\lim_{t\to-\infty}\gamma(t) = 0$. Similarly to the proof of Step 5, we show that there exist two constants C_1, $C_2 > 0$ such that

$$1 \le \dot{G}(t) \le 1 + C_1 e^{C_2 t} \quad \text{for all } t \in (-\infty, 0].$$

We conclude as in Step 5 that $\dot{\alpha}_0(0)$ exists and is positive. This achieves the proof of Theorem 2. $\square$

Proof Proposition 1. – If the map F_α is a solution of Dir (ρ, m), then the study of the Lyapunov function $V_p(A)$ shows that

$$0 < A(t) < \pi \quad \text{for all } t \in \mathbb{R} \quad \text{and} \quad \lim_{t\to+\infty} A(t) = \pi/2, \ \lim_{t\to+\infty}\dot{A}(t) = 0.$$

Set $Y = \dot{A}$ and $X = A - \pi/2$. Since A is a solution of (2.8), Taylor's expansion centered at the point $X = 0$, $Y = 0$ shows that (2.8) becomes

$$\dot{X} = Y$$
$$\dot{Y} = -\frac{m-1}{b^2} X + (p-m) Y + o(\sqrt{X^2+Y^2}).$$

From the study of the eigenvalues of the matrix associated to the linear part and from the general theory of perturbed linear systems (see [9]), it follows that: if $b^2 < 4\,\frac{m-1}{(m-p)^2}$, then the eigenvalues are complex with negative real part, and so the point (0, 0) is a focus; by contrast if $b^2 > 4\,\frac{m-1}{(m-p)^2}$ then the eigenvalues are real and negative, and so the point (0, 0) is an improper node (if $b^2 = 4\,\frac{m-1}{(m-p)^2}$, there is a double eigenvalue and the point (0, 0) is a proper node).

Case (i): suppose that Dir (ρ, m) admits a solution F_α for $\rho \geq \pi/2$. Then, $A(t) = \alpha(e^t)$ is a solution of (2.8) with $A(0) = \rho$ and $\lim_{t \to -\infty} A(t) = 0$. Let t' be the first point such that $A(t') = \pi/2$ and set $B(t) = A(t+t')$, $\alpha_1(r) = B(\operatorname{Log} r)$. It follows that F_{α_1} is a solution of Dir $(\pi/2, m)$. This contradicts part (i) of Theorem 1.

Now, suppose $0 < \rho < \pi/2$. A similar study to Theorem 1 shows that Dir (ρ, m) admits a solution F_α (α minimises J_p over $\{\alpha \in X : 0 \leq \alpha(r) \leq \rho;\ \alpha(1) = \rho\}$). Moreover, the function α is increasing on $[0, +\infty)$.

Case (ii): let F_{α_0} be the solution of Dir $(\pi/2, m)$ which satisfies (1.2) and set $A_0(t) = \alpha_0(e^t)$, $t \in \mathbb{R}$. Let F_α be another solution of Dir (ρ, m) and $A(t) = \alpha(e^t)$ (we can suppose that $t = 0$ is the first point such that $A(0) = \pi/2$). Set $M = \sup\{A(t),\ t \in \mathbb{R}\}$ $(\pi/2 < M < \pi)$. Let t'' such that $A(t'') = M$. Since F_{α_0} satisfies (1.2) it follows from (2.20) and the fact that the Lyapunov function is nonincreasing that

$$V_p(A(t'')) < V_p(A(0)) < V_p(A_0(0)) < 0. \tag{2.70}$$

On the other hand, from $\dot{A}(t'') = 0$ and (2.70), we deduce

$$-(m-1)^{\frac{p}{2}} \sin^p M < V_p(A_0(0)) < 0.$$

Then M cannot be close to π, a fact which implies the existence of the required σ. □

Proof of Proposition 2. – Let F_α be the solution of Dir $(\pi/2, m)$ which satisfies (1.2) and set $A(t) = \alpha(e^t)$.

(i) From the equality

$$h^2(A)\,\dot{A}^2 = \sin^2(A),$$

it follows that $\dot{A}(t) > 0$ and $0 < A(t) < \pi$ for all $t \in \mathbb{R}$ (for otherwise, there would exist $t' > 0$ such that $\dot{A}(t') = 0$ and so $A(t') = \pi$. Next, replacing A by $A - \pi$, it is easy to check that our solution satisfy the unique continuation principle, so $A \equiv \pi$). Then, $\lim_{t \to +\infty} A(t) = L \leq \pi$ and since there exists a sequence $t_n \to +\infty$ such that $\dot{A}(t_n) \to 0$ and $A(t_n) \to L$, it follows from the previous equality that $L = \pi$.

(ii) From the inequality $V_p(A(t)) > 0$ for all $t \in \mathbb{R}$, we deduce that $\dot{A}(t) > 0$ for all $t \in \mathbb{R}$ (the proof is similar to the case (i)). Now, suppose that $\lim_{t \to +\infty} A(t) = L < +\infty$: then there exists a sequence $t_n \to +\infty$ such that $\dot{A}(t_n) \to 0$ and $A(t_n) \to L$. This contradicts the fact that $V_p(A)$ is an increasing function and $\lim_{t \to -\infty} V_p(A(t)) = 0$. It follows that $\lim_{t \to +\infty} A(t) = +\infty$. $\square$

Proof of Theorem 3. – The proof of Theorem 3 is based on the Karcher-Wood identity (see [10]), for a 1-form w on B^m with values in $u^* TN$ and which is not necessarily harmonic. Keeping notation as in [10], we let V be a vector field of class C^1 defined on B^m, V^{tang} its tangential component on S^{m-1}, ν the unit normal vector. The relevant identity is

$$\begin{aligned} (2.71) \quad & \int_{S^{m-1}} \langle V, \nu\rangle\, |w|^2 - 2\int_{S^{m-1}} \langle V, \nu\rangle |\, w^{\text{norm}}|^2 \\ & - 2\int_{B^m} \langle w_{V^{\text{tang}}}, w_\nu\rangle + 2\int_{B^m} \langle w^{\nabla V}, w\rangle \\ & = \int_{B^m} |w|^2 \operatorname{div}(V) + 2\int_{B^m} \langle (dw)_V, w\rangle + 2\int_{B^m} \langle w_V, \delta w\rangle, \end{aligned}$$

where div denotes the divergence.

The p-tension of u is the field $\tau_p(u)$ given by

$$(2.72) \qquad \tau_p(u) = |du|^{p-2}\, \tau_2(u) + du\,(\text{grad}\, |du|^{p-2}),$$

where grad is the gradient and $\tau_2(u)$ is the usual tension field of u. The map u is p-harmonic if and only if $\tau_p(u) = 0$. (By the Nash Theorem, the target manifold N can be isometrically embedded in $\mathbb{R}^{n+k}$, where n is the dimension of N; let A be the second fundamental form of N in $\mathbb{R}^{n+k}$, then (0.3) is equivalent to $\tau_p(u) = 0$.)

Now, we take $w = |du|^{\frac{p-2}{2}} du$ and $V = r \frac{\partial}{\partial r}$ (where r, θ are the polar coordinates on B^m). From (2.71) and (2.72), we get the following equality

$$\begin{aligned}(2.73)\quad & \int_{S^{m-1}} |du|^p \, d\theta \\ & = (m-p) \int_{B^m} |du|^p \, dx + p \int_{S^{m-1}} |du|^{p-2} \left| du \left(\frac{\partial}{\partial r} \right) \right|^2 d\theta.\end{aligned}$$

Next, since $p < m$, the proof of Theorem 3 is an immediate consequence of (2.73) and the following equality: $|du|^p_{S^{m-1}} = [|du|_{S^{m-1}}|^2 + |du(\frac{\partial}{\partial r})|^2_{r=1}]^{\frac{p}{2}}$. □

3. p-HARMONIC MAPS BETWEEN SPHERES AND ELLIPSOIDS

In this section, we establish sufficient conditions for the existence of a p-harmonic map between spheres or ellipsoids. For a, $b > 0$ and m, $q \geq 1$, we introduce the ellipsoids

$$Q^{m+q+1}(a, b) = \left\{ (x, y) \in \mathbb{R}^{m+1} \times \mathbb{R}^{q+1} : \frac{|x|^2}{a^2} + \frac{|y|^2}{b^2} = 1 \right\}.$$

We parametrize the points of $Q^{m+q+1}(a, b)$ by

$$(3.1) \qquad (a \sin s \cdot x, \; b \cos s \cdot y)$$

where

$$x \in S^m, \qquad y \in S^q \qquad \text{and} \qquad 0 \leq s \leq \pi/2.$$

The Riemannian metric on $Q^{m+q+1}(a, b)$ induced from its embedding in $\mathbb{R}^{m+q+2}$ is

$$(3.2) \qquad g = a^2 \sin^2 s \, g_m + b^2 \cos^2 s \, g_q + h^2(s) \, ds^2,$$

where

$h^2(s) = a^2 \cos^2 s + b^2 \sin^2 s$ and g_n denotes the standard metric of S^n.

A map $u : S^m \to S^r$ is *an eigenmap with eigenvalue* λ_u, if u is harmonic and $|du|^2 = \lambda_u$. It is well known that the components of u

(considered as a map in $\mathbb{R}^{r+1}$) are the restrictions to S^m of $(r+1)$-harmonic homogeous polynomial of a common degree k; so its associated eigenvalue is $\lambda_u = k(k+m-1)$. Examples are the Hopf map $h : S^3 \to S^2$ with eigenvalue $\lambda = 8$, the identity map $\mathrm{Id}_{S^q} : S^q \to S^q$ and the map $u_k : S^1 \to S^1 (e^{i\theta} \to e^{ik\theta},\ k \in \mathbb{Z})$ (see [6] for a more complete list of examples). Let $\alpha : [0, \pi/2] \to [0, \pi/2]$ be a smooth function satisfying the boundary conditions

$$\alpha(0) = 0, \qquad \alpha(\pi/2) = \pi/2. \tag{3.3}$$

For given $a, b, c, d > 0$, the equivariant α-join of two eigenmaps $u : S^m \to S^r$ and $v : S^q \to S^s$ is the map

$$F_\alpha = u * v : Q^{m+q+1}(a, b) \to Q^{r+s+1}(c, d)$$

given by

$$(a \sin s{\cdot}x,\ b \cos s{\cdot}y) \to (c \sin \alpha(s){\cdot}u(x),\ d \cos \alpha(s){\cdot}v(y)). \tag{3.4}$$

Our main result in this context is the following Theorem

THEOREM 4. – *Suppose $p \geq 2$. Let $u : S^m \to S^r$ and $v : S^q \to S^s$ be two eigenmaps with eigenvalues λ_u and λ_v respectively. If one of the following hypotheses $(H_i)\,(i = 1, 2)$ hold*

$$c(q-p+1) < 2d\sqrt{\lambda_v} \quad \text{and} \quad d(m-p+1) < 2c\sqrt{\lambda_u}; \tag{H$_1$}$$

or

$$\begin{aligned} &m \leq q,\ p < m+1,\ a \geq b,\ c(q-p+1) < 2d\sqrt{\lambda_v} \\ &\text{and} \quad \lambda_u \frac{c^2}{d^2}(q-p+1) \geq \lambda_v \frac{a^2}{b^2}(m-1) \end{aligned} \tag{H$_2$}$$

then there exists an equivariant p-harmonic α-join $F_\alpha = u * v : Q^{m+q+1}(a, b) \to Q^{r+s+1}(c, d)$.

Remark 6. – The hypotheses (H_1) and (H_2) are independent. Indeed, if $a = b = c = d = 1$, $p = 2$, $q = 3$, $m = 2$, $\lambda_u = 2$ and $\lambda_v = 8$ (for example, take v equal to the Hopf map $h : S^3 \to S^2$ and $u = \mathrm{Id}_{S^2}$) then (H_1) is satisfied but (H_2) does not hold. By contrast, if $m = 7$, $\lambda_u = 55$, $q = 7$, $\lambda_v = 7$, $a = b = 1$, $\frac{c}{d} = \frac{q-p+1}{2\sqrt{55}}$ and $p = 2$ (for instance, this situation occurs when $u : S^7 \to S^7$ is the gradient of Cartan's harmonic eiconal of polynomial degree $k = 6$ (see [6]) and $v = \mathrm{Id}_{S^7}$) then (H$_2$) is satisfied but (H$_1$) does not hold.

By using the different eigenmaps, we can deduce from Theorem 4 the existence of new p-harmonic maps. For instance, if we take the join of u_k and Id_{S^q} and $a = b = c = d = 1$ in Theorem 4, we get the following result which generalize the results of Smith [16] ($p = 2$) and Xu and Yang [17] ($p = q + 2$).

COROLLARY. – *Suppose $q \geq 1$. If $p \geq 2$ and $p > (\sqrt{q} - 1)^2$, then there exists a p-harmonic α-join $u_k * \mathrm{Id}_{S^q} : S^{q+2} \to S^{q+2}$.*

(*This p-harmonic map represents the element $k \in \pi_{q+2}(S^{q+2}) = \mathbb{Z}$.*)

Proof of Theorem 4. – The p-energy of the α-join F_α is equal to

$$E_p(F_\alpha) = a^m\, b^q \operatorname{vol}(S^m) \operatorname{vol}(S^q)\, J_p(\alpha),$$

where

$$J_p(\alpha) = \frac{1}{p} \int_0^{\pi/2} \left[\dot{\alpha}^2 \frac{k^2(\alpha)}{h^2(s)} + \frac{\lambda_u\, c^2 \sin^2 \alpha}{a^2 \sin^2 s} + \frac{\lambda_v\, d^2 \cos^2 \alpha}{b^2 \cos^2 s}\right]^{p/2} h(s)\, \nu\, ds, \tag{3.5}$$

with

$$\nu = \sin^m s \cos^q s \qquad \text{and} \qquad k^2(\alpha) = c^2 \cos^2 \alpha + d^2 \sin^2 \alpha.$$

Similarly to Theorem 1, it is convenient to introduce the following function spaces

$$Y = \left\{\alpha \in H^{1,p}([0, \pi/2]; \mathbb{R}) : \|\alpha\|^p = \int_0^{\pi/2} [|\dot{\alpha}|^p + |\alpha|^p]\, h(s)\, \nu\, ds < \infty\right\}$$

$$Y_0 = \{\alpha \in Y : 0 \leq \alpha(s) \leq \pi/2\} \qquad \overset{\circ}{B} = \overset{\circ}{H}{}^{1,p}([0, \pi/2]; \mathbb{R}).$$

We denote by $\alpha_{\pi/2}$ (respectively α_0) the constant critical point $\alpha \equiv \pi/2$ (respectively $\alpha \equiv 0$). Theorem 3 is obtained essentially by the minimisation of the functional $J_p(\alpha)$ on Y_0. We prove that if one of the hypotheses (H_i) ($i = 1, 2$) is satisfied, the the minimum $\overline{\alpha}$ is different from $\alpha_{\pi/2}$ and α_0. (Indeed, if (H_1) is satisfied then $\alpha_{\pi/2}$ and α_0 are unstable critical points. If (H_2) is satisfied, then $J_p(\alpha_{\pi/2}) \geq J_p(\alpha_0)$ and α_0 is an unstable critical point.) Next, we prove that $\overline{\alpha}$ is smooth on $(0, \pi/2)$ and $\lim_{s \to 0} \overline{\alpha}(s) = 0$, $\lim_{s \to \pi/2} \overline{\alpha}(s) = \pi/2$. Finally, we prove the regularity of $F_{\overline{\alpha}}$: for this purpose, we need to study the existence of the first order derivative of $\overline{\alpha}$ at the points 0 and $\pi/2$. The proof of Theorem 4 is divided into 7 steps.

STEP 1. – *There exists a map $\overline{\alpha} \in Y_0$ which minimises J_p on Y_0 and satisfies*

$$\begin{aligned}(3.6)\quad \int_0^{\pi/2} F_{p-2}(s, \overline{\alpha}, \dot{\overline{\alpha}}) \Bigg\{ \frac{k^2(\overline{\alpha})}{h^2(s)} \dot{\overline{\alpha}} \dot{\zeta} + \bigg(\frac{k(\overline{\alpha})\, \dot{k}(\overline{\alpha})}{h^2(s)} \dot{\overline{\alpha}}^2 \\ + \cos\overline{\alpha} \sin\overline{\alpha} \left(\frac{\lambda_u c^2}{a^2 \sin^2 s} - \frac{\lambda_v d^2}{b^2 \cos^2 s} \right) \bigg) \zeta \Bigg\} h(s)\, \nu\, ds = 0 \\ \text{for all } \zeta \in \overset{\circ}{B},\end{aligned}$$

where

$$F_p(s, \overline{\alpha}, \dot{\overline{\alpha}}) = \left[\dot{\overline{\alpha}}^2 \frac{k^2(\overline{\alpha})}{h^2(s)} + \lambda_u \frac{c^2 \sin^2 \overline{\alpha}}{a^2 \sin^2 s} + \lambda_v \frac{d^2 \cos^2 \overline{\alpha}}{b^2 \cos^2 s} \right]^{\frac{p}{2}}.$$

Proof. – The proof of this step is similar to the proof of Step 1 of Theorem 1 and so we omit it.

STEP 2. – *Suppose $p \geq 2$. If one of the hypotheses (H_i) ($i = 1, 2$) is satisfied then $\overline{\alpha} \neq \alpha_{\pi/2}$ and $\overline{\alpha} \neq \alpha_0$.*

Proof. – First, we note that if $p \geq q+1$ (respectively $p \geq m+1$) then $J_p(\alpha_0) = +\infty$ (respectively $J_p(\alpha_{\pi/2}) = +\infty$). Therefore, we may restrict attention to the case that $p < q+1$ and $p < m+1$.

Now, suppose that (H_1) is satisfied. If we prove that α_0 and $\alpha_{\pi/2}$ are unstable critical points then $\overline{\alpha} \neq \alpha_{\pi/2}$ and $\overline{\alpha} \neq \alpha_0$. For this purpose, let $\zeta(s) = \sin^n s \cos^{-r} s$ where $n > 0$ is to be taken sufficiently large and $r \in (0, \frac{q-p+1}{2})$ is to be determined. Set

$$(3.7)\quad \zeta_M(s) = \zeta(s) \quad \text{if } \zeta(s) < M \qquad \text{and} \qquad \zeta_M(s) = M \text{ otherwise.}$$

($\zeta_M \notin \overset{\circ}{B}$, so (3.6) is not necessarily satisfied for variations of the form (3.7).) We study

$$J_p(\alpha_0 + t\,\zeta_M) = J_p(t\,\zeta_M) = \frac{1}{p} \int_0^{\pi/2} F_p(s, t\,\zeta_M(s), t\,\dot{\zeta}_M(s))\, h(s)\, \nu\, ds,$$

as a function of t. A short computation shows that

$$\left| h(s)\,\nu \frac{d}{dt} F_p(s,\, t\,\zeta_M(s),\, t\,\dot{\zeta}_M(s)) \right|$$

and
$$\left| h(s)\,\nu \frac{d^2}{dt^2} F_p(s,\, t\,\zeta_M(s),\, t\,\dot{\zeta}_M(s)) \right|$$

are dominated independently of t by functions in $L^1([0,\, \pi/2])$. It follows by the Lebesgues dominated convergence Theorem that

$$\frac{d}{dt} J_p(t\,\zeta_M)|_{t=0} = \int_0^{\pi/2} \frac{d}{dt} F_p(s,\, t\,\zeta_M(s),\, t\,\dot{\zeta}_M(s))|_{t=0}\, h(s)\,\nu\, ds = 0$$

and

$$\begin{aligned} Q(\zeta_M) &= \frac{d^2}{dt^2} J_p(t\,\zeta_M)|_{t=0} \\ &= \int_0^{\pi/2} \frac{d^2}{dt^2} F_p(s,\, t\,\zeta_M(s),\, t\,\dot{\zeta}_M(s))|_{t=0}\, h(s)\,\nu\, ds \\ &= \left(\lambda_v \frac{d}{b}\right)^{\frac{p-2}{2}} \int_0^{\pi/2} \left\{ \dot{\zeta}_M^2(s) \frac{c^2}{h^2(s)} \right. \\ &\quad \left. + \zeta_M^2(s) \left(\frac{\lambda_u\, c^2}{a^2 \sin^2 s} - \frac{\lambda_v\, d^2}{b^2 \cos^2 s} \right) \right\} \frac{1}{\cos^{\frac{p-2}{2}} s}\, h(s)\,\nu\, ds. \end{aligned}$$

Now, a short computation shows that if $c(q-p+1) < 2\,d\,\sqrt{\lambda_v}$, and if r tends to $\frac{q-p+1}{2}$ then $Q(\zeta)$ tends to $-\infty$. By the monotone convergence Theorem, we get $\lim\limits_{M \to +\infty} Q(\zeta_M) = Q(\zeta)$ and so we take M large enough to insure that $Q(\zeta_M) < 0$. Finally, for t positive and close to 0, $t\,\zeta_M \in Y_0$; and we obtain $J_p(\alpha_0) > J_p(t\,\zeta_M)$. So, α_0 is an unstable critical point. Similarly, we prove that if $d(m-p+1) < 2\,c\,\sqrt{\lambda_u}$, then $\alpha_{\pi/2}$ is an unstable critical point. Now, suppose that (H_2) is satisfied. As it has been shown in the previous case, α_0 is an unstable critical point. Now, it suffices to prove that $J_p(\alpha_{\pi/2}) \geq J_p(\alpha_0)$ in order to conclude that $\overline{\alpha}$ is different from $\alpha_{\pi/2}$ and α_0. Towards this end, set $f(s) = \sin^{-2} s \cos^{2-p} s$. Thanks to Hölder's inequality, we get

$$\begin{aligned} (3.8) \qquad \int_0^{\pi/2} f(s)\,h(s)\,\nu\, ds \leq & \left(\int_0^{\pi/2} \sin^{-p} s\, h(s)\,\nu\, ds \right)^{2/p} \\ & \times \left(\int_0^{\pi/2} \cos^{-p} s\, h(s)\,\nu\, ds \right)^{\frac{p-2}{p}}. \end{aligned}$$

Next, an integration by parts gives

$$\int_0^{\pi/2} f(s)\, h(s)\, \nu\, ds = \int_0^{\pi/2} \sin^{m-2} s \cos^{q-p+2} s\, h(s)\, ds$$
$$= \frac{q-p+1}{m-1} \int_0^{\pi/2} \cos^{-p} s\, h(s)\, \nu\, ds$$
$$+ \frac{a^2-b^2}{m-1} \int_0^{\pi/2} \frac{\sin^m s \cos^{q-p+2} s}{h(s)}\, ds.$$

From this equality and from (3.8), it follows, since $a \geq b$, that

$$\int_0^{\pi/2} \sin^{-p} s\, h(s)\, \nu\, ds \geq \left(\frac{q-p+1}{m-1}\right)^{p/2} \int_0^{\pi/2} \cos^{-p} s\, h(s)\, \nu\, ds.$$

Finally, if $\lambda_u \frac{c^2}{d^2}(q-p+1) \geq \lambda_v \frac{a^2}{b^2}(m-1)$ we get the inequality

$$(3.9)\quad J_p(\alpha_{\pi/2}) - J_p(\alpha_0) \geq \frac{1}{p}\left[\left(\frac{q-p+1}{m-1}\lambda_u \frac{c^2}{a^2}\right)^{p/2} - \left(\lambda_v \frac{d^2}{b^2}\right)^{p/2}\right]$$
$$\times \int_0^{\pi/2} \cos^{-p} s h(s)\, \nu\, ds \geq 0.$$

This completes the proof of Step 2.

STEP 3. – $\overline{\alpha} \in C^\infty((0, \pi/2))$. *Moreover* $\dot{\overline{\alpha}}(s) > 0$ *for all* $s \in (0, \pi/2)$ *and* $\overline{\alpha}$ *satisfies the boundary conditions* (3.3).

Proof. – A straightforward modification of the arguments used in the proof of Theorem 1 permit us to show that $\overline{\alpha} \in C^1((0, \pi/2))$. Now, since $F_2(s, \overline{\alpha}(s), \dot{\overline{\alpha}}(s)) > 0$ for all $s \in (0, \pi/2)$ and $\overline{\alpha}$ satisfies strongly the Euler-Lagrange equation associated to (3.5) on $(0, \pi/2)$, that is

$$(3.10)\quad \ddot{\overline{\alpha}}\, k^2(\overline{\alpha}) + \left(-\frac{\lambda_u c^2}{a^2 \sin^2 s} + \frac{\lambda_v d^2}{b^2 \cos^2 s}\right) \sin \overline{\alpha} \cos \overline{\alpha}\, h^2(s)$$
$$+k^2(\overline{\alpha})\, \dot{\overline{\alpha}} \left(m \operatorname{cotg} s - q \operatorname{tg} s - \frac{\dot{h}(s)}{h(s)}\right)$$
$$-\dot{\overline{\alpha}}^2 \sin \overline{\alpha} \cos \overline{\alpha}\,(c^2-d^2) + \frac{p-2}{2} k^2(\overline{\alpha})\, \dot{\overline{\alpha}}\, \frac{\frac{d}{ds} F_2(s, \overline{\alpha}, \dot{\overline{\alpha}})}{F_2(s, \overline{\alpha}, \dot{\overline{\alpha}})} = 0.$$

It follows by simple inspection of (3.10) that $\overline{\alpha} \in C^\infty((0, \pi/2))$.

Now, we will suppose for simplicity that $a = b = 1$ (the general case is similar). To prove that $\dot{\overline{\alpha}}(s) > 0$ for all $s \in (0, \pi/2)$, we proceed similarly to [17]. Set $s = \operatorname{Arctg}(e^t)$ and $A(t) = \overline{\alpha}(\operatorname{Arctg}(e^t))$. Then (3.10) becomes equivalent to the following differential equation

$$\begin{aligned}(3.11)\quad \ddot{A} + \dot{A}\Bigg\{\left(\frac{m-q}{2}\right) &+ \frac{2p-m-q-2}{2}\,\frac{e^t - e^{-t}}{e^t + e^{-t}} \\ &+ \frac{p-2}{2}\,\frac{\dot{K}}{K} + \frac{(d^2-c^2)}{k^2(A)}\,\dot{A}\,\sin A\,\cos A\Bigg\} \\ &= \frac{\sin A\,\cos A}{k^2(A)}\left\{\frac{\lambda_u\,c^2\,e^{-t}}{e^t+e^{-t}} - \frac{\lambda_v\,d^2\,e^t}{e^t+e^{-t}}\right\}, \qquad t \in \mathbb{R}\end{aligned}$$

where

$$\begin{aligned}K(t) =& k^2(A(t))\,\dot{A}^2(t) + \sin^2 A(t)\,\frac{\lambda_u\,c^2\,e^{-t}}{e^t+e^{-t}} \\ &+ \cos^2 A(t)\,\frac{\lambda_v\,d^2\,e^t}{e^t+e^{-t}}.\end{aligned}$$

Now, an analogous proof to that of [17] shows that $\dot{A}(t) > 0$ on $\mathbb{R}$. Finally, $\lim_{t \to -\infty} A(t)$ and $\lim_{t \to +\infty} A(t)$ exist and are finited. Since there exist two sequences $t_n \to -\infty$ and $t_n \to +\infty$ such that $\ddot{A}(t_n) \to 0$ and $\dot{A}(t_n) \to 0$, by passing to the limit in (3.11), we get that $\lim_{t \to -\infty} A(t) = 0$ and $\lim_{t \to +\infty} A(t) = \pi/2$. This completes the proof of Step 3.

STEP 4. – $\lim_{t \to -\infty} \dot{A}(t) = 0$.

Proof. – We rewrite the equation (3.11) as

$$(3.12)\qquad \ddot{A} = H(t, A, \dot{A}).$$

A simple inspection of the explicit expression of $H(t, A, \dot{A})$ shows that there exist three constants $C_1, C_2, C_3 > 0$ such that

$$|\ddot{A}| \leq C_1 + C_2\,|\dot{A}| + C_3\,\dot{A}^2.$$

Set

$$\varphi(s) = C_1 - C_2\,s + C_3\,s^2,$$

and note that

$$\varphi(s) \geq 0 \quad \text{on}(-\infty, 0), \qquad \int_{-\infty}^{0} \frac{s}{\varphi(s)}\,ds = -\infty \quad \text{and} \quad |\ddot{A}| \leq \varphi(-\dot{A}).$$

Since $\lim_{t \to -\infty} A(t) = 0$, we see that for any $\varepsilon > 0$ there exists $T_\varepsilon \ll 0$ such that $0 \leq A(t) \leq \varepsilon$ for all $t \leq T_\varepsilon$. Now, suppose that the hypotheses "$\lim_{t \to -\infty} \dot{A}(t) = 0$" is not satisfied: then there exist a nonincreasing sequence t_n and $l > 0$ and a natural integer N such that $\dot{A}(t_n) > l$ for all $n \geq N$. Set

$$\psi(t) = \int_t^{\frac{2\varepsilon}{T_\varepsilon}} \frac{s}{\varphi(s)}\, ds \qquad (t < 0).$$

Then it is easy to check that there exists $M < \frac{2\varepsilon}{T_\varepsilon}$ such that $\psi(M) = -2\varepsilon$. Moreover, we take ε close to 0 such that $|M| < l$. Next, fix n such that $t_{2n} < t_n < T_\varepsilon$. If we prove that $\dot{A}(t) \leq |M|$ on $[t_{2n}, T_\varepsilon]$, then we would contradict the fact that $\dot{A}(t_n) > l$. Since the function $\dot{A}$ is continuous on $[t_{2n}, T_\varepsilon]$, then its maximum is achieved at a point T_1 of this compact set. If $\dot{A}(T_1) \leq \frac{2\varepsilon}{|T_\varepsilon|} < |M|$, this leads us to a contradiction. For otherwise, we claim that there exists $t_1 < T_\varepsilon$ such that

$$\dot{A}(t_1) \leq \frac{2\varepsilon}{|t_{2n}|} < \frac{2\varepsilon}{|T_\varepsilon|}. \tag{3.13}$$

But we have

$$\dot{A}(t) > \frac{2\varepsilon}{|t_{2n}|} \qquad \text{for all} \quad t \leq T_\varepsilon.$$

We integrate this inequality between t and T_ε and let t tend to $-\infty$ we get: $\lim_{t \to -\infty} A(t) = -\infty$. This is not possible. Now, it follows from (3.13) that there exists $t' < T_\varepsilon$, such that $\dot{A}(t') = \frac{2\varepsilon}{|T_\varepsilon|}$ (we will suppose that t' is the first point having this property such that $t' < T_1$). From the definition of t', we have the following inequality

$$\dot{A}(t') \leq \dot{A}(t) \leq \dot{A}(T_1) \quad \text{for all } t \in (t', T_1). \tag{3.14}$$

Thanks to the inequality $\frac{\dot{A}\,|\ddot{A}|}{\varphi(-\dot{A})} \leq \dot{A}$ and to (3.14), we get

$$\Big| \int_{t'}^{T_1} \frac{\dot{A}(t)\,\ddot{A}(t)}{\varphi(-\dot{A}(t))}\, dt \Big| < 2\,\varepsilon. \tag{3.15}$$

Finally, using that $\big| \int_{t'}^{T_1} \frac{\dot{A}(t)\,\ddot{A}(t)}{\varphi(-\dot{A}(t))}\, dt \big| = -\int_{-\dot{A}(T_1)}^{-\dot{A}(t')} \frac{s}{\varphi(s)}\, ds$ and (3.15) and since ψ is an increasing function, it follows that $\dot{A}(T_1) < |M|$. This completes the proof of Step 4.

STEP 5. – *If $\lambda_u = m$ (respectively $\lambda_v = q$) then $\dot{\alpha}(0)$ (respectively $\dot{\alpha}(\pi/2)$) exists and is positive.*

If $\lambda_u > m$ (respectively $\lambda_v > q$) then $\dot{\overline{\alpha}}(0)$ (respectively $\dot{\overline{\alpha}}(\pi/2)$) exists and is equal to 0.

Proof. – We shall prove the results stated at the point 0. By symmetry, we get the same results at the point $\pi/2$. A simple inspection of the explicit expression of $H(t, A, \dot{A})$ shows that when t tends to $-\infty$, $H(t, A, \dot{A})$ tends uniformely to $\overline{H}(A, \dot{A})$, where

$$\begin{aligned}\overline{H}(A, \dot{A}) = & \frac{k^2(A)\dot{A}^2 + \sin^2 A\,\lambda_u c^2}{(p-1)k^2(A)\dot{A}^2 + \sin^2 A\,\lambda_u c^2} \\ & \times \left\{\frac{\sin A \cos A\,\lambda_u c^2}{k^2(A)} + (p-m-1)\dot{A}\right\} \\ & - (d^2 - c^2)\frac{\dot{A}^2 \sin A \cos A}{k^2(A)} \\ & - (p-2)\lambda_u c^2 \frac{\dot{A}^2 \sin A \cos A}{(p-1)k^2(A)\dot{A}^2 + \sin^2 A\,\lambda_u c^2}.\end{aligned}$$

Then, the solution A satisfies

$$\ddot{A} = \overline{H}(A, \dot{A}) + \phi_1(t), \quad \text{where} \quad \lim_{t\to-\infty} \phi_1(t) = 0. \tag{3.16}$$

Similarly to Step 5 of Theorem 1, set

$$\dot{G} = \frac{\dot{A}\,k(A)}{\sin A}.$$

Then equation (3.16) becomes equivalent to

$$\begin{aligned}\ddot{G} = & -\frac{[\dot{G}^2 + \lambda_u c^2]}{[(p-1)\dot{G}^2 + \lambda_u c^2]}\left\{(p-1)\frac{\dot{G}^2}{c} - (p-m-1)\dot{G} - \lambda_u c\right\} \\ & - \left(\frac{1}{c} - \frac{\cos A}{k(A)}\right)((p-1)\dot{G}^2 - \lambda_u c^2) + \phi_1(t)\end{aligned} \tag{3.17}$$

Taylor's expansion centered at $A(-\infty) = 0$ shows that there exists a function g such that

$$\left(\frac{1}{c} - \frac{\cos A}{k(A)}\right) = \frac{d^2 A^2}{2c^3} + A^2 g(A) \quad \text{and} \quad \lim_{t\to-\infty} g(A(t)) = 0. \tag{3.18}$$

From (3.18) and $\lim_{t\to-\infty} \dot{A}(t) = 0$, we get

$$\lim_{t\to-\infty} \left(\frac{1}{c} - \frac{\cos(A(t))}{k(A(t))}\right)\left((p-1)\dot{G}^2(t) - \lambda_u c^2\right) = 0. \tag{3.19}$$

From (3.19), it follows that $\dot{G}$ is a solution of the following differential equation:

(3.20)

$$\ddot{G} = -\frac{[\dot{G}^2 + \lambda_u c^2]}{[(p-1)\dot{G}^2 + \lambda_u c^2]}\left\{(p-1)\frac{\dot{G}^2}{c} - (p-m-1)\dot{G} - \lambda_u c\right\} + \phi_2(t)$$
$$\text{where} \quad \lim_{t\to-\infty} \phi_2(t) = 0.$$

First, we study the asymptotic behavior of $\dot{G}$. For this purpose, we compare solutions of (3.20) with solutions of the following differential equation

$$\ddot{G} = -\frac{[\dot{G}^2 + \lambda_u c^2]}{[(p-1)\dot{G}^2 + \lambda_u c^2]}\left\{(p-1)\frac{\dot{G}^2}{c} - (p-m-1)\dot{G} - \lambda_u c\right\}. \tag{3.21}$$

Thus, we need to investigate the qualitative behavior of solutions to (3.21). This equation has two constant solutions $\dot{G}_1 \equiv -\lambda_1 < 0$ and $\dot{G}_2 \equiv \lambda_2 > 0$ (if $\lambda_u = m$ then $\lambda_2 = c$, and if $\lambda_u > m$ then $\lambda_2 > c$). Let $\dot{G}_r$ be a nonconstant solution of (3.21). Then a study similar to one which we performed for the differential equation (2.25) shows the following statement: if there exists t_1 such that $\dot{G}_r(t_1) > \lambda_2$ (respectively $\dot{G}_r(t_1) < \lambda_2$), then $\dot{G}_r(t) > \lambda_2$ for all $t \in \mathbb{R}$, moreover $\dot{G}_r$ is a nonincreasing function and $\lim_{t\to-\infty} \dot{G}_r(t) = +\infty$ $\lim_{t\to+\infty} \dot{G}_r(t) = \lambda_2$ (respectively $-\lambda_1 < \dot{G}_r(t) < \lambda_2$, $\dot{G}_r$ is an increasing function and $\lim_{t\to-\infty} \dot{G}_r(t) = -\lambda_1$, $\lim_{t\to+\infty} \dot{G}_r(t) = \lambda_2$). Now we use these qualitative facts about to (3.21) in order to show that $\lim_{t\to-\infty} \dot{G}(t) = \lambda_2$. We argue by contradiction: then there exist $\varepsilon > 0$ and a sequence $t_n \to -\infty$ such that either

$$\dot{G}(t_n) \geq \lambda_2 + \varepsilon, \tag{3.22}$$

or

$$\dot{G}(t_n) \leq \lambda_2 - \varepsilon. \tag{3.23}$$

First, suppose that (3.22) is not satisfied. We claim that $\dot{G}$ is unbounded on $(-\infty, 0]$. For otherwise, there would exist $M > 0$ such that $\dot{G}(t) \leq M$ for all $t \in (-\infty, 0]$. Let $B < 0$. If $\dot{G}_s$ is a solution of (3.21) with $\dot{G}_s(B) \geq \lambda_2 + \varepsilon > \lambda_2$, then it follows, from the previous study of the solutions of (3.21), that

$$\dot{G}_s(t) > \lambda_2 \quad \text{for all } B_0 < B \quad \text{and } t \in [B_0, B].$$

$\dot{G}_s$ is a nonincreasing function, so

$$\dot{G}_s(t) \geq \dot{G}_s(B) \geq \lambda_2 + \varepsilon \quad \text{for all } t \in [B_0, B];$$

Now, a simple inspection of (3.21) shows that there exists $C_\varepsilon > 0$ such that

$$\ddot{G}_s(t) \leq -C_\varepsilon \quad \text{for all } t \in [B_0, B].$$

We integrate this inequality between B and B_0, and we get

$$\dot{G}_s(B_0) - \dot{G}_s(B) \geq C_\varepsilon (B - B_0) \geq M + 1$$

(in particular, we fix $B_0 = B - \frac{M+1}{C_\varepsilon}$).

Next, we take $B = t_n$ and $\dot{G}_s(B) = \dot{G}(B) \geq \lambda_2 + \varepsilon$ and we take n sufficiently large such that $|\dot{G}_s(t) - \dot{G}(t)| < 1/2$ (this is possible because $\lim\limits_{t \to -\infty} \phi_2(t) = 0$). Then we get $\dot{G}(B_0) > M + 1/2$. This leads us to a contradiction.

Next, we claim that $\lim\limits_{t \to -\infty} \dot{G}(t) = +\infty$. For otherwise, there would exist a sequence $t_n \to -\infty$ where $\ddot{G}(t_n) = 0$ and $\lim\limits_{n \to +\infty} \dot{G}(t_n) = +\infty$. If we take $t = t_n$ in (3.20), a simple inspection shows that it is not possible. Now, we write (3.20) as

(3.24)
$$\ddot{G} = \dot{G}^2 \left(- \frac{[\dot{G}^2 + \lambda_u c^2]}{[(p-1)\dot{G}^2 + \lambda_u c^2]} \left\{ \frac{(p-1)}{c} - \frac{(p-m-1)}{\dot{G}} - \frac{\lambda_u c}{\dot{G}^2} \right\} + \frac{\phi_2(t)}{\dot{G}^2} \right).$$

From $\lim\limits_{t \to -\infty} \dot{G}(t) = +\infty$ and (3.24), it follows that for any $\varepsilon > 0$ there exists $T_\varepsilon < 0$ such that the following inequality is satisfied

$$\frac{\ddot{G}(t)}{\dot{G}^2(t)} \leq -\left(\frac{1}{c} - \varepsilon\right) \quad \text{for all } t \leq T_\varepsilon. \tag{3.25}$$

For a fixed $\varepsilon \, (\varepsilon < \frac{1}{c})$, integrating (3.25) between T_ε and t, we obtain the inequality

$$\frac{1}{\dot{G}(t)} \leq \left(\frac{1}{c} - \varepsilon\right)(t - C) \quad \text{where } C = \left(T_\varepsilon - \frac{1}{((\frac{1}{c} - \varepsilon))\,\dot{G}(T_\varepsilon)}\right) < T_\varepsilon.$$

Then $\lim_{t \to C^+} \dot{G}(t) = +\infty$. This leads us to a contradiction because $\dot{G}$ is a continuous function on $\mathbb{R}$. Now, suppose that (3.23) is satisfied. We fix $B < 0$ and $B_0 = B - \frac{\lambda_2}{C_\varepsilon} < 0\,(C_\varepsilon = \lambda_1\,\varepsilon)$ and a solution $\dot{G}_s$ of (3.21) which satisfies $\dot{G}_s(B) \le \lambda_2 - \varepsilon < \lambda_2$. From the study of the solutions of (3.21), it follows that

$$\dot{G}_s(t) < \lambda_2 \quad \text{for all } t \in [B_0, B].$$

$\dot{G}_s$ is an increasing function, so

$$\dot{G}_s(t) \le \dot{G}_s(B) \le \lambda_2 - \varepsilon \quad \text{for all } t \in [B_0, B].$$

Then, a simple inspection of (3.21) shows that

$$\ddot{G}_s(t) \ge C_\varepsilon \quad \text{for all } t \in [B_0, B].$$

We integrate this inequality between B and B_0, so we get $\dot{G}_s(B_0) \le \dot{G}_s(B) + C_\varepsilon(B_0 - B) \le -\varepsilon$. Finally, we take $B = t_n$ and $\dot{G}_s(B) = \dot{G}(B)$ and we take n sufficiently large such that $|\dot{G}_s(t) - \dot{G}(t)| < \frac{\varepsilon}{2}$ for all $t \in [B_0, B]$. Then, we get $\dot{G}(B_0) < -\frac{\varepsilon}{2}$. This contradicts the fact that the solution $\dot{G}$ is positive.

Finally, if $\lambda_u > m$ we have

$$\lim_{t \to -\infty} \dot{G}(t) = \lim_{t \to -\infty} \frac{\dot{A}(t)\,k(A(t))}{\sin(A(t))} = \lambda_2 > 0$$

$$\text{so} \quad \lim_{t \to -\infty} \frac{\dot{A}(t)}{A(t)} = \frac{\lambda_2}{c} > 1.$$

A short calculation gives

$$\lim_{t \to -\infty} e^{-t} A(t) = \lim_{s \to 0} \frac{\overline{\alpha}(s)}{s} = \dot{\overline{\alpha}}(0) = 0.$$

And, if $\lambda_u = m$, an argument similar to that of Step 5 of Theorem 1 shows that $\dot{\overline{\alpha}}(0)$ exists and is positive.

STEP 6. – *$F_{\overline{\alpha}}$ is a weakly p-harmonic map.*

Proof. – Set S_p^ε be the manifold parametrized by

$$(a \sin s.\, x,\ b \cos s{\cdot}y) \quad \text{where } x \in S^m,\ y \in S^q \quad \text{and} \quad 0 \le s \le \varepsilon.$$

Similarly, let S_q^ε be the manifold parametrized by

$$(a \sin s.\, x,\ b \cos s{\cdot}y) \quad \text{where} x \in S^m, y \in S^q \quad \text{and} \quad \pi/2 - \varepsilon \le s \le \pi/2.$$

Let dv_g be the volume element of the metric g. Then $F_{\overline{\alpha}}$ is a weakly p-harmonic map if it satisfies,

$$
\begin{aligned}
(3.26)\quad & \int_{Q^{m+q+1}(a,b)} |dF_{\overline{\alpha}}|^{p-2}\{dF_{\overline{\alpha}}\, d\phi + \phi.A(F_{\overline{\alpha}})(dF_{\overline{\alpha}}, dF_{\overline{\alpha}})\}\, dv_g \\
&= \lim_{\varepsilon\to 0} \int_{Q^{m+q+1}(a,b)-S_p^\varepsilon-S_q^\varepsilon} |dF_{\overline{\alpha}}|^{p-2} \\
&\times \{dF_{\overline{\alpha}}\, d\phi + \phi.A(F_{\overline{\alpha}})(dF_{\overline{\alpha}}, dF_{\overline{\alpha}})\}\, dv_g = 0 \\
&\text{for all} \quad \phi \in C^\infty(Q^{m+q+1}(a,b), \mathbb{R}^{s+r+2}).
\end{aligned}
$$

Next, similarly to the equality (2.45) we get,
(3.27)

$$
\begin{aligned}
&\operatorname{div}[(|dF_{\overline{\alpha}}|^{p-2}\, dF_{\overline{\alpha}}).\phi] = \langle \operatorname{div}(|dF_{\overline{\alpha}}|^{p-2}\, dF_{\overline{\alpha}}), \phi\rangle + \langle |dF_{\overline{\alpha}}|^{p-2}\, dF_{\overline{\alpha}}, d\phi\rangle \\
&\qquad \text{for all } \phi \in C^\infty(Q^{m+q+1}(a,b), \mathbb{R}^{s+r+2}).
\end{aligned}
$$

Moreover, we have the following equality

$$
\begin{aligned}
(3.28)\quad & -\operatorname{div}(|dF_{\overline{\alpha}}|^{p-2}\, dF_{\overline{\alpha}}) + |dF_{\overline{\alpha}}|^{p-2} A(F_{\overline{\alpha}})(dF_{\overline{\alpha}}, dF_{\overline{\alpha}}) = 0 \\
&\qquad \text{on } Q^{m+q+1}(a,b) - S_p^\varepsilon - S_q^\varepsilon,
\end{aligned}
$$

because $\overline{\alpha}$ is smooth on $[\varepsilon, \pi/2-\varepsilon]$, so $\overline{\alpha}$ is a strong solution of the Euler-Lagrange equation associated to (3.5), that is (3.28). From (3.27) and (3.28), it follows that (3.26) is equivalent to

$$
\begin{aligned}
(3.29)\quad & \lim_{\varepsilon\to 0} \int_{Q^{m+q+1}(a,b)-S_p^\varepsilon-S_q^\varepsilon} \operatorname{div}[(|dF_{\overline{\alpha}}|^{p-2}\, dF_{\overline{\alpha}}).\phi]\, dv_g = 0 \\
&\qquad \text{for all } \phi \in C^\infty(Q^{m+q+1}(a,b), \mathbb{R}^{s+r+2}).
\end{aligned}
$$

Let ν_1 (respectively ν_2) be the unit normal to ∂S_p^ε (respectively to ∂S_q^ε) and $d\theta_\varepsilon^1$ (respectively $d\theta_\varepsilon^2$) be the volume element of the metric of ∂S_p^ε (respectively of ∂S_q^ε). Using the Stokes Theorem, the left hand side of (3.29) is equal to

$$
\lim_{\varepsilon\to 0} \int_{\partial S_p^\varepsilon} |dF_{\overline{\alpha}}|^{p-2}\langle dF_{\overline{\alpha}}.\phi, \nu_1\rangle\, d\theta_\varepsilon^1 + \lim_{\varepsilon\to 0} \int_{\partial S_q^\varepsilon} |dF_{\overline{\alpha}}|^{p-2}\langle dF_{\overline{\alpha}}.\phi, \nu_2\rangle\, d\theta_\varepsilon^2.
$$

Now, let $M = \sup\{\phi(x) : x \in Q^{m+q+1}(a, b)\}$, we get the following inequalities

$$\left| \int_{\partial S_p^\varepsilon} |dF_{\overline{\alpha}}|^{p-2} \langle dF_{\overline{\alpha}} \,.\, \phi,\, \nu_1 \rangle \, d\theta_\varepsilon^1 \right| \leq M \int_{\partial S_p^\varepsilon} |dF_{\overline{\alpha}}|^{p-1} \, d\theta_\varepsilon^1$$
$$= F_{p-1}(\varepsilon,\, \overline{\alpha},\, \dot{\overline{\alpha}})\, a^m\, b^q \operatorname{vol}(S^m) \operatorname{vol}(S^q)\, \nu(\varepsilon)\, M$$

and

$$\left| \int_{\partial S_q^\varepsilon} |dF_{\overline{\alpha}}|^{p-2} \langle dF_{\overline{\alpha}} \,.\, \phi,\, \nu_2 \rangle \, d\theta_\varepsilon^2 \right| \leq M \int_{\partial S_q^\varepsilon} |dF_{\overline{\alpha}}|^{p-1} \, d\theta_\varepsilon^2$$
$$= F_{p-1}\left(\frac{\pi}{2} - \varepsilon,\, \overline{\alpha},\, \dot{\overline{\alpha}}\right) a^m\, b^q \operatorname{vol}(S^m) \operatorname{vol}(S^q)\, \nu\left(\frac{\pi}{2} - \varepsilon\right) M.$$

If we let tend ε to 0, the right hand side tends to 0; it follows that (3.29) is satisfied. This completes the proof of Step 6.

STEP 7. – *(Conclusion of Theorem 4). – Using Step 5, it is easy to check that the map $F_{\overline{\alpha}}$ is of class C^1 on $Q^{m+q+1}(a, b)$ and since $|dF_{\overline{\alpha}}| > 0$ we conclude, thank to Lemma 2.1 of [3], that $F_{\overline{\alpha}}$ is of class C^∞ on $Q^{m+q+1}(a, b)$.* □

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REFERENCES

[1] A. BALDES, Stability and uniqueness properties of the equator map from a ball into an ellipsoid, *Math. Z.*, Vol. **185**, 1984, pp. 505-516.
[2] W. Y. DING, Symmetric harmonic maps between spheres, *Comm. Math. Phys.*, 1988, pp. 641-649.
[3] F. DUZAAR, Regularity of p-harmonic maps into a closed hemisphere, *Bolletino U.M.I.*, Vol. **7**, 5-B, 1991, pp. 157-170.
[4] F. DUZAAR and M. FUCHS, On removable singularities of p-harmonic maps, *Ann. Inst. Poincaré Anal. Non Linéaire*, Vol. **7**, 1990, pp. 385-405.
[5] J. EELLS and L. LEMAIRE, Anoter report on harmonic maps, *Bull. London Math. Soc.*, Vol. **20**, 1988, pp. 385-524.

[6] J. Eells and A. Ratto, Harmonic maps and minimal immersions with symmetries, *Annals of Math. Studies,* Vol. **130**, Princeton Univ. Press, 1993.
[7] A. Fardoun, Sur le problème de Dirichlet pour les applications p-harmoniques de la boule dans l'ellipsoide, *C. R. Acad. Sci., Paris,* t. **321**, Série I, 1995, pp. 1189-1194.
[8] R. Hardt and F. H. Lin, Mapping minimizing the L^p norm of the gradient, *Comm. Pure Appl. Math.,* Vol. **40**, 1987, pp. 555-588.
[9] P. Hartman, *Ordinary differential equations*, J. Wiley, New York, 1964.
[10] J. Karcher and J. C. Wood, Non existence results and growth properties for harmonic maps and forms, *J. Reine. Angew. Math.,* Vol. **353**, 1984, pp. 165-180.
[11] W. Jäger and H. Kaul, Rotationally symmetric harmonic maps from a ball into a sphere and the regularity problem for weak solutions of elliptic systems, *J. Reine. Angew. Math.,* Vol. **343**, 1983, pp. 146-161.
[12] O. A. Ladyzhenskaya and N. N. Uraltseva, *Linear and quasilinear elliptic equations,* Academic Press, New York, 1968.
[13] J. P. Lasalle, Stability theory for ordinary differential equations, *J. Differential Equations*, Vol. **4**, 1968, pp. 57-65.
[14] L. Lemaire, Applications harmoniques de surfaces riemanniennes, *J. Differential Geometry,* Vol. **13**, 1978, pp. 51-78.
[15] A. Ratto, On the Dirichlet problem for harmonic maps into spheres or ellipsoids and equivariant harmonic maps from warped products, *Bull. Soc. Math. Belgique,* Vol. **41**, 1989, 2-SERB, pp. 145-168.
[16] R. T. Smith, Harmonic mappings of spheres, *Amer. J. Math.,* 1975, pp. 364-384.
[17] X. W. Xu and P. C. Yang, A construction of m-harmonic maps of spheres, *International Journal of Math.,* Vol. **4**, No. 3, 1993, pp. 521-533.

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