ANNALES DE L'I. H. P., SECTION C

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*Annales de l'I. H. P., section C*, tome 14, nº 5 (1997), p. 597-613
<http://www.numdam.org/item?id=AIHPC_1997__14_5_597_0>

*Ann. Inst. Henri Poincaré,*

Vol. 14, n° 5, 1997, p. 597-613 *Analyse non linéaire*

# Proof of the De Gennes formula for the superheating field in the weak $\kappa$ limit

by

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ABSTRACT. – In continuation with our preceding paper [3] concerning the superconducting film, we present in this article new estimates for the superheating field in the weak $\kappa$ limit. The principal result is the proof of the existence of a finite superheating field $h^{sh,+}(\kappa)$ (obtained by restricting the usual definition of the superheating field to solutions of the Ginzburg-Landau system $(f, A)$ with $f$ positive) in the case of a semi-infinite interval. The bound is optimal in the limit $\kappa \to 0$ and permits to prove (combining with our previous results) the De Gennes formula

$$2^{-\frac{3}{4}} = \lim_{\kappa \to 0} \kappa^{\frac{1}{2}}\, h^{sh,+}(\kappa)$$

The proof is obtained by improving slightly the estimates given in [3] where an upper bound was found but under the additional condition that the function $f$ was bounded from below by some fixed constant $\rho > 0$.

*A.M.S. Classification:* 34 A 34, 82 D 55.

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RÉSUMÉ. – Poursuivant notre étude précédente [3] sur les films supraconducteurs, nous présentons dans cet article de nouvelles estimations sur le champ de surchauffe lorsque $\kappa$ tend vers 0. Le résultat principal est la démonstration, dans le cas d'un intervalle semi-infini, de l'existence d'un champ de surchauffe fini $h^{sh,+}(\kappa)$ (obtenue en restreignant la définition usuelle du champ de surchauffe aux solutions $(f, A)$ du système de Ginzburg-Landau telles que $f$ soit positive). La majoration est optimale à la limite $\kappa \longrightarrow 0$ et permet de montrer (en regroupant avec nos résultats précédents) la formule de De Gennes

$$2^{-\frac{3}{4}} = \lim_{\kappa \longrightarrow 0} \kappa^{\frac{1}{2}}\, h^{sh,+}(\kappa)$$

La démonstration est obtenue en améliorant légèrement les estimations de l'article [3] dans lequel nous donnions une majoration, mais en supposant la fonction $f$ minorée par une constante $\rho > 0$ fixée.

## 1. INTRODUCTION

This paper is devoted to the analysis of the local minima of the Ginzburg-Landau functional in a superconducting film. When the Ginzburg-Landau parameter $\kappa$ is small and when the width of the film $d$ is large (in the sense that $\kappa d$ is large), a natural approximation (cf. [11]) consists in reducing the problem to the study of local minima of the functional

$$\begin{aligned} &\varepsilon_\infty(f, A; h) \\ &\quad = \int_0^{+\infty} \left[\frac{1}{2}(f^2 - 1)^2 + \kappa^{-2} f'^2 + f^2 A^2 + A'^2\right] dx + 2h\, A(0), \quad (1.1) \end{aligned}$$

defined on the set of pairs $(f, A)$ such that $(1 - f) \in H^1(\mathbb{R}^+)$, $A \in H^1(\mathbb{R}^+)$. The corresponding Ginzburg-Landau equations expressing the necessary conditions to have minima are then:

$$-\kappa^{-2} f'' - f + f^3 + A^2 f = 0 \quad \text{on } ]0, +\infty[ \qquad (1.2)$$

$$-A'' + f^2 A = 0 \quad \text{on } ]0, +\infty[ \qquad (1.3)$$

$$f'(0) = 0; \qquad \lim_{x \longrightarrow +\infty} f(x) = 1, \qquad (1.4)$$

$$A'(0) = h; \qquad \lim_{x \longrightarrow +\infty} A(x) = 0, \qquad (1.5)$$

with $A \in H^2(]0, +\infty[)$, $(1 - f) \in H^2(]0, +\infty[)$.

This system (1.2)-(1.5) will be denoted by $(GL)_\infty$ in the following sections.

The main result of this article concerns the asymptotic behavior of the superheating field as $\kappa$ tends to 0 in this context.

We recall that it was defined as

$$h^{sh}(\kappa, \infty) = \sup A'(0),$$

where the sup was considered over all the solutions $(f, A)$ of the above Ginzburg-Landau system (1.2)-(1.5). Of course, it would have been more suitable to define it as the sup over local minima of the functional introduced before in (1.1). It might be smaller but we were unable to attack this point, which in any case does not affect the results which are presented in this article.

We have considered in [3] the problem of proving the following asymptotic formula given by De Gennes and the Orsay group [8]:

$$\lim_{\kappa \to 0} \kappa^{1/2} h^{sh}(\kappa, \infty) = 2^{-3/4}. \tag{1.6}$$

This asymptotic limit was obtained by De Gennes by looking for an approximate model, only valid in the region $f(0) > \rho > 0$, giving the following relation between $f(0)$ and $h = A'(0)$ for a local minimum of the $GL$-functional:

$$\kappa h^2 \sim \sqrt{2}\, f(0)^2 (1 - f(0)^2), \tag{1.7}$$

from which (1.6) was deduced.

This approximate model was analyzed in [4] and an approximate model valid for any value of $f(0)$ was also proposed.

On the other hand, we have proved in [3] that

$$\liminf_{\kappa \to 0} \kappa^{1/2} h^{sh}(\kappa, \infty) = 2^{-3/4} \tag{1.8}$$

Our aim is to prove here the corresponding result for the $\limsup$ and for a slightly modified but natural version of the superheating field.

Here we define $h^{sh,+}(\kappa, \infty)$ in the following way.

We first consider the set $\mathcal{H}^+ = \mathcal{H}^{sh,+}(\kappa)$ in $\mathbb{R}^+$ of the $h$'s such that there exist superconducting solutions with $f > 0$. We have seen in [3] that this is an interval $[0, h^+)$ and we then introduce the definition:

DEFINITION 1.1. – *The superheating field $h^+ := h^{sh,+}(\kappa)$ is defined as the supremum of the interval $\mathcal{H}^{sh,+}(\kappa)$.*

We shall indeed prove the analogous of (1.6) for the $\limsup$ for this restricted definition.

This will be obtained by establishing first in Section 2 fine inequalities on the solutions as a consequence of the maximum principle and the energy conservation.

The main results will be established in Section 3.

In Section 4, we shall prove the "main proposition". This corresponds to an improvement of previous results obtained in [3]. It seems rather optimal and exhibits the approximate formula (1.7) relating $f(0)$ and $h$ in the form of an upper bound for $\kappa h^2$ but with a remainder term which is asymptotically good only under the restrictive condition that $f(0) \geq \rho > 0$.

This still leaves the question of a general upper bound for the superheating field open, unless we find a control of the $h$'s such that $f(0)$ is small.

In Section 5, we use the same idea in order to get this time a lower bound for $\kappa h^2$. The formula given by De Gennes appears here very simply and no condition on $f(0)$ is involved.

It remains to eliminate the unfortunate condition on $f(0)$ for the upper bound. Here our idea will be that, modulo a controlled error, one can replace $f(0)$ by $f(\alpha)$ for a suitable $\alpha > 0$. This idea will be developed in Section 6 where we essentially prove that in the case when $\kappa h^2$ is bounded from below, one can get a corresponding lower bound for $f(\alpha) - f(0)$, and consequently for $f(\alpha)$, which will be independent of $f(0)$.

This permits the proof of the main theorem which will be given in Section 7.

## 2. GENERAL PROPERTIES OF THE GINZBURG-LANDAU EQUATIONS

We first recall one useful version of the maximum principle:

LEMMA 2.1. – *Let $C$ be a bounded function on $]0, +\infty[$ such that:*

$$\forall x \in ]0, +\infty[ : \; C(x) \geq c_0 > 0$$

*and $u \in C^2_{bdd}([0, +\infty[) \cap H^1(]0, +\infty[)$ a function such that:*

$$u'(x) \to 0 \; as \; x \to +\infty,$$

$$-u'' + C(x)\, u \; \leq \; 0 \; in \; ]0, +\infty[,$$

$$u'(0) \geq 0,$$

*then:*

$$u \leq 0 \; in \; ]0, +\infty[.$$

Let us also recall:

PROPOSITION 2.2. – *Let $(f, A)$ be a solution of $(GL)_\infty$; then we have:*

(*a*)

$$|f(x)| \leq 1, \; \forall x \in [0, +\infty[ \tag{2.1}$$

(*b*) *The function $A$ is strictly increasing on the interval. Moreover we have the inequality:*

$$0 \leq A'(x) \leq h, \tag{2.2}$$

(*c*) *If $f$ is positive, then $f$ is strictly increasing on $[0, +\infty[$.*

Let us also recall the energy conservation for a solution $(f, A)$ satisfying $(GL)_\infty$:

$$\kappa^{-2} f'(x)^2 + A'(x)^2 = f(x)^2 \cdot A(x)^2 + \frac{1}{2}(1 - f(x)^2)^2. \tag{2.3}$$

As a side remark we observe in particular that at 0

$$h^2 = f(0)^2.A(0)^2 + \frac{1}{2}(1 - f(0)^2)^2, \tag{2.4}$$

which will be interesting to compare with (2.12).

It can also be useful to get estimates on $f'$ and this will be the object of:

LEMMA 2.3. – *If $f$ is a solution of $(GL)_\infty$, then we have:*

$$|f'(x)| \leq \frac{\kappa}{\sqrt{2}}. \tag{2.5}$$

Using again the conservation of the energy and the preceding lemma, we get the estimate:

$$f(x)^2 \, A(x)^2 \leq h^2 + \frac{1}{2}, \tag{2.6}$$

which will be improved soon under the additional condition $f > 0$.

We now give one important improvement:

PROPOSITION 2.4. – *If $(f, A)$ is a solution of $(GL)_\infty$ and if $f$ is positive then we have*

- (*a*)

$$0 < -A(x)f(x) \leq A'(x) \leq -A(x), \tag{2.7}$$

- (*b*)

$$0 \leq \kappa^{-1} f'(x) \leq \frac{1}{\sqrt{2}}(1 - f(x)^2), \tag{2.8}$$

- (*c*) *when* $f \not\equiv 1$:

$$0 < f(x) \leq \tanh(\frac{\kappa x}{\sqrt{2}} + x_f), \tag{2.9}$$

*with* $x_f = \arg\tanh f(0)$,

- (*d*)

$$0 < -A(x) \leq -A(0) \exp - \int_0^x f(t)dt. \tag{2.10}$$

*Proof.*

*Proof of* (*a*). – Let $(f, A)$ be our solution. For any $\alpha > 0$, we compare in $[\alpha, +\infty[$ the solution $A(x)$ which satisfies

$$\begin{gathered} -A''(x) + f(x)^2 A(x) = 0 \quad \text{for } x \in [\alpha, +\infty[ \\ A'(\alpha) := h_\alpha \\ \lim_{x \to \infty} A'(x) = 0 \end{gathered}$$

with the solution $A_\alpha(x)$ of

$$\begin{gathered} -A''_\alpha(x) + f(\alpha)^2 A_\alpha(x) = 0 \quad \text{for } x \in [\alpha, +\infty[ \\ A'_\alpha(\alpha) := h_\alpha \\ \lim_{x \to \infty} A'_\alpha(x) = 0, \end{gathered}$$

which can be explicitely computed as

$$A_\alpha(x) = -\frac{A'(\alpha)}{f(\alpha)} \exp(-f(\alpha)(x - \alpha)).$$

Applying Lemma 2.1 with $C(x) = (f(\alpha))^2$ and $u(x) = A_\alpha(x) - A(x)$ in the interval $[\alpha, +\infty[$, we get, for any $(\alpha, x)$ satisfying $x \geq \alpha \geq 0$,

$$-A(x) \leq \frac{A'(\alpha)}{f(\alpha)} \exp(-f(\alpha)(x - \alpha)) \tag{2.11}$$

and in particular when $x = \alpha$

$$-A(\alpha) \leq \frac{A'(\alpha)}{f(\alpha)}.$$

We have consequently ([1]) proved (a) (the other inequality was already obtained in [3]). Combining with an already given estimate on $A'(x)$, we note as a byproduct that

$$0 \leq -A(x) \leq \frac{h}{f(x)} \quad \text{for } x \in [0, +\infty[. \tag{2.12}$$

*Proof of* (*b*). – Inspired by similar arguments given by J. Chapman [6], we write the conservation of the energy (2.3) in the form

$$(A' - fA)(A' + fA) = \left(\frac{1}{\sqrt{2}}(1 - f^2) - \kappa^{-1} f'\right)\left(\frac{1}{\sqrt{2}}(1 - f^2) + \kappa^{-1} f'\right). \tag{2.13}$$

The statement (b) is then an immediate consequence of (a) if we think of the properties $A' > 0$, $f > 0$, $A < 0$, $f < 1$ and $f' > 0$. We observe that this is an improvement of (2.5).

*Proof of* (*c*). – This is an immediate consequence of (b) by integration of this inequation.

*Proof of* (*d*). – This is an immediate consequence of (a) by integration of this inequation. We observe that it is slightly better than what we have stated previously.

## 3. MAIN RESULTS

The following proposition will be a small improvement of a similar proposition given in [3]:

PROPOSITION 3.1. *Main Proposition – For any pair* $(f, A)$ *solution of* $(GL)_\infty$ *with* $f > 0$, *the following estimate is true:*

$$A'(0)^2 \leq \sqrt{2}\,[(1 - f(0)^2) f(0)^2 \kappa^{-1} + 5\, A'(0) f(0)^{-1}]. \tag{3.1}$$

([1]) We met quite recently in a preprint by S. P. Hastings, M. K. Kwong and W. C. Troy [12] a similar inequality in the case of a finite interval.

COROLLARY 3.2. – *In particular, as $\kappa \to 0$, and if $f(0) \geq \rho > 0$, one has*

$$\kappa h^2 \leq \frac{\sqrt{2}}{4} + \mathcal{O}_\rho(\kappa^{1/2}). \tag{3.2}$$

REMARK 3.3. – *This is interesting to compare with the De Gennes formula. We recall that for the critical h, we found $\rho^2 \sim \frac{1}{2}$ (corresponding here to the value of $f(0)$ for which $f(0)^2(1 - f(0)^2)$ is maximal) and so, if we can relax the condition $f(0) \geq \rho$, the result is optimal.*

We shall actually relax the restriction $f \geq \rho > 0$ and shall get

THEOREM 3.4. – *There exists $\kappa_0$ such that for all $\kappa$ in $]0, \kappa_0]$, for all $h \in \mathcal{H}^{sh,+}(\kappa)$, the following estimate is true as $\kappa \to 0$,*

$$\kappa h^2 \leq \frac{\sqrt{2}}{4} + \mathcal{O}(\kappa^{\frac{1}{2}}). \tag{3.3}$$

Combining ([2]) with the results obtained in [3], we get

COROLLARY 3.5. – *De Gennes Formula:*

$$\lim_{\kappa \to 0} \kappa\, h^{sh,+}(\kappa)^2 = \frac{\sqrt{2}}{4}. \tag{3.4}$$

## 4. PROOF OF THE MAIN PROPOSITION

Our starting point will be the identity

$$h^2 = A'(0)^2 = -2 \int_0^\infty A'(t)\, A''(t) dt. \tag{4.1}$$

Using (1.3), we get

$$h^2 = A'(0)^2 = 2 \int_0^\infty A'(t)\, (-A(t))\, f(t)^2 dt. \tag{4.2}$$

We now use (2.7) and obtain

$$h^2 \leq 2 \int_0^\infty f(t)\, A'(t)^2\, dt. \tag{4.3}$$

([2]) We observe here that our construction of subsolutions leads actually, to the existence of solutions with $f > 0$ permitting us to replace $h^{sh}$ by $h^{sh,+}$ in the statement.

We remark here that, in ([3]), we only used the weaker

$$h^2 \leq 2 \int_0^\infty A'(t)^2\, dt. \tag{4.4}$$

We have consequently gained some $f$ which will be decisive in order to avoid the $\frac{1}{\rho}$ appearing in the proof in ([3]) and in the proposition stated above.

Using now the energy conservation, we get for any $k \in \mathbb{N}$

$$\kappa^{-2} f(x)^k\, f'(x)^2 + f(x)^k A'(x)^2 = f(x)^{k+2}\, A(x)^2 + f(x)^k\, (1 - f(x)^2)^2/2, \tag{4.5}$$

and integrating over $[0, \infty[$

$$\begin{aligned}\int_0^\infty f(x)^k A'(x)^2\, dx = & \int_0^\infty f(x)^{k+2}\, A(x)^2\, dx \\ & + \frac{1}{2} \int_0^\infty f(x)^k\, (1 - f(x)^2)^2\, dx \\ & - \kappa^{-2} \int_0^\infty f(x)^k\, f'(x)^2\, dx.\end{aligned} \tag{4.6}$$

We now multiply by $f^{k+1}$ the first GL equation (1.2)

$$\begin{aligned}& -\kappa^{-2} \int_0^{+\infty} f''(x)\, f(x)^{k+1}\, dx \\ & + \int_0^{+\infty} [-1 + f(x)^2 + A(x)^2]\, f(x)^{k+2}\, dx = 0.\end{aligned}$$

or, using the boundary conditions (1.4) on $f$:

$$\begin{aligned}\int_0^{+\infty} A(x)^2\, f(x)^{k+2}\, dx = & + \int_0^{+\infty} f(x)^{k+2} (1 - f(x)^2)\, dx \\ & - (k+1) \cdot \kappa^{-2} \int_0^{+\infty} f^k(x) f'(x)^2\, dx.\end{aligned} \tag{4.7}$$

We now use (4.6) and (4.7) in order to obtain

$$\begin{aligned}\int_0^\infty f(x)^k A'(x)^2\, dx = & \frac{1}{2} \int_0^\infty f(x)^k\, (1 - f(x)^4)\, dx \\ & - (k+2) \cdot \kappa^{-2} \int_0^{+\infty} f^k(x) f'(x)^2\, dx.\end{aligned} \tag{4.8}$$

Finally, we obtain with $k = 1$ ([3]) using (4.3)

$$h^2 \leq \int_0^{+\infty} f(x)(1 - f(x)^4)\, dx \ - \ 6 \cdot \kappa^{-2} \int_0^{+\infty} f(x) f'(x)^2\, dx. \quad (4.9)$$

In order to get the control of the r.h.s., we come back again to the identity expressing the conservation of the energy and we deduce (using also the monotonicity of $f$ proved in Section 2) the following inequality:

$$\kappa^{-1} f'(x) + A'(x) \geq \frac{1}{\sqrt{2}}(1 - f(x)^2). \quad (4.10)$$

We first use (4.10) in order to get

$$\int_0^{+\infty} f(x)(1 - f(x)^4)\, dx \leq \frac{1}{2\sqrt{2}} \cdot \kappa^{-1}(1 - f(0)^2)(3 + f(0)^2) + \sqrt{2} \int_0^{\infty} f(x)(1 + f(x)^2) A'(x) dx, \quad (4.11)$$

and finally

$$\int_0^{+\infty} f(x)(1 - f(x)^4)\, dx \leq \frac{1}{2\sqrt{2}} \cdot \kappa^{-1}(1 - f(0)^2)(3 + f(0)^2) \ + \ 2\sqrt{2}(-A(0)). \quad (4.12)$$

We shall now use again (4.10) and (2.5) in order to find a lower bound for $\int_0^{+\infty} f(x) f'(x)^2\, dx$ by

$$\begin{aligned} \int_0^{+\infty} f(x) f'(x)^2\, dx &\geq \frac{\kappa}{\sqrt{2}} \int_0^{+\infty} (1 - f(x)^2) f(x) f'(x)\, dx - \kappa \int_0^{+\infty} A'(x) f(x) f'(x)\, dx \\ &\geq \frac{\kappa}{4\sqrt{2}}(1 - f(0)^2)^2 + \frac{\kappa^2}{\sqrt{2}} A(0). \end{aligned} \quad (4.13)$$

Here we have used the simplest upper bounds for $f$ and $f'$. We finally get

$$h^2 \leq \sqrt{2}\kappa^{-1}(1 - f(0)^2) f(0)^2 - 5\sqrt{2} A(0) \quad (4.14)$$

and finally (3.1).

This ends the proof of the proposition.

([3]) In [3], we were only playing with $k = 0$.

## 5. ABOUT THE LOWER BOUND

We follow the same idea as in the preceding section, but look now for an inequality for $h$ going in the opposite direction. We shall prove

PROPOSITION 5.1. – *For any pair $(f, A)$ solution of $(GL)_\infty$ with $f > 0$, the following estimate is true:*

$$\kappa\, A'(0)^2 \ \geq\ \sqrt{2}\,[(1 - f(0)^2)f(0)^2]. \tag{5.1}$$

The proof is in the same vein but finally simpler than the main proposition. We emphasize that no condition on a lower bound for $f(0)$ is present.

Starting from (4.2), we first get using (2.7)

$$\begin{aligned} A'(0)^2 &= 2\int_0^\infty A'(t)\,(-A(t))\,f(t)^2 dt \\ &\geq 2\int_0^\infty f(t)^3\,A(t)^2\,dt. \end{aligned} \tag{5.2}$$

We then use (4.7) and get

$$\begin{aligned} A'(0)^2 &\geq 2\int_0^\infty f(t)^3\,(1 - f(t)^2)\,dt - 4\kappa^{-2}\int_0^\infty f(t)f'(t)^2 dt \\ &\geq \frac{4}{\kappa\sqrt{2}}\left[\int_0^\infty f(t)^3 f'(t)dt - \int_0^\infty f(t)(1 - f(t)^2)f'(t)dt\right] \\ &\geq \sqrt{2}\,\kappa^{-1}\cdot[(1 - f(0)^2)\,f(0)^2]. \end{aligned} \tag{5.3}$$

Here we have used two times the inequality (2.8).

We remark that this proposition does not replace what was proved in [3] concerning $\liminf_{\kappa\to 0} h^{sh,+}(\kappa)$ but gives a complementary information. We have indeed proved in [3] by using suitable constructions of subsolutions that if $h$ satisfies

$$h^2 \leq \frac{\sqrt{2}}{4}\,\kappa^{-1} - C$$

for a suitably large constant $C > 0$ and for sufficiently small $\kappa$, then one can find a solution of the above Ginzburg-Landau system for a suitable $f(0) \geq \frac{1}{\sqrt{2}} - C\kappa$.

As a corollary of the proposition, we know that this solution satisfies necessarily

$$\kappa\, h^2 \ \geq\ \sqrt{2}\cdot[(1 - f(0)^2)\,f(0)^2] \tag{5.4}$$

and combining with the theorem 3.4 we obtain moreover

$$|\kappa h^2 - \sqrt{2}\cdot[(1-f(0)^2)\,f(0)^2]| \leq C\kappa^{\frac{1}{4}}. \tag{5.5}$$

This strongly supports the idea that in the region

$$1 \geq f_0 \geq \frac{1}{\sqrt{2}} + C\kappa$$

there exists a unique solution $(f, A)$ such that $A'(0) = h$, $f(0) = f_0$. Moreover, this solution satisfies (5.5). What is missing here is a proof of the uniqueness.

This strongly supports also the heuristic curve produced by De Gennes (except near $f(0) = 0$).

## 6. LOWER BOUND FOR $f(x)$

In order to control this function, we want to estimate from below $f(\alpha)$, for some suitable $\alpha > 0$. This estimate is needed only in the case when $f(0)$ is small.

What is of course important is to get a $f(0)$−independent lower bound for $f(\alpha)$.

On the other hand, because we want to estimate the superheating field, there is no restriction to assume that we are in the case when

$$\kappa \cdot h^2 \geq \beta > 0, \tag{6.1}$$

for some $\beta$ small enough. The choice of $\beta$ will be given later.

Starting from the first (GL)-equation, we obtain the following estimate from below for $f''$

$$f''(x) \geq \kappa^2 f(x) A(x)^2 - \kappa^2. \tag{6.2}$$

We shall now find a lower bound for $f(x)A(x)^2$. Using (2.3), we have

$$f(x)^2\, A(x)^2 \geq A'(x)^2 - \frac{1}{2} \tag{6.3}$$

which will imply

$$f(x)\, A(x)^2 \geq \frac{A'(x)^2 - \frac{1}{2}}{f(x)} \tag{6.4}$$

for $x \in [0, \alpha]$.

We now control the variation of $A'(x)$ in $[0,\alpha]$. We recall that $A'$ is monotonically decreasing and that we have, using the second $(GL)$–equation:

$$A'(0) - A'(\alpha) = \int_0^\alpha f(t)^2 \cdot (-A(t))\, dt \leq \alpha f(\alpha) h. \tag{6.5}$$

If we find consequently $\alpha$ such that

$$\alpha \cdot f(\alpha) \leq \frac{1}{2}, \tag{6.6}$$

we shall have the property

$$h \geq A'(x) \geq A'(\alpha) \geq \frac{h}{2} \text{ for } x \in [0,\alpha]. \tag{6.7}$$

We then have, under the conditions (6.6), (6.7),

$$f(x)\, A(x)^2 \geq \frac{h^2 - 2}{4f(x)},$$

and, using also (6.1), we get, for $\kappa \leq \dfrac{\beta}{10}$

$$\begin{aligned} f(x)\, A(x)^2 &\geq h^2\, \frac{1 - 2\frac{\kappa}{\beta}}{4f(x)} \\ &\geq \frac{h^2}{5f(\alpha)}, \text{ for } x \in [0,\alpha]. \end{aligned}$$

Coming back to (6.2), we obtain the following lower bound for $f$ in $[0,\alpha]$

$$\begin{aligned} f(x) &\geq [\inf_{t\in[0,x]} f''(t)]\, \frac{x^2}{2} \\ &\geq \left[\kappa^2 h^2 \frac{1}{5f(\alpha)} - \kappa^2\right] \frac{x^2}{2}. \end{aligned} \tag{6.8}$$

Using again the lower bound (6.1) for $\kappa h^2$, we get, when $\kappa \leq \dfrac{\beta}{10}$ and $\alpha f(\alpha) \leq \dfrac{1}{2}$:

$$f(x) \geq \left[\frac{\beta}{10 f(\alpha)}\right] \cdot \frac{\kappa x^2}{2}, \text{ for } x \in [0,\alpha]. \tag{6.9}$$

## 7. PROOF OF THEOREM 3.4

We start from

$$\frac{1}{2}h^2 \leq \int_0^\infty f(t)A'(t)^2 dt, \tag{7.1}$$

and from

$$\begin{aligned}\int_0^\infty f(t)A'(t)^2 dt =& \frac{1}{2}\int_0^{+\infty} f(x)(1-f(x)^4)\,dx \\ &- 3\cdot\kappa^{-2}\int_0^{+\infty} f(x)f'(x)^2\,dx.\end{aligned} \tag{7.2}$$

We then use (4.10) and get

$$\begin{aligned}\int_0^\infty f(t)A'(t)^2 dt \leq& \frac{\sqrt{2}}{2}\kappa^{-1}\int_0^\infty f'(x)f(x)\,(1+f(x)^2)\,dx \\ - 3\cdot\kappa^{-2}&\int_0^{+\infty} f(x)f'(x)^2\,dx + \sqrt{2}\int_0^{+\infty} f(x)\,A'(x)\,dx.\end{aligned} \tag{7.3}$$

A variant of our preceding lower bound for $\int_0^{+\infty} f(x)f'(x)^2\,dx$ then gives

$$\int_0^\infty f(t)A'(t)^2 dt \leq \frac{\sqrt{2}}{2}\kappa^{-1}(1-f(0)^2)f(0)^2 + \frac{5}{\sqrt{2}}\int_0^{+\infty} f(x)\,A'(x)dx. \tag{7.4}$$

In the proof of Proposition 3.1, we were estimating from above $f$ by 1 and we got the proposition, but this leads to an upper bound with $\frac{1}{f(0)}$ which is bad for our purpose. We now concentrate our efforts on an estimate of

$$\int_0^\infty f(x)\,A'(x)\,dx.$$

By cutting the integral in two parts we obtain for any $\alpha \geq 0$

$$\int_0^\infty f(x)\,A'(x)\,dx \leq (\alpha\cdot f(\alpha))^{\frac{1}{2}}\cdot\left(\int_0^\infty f(t)A'(t)^2\,dt\right)^{\frac{1}{2}} - A(\alpha). \tag{7.5}$$

We were using before the case $\alpha = 0$. We now deduce

$$\int_0^\infty f(x)\,A'(x)\,dx \leq \frac{1}{\kappa^{\frac{1}{4}}}(\alpha\cdot f(\alpha)) + \frac{\kappa^{\frac{1}{4}}}{4}\left(\int_0^\infty f(t)A'(t)^2\,dt\right) - A(\alpha). \tag{7.6}$$

We now choose

$$\alpha = \gamma\kappa^{-\frac{1}{4}}, \tag{7.7}$$

and we deduce from (6.9):

$$f(\alpha) \geq \left[\gamma^2 \frac{\beta}{20 f(\alpha)}\right] \kappa^{\frac{1}{2}}.$$

This implies that:

$$f(\alpha) \geq \frac{\gamma\beta^{\frac{1}{2}}}{6}\, \kappa^{\frac{1}{4}}. \tag{7.8}$$

On the other hand, we also need a weak control of $f$ from above. We recall that we can assume that

$$f(0) \leq \gamma\kappa^{\frac{1}{4}}. \tag{7.9}$$

We then immediately get

$$0 \leq f(\alpha) \leq \tanh\left(\kappa^{\frac{3}{4}} \frac{\gamma}{\sqrt{2}} + x_f\right) \leq 2\gamma\,\kappa^{\frac{1}{4}}, \tag{7.10}$$

for $\kappa$ small enough.

For $\gamma$ small enough, we obtain that $\alpha f(\alpha) \leq \frac{1}{2}$.

We finally get when $\kappa$ is small:

$$\left(1 - \frac{5\sqrt{2}}{8}\kappa^{\frac{1}{4}}\right) \int_0^{+\infty} f(x) A'^2(x)dx \leq \frac{\sqrt{2}}{2}\,\kappa^{-1}\,(1 - f(0)^2) f(0)^2 + \frac{5\sqrt{2}}{2}\left(\kappa^{-\frac{1}{4}}\alpha f(\alpha) + \frac{h}{f(\alpha)}\right).$$

We now choose

$$\beta = 2^{-\frac{3}{2}},$$

and finally get, for some suitable constants $C_1$ and $C_2$

$$h^2 \leq \sqrt{2}\kappa^{-1}(1 - f^2(0)) \cdot f^2(0) + C_1 h\kappa^{-\frac{1}{4}} + C_2\kappa^{-\frac{3}{4}}. \tag{7.11}$$

Then this gives

$$h \leq 2^{\frac{1}{4}}\,\kappa^{-\frac{1}{2}} \cdot (1 - f(0)^2)^{\frac{1}{2}} \cdot f(0) + \mathcal{O}(\kappa^{-\frac{1}{4}}). \tag{7.12}$$

We have finally obtained the theorem with a slightly worse remainder term than announced.

There was indeed three cases.

• The case $\kappa h^2 \leq \beta$, which is precisely the estimate announced by the theorem.

• The case $\kappa h^2 \geq \beta$ and $f(0) \geq \gamma\kappa^{\frac{1}{4}}$, where we can apply the main proposition,

• The case $\kappa h^2 \geq \beta$ and $f(0) \leq \gamma\kappa^{\frac{1}{4}}$ which we just treated.

In order to get the theorem with the best remainder term, we observe that the coefficient $f(0)^2(1 - f(0)^2)$ of $\kappa^{-1}$ is maximum for $f(0) = \frac{1}{\sqrt{2}}$, and we can consequently use the main proposition in order to get a remainder in $\mathcal{O}(\kappa^{\frac{1}{2}})$ in (3.3). The formula given by H. Parr [15] and confirmed by our numerical computations in [4] suggest a second term of order $\kappa$ for the superheating field.

## 8. CONCLUSION

We have consequently proved a rather satisfactory version of the De Gennes formula. Two points remain to be analyzed in the future in order to be complete. The first point is to analyze possibly vanishing solutions. The second point will be to prove that the solutions whose existence was obtained in [3] correspond actually to local minima.

Here we have especially paid attention to the case $\kappa \to 0$. But the techniques developed here permit also to treat other asymptotic regims as we shall explain in another paper [5].

We hope also to analyze more precisely the situation near $f(0) = 0$ where $h$ is hoped to approach $\frac{1}{\sqrt{2}}$.

Another open problem is the analysis of the problem in an interval $[-d, +d]$ in the regime $\kappa d$ large and the proof that, as $d \to \infty$ and $\kappa \to 0$, the formula given here for the semi-infinite case is a good asymptotic value, at least if one considers solutions $(f, A)$ with $f > 0$.

## REFERENCES

[1] C. Bolley and B. Helffer, Rigorous results on the Ginzburg-Landau models in a film submitted to an exterior parallel magnetic field, *Nonlinear Studies*, Part I: Vol. **3**, n° 1, 1996, pp. 1-29, Part II: Vol. **3**, n° 2, 1996, pp. 1-32.

[2] C. Bolley and B. Helffer, *Sur les asymptotiques des champs critiques pour l'équation de Ginzburg-Landau.* Séminaire Equations aux dérivées partielles de l'Ecole Polytechnique, November 1993.

[3] C. Bolley and B. Helffer, Rigorous results for the Ginzburg-Landau equations associated to a superconducting film in the weak $\kappa$-limit, *Reviews in Math. Physics*, Vol. **8**, n° 1, 1996, pp. 43-83.
[4] C. Bolley and B. Helffer, Superheating in a film in the weak $\kappa$ limit: numerical results and approximate models, Dec. 1994 (Part I to appear in $M^2AN$).
[5] C. Bolley and B. Helffer, In preparation.
[6] S. J. Chapman, *Asymptotic analysis of the Ginzburg-Landau model of superconductivity: reduction to a free boundary model*, Preprint, 1992.
[7] V. P. Galaiko, Superheating critical field for superconductors of the first kind, *Soviet Physics JETP*, Vol. **27**, n° 1, July 1968.
[8] P. G. de Gennes, *Superconductivity, selected topics in solid state physics and theoretical Physics*, Proc. of 8th Latin american school of physics, Caracas, 1966.
[9] V. L. Ginzburg, On the theory of superconductivity, *Nuovo Cimento*, Vol. **2**, 1955, p. 1234.
[10] V. L. Ginzburg, On the destruction and the onset of superconductivity in a magnetic field, *Soviet Physics JETP*, Vol. **7**, 1958, p. 78.
[11] V. L. Ginzburg and L. D. Landau, On the theory of superconductivity, *Zh. Eksperim. i teor. Fiz.*, Vol. **20**, 1950, pp. 1064-1082. *English translation Men of Physics*, L. D. Landau, I, Ed. by D. TerHaar, Pergamon oxford, 1965, pp. 138-167.
[12] S. P. Hastings, M. K. Kwong and W. C. Troy, *The existence of multiple solutions for a Ginzburg-Landau type model of superconductivity*, Preprint May 1995.
[13] D. Saint James and P. G. de Gennes, Onset of superconductivity in decreasing fields, *Phys. Lett.*, Vol. **7**, 1963, p. 306.
[14] D. Saint James, G. Sarma and E. J. Thomas, *Type II Superconductivity*, Pergamon Press, 1969.
[15] H. Parr, Superconductive superheating field for finite $\kappa$, *Z. Physik*, Vol. **B25**, 1976, pp. 359-361.

*(Manuscript received September 1, 1995.)*