I. BIRINDELLI
I. CAPUZZO DOLCETTA
A. CUTRÌ

Liouville theorems for semilinear equations on the Heisenberg group


<http://www.numdam.org/item?id=AIHPC_1997__14_3_295_0>
Liouville theorems for semilinear equations on the Heisenberg group

by

I. BIRINDELLI, I. CAPUZZO DOLCETTA

Università “La Sapienza”, Dip. Matematica, P.zza A.Moro, 2-00185 Roma, Italy.

and

A. CUTRÌ

Università “Tor Vergata”, Dip. Matematica, V.le Ricerca Scientifica, 00133 Roma, Italy.

ABSTRACT. – In this paper we consider problems of the type

\[
\begin{cases}
\Delta_H u + h(x)u^p \leq 0, & \text{in } D \subset \mathbb{R}^{2n+1}, \\
u \geq 0 & \text{in } D,
\end{cases}
\] (1)

where $\Delta_H$ is the Heisenberg Laplacian, $D$ is an unbounded domain and $h$ is a non negative function.

We prove that, under suitable conditions on $h$, $p$ and $D$, the only solution of (1) is $u \equiv 0$.

Key words: Liouville property, Heisenberg group.

RÉSUMÉ. – Dans ce travail nous considérons des problèmes du type

\[
\begin{cases}
\Delta_H u + h(x)u^p \leq 0, & \text{dans } D \subset \mathbb{R}^{2n+1}, \\
u \geq 0 & \text{dans } D,
\end{cases}
\] (1)

où $\Delta_H$ est le Laplacien de Heisenberg, $D$ est un domaine non borné et $h$ est une fonction positive.

Nous démontrons que sous certaines hypothèses sur $h$, $p$ et $D$, la seule solution de (1) est $u \equiv 0$.

A.M.S. Classification: 35 J 60, 35 J 70.
1. INTRODUCTION

In this paper we establish some Liouville type theorems for positive functions $u$ satisfying, for example,

$$\begin{align*}
\Delta_H u + h(\xi)u^p &\leq 0 \quad \text{in } D, \\
u &\geq 0 \quad \text{in } D,
\end{align*}$$

(1.1)

where $D$ is an unbounded domain of the Heisenberg group $H^n$. We recall that $H^n$ is the Lie group $(\mathbb{R}^{2n+1}, o)$ equipped with the group action

$$\xi_0 \circ \xi = \left( x + x_0, y + y_0, t + t_0 + 2 \sum_{i=1}^{n} (x_i y_0_i - y_i x_0_i) \right),$$

(1.2)

for $\xi := (x_1, \ldots, x_n, y_1, \ldots, y_n, t) := (x, y, t) \in \mathbb{R}^{2n+1}$ and $\Delta_H$ is the subelliptic Laplacian on $H^n$ defined by

$$\Delta_H = \sum_{i=1}^{n} X_i^2 + Y_i^2$$

with

$$\begin{align*}
X_i &= \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t}, \\
Y_i &= \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t}.
\end{align*}$$

It is easy to check that $\Delta_H$ is a degenerate elliptic operator satisfying the Hormander condition of order one (see Section 2).

As an example of our results for the case where $D = H^n$ we prove that, under some conditions on the non negative coefficient $h$ and suitable restriction on the power $p$, any non negative smooth solution $u$ of (1.1) is identically zero. More precisely, denoting by $Q = 2n + 2$ the homogeneous dimension of $H^n$ and by $|\xi|_H$ the intrinsic distance of the point $\xi$ to the origin (see [6], [7]), namely

$$|\xi|_H = \left( \sum_{i=1}^{n} (x_i^2 + y_i^2)^2 + t^2 \right)^{\frac{1}{2}},$$

(1.3)

we have:

**Theorem 1.1.** – Let $u$ be a non negative solution of

$$\Delta_H u(\xi) + a|\xi|_H^q u^p(\xi) \leq 0 \quad \text{in } H^n,$$

(1.4)
where \( a \) is a positive constant and \( \gamma > -2 \).

Then, if \( 1 < p \leq \frac{\gamma + 2}{2 - \gamma} \), \( u \equiv 0 \).

A generalized version of this theorem is proved in section 3 below, where also several variants covering the cases when the equation holds in a half space or some “cone” in \( H^n \) are considered (see Theorem 3.2, 3.3, 3.4).

Let us point out that a common feature of our results is that we do not impose any condition on the behaviour of \( u \) for large \( |\xi|_H \), thus allowing \( u \) to be, \( a \) priori, singular at infinity.

Therefore our results can be viewed as the analogues, in the present degenerate elliptic setting, of previous ones due to Gidas-Spruck [10] for the uniformly elliptic case. However, our method of proof is rather inspired by [1], where Liouville type results are established for non negative solutions of

\[
\Delta u + a |x|^{\gamma} u^p \leq 0
\]

in a cone of \( \mathbb{R}^n \).

We wish to mention that non existence results for non negative solutions of semilinear equations on the Heisenberg group have been obtained previously by Garofalo-Lanconelli in [8]. Note, however, that the theorems in [8], based on Rellich-Pohozaev identities, differ considerably from those in the present paper since they require global integrability conditions on \( u \) and on the gradient of \( u \). (see also [5] for similar results in the uniformly elliptic case).

Finally, we point out that the Liouville theorems presented here are the basic tools for obtaining an \( a \) priori bound in the sup norm for solutions of the Dirichlet problem

\[
\begin{cases}
\Delta_H u + f(\xi, u) = 0 & \text{in } \Omega \subset \mathbb{R}^{2n+1}, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

under some growth conditions on \( f \). This can be done using a blow up technique on the lines of [10], [1], [2] and will be the object of a separate paper [3].

2. PRELIMINARY FACTS

In this section we collect for the convenience of the reader some known facts about the Heisenberg group \( H^n \) and the operator \( \Delta_H \) which will be useful later on. For their proof and more informations we refer for example to [6], [7], [8], [12], [13].
As mentioned in the introduction the Heisenberg group $H^n$ is the Lie group whose underlying manifold is $\mathbb{R}^{2n+1}$ $(n \geq 1)$, endowed with the group action,

$$\xi_0 \circ \xi = \left( x + x_0, y + y_0, t + t_0 + 2 \sum_{i=1}^{n} (x_i y_0, -y_i x_0,) \right),$$

for $\xi = (x_1, \ldots, x_n, y_1, \ldots, y_n, t) := (x, y, t)$.

The corresponding Lie Algebra of left-invariant vector fields is generated by $X_i, Y_i$ for $i = 1, \ldots, n$, and $T = \frac{\partial}{\partial t}$.

It is easy to check that $X_i$ and $Y_i$ satisfy $[X_i, Y_j] = -4t \delta_{i,j}$, $[X_i, X_j] = [Y_i, Y_j] = 0$ for any $i, j \in \{1, \ldots, n\}$. Therefore, the vector fields $X_i, Y_i$ $(i = 1, \ldots, n)$ and their first order commutators span the whole Lie Algebra. Hence, the Hormander condition of order one holds true for $\Delta_H$ (see [13]); this implies its hypoellipticity (i.e. if $\Delta_H u \in C^\infty$ then $u \in C^\infty$ (see [13])) and the validity of the maximum principle (see [4]).

An intrinsic metric can be defined on $H^n$ by setting

$$d_H(\xi, \eta) = |\eta^{-1} \circ \xi|_H$$

where $|\cdot|_H$ has been defined in (1.3), see [6]. Clearly in this metric the open ball of radius $R$ centered at $\xi_o$ is the set:

$$B_H(\xi_o, r) = \{ \eta \in H^n : d_H(\eta, \xi_o) < r \}.$$

It is also important to observe that $\xi \to |\xi|_H$ is homogeneous of degree one with respect to the natural group of dilations (see [6], [7]):

$$\delta_\lambda(\xi) = (\lambda x, \lambda y, \lambda^2 t). \quad (2.1)$$

Since the base $\{X_i, Y_i, T\}$ is obtained by the standard one $\{ \frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}, \frac{\partial}{\partial t} \}$, using the transformation

$$B = \begin{pmatrix} I_n & 0 & 2y \\ 0 & I_n & -2x \\ 0 & 0 & 1 \end{pmatrix}$$

whose determinant is identically 1, it follows that the Lebesgue measure is the Haar measure on $H^n$.

This fact, together with the homogeneity property of $|\xi|_H$ described above, implies that

$$|B_H(\xi_o, R)| = |B_H(0,1)| R^Q, \quad (2.2)$$
where \( Q = 2n + 2 \) is the homogeneous dimension of \( H^n \) (see [12]) and \( | \cdot | \) denotes the Lebesgue measure.

To conclude this section we recall some simple properties of \( \Delta_H \). Observe first that

\[
\Delta_H = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} + 4y_i \frac{\partial^2}{\partial x_i \partial t} - 4x_i \frac{\partial^2}{\partial y_i \partial t} + 4(x_i^2 + y_i^2) \frac{\partial^2}{\partial t^2}.
\]

It is easy to check that the operator \( \Delta_H \) is homogeneous of degree 2 with respect to the dilation \( \delta_\lambda \) defined in (2.1), namely

\[
\Delta_H(\delta_\lambda) = \lambda^2 \delta_\lambda(\Delta_H);
\]
also, for any fixed \( \xi^o \), by the left invariance of the vector fields \( X_i, Y_i \) with respect to the group action we have:

\[
\Delta_H(u(\xi^o \circ \xi)) = (\Delta_H u)(\xi^o \circ \xi) \quad \forall \xi \in H^n.
\]

The next remark concerns the action of \( \Delta_H \) on functions \( u \) depending only on \( \rho := |\xi|_H \). It is easy to show that

\[
\Delta_H u(\rho) = \psi \left[ \frac{\partial^2 u}{\partial \rho^2} + \frac{Q - 1}{\rho} \frac{\partial u}{\partial \rho} \right],
\]

where the function \( \psi \) is defined by

\[
\psi(\xi) = \frac{\sum_{i=1}^{n} (x_i^2 + y_i^2)}{\rho^2} = |\nabla_H \rho|^2 \quad \text{for} \ \xi \neq 0,
\]

where with \( \nabla_H u \) we denote the vector field \( (X_i u, Y_i u) \), for \( i = 1, \ldots, n \).

It is useful to observe that

\[
\Delta_H = \text{div}(\sigma^T \sigma \nabla) \quad \text{with} \ \sigma = \begin{pmatrix} I_n & 0 & 2y \\ 0 & I_n & -2x \end{pmatrix}.
\]

3. LIOUVILLE TYPE THEOREMS

In this section we will generalize to the Heisenberg group some Liouville type results which hold for positive solutions of superlinear equations associated to the laplacian, see [1], [2], [10].

**Theorem 3.1.** – Let \( u \) be a non negative solution of

\[
\Delta_H u(\xi) + f(\xi, u(\xi)) \leq 0 \ \text{in} \ H^n,
\]

where $f$ is a non negative function satisfying
\[ f(\xi, u) \geq h(\xi)u^p \] \tag{3.2}
for some function $h(\xi) \geq 0$ such that, for $|\xi|_H$ large,
\[ h(\xi) \geq K |\xi|_H^\gamma \]
for some $K > 0$ and $\gamma > -2$.
If $1 < p \leq \frac{Q+\gamma}{Q-2}$, then $u \equiv 0$.
Before the proof let us introduce a cut-off function $\phi_R$ which will be used throughout this section. Consider $\phi_R(\rho) := \phi\left(\frac{\rho}{R}\right)$, where $\rho := |\xi|_H$, $R > 0$, and $\phi$ satisfies:
\[
\begin{align*}
\phi & \in C^\infty[0, +\infty), \quad 0 \leq \phi \leq 1, \\
\phi & \equiv 1 \quad \text{on } \left[0, \frac{1}{2}\right], \\
\phi & \equiv 0 \quad \text{on } [1, +\infty), \\
-\frac{C}{R} & \leq \frac{\partial \phi_R}{\partial \rho} \leq 0, \\
\text{and} \quad \frac{\partial^2 \phi_R}{\partial \rho^2} & \leq \frac{C}{R^2} \quad \text{for some constant } C > 0.
\end{align*}
\tag{3.3}
\]

Proof. Set, for $R > 0$,
\[ I_R := \int_{B_H(0,R)} h(\xi)u^p \phi_R^{q} d\xi \quad \text{with } \frac{1}{p} + \frac{1}{q} = 1. \tag{3.4} \]
Observe that $I_R \geq 0$. Moreover, by equation (3.1) and (3.2)
\[ I_R \leq \int_{B_H(0,R)} f(\xi, u)\phi_R^{q} d\xi \leq -\int_{B_H(0,R)} \Delta_H u \phi_R^{q} d\xi; \tag{3.5} \]

hence an integration by parts yields,
\[
\begin{align*}
I_R & \leq -\int_{B_H(0,R)} u \Delta_H (\phi_R^{q}) d\xi + \int_{\partial B_H(0,R)} u \nabla_H (\phi_R^{q}) \cdot \nu_H dH_{2n} \\
& \quad - \int_{\partial B_H(0,R)} \phi_R^{q} \nabla_H u \cdot \nu_H dH_{2n} \leq -\int_{B_H(0,R)} u \Delta_H (\phi_R^{q}) d\xi \\
& \quad + \int_{\partial B_H(0,R)} u q \phi_R^{q-1} \phi_R' \nabla_H \rho \cdot \nu_H dH_{2n} \leq -\int_{B_H(0,R)} u \Delta_H (\phi_R^{q}) d\xi,
\end{align*}
\]
where \( \nu_H(\xi) = \sigma(\xi)\nu(\xi) \) and \( \nu \) is the normal to \( \partial \Omega \); \( dH_{2n} \) denotes the \( 2n \)-dimensional Hausdorff measure. On the other hand, as observed in Section 2 (see (2.3)),

\[
\Delta_H(\phi_R^q) = \psi \left[ \frac{\partial^2}{\partial\rho^2}(\phi_R^q) + \frac{Q-1}{\rho} \frac{\partial}{\partial\rho}(\phi_R^q) \right].
\] (3.6)

Thus we get, using the hypotheses on \( \phi_R \) and denoting by \( \Sigma_R := B_H(0, R) \backslash B_H(0, \frac{R}{2}) \),

\[
I_R \leq -\int_{\Sigma_R} \psi \left[ q\phi_R^{q-1}\phi_R'' + \frac{Q-1}{\rho} q\phi_R^{q-1}\phi_R' \right] d\xi \\
\leq \frac{C}{R^2} \int_{\Sigma_R} \psi\phi_R^{q-1} d\xi.
\]

Hence, the Hölder inequality yields:

\[
I_R \leq \frac{C}{R^2} \left[ \int_{\Sigma_R} u^p \rho^{-1}(\psi\phi_R^{q-1})^p d\xi \right]^{\frac{1}{p}} \left[ \int_{B_H(0,R)} \psi \rho^{-\frac{2n}{q}} d\xi \right]^{\frac{1}{q}}.
\] (3.7)

Choosing \( R > 0 \) sufficiently large, in \( \Sigma_R \), \( h \) satisfies \( h \geq \psi K \rho^{\gamma} \). Therefore,

\[
I_R \leq C \left[ \int_{\Sigma_R} u^p h \phi_R^q d\xi \right]^{\frac{1}{p}} R^{\left( \frac{-\gamma}{\rho} + \frac{Q}{q} - 2 \right)},
\] (3.8)

as \( 0 \leq \psi \leq 1 \). Then,

\[
I_R^{1-\frac{1}{p}} \leq CR^{\left( \frac{-\gamma}{\rho} + \frac{Q}{q} - 2 \right)}.
\]

Hence, if \( 1 < p < \frac{Q+\gamma}{Q-2} \), letting \( R \to +\infty \), we obtain

\[
I := \int_{\mathbb{H}^n} hu^p d\xi = 0.
\]

This implies \( u \equiv 0 \) for \( \rho \) large, since \( h \) is strictly positive outside of a set of measure zero and \( u \) is a priori non negative.

The claim follows now by the maximum principle (see [4]). In fact, choose \( \bar{R} > 0 \) in such a way that, for \( \rho \geq \bar{R} \), \( h > 0 \). Then, \( u \equiv 0 \) on the complementary of \( B_H(0, \bar{R}) \), as we proved. Hence, \( u \) satisfies:

\[
\begin{cases} 
  u \geq 0 & \text{in } B_H(0, \bar{R} + \delta), \\
  \Delta_H u \leq 0 & \text{in } B_H(0, \bar{R} + \delta), \\
  u \equiv 0 & \text{for } \bar{R} \leq \rho \leq \bar{R} + \delta,
\end{cases}
\]

for some $\delta > 0$. Therefore, by the maximum principle, since $u$ is not strictly positive, $u$ has to be identically zero.

If $p = \frac{Q+\gamma}{Q-2}$, we obtain, by (3.7), that $I$ is finite and that the right hand side of (3.7) tends to zero when $R$ goes to infinity. This yields $I = 0$ and we can conclude as above.

**Remark 3.1.** – If $h = K > 0$, we get by the previous theorem that, for $1 < p \leq \frac{Q}{Q-2}$, the unique solution of

$$
\Delta_H u + Ku^p \leq 0 \text{ in } H^n
$$

(3.9)
is $u \equiv 0$.

**Remark 3.2.** – The upper bound of the exponent $p$ is optimal. Indeed, we claim that the function $v(p) = C_\varepsilon (1 + \rho^2)^{-\frac{Q}{2}}$ with $\alpha = Q - 2 - \varepsilon$ and a suitable choice of $C_\varepsilon$ is a positive solution of

$$
\Delta_H u(\xi) + \psi(\xi)\rho^\gamma u^p(\xi) \leq 0 \text{ in } H^n,
$$

(3.10)
for $p \geq \frac{Q+\gamma-\varepsilon}{Q-2-\varepsilon}$.

Indeed, let $u(\rho) = (1 + \rho^2)^{-\frac{Q}{2}}$. Then $u$ satisfies:

$$
-\Delta_H u = -\psi \left[ \frac{\partial^2 u}{\partial \rho^2} + \frac{Q - 1}{\rho} \frac{\partial u}{\partial \rho} \right] \\
= \psi \alpha (1 + \rho^2)^{-(\frac{\gamma}{2}+2)} [Q(1 + \rho^2) - (\alpha + 2)\rho^2] \\
= \psi \alpha (1 + \rho^2)^{-(\frac{\gamma}{2}+2)} [\rho^2(Q - \alpha - 2) + Q] \\
\geq \psi \alpha (Q - \alpha - 2)(1 + \rho^2)^{-(\frac{\gamma}{2}+1)}. \quad (3.11)
$$

Hence, if we impose that

$$
Q - 2 > \alpha, \quad \frac{\alpha}{2} - \frac{\gamma}{2} \geq \left( \frac{\alpha}{2} + 1 \right), \quad (3.12)
$$

we can choose $c = (\alpha(Q - \alpha - 2))^{\frac{1}{p-1}}$ and $v = cu$ satisfies:

$$
-\Delta_H v \geq \psi(\alpha(Q - \alpha - 2))^{\frac{1}{p-1}} (1 + \rho^2)^{-p \frac{Q}{2} + \frac{\gamma}{2}} \geq \psi \rho^\gamma v^p.
$$

Now just choose $\alpha = Q - 2 - \varepsilon$ then (3.12) holds if $p \geq \frac{Q+\gamma-\varepsilon}{Q-2-\varepsilon}$ for any $\varepsilon$ positive.

The idea of the function $v$ was taken from Ramon Soranzo (personal communication to I.B.) who gave a similar counterexample for the Laplacian.
The next result concern the case where the unbounded domain \( D \) is an half-space.

**THEOREM 3.2.** - Let \( D \subset H^n \) be the set

\[
D = \left\{ \xi \in H^n : \sum_{i=1}^{n} a_i x_i + b_i y_i + \theta > 0, \right. \\
\left. \text{with } (a, b) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}, d \in \mathbb{R} \right\}.
\]

Let \( u \) be a non negative solution of

\[
\Delta_H u(\xi) + f(\xi, u(\xi)) \leq 0 \text{ in } D, \tag{3.13}
\]

where \( f \) is as in Theorem 3.1 with \( \gamma > -1 \).

If \( 1 < p \leq \frac{q+\gamma}{q-1} \), then \( u \equiv 0 \) in \( D \).

A similar result is valid for half-spaces which do not contain the \( t \)-direction or for particular cones. However, the upper bound of the exponent \( p \) is lower than in the previous case.

The following results hold:

**THEOREM 3.3.** - Let \( D \subset H^n \) be the set

\[
D = \{ \xi \in H^n : \sum_{i=1}^{n} a_i x_i + b_i y_i + ct + d > 0 \},
\]

for \( a, b \in \mathbb{R}^n, c \in \mathbb{R} \setminus \{0\}, d \in \mathbb{R}, \)

and let \( u \) be a non negative solution of

\[
\Delta_H u(\xi) + f(\xi, u(\xi)) \leq 0 \text{ in } D, \tag{3.14}
\]

with \( f \) as in theorem 3.1 and \( \gamma > 0 \).

Then, if \( 1 < p \leq \frac{q+\gamma}{q-1} \), \( u \equiv 0 \) in \( D \).

**THEOREM 3.4.** - Let \( \Sigma \) be the cone

\[
\Sigma = \left\{ \xi \in H^n : \sum_{i=1}^{n} (a_i x_i - b_i y_i)(b_i x_i + a_i y_i) > 0 \right\},
\]

and let \( u \) be a non negative solution of

\[
\Delta_H u(\xi) + f(\xi, u(\xi)) \leq 0 \text{ in } \Sigma, \tag{3.15}
\]

with \( f \) as in theorem 3.1 and \( \gamma > 0 \).
If \( 1 < p \leq \frac{Q+1}{Q} \), \( u \equiv 0 \) in \( \Sigma \).

The proofs of theorems 3.2, 3.3, 3.4 follow from the next lemma.

**Lemma 3.1.** Let \( D \subset H^n \) be an unbounded domain. Assume that \( \eta \) satisfies:

\[
\begin{cases}
\eta > 0 & \text{in } D, \\
\Delta_H \eta \geq 0 & \text{in } D, \\
\eta = 0 & \text{on } \partial D,
\end{cases}
\]

and let \( u \) be a non negative solution of

\[
\Delta_H u(\xi) + f(\xi, u(\xi)) \leq 0 \quad \text{in } D,
\]

with \( f \) as in Theorem 3.1. Then, for \( R > 0 \) large enough, where \( D_R := B_H(0, R) \cap D, \Omega_R := (B_H(0, R) \setminus B_H(0, \frac{R}{2})) \cap D, \) and \( q \) is the conjugate exponent of \( p \).

\[
I_R := \int_{D_R} h(\xi) u^p \phi_R^q \eta^q d\xi,
\]

the following estimate holds

\[
I_R \leq I_R^\frac{1}{q} \left( \frac{C}{R^2} \left[ \int_{\Omega_R} \eta^q \psi \frac{\nabla H \eta}{p} \cdot \nabla H \rho^{\frac{Q}{p}} d\xi \right] \right)^\frac{1}{q} + C \left( \int_{\Omega_R} \psi \left[ \nabla H \eta \cdot \nabla H \rho^{\frac{Q}{p}} d\xi \right] \right)^\frac{1}{q}
\]

(3.17)

for \( R > 0 \) large enough, where \( D_R := B_H(0, R) \cap D, \Omega_R := (B_H(0, R) \setminus B_H(0, \frac{R}{2})) \cap D, \) and \( q \) is the conjugate exponent of \( p \).

**Proof.** From equation (3.16), assumption (3.2) and the divergence’s theorem we get:

\[
I_R \leq - \int_{D_R} u \Delta_H (\eta^q \phi_R^q) d\xi + \int_{\partial D_R} u \nabla_H (\eta^q \phi_R^q) \cdot \nu_H dH_{2n}
\]

\[- \int_{\partial D_R} \eta^q \phi_R^q \nabla_H u \cdot \nu_H dH_{2n}.
\]

Moreover, since \( \phi_R = 0 \) on \( \partial B_H(0, R) \), \( \eta = 0 \) on \( \partial D \), and \( q > 1 \), the integrals on the boundary of \( D_R \) vanish and therefore,

\[
I_R \leq - \int_{D_R} u \Delta_H (\eta \phi_R^q) d\xi.
\]

Thus, using the properties of \( \phi_R \) and observing that, by the hypotheses made on \( \eta \),

\[
\Delta_H (\eta^q) = q(q - 1) \eta^{q-2} |\nabla H \eta|^2 + q \eta^{q-1} \Delta_H \eta > 0
\]

(3.18)
it results:

\[ I_R \leq - \int_{\Omega_R} u\eta^q \Delta_H(\phi_R^q) \, d\xi - 2 \int_{\Omega_R} u \nabla_H(\eta^q) \cdot \nabla_H(\phi_R^q) \, d\xi. \]

Using the properties of \( \phi_R \), as in the proof of Theorem 3.1 we obtain

\[ I_R \leq \frac{C}{R^2} \int_{\Omega_R} u\eta^q \psi \phi_R^{q-1} \, d\xi + \frac{C}{R} \int_{\Omega_R} u\eta^{q-1} \psi \phi_R^{q-1} \nabla_H \eta \cdot \nabla_H \rho \, d\xi. \quad (3.19) \]

Thus, the Hölder inequality yields:

\[
I_R \leq \frac{C}{R^2} \left[ \int_{\Omega_R} \psi \rho^\gamma u^p (\eta \phi_R^{q-1}) \, d\xi \right]^\frac{1}{p} \left[ \int_{\Omega_R} \eta^q \psi \rho^\frac{-\gamma}{p} \, d\xi \right]^\frac{1}{q} \\
+ \frac{C}{R} \left[ \int_{\Omega_R} \psi \rho^\gamma u^p (\eta \phi_R^{q-1}) \, d\xi \right]^\frac{1}{p} \left[ \int_{\Omega_R} \nabla_H \eta \cdot \nabla_H \rho |\eta^q \psi \rho^\frac{-\gamma}{p} \, d\xi \right]^\frac{1}{q} \\
\leq I_R^\frac{1}{p} \left( \frac{C}{R^2} \left[ \int_{\Omega_R} \eta^q \psi \rho^\frac{-\gamma}{p} \, d\xi \right]^\frac{1}{q} \\
+ \frac{C}{R} \left[ \int_{\Omega_R} \psi \nabla_H \eta \cdot \nabla_H \rho |\eta^q \rho^\frac{-\gamma}{p} \, d\xi \right]^\frac{1}{q} \right), \quad (3.20) \]

for \( R > 0 \) large enough. The statement is proved.

**Proof of Theorem 3.2.** – Consider, without loss of generality, the half space \( \{x_1 > 0\} \).

The claim is proved by using the estimate (3.17) applied to \( D = \{x_1 > 0\} \) and \( \eta = x_1 \).

Indeed, by the maximum principle, to show that \( u \equiv 0 \), it is enough to check that

\[
I_R := \int_{\{x_1 > 0\}} hu^p \phi_R^q x_1^q \, d\xi \to 0 \quad \text{when} \ R \to \infty, \quad (3.21)
\]

where \( \phi_R \) is as in (3.3).

If \( D_R := B_H(0, R) \cap \{x_1 > 0\} \), then (3.17) becomes:

\[
I_R \leq I_R^\frac{1}{p} \left( \frac{C}{R^2} \left[ \int_{\Omega_R} x_1^q \psi \rho^\frac{-\gamma}{p} \, d\xi \right]^\frac{1}{q} + \frac{C}{R} \left[ \int_{\Omega_R} \psi |\nabla_H \rho| q \rho^\frac{-\gamma}{p} \, d\xi \right]^\frac{1}{q} \right). 
\]

Therefore, as \( 0 \leq \psi \leq 1 \) and \( x_1 \leq CR \) in \( \Omega_R \), for \( p \leq \frac{Q+\gamma}{Q-1} \) we get:

\[
I_R \leq CI_R^\frac{1}{p} R^\left( \frac{Q+\gamma}{Q-1} \right), \quad (3.22)
\]
and we can conclude using the same arguments as in Theorem 3.1.

Proof of Theorem 3.3. – As in the proof of Theorem 3.2, the claim is proved using the estimate (3.17) of Lemma 3.1 with \( \eta = A \cdot x + B \cdot y + ct + d \) and \( D_R := B_H(0, R) \cap D \).

Let us consider the integral

\[
I_R := \int_D h^\rho \phi_R^q \eta^q d\xi, \quad (3.23)
\]

where \( \phi_R \) is as in (3.3). By (3.17), using the fact that

\[
\eta \leq CR^2
\]

\[
|\nabla_H \eta| = |(A + 2cy, B - 2cx)| \leq CR
\]

we obtain:

\[
I_R \leq I_R^{\frac{1}{2}} \left( \frac{C}{R^2} \left[ \int_{\Omega_R} |\eta^q \psi \rho \frac{-2q}{\rho} d\xi \right]^{\frac{1}{q}} + \frac{C}{R} \left[ \int_{\Omega_R} \psi |\nabla_H \eta \cdot \nabla_H \rho|^q \rho \frac{-2q}{\rho} d\xi \right]^{\frac{1}{q}} \right) \leq C I_R^{\frac{1}{2}} R^{\frac{Q + \frac{Q}{p}}{2}}, \quad (3.25)
\]

If \( 1 < p \leq \frac{Q + \gamma}{Q} \) we can conclude as in the previous cases.

Proof of Theorem 3.4. – This result follows from the estimate (3.17) by choosing \( \eta := \sum_{i=1}^n (a_i x_i - b_i y_i) (b_i x_i + a_i y_i) \) and \( D := \Sigma \). Since the function \( \eta \) has the same behaviour as the function \( \eta \) chosen in the proof of Theorem 3.3, we can conclude in the same way.

Remark 3.3. – Let us observe that, instead of inequality (3.17), one can similarly obtain

\[
I_R \leq I_R^{\frac{1}{2}} \left( \frac{1}{R^2} \left[ \int_{\Omega_R} \eta^q \psi h^{-\frac{q}{p}} d\xi \right]^{\frac{1}{q}} + \frac{1}{R} \left[ \int_{\Omega_R} \psi h^{-\frac{q}{p}} |\nabla_H \eta \cdot \nabla_H \rho|^q d\xi \right]^{\frac{1}{q}} \right), \quad (3.26)
\]

provided \( f \) satisfies (3.2) for some \( h \geq 0 \) such that the right hand side of (3.26) exists.

Consequently, if \( h \) verifies:

\[
\lim_{R \to +\infty} \frac{1}{R^q} \int_0^R h^{-\frac{q}{p}}(\rho \omega) \rho^{Q-1} d\rho = 0
\]

where \( \omega = \frac{\xi}{|\xi|^p} \), then the conclusion of Theorem 3.2 holds true. Similar conditions on \( h \) and \( p \) can be given for Theorems 3.3 and 3.4.
For the sake of completeness, we will also prove a Liouville theorem for bounded solutions of $\Delta_H u = 0$ in the whole space $H^n$.

**Theorem 3.5.** - If $u$ is a bounded function such that $\Delta_H u = 0$ in the whole space $H^n$, then $u$ is a constant.

The proof is based on the following representation formula for Heisenberg harmonic functions. This formula can be proved easily by using the divergence’s theorem, see e.g. Gaveau ([9]) for details.

**Lemma 3.2.** - Let $w$ satisfy $\Delta_H w = 0$ in $H^n$. Then, for any $\xi \in H^n$,

$$w(\xi) = \frac{C_Q}{R^Q} \int_{B_H(\xi, R)} w(\eta)\psi(\eta)d\eta,$$

where $\psi$ is defined in (2.4), and $C_Q = |B_H(\xi, 1)|^{-1}$.

**Proof of Theorem 3.5.** - Let us first prove that $\frac{\partial w}{\partial t} \equiv 0$. Observe that, in view of the Hormander condition, the vector field $T = \frac{\partial}{\partial t}$ commutes with $X_i$ and $Y_i$, i.e. $T(X_i) = X_i(T)$ and $T(Y_i) = Y_i(T)$. Hence,

$$\Delta_H(Tw) = T(\Delta_H w) = 0.$$

Therefore, applying the previous lemma, we get:

$$\frac{\partial w}{\partial t}(\xi) = \frac{C_Q}{R^Q} \int_{B_H(\xi, R)} \frac{\partial w}{\partial t}(\eta)\psi(\eta)d\eta$$

$$= -\frac{C_Q}{R^Q} \int_{B_H(\xi, R)} \frac{\partial \psi}{\partial t}(\eta)w(\eta)d\eta + \frac{C_Q}{R^Q} \int_{\partial B_H(\xi, R)} \psi \nu_i dH_{2n},$$

where $\nu_i$ is the $t$-component of the exterior unit normal vector to $B_H(\xi, R)$. Since

$$\left| \frac{\partial \psi}{\partial t} \right| = \frac{|\psi|}{\rho^4} \leq \frac{1}{\rho^2}$$

$$|\nu_t| = \frac{|t|}{2\rho^3} \leq \frac{1}{\rho|\nabla \rho|},$$

from (2.2) we obtain that

$$\left| \frac{\partial w}{\partial t}(\xi) \right| \leq \frac{C||w||_{L^\infty}}{R^2}$$

for any $\xi \in H^n$ and for any $R > 0$. Thus, letting $R$ go to infinity, we get $\frac{\partial w}{\partial t}(\xi) = 0$ for any $\xi \in H^n$. Then, $w$ is a bounded solution of

$$\sum_{i=1}^n \frac{\partial^2 w}{\partial x_i^2} + \frac{\partial^2 w}{\partial y_i^2} + \frac{\partial^2 w}{\partial t^2} = 0 \quad \text{in } \mathbb{R}^{2n+1}.$$

Therefore it has to be constant by the classical Liouville theorem (see e.g. [11]).

ACKNOWLEDGEMENTS

This paper was completed during a visit of I.B. and I.C.D. at the Université Paris-Dauphine. They wish to thank the CEREMADE for the kind hospitality.

REFERENCES


(Manuscript received April 18, 1995.)