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Addenda to the book "Critical points at infinity in some variational problems" and to the paper "The scalar-curvature problem on the standard three-dimensional sphere"

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Addenda to the book "Critical points at infinity in some variational problems" and to the paper "The scalar-curvature problem on the standard three-dimensional sphere"

by

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In the sequel, [1] stands for the first reference (the Pitman Research Notes n° 182) while [2] stands for the second reference (*Journal of Functional Analysis*, Vol. **95**, n° 1, pp. 106-172).

In [2], the arguments are more specific, since $n = 3$.

However at least for the first addendum, the modifications for [1] and [2], are the same, up to the numeration of the formulae; therefore, we present them together.

ADDENDUM 1

The argument fiven in (3.78)-(3.82), page 76, of [1] and in (B.35)-(B.41), pages 169-170 of [2], should be modified, although the general line of proof of Lemma 3.2 of [1) and of A.18 of Lemma A2 in [2] remains unchanged.

There is a misprint in (3.46) of [1], which should read:

$$\int_{\partial B_1} \bar{h} = \frac{1}{\lambda^{n/2}} \int_{\partial B_\lambda} h.$$

Also in (3.77) of [1] and (B.35) of [2], which should read:

$$\int_{\tilde{W}} \bar{\delta}_1^5 \,|\, \bar{\phi} - \tilde{\psi}_1 \,| \leq C \sup_{\partial \tilde{W}} \left| \frac{\partial}{\partial n} (\bar{\delta}_1 - \bar{\theta}_1) \right| \frac{1}{\bar{\lambda}_2^{n/2} \, \varepsilon_{12}^{n/2\,(n-2)}} \left(\int |\nabla \phi|^2 \right)^{1/2}.$$

A related misprint is completed in, (3.82) of [1] and (B.40) of [2], where $r^{n/2}$ should replace $r^{(n-2)/2}$ in the definition of ρ.

The conclusion of the proofs remain unchanged, up to this change of exponent in the definition of ρ $\Big($when $|\, x_2 \,| \geq 1$, ρ is now upperbounded by $\left(\frac{C}{|\, x_2 \,|^{n-1}} \times \frac{|\, x_2 \,|^{n/2}}{\bar{\lambda}_2^{n/4}} \right) \cdot \Big)$:

These are minor modifications.

However, we need a more substantial change in the proof because (3.78) of [1] and (B.36) of [2] are difficult to prove. It is not easy to upperbound – although true – $\sup\limits_{\partial \tilde{W}} \left| \frac{\partial}{\partial n} \tilde{\theta}_1 \right|$ by $C \sup\limits_{\partial \tilde{W}} \left| \frac{\partial}{\partial n} \tilde{\delta}_1 \right|$.

We thus introduce the following modification in the proof :

Let $G_{\tilde{W}}$ be the Dirichlet Green's function on $\tilde{W}$.

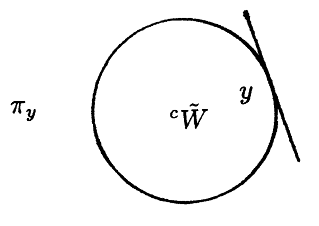
Then,

$$\bar{\delta}_1 - \bar{\theta}_1 = \int_{\tilde{W}} G_{\tilde{W}}(x, y) \, \bar{\delta}_1^{(n+2)/(n-2)}(x) \, dx$$

and

$$\frac{\partial}{\partial n} (\bar{\delta}_1 - \bar{\theta}_1) = \int_{\tilde{W}} \frac{\partial}{\partial n_y} G_{\tilde{W}}(x, y) \, \bar{\delta}_1^{(n+2)/(n-2)}(x) \, dx.$$

${}^c\tilde{W}$ is a ball of radius r. Therefore, for any y $\partial \tilde{W}$, ${}^c\tilde{W}$ is contained in the half-space π_y, whose boundary $\partial \pi_y$ is tangent to $\partial \tilde{W}$ at y.



Thus,

$$\left| \frac{\partial}{\partial n_y} (G_{\tilde{W}^c}(x, y)) \right| \leq \left| \frac{\partial}{\partial n_y} G_{\pi_y}(x, y) \right| \leq \frac{C}{|\, x - y \,|^{n-1}}.$$

On the other hand, computing as if ${}^c\tilde{W}$ was centered at zero:

$$G_{\tilde{W}}(x, y) = \frac{(r^{n-2})^2}{|x|^{n-2}|y|^{n-2}} G_{\tilde{W}^c}\left(\frac{r^2 x}{|x|^2}, \frac{r^2 y}{|y|^2}\right)$$

(r is the radius of ${}^c\tilde{W}$). For the unit ball

$$B,\ G_{B^c}(x, z) = \frac{1}{|z|^{n-2}|x|^{n-2}} \cdot G_B\left(\frac{x}{|x|^2}, \frac{z}{|z|^2}\right).$$

Since $\tilde{W} = rB^c$,

$$\begin{aligned} G_{\tilde{W}}(x, y) &= \frac{1}{r^{n-2}} G_{B^c}\left(\frac{x}{r}, \frac{y}{r}\right) = \frac{r^{n-2}}{|y|^{n-2}|x|^{n-2}} G_B\left(\frac{rx}{|x|^2}, \frac{ry}{|y|^2}\right) \\ &= \frac{(r^{n-2})^2}{|y|^{n-2}|x|^{n-2}} G_{{}^c\tilde{W}}\left(\frac{r^2 x}{|x|^2}, \frac{r^2 y}{|y|^2}\right)\Bigg). \end{aligned}$$

Thus, since $|y| = r$, since $|G_{c_{\tilde{W}}}(x', y')| \leq \frac{c}{|x'-y'|^{n-2}}$ and $\frac{r^2 y}{|y|^2} = y$:

$$\begin{aligned} \left|\frac{\partial}{\partial n_y} G_{\tilde{W}}(x, y)\right| &\leq \frac{\bar{c} r^{n-2}}{|x|^{n-2}} \frac{1}{\left|\frac{r^2 x}{|x|^2} - y\right|^{n-1}} + \frac{r^{n-2}}{r|x|^{n-2}} \frac{1}{\left|\frac{r^2 x}{|x|^2} - y\right|^{n-2}} \\ &= \frac{\bar{c}}{r|x|^{n-2}\left|\frac{rx}{|x|^2} - y\right|^{n-1}} + \frac{\bar{c}}{r|x|^{n-2}\left|\frac{rx}{|x|^2} - \frac{y}{r}\right|^{n-2}}. \end{aligned}$$

Since $|y| = r$:

$$\left|\frac{rx}{|x|^2} - \frac{y}{r}\right| = \frac{|x-y|}{|x|}.$$

Therefore:

$$\begin{aligned} \left|\frac{\partial}{\partial n_y} G_{\tilde{W}}(x, y)\right| &\leq \frac{\bar{C}|x|}{r|x-y|^{n-1}} + \frac{\bar{C}}{r|x-y|^{n-2}} \\ &\leq C\left(\frac{1}{|x-y|^{n-1}} + \frac{1}{r|x-y|^{n-2}}\right). \end{aligned}$$

This inequality is translation invariant and therefore holds whatever the center of $\tilde{W}$ is (not necessary zero, anymore).

Thus, since $-\Delta\bar{\delta}_1 = \bar{\delta}_1^{(n+2)/(n-2)}$ in R^n :

$$\left|\frac{\partial}{\partial n}(\bar{\delta}_1 - \bar{\theta}_1)\right| \le C\left(\int_{R^n}\frac{1}{|x-y|^{n-1}}\,\bar{\delta}_1^{(n+2)/(n-2)}(x)\,dx + \frac{1}{r}\,\bar{\delta}_1(y)\right)$$

$$= C\left(\int_{R^n}\frac{1}{|x|^{n-1}}\,\bar{\delta}_1(x-y)^{(n+2)/(n-2)}\right)$$

$$+ \frac{1}{r}\,\bar{\delta}_1(y) < \left(\frac{10\,C}{\sqrt{1+|y|^2}} + \frac{C}{r}\right)\bar{\delta}_1(y)$$

$$+ C\int_{|x|\le\frac{1}{10}\sqrt{1+|y|^2}}\frac{1}{|x|^{n-1}}\,\bar{\delta}_1(x-y)^{(n+2)/(n-2)}.$$

Observe now that, if $|x| < \frac{1}{10}\sqrt{1+|y|^2}$, either $|y| \le 1$–then $|x| \le 2$– and

$$c_2 \le \bar{\delta}_1(x-y) \le c_1; \qquad c_2 \le \bar{\delta}_1(y) \le c_1$$

where c_1 and c_2 are universal positive constants.

Thus:

$$\int_{|x|\le\frac{1}{10}\sqrt{1+|y|^2}}\frac{1}{|x|^{n-1}}\,\bar{\delta}_1(x-y)^{(n+2)/(n-2)} \le C\,\bar{\delta}_1(y)^{n+2)/(n-2)}.$$

Or $|y| > 1$, then $|x| \le \frac{1}{5}|y|$ and $\bar{\delta}_1(x-y) \le c_1\,\bar{\delta}_1(y)$. Thus,

$$\int_{|x|\le\frac{1}{10}\sqrt{1+|y|^2}}\frac{1}{|x|^{n-1}}\,\bar{\delta}_1(x-y)^{(n+2)/(n-2)}dx$$

$$\le C\,\bar{\delta}_1(y)^{n+2)/(n-2)}(1+|y|^2)^{1/2} \le \frac{1}{(1+|y|^2)^{(n+1)/2}}.$$

Combining both estimates, we derive:

$$\sup_{\partial\tilde{W}}\left|\frac{\partial}{\partial n}(\bar{\delta}_1 - \bar{\theta}_1)\right|$$

$$\le C\left(\sup_{\partial\tilde{W}}\frac{1}{(1+|x|^2)^{(n-1)/2}} + \frac{1}{r}\sup_{\partial\tilde{W}}\frac{1}{(1+|x|^2)^{(n-2)/2}}\right).$$

Thus,

$$\left|\int_{\tilde{W}}\bar{\delta}_1^{(n+2)/(n-2)}(\bar{\phi} - \tilde{\psi}_1)\right| \le C\left(\frac{r}{(1+r^2+|x_2|^2-2\,r\,|x_2|)^{1/2}} + 1\right)$$

$$\times\frac{r^{(n-2)/2}}{(1+r^2+|x_2|^2-2\,r\,|x_2|)^{(n-2)/2}}\left(\int|\nabla\phi|^2\right)^{1/2}.$$

We set then:

$$\rho = \frac{r^{(n-2)/2}}{(1 + r^2 + |\, x_2 \,|^2 - 2\, r \,|\, x_2 \,|)^{(n-2)/2}} \times \left(1 + \frac{r}{(1 + r^2 + |\, x_2 \,|^2 - 2\, r \,|\, x_2 \,|)^{1/2}}\right).$$

The remainder of the argument is unchanged.

The proof of Lemma 3.2 of [1] and Lemma A2 of [2] is, now, fully transparent.

ADDENDUM 2

In [1], F22 and F23 have not been established. Instead, slightly weaker estimates, F22′ and F23′ have been established.

We neverthless used F22 and F23 when we described the normal form of the dynamical system near infinity.

Checking F22 and F23 is a quite long process, that we never completed, although the proof should be quite similar to previous estimates.

If we only use F22′ and F23′, then the early estimates for the matrices A and A', in the section 4 of [1], are slightly changed, – the estimates are numbered (4.16)-(4.22) – by the introduction of a logarithmic factor $\log \varepsilon_{ij}^{-1}$ in certain terms, namely those corresponding to $\frac{1}{\lambda_i \lambda_j} \int \nabla \frac{\partial \delta_i}{\partial x_i} \nabla \frac{\partial \delta_j}{\partial x_j}$ and $\frac{1}{\lambda_i} \int \nabla \frac{\partial \delta_i}{\partial x_i} \nabla \lambda_j \frac{\partial \delta_j}{\partial x_j}$.

Observe that – by very easy estimates – both terms are $0\,(\varepsilon_{ij})$.

Therefore, the remainder of the estimates on A and A', in particular in (4.53)-(4.54), holds without change. The remainder of section 4, in particular Lemmas 4.1 and 4.2, is unchanged.

ADDENDUM 3

To the regret of the author, the misprints of [1] are many. Most are meaningless and can be easily corrected.

A misprint in (7.21) of [1] has nevertheless obscured the proof of Proposition 7.2. There is a misprint in the statement of Proposition 7.2 where $-L$ should be replaced by $-\Delta$. This holds also for the statement of Theorem 1 of [2].

Proposition 7.3 holds only for $n = 3$; it is used only in this case in [1] and the proof of this Proposition-provided in the Appendix of [1]–displays this fact clearly.

(ii) of Proposition 7.2 has been proved in [2], Lemma 5. We do not need to repeat the proof here.

However, for (i) of Proposition 7.2, the only proof is in [1].

Unifortunately, in (7.21), the best estimate on $|v|_H$ we can derive from Lemma 4.1, in dimension $n \geq 6$, is:

$$|v|_H < O(|\partial J(\Sigma \alpha_k \delta_k + v)| + \Sigma \varepsilon_{ij}^{(n+2)/2(n-2)} (\log \varepsilon_{ij}^{-1})^{(n+2)/2n})$$

which is only $o(\sum_{k \neq r} \varepsilon_{kr}^{1/2}) + O(|\partial J(\Sigma \alpha_k \delta_k + v)|)$.

This is, in fact, only a misprint and the statement of (i) Proposition 7.2 (namely that the Yamabe flow satisfies the Palais-Smale condition on decreasing flow-lines for the Yamabe functional on S^n equipped with its standard metric) remains unchanged, as well as the essential argument.

Since the argument might have been obscured by the misprint, we provide herre a slight modification, which clarifies the line of proof:

We first observe that the first part of the proof of Lemma 5 of [2] holds in any dimension–as well as Lemma A1 of [2].

In particular λ (100) of [2] holds. Using Lemma A1 of [2], and the fact that $|v|^2 = o(\Sigma \varepsilon_{kr})$, we easily derive from (100):

$$\dot{\alpha}_i = \frac{1}{C_0} (\partial J(u), \delta_i)_{-L} + o(\Sigma \varepsilon_{kr}) + O(|\partial J(u)|^2)$$

$$\lambda_i \dot{x}_i = -\frac{1}{C_1 \alpha_i} \left(\partial J(u), \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial x_i}\right)_{-L} + o(\Sigma \varepsilon_{kr}) + O(|\partial J(u)|^2)$$

$$\frac{\dot{\lambda}_i}{\lambda_i} = -\frac{1}{C_2 \alpha_i} \left(\partial J(u), \lambda_i \frac{\partial \delta_i}{\partial x_i}\right)_{-L} + o(\Sigma \varepsilon_{kr}) + O(|\partial J(u)|^2).$$

J is the Yamabe functional on (S^n, c).

Using estimate $G7$ of [1] – observe that, since K is constant, the term O_K in $G7$ can be dropped out here; also, there is no boundary, therefore other terms drop – we derive that

$$\left(\alpha_i^{4/(n-2)}/\alpha_j^{4/(n-2)} = 1 + o(1); \quad \left(v, \lambda \frac{\partial \delta_i}{\partial \lambda_i}\right)_{-L} = 0\right)$$

$$\begin{aligned}
\left(\partial J(u), \lambda \frac{\partial \delta_i}{i\partial \lambda_i}\right)_{-L} &= \left(\partial J(\Sigma \alpha_j \delta_j), \lambda \frac{\partial \delta_i}{\partial \lambda_i}\right)_{-L} + o(\Sigma \varepsilon_{kr}) + O(|v|^2) \\
&\quad + O_\varepsilon\left(\int_{|v| \geq \varepsilon \Sigma \alpha_j \delta_j} \delta_i |v|^{(n+2)/(n-2)}\right) \\
&= \left(\partial J(\Sigma \alpha_j \delta_j), \lambda_i \frac{\partial \delta_i}{\partial \lambda_i}\right)_{-L} \\
&\quad + o(\Sigma \varepsilon_{kr}) + O(|v|^2) + O_\varepsilon\left(\int |v|^{2n/(n-2)}\right) \\
&= \left(\partial J(\Sigma \alpha_j \delta_j), \lambda_i \frac{\partial \delta_i}{\partial \lambda_i}\right)_{-L} + o(\Sigma \varepsilon_{kr}) + O\left(|\partial J(u)|^2\right)
\end{aligned}$$

The F-estimates of [1] allow then to derive (7.24) (just as in (4.10)-(4.11) of [1]). Proposition 7.3 is not needed for this purpose, contrary to what is written in [1], and this is quite obvious. The remainder of the argument of Proposition 7.2 of [1] is unchanged. It is quite similar to the proof of Lemma 5 of [2]. Q.E.D.

REFERENCES

[1] A. Bahri, Critical points at infinity in some variational problems, *Pitman Research Notes in Mathematics*, n° 182.

[2] A. Bahri, J. M. Coron, The scalar curvature problem on the standard three dimensional sphere, *Journal of Functional Analysis,* Vol. **95**, n° 1, pp. 106-172.