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Multiplicity of positive and nodal solutions for nonlinear elliptic problems in $\mathbb{R}^N$

by

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ABSTRACT. – We are concerned with the multiplicity of positive and nodal solutions of

$$\begin{cases} -\Delta u + \mu u = Q(x)|u|^{p-2}u & \text{in} \quad \mathbb{R}^N \\ u \in H^1(\mathbb{R}^N) \end{cases}$$

where $2 < p < \dfrac{2N}{N-2}$, $N \geq 3, \mu > 0$, $Q \in C(\mathbb{R}^N)$ and $Q(x) \geq 0$ for $x \in \mathbb{R}^N$. We show how the "shape" of the graph of $Q(x)$ affects the number of both positive and nodal solutions.

Key words: Nonlinear elliptic problems, multiplicity of solutions

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1. INTRODUCTION

We consider the multiplicity question of both positive and nodal solutions (solutions which change sign) of the following problem

$$(1.1) \qquad \begin{cases} -\Delta u + \mu u = Q(x)|u|^{p-2}u & \text{in} \quad \mathbb{R}^N \\ u \in H^1 \end{cases}$$

where $N \geq 3$, $2 < p < \dfrac{2N}{N-2}$, $\mu > 0$, H^1 denotes the usual Sobolev space $H_0^1(\mathbb{R}^N)$, and $Q \in C(\mathbb{R}^N)$ is assumed to satisfy the following condition:

Condition (Q)

$Q(x) \geq 0$ in $\mathbb{R}^N$ and there exist some points $a^1, \ldots, a^k$ in $\mathbb{R}^N$ such that $Q(a^j)$ are strict maximums and satisfy

$$Q(a^j) = Q_{\text{Max}} \equiv \text{Max}\{Q(x) : x \in \mathbb{R}^N\} > 0, \quad j = 1, \ldots, k.$$

Our objective is to establish the existence of at least k positive solutions and k nodal solutions of problem (1.1). Our main result is:

THEOREM A. – *Assume condition* (Q) *holds. Then there exists* $\mu_o > 0$ *such that problem* (1.1) *has at least k positive and k nodal solutions for each* $\mu \geq \mu_o$.

It is known that if Q is a positive constant, (1.1) has a unique positive solution for each $\mu > 0$ [10], and infinitely many radially symmetric nodal solutions. When $Q(x)$ is not a positive constant, the existence of a positive solution has been established by several authors under various conditions. We mention, in particular, results by A. Bahri and P. L. Lions [3], P. L. Lions [13], Yi Li [11], A. Bahri and Y. Y. Li [2], D. M. Cao [6]. In [2], [3], [11], [13], $Q(x)$ is required to satisfy

$$\begin{aligned} &Q(x) \to \overline{Q} \quad \text{as } |x| \to \infty \text{ and} \\ &Q(x) - \overline{Q} \geq -C\exp(-\delta|x|) \quad \text{as} \quad |x| \to \infty, \end{aligned}$$

for some constants $C, \delta > 0$.

In [6], $Q(x)$ is required to satisfy

$$Q(x) \geq 2^{(2-p)/2}\overline{Q} \quad \text{for} \quad x \in \mathbb{R}^N.$$

Regarding nodal solutions we mention a result by X.P. Zhu [17] where existence of at least one nodal solution is established provided $Q(x)$ satisfies

$$Q(x) - \overline{Q} \geq \frac{C}{|x|^m} \quad \text{as} \quad |x| \to \infty$$

for some constants $C, m > 0$.

In this paper, we do not require $Q(x)$ to satisfy any asymptotic property, all of the above conditions may fail in our case. In fact, it is easy to construct examples of $Q(x)$ for which none of the above criteria are satisfied and condition (Q) holds. When $Q(x)$ is a positive constant, $\mathbb{R}^N$ is replaced by a bounded or an exterior domain, V. Benci and G. Cerami [4], G. Cerami and D. Passaseo [7], A. Bahri and P.L. Lions [3] have considered the effect of domain topology on the existence of positive solutions. Roughly speaking, if Ω has a "rich" topology then the problem

$$\begin{cases} -\Delta u + \mu u = |u|^{p-1} u & x \in \Omega \\ \quad u|_{\partial\Omega} = 0 \end{cases}$$

has many positive solutions for larger μ.

In this paper $\mathbb{R}^N$ has a trivial topology. Our emphasis here is on the "shape" of $Q(x)$. Our result shows how the "shape" of the graph of $Q(x)$ affects the number of both positive and nodal solutions.

Our arguments are based on a combination of the concentration-compactness principle of P. L. Lions [12], and Ekeland's variational principle [9].

In Section 2, we give some notations and preliminary results. In Section 3, we first establish two results concerning the compactness of minimizing sequences and then give a proof of Theorem A.

2. NOTATIONS AND PRELIMINARY RESULTS

For $a > 0$, let $C_a(a^j)$ denote the hypercube $\prod_{i=1}^{N} (a_i^j - a,\ a_i^j + a)$ centred at $a^j = (a_i^j)$, $j = 1, \ldots, k$; $i = 1, \ldots, N$. Let $\overline{C}_a(a^j)$ and $\partial C_a(a^j)$ denote the closure and the boundary of $C_a(a^j)$ respectively.

Set $\lambda = \dfrac{1}{\sqrt{\mu}}$, $v(x) = \lambda^{2/(p-2)} u(\lambda x)$. Then (1.1) becomes

$$\begin{cases} -\Delta v + v = Q(\lambda x)|v|^{p-2} v \quad \text{in} \quad \mathbb{R}^N \\ v \in H^1. \end{cases} \tag{2.1}$$

Set $Q_\lambda = Q(\lambda x)$.

For $u \in H^1$, $c \in \mathbb{R}$ and nonnegative bounded functions $b \in C(\mathbb{R}^N)$, define

$$
\begin{aligned}
& I_{b(x)}(u) = \frac{1}{2}\int |\nabla u|^2 + u^2 - \frac{1}{p}\int b(x)|u|^p \\
(2.2) \qquad & M^c_{b(x)} = \{u \in H^1 : u \not\equiv 0 \text{ and } \langle I'_{b(x)}(u), u\rangle = c\} \\
& I^c_{b(x)} = \inf\{I_{b(x)}(u) : u \in M^c_{b(x)}\}
\end{aligned}
$$

where $I'_{b(x)}$ denotes the Fréchet derivative of $I_{b(x)}$.

We will write $I_{b(x)}(u)$, $M^c_{b(x)}$, $I^c_{b(x)}$ and $I'_{b(x)}$ simply as $I_b(u)$, M^c_b, I^c_b and I'_b if there is no confusion. We will also write M^0_b, I^0_b as M_b and I_b.

Choose numbers $K, \ell > 0$, so that $\overline{C}_\ell(a^j)$ are disjoint, $Q(x) < Q(a^j)$ for $x \in \partial C_\ell(a^j)$ for $j = 1, \ldots, k$, and $\bigcup\limits_{j=1}^{k} C_\ell(a^j) \subset \prod\limits_{i=1}^{N}(-K, K)$. This is possible by the assumptions on Q.

Define $\phi_\lambda \in C(\mathbb{R}, \mathbb{R})$, $g_\lambda \in C(H^1, \mathbb{R}^N)$ by

$$
(2.3) \qquad \phi_\lambda(t) = \begin{cases} \dfrac{2K}{\lambda} & t > \dfrac{2K}{\lambda} \\ t & -\dfrac{2K}{\lambda} \le t \le \dfrac{2K}{\lambda} \\ -\dfrac{2K}{\lambda} & t < -\dfrac{2K}{\lambda} \end{cases}
$$

$$
\begin{aligned}
(2.4) \qquad & g^i_\lambda(u) = \frac{\int \phi_\lambda(x_i)|u|^p}{\int |u|^p} \qquad i = 1, 2, \ldots, N \qquad \text{and} \\
& g_\lambda(u) = (g^i_\lambda(u)).
\end{aligned}
$$

All our integrals are over $\mathbb{R}^N$ unless otherwise stated.

Let $C^j_{\ell/\lambda} \equiv C_{\ell/\lambda}(\frac{a^j}{\lambda})$, and for $j = 1, \ldots, k$, let

$$
\begin{aligned}
& N^j_\lambda = \{u \in H^1 : u \in M_{Q_\lambda} \text{ and } g_\lambda(u) \in C^j_{\ell/\lambda}\} \\
(2.5) \qquad & O^j_\lambda = \{u \in H^1 : u \in M_{Q_\lambda} \text{ and } g_\lambda(u) \in \partial C^j_{\ell/\lambda}\} \\
& \Lambda^j_\lambda = \{u \in H^1 : u^\pm \in N^j_\lambda\} \\
& \partial\Lambda^j_\lambda = \{u \in H^1 : u^\pm \in N^j_\lambda \cup O^j_\lambda \text{ and } u^+ \text{ or } u^- \in O^j_\lambda\}
\end{aligned}
$$

where $u^+ = \max\{u, 0\}$, $u^- = u^+ - u$.

It is easy to verify that $N_\lambda^j, O_\lambda^j, \Lambda_\lambda^j$ and $\partial\Lambda_\lambda^j$ are non-empty sets for $j = 1, \dots, k$. Define for $j = 1, \dots, k$

$$\begin{aligned} J_\lambda^j &= \inf\{I_{Q_\lambda}(u) : u \in \Lambda_\lambda^j\} \\ \tilde{J}_\lambda^j &= \inf\{I_{Q_\lambda}(u) : u \in \partial\Lambda_\lambda^j\} \\ P_\lambda^j &= \inf\{I_{Q_\lambda}(u) : u \in N_\lambda^j\} \\ \tilde{P}_\lambda^j &= \inf\{I_{Q_\lambda}(u) : u \in O_\lambda^j\} \end{aligned} \tag{2.6}$$

The main results of this section is included in the following proposition:

PROPOSITION 2.1. – *Assume condition (Q) holds. Then there exists $\lambda_0 > 0$ such that for each $\lambda \in (0, \lambda_0)$, $j = 1, \dots, k$*

(1) *$J_\lambda^j < 3I_{Q_{\mathrm{Max}}}$ and J_λ^j has a minimizing sequence $\{u_n^j\} \subset \Lambda_\lambda^j$ satisfying*

$$\begin{aligned} &I_{Q_\lambda}(u_n^j) \to J_\lambda^j \\ &I'_{Q_\lambda}(u_n^j) \to 0 \qquad in \quad H^{-1} \end{aligned}$$

as $n \to \infty$.

(2) *$P_\lambda^j < 2I_{Q_{\mathrm{Max}}}$, and P_λ^j has a minimizing sequence $\{v_n^j\} \subset N_\lambda^j$ satisfying*

$$\begin{aligned} &I_{Q_\lambda}(v_n^j) \to P_\lambda^j \\ &I'_{Q_\lambda}(v_n^j) \to 0 \qquad in \ H^{-1} \end{aligned}$$

as $n \to \infty$.

The proof will be accomplished by a series of lemmas.

LEMMA 2.2. – *$I_b^c = \frac{c}{2}$ for $c > 0$, and*

$$I_b^o \leq I_b^c + I_b^{-c} - \frac{p-2}{2p}|c| \quad for\ any \quad c \in \mathbb{R}$$

where b is bounded, $b \in C(\mathbb{R}^N)$ and $b(x) \geq 0$ in $\mathbb{R}^N$.

Proof. – Let $u \in M_b^c$ and $c > 0$. Then

$$\int |\nabla u|^2 + u^2 = \int b|u|^p + c \geq c$$

$$\begin{aligned} I_b(u) &= \frac{p-2}{2p}\left(\int |\nabla u|^2 + u^2\right) + \frac{c}{p} \\ &\geq \frac{p-2}{2p}c + \frac{c}{p} = \frac{c}{2}. \end{aligned}$$

To show that equality holds, let $v \in H^1$, $\int |\nabla v|^2 = c$, and define

$$u_\sigma(x) = \sigma^{(N-2)/2} v(\sigma x),\ w_\sigma(x) = (1+\delta) u_\sigma$$

with $\delta > 0$ being selected so that $w_\sigma \in M_b^c$.

Since $\int |\nabla u_\sigma|^2 = c$, $\int |u_\sigma|^q = \sigma^{(N-2)q/2-N} \int |v|^q \to 0$ as $\sigma \to +\infty$, for $q \in \left(2, \dfrac{2N}{N-2}\right)$, it is easy to see that such a $\delta = \delta(\sigma)$ exists and $\delta \to 0$ as $\sigma \to +\infty$. Therefore

$$I_b(w_\sigma) = \frac{1}{2} \int |\nabla w_\sigma|^2 + |w_\sigma|^2 - \frac{1}{p} \int b |w_\sigma|^p \to \frac{c}{2} \quad \text{as} \quad \sigma \to +\infty.$$

Hence $I_b^c = \dfrac{c}{2}$.

To complete the proof of Lemma 2.2, let $c > 0$ and $u \in M_b^{-c}$. Then

$$\int |\nabla u|^2 + u^2 = \int b|u|^p - c < \int b|u|^p.$$

Let $v = tu$, where $t > 0$ is selected so that $v \in M_b$. It is easy to see that $t \in (0,1)$. We have

$$\begin{aligned} I_b(v) &= \left(\frac{1}{2} - \frac{1}{p}\right) \int |\nabla v|^2 + v^2 \\ &= \left(\frac{1}{2} - \frac{1}{p}\right) t^2 \int |\nabla u|^2 + u^2 \\ &< \left(\frac{1}{2} - \frac{1}{p}\right) \left(\int |\nabla u|^2 + u^2\right) + \frac{c}{p} - \frac{c}{p} \\ &= I_b(u) + \frac{c}{2} + \left(\frac{1}{p} - \frac{1}{2}\right) c \\ &= I_b(u) + I_b^c - \frac{p-2}{2p} c. \end{aligned}$$

The required inequality then follows by taking the infimum over M_b^{-c}.

LEMMA 2.3. – *Assume condition* (Q) *holds. Then for any* $\epsilon > 0$, *there exists a* $\lambda_\epsilon > 0$ *such that*

(1) $J_\lambda^j < 2 I_{Q_{\mathrm{Max}}} + \epsilon$,

(2) $P_\lambda^j < I_{Q_{\mathrm{Max}}} + \epsilon$.

for $j = 1, \ldots, k$, *and* $\lambda \in (0, \lambda_\epsilon)$

Proof. – We prove (1) by constructing functions $w_\lambda, \lambda > 0$, with $w_\lambda \in \Lambda_\lambda^j$ such that $I_{Q_\lambda}(w_\lambda) \to 2I_{Q_{\mathrm{Max}}}$ as $\lambda \to 0$. Let j be fixed and u denote the ground state solution of

$$\begin{cases} -\Delta u + u = Q_{\mathrm{Max}}|u|^{p-2}u & \text{in} \quad \mathbb{R}^N \\ u > 0 \quad \text{in} \quad \mathbb{R}^N \\ u \in H^1 \end{cases} \tag{2.7}$$

Define for small $\lambda > 0$ such that $2\sqrt{\lambda} < 1$

$$\psi_\lambda(x) = \begin{cases} 1 & |x| < \frac{1}{2\sqrt{\lambda}} - 1 \\ 0 & |x| > \frac{1}{2\sqrt{\lambda}} \end{cases}$$

$\psi_\lambda(x) \in C^1(\mathbb{R}^N)$ and $|\nabla\psi_\lambda| \leq 2$ in $\mathbb{R}^N$.

Let $x^\lambda = \dfrac{1}{2\sqrt{\lambda}}(1, 1, \ldots, 1) \in \mathbb{R}^N$ and

$$\begin{aligned} w_\lambda(x) =& t_\lambda^+ u\Big(x - \frac{a^j}{\lambda} + x^\lambda\Big)\psi_\lambda\Big(x - \frac{a^j}{\lambda} + x^\lambda\Big) \\ &- t_\lambda^- u\Big(x - \frac{a^j}{\lambda} - x^\lambda\Big)\psi\Big(x - \frac{a^j}{\lambda} - x^\lambda\Big), \end{aligned}$$

where $t_\lambda^\pm > 0$ are selected so that $\langle I'_{Q_\lambda}(w_\lambda^\pm),\ w_\lambda^\pm\rangle = 0$. That is

$$(t_\lambda^\pm)^{p-2} = \frac{\left\{\begin{array}{c} \int |\nabla(u(x - \frac{a^j}{\lambda} \pm x^\lambda)\psi_\lambda(x - \frac{a^j}{\lambda} \pm x^\lambda))|^2 \\ +|u(x - \frac{a^j}{\lambda} \pm x^\lambda)\psi_\lambda(x - \frac{a^j}{\lambda} \pm x^\lambda)|^2 \end{array}\right\}}{\int Q(\lambda x)|u(x - \frac{a^j}{\lambda} \pm x^\lambda)\psi_\lambda(x - \frac{a^j}{\lambda} \pm x^\lambda)|^p}$$

It is easy to see from the definitions of u, ψ_λ and x^λ that $t_\lambda^\pm$ exist and $t_\lambda^\pm \to 1$ as $\lambda \to 0$.

We show next that $g_\lambda(w_\lambda^\pm) \in C_{\ell/\lambda}^j$:

$$g_\lambda^i(w_\lambda^\pm) = \frac{\int \phi_\lambda(x_i)u^p(x - \frac{a^j}{\lambda} \pm x^\lambda)\psi_\lambda^p(x - \frac{a^j}{\lambda} \pm x^\lambda)}{\int u^p(x - \frac{a^j}{\lambda} \pm x^\lambda)\psi_\lambda^p(x - \frac{a^j}{\lambda} \pm x^\lambda)}$$

Since $\psi_\lambda(x - \frac{a^j}{\lambda} \pm x^\lambda) = 0$ if $|x_i - \frac{a_i^j}{\lambda}| > \frac{1}{\sqrt{\lambda}}$ by the definition of ψ_λ, we have

$$g_\lambda^i(w_\lambda^\pm) = \frac{\int_{C_{\ell/\lambda}^j} \phi_\lambda(x_i)u^p(x - \frac{a^j}{\lambda} \pm x^\lambda)\psi_\lambda^p(x - \frac{a^j}{\lambda} \pm x^\lambda)}{\int_{C_{\ell/\lambda}^j} u^p(x - \frac{a^j}{\lambda} \pm x^\lambda)\psi_\lambda^p(x - \frac{a^j}{\lambda} \pm x^\lambda)}$$

provided $\frac{1}{\sqrt{\lambda}} < \frac{\ell}{\lambda}$, and from the definition of ϕ_λ we conclude that $g_\lambda^i(w_\lambda^\pm) \in C_{\ell/\lambda}^j$. Thus $w_\lambda^\pm \in N_\lambda^j$.

We also have

$$
\begin{aligned}
(2.8)\quad I_{Q_\lambda}(w_\lambda) = {} & \frac{(t_\lambda^+)^2}{2}\Big[\int |\nabla(u(x - \frac{a^j}{\lambda} + x^\lambda)\psi_\lambda(x - \frac{a^j}{\lambda} + x^\lambda))|^2 \\
& + \int |u(x - \frac{a^j}{\lambda} + x^\lambda)\psi_\lambda(x - \frac{a^j}{\lambda} + x^\lambda)|^2\Big] \\
& - \frac{(t_\lambda^+)^p}{p}\int Q(\lambda x)|u(x - \frac{a^j}{\lambda} + x^\lambda)\psi_\lambda(x - \frac{a^j}{\lambda} + x^\lambda)|^p \\
& + \frac{(t_\lambda^-)^2}{2}\Big[\int |\nabla(u(x - \frac{a^j}{\lambda} - x^\lambda)\psi_\lambda(x - \frac{a^j}{\lambda} - x^\lambda))|^2 \\
& + \int |u(x - \frac{a^j}{\lambda} - x^\lambda)\psi_\lambda(x - \frac{a^j}{\lambda} - x^\lambda)|^2\Big] \\
& - \frac{(t_\lambda^-)^p}{p}\int Q(\lambda x)|u(x - \frac{a^j}{\lambda} - x^\lambda)\psi_\lambda(x - \frac{a^j}{\lambda} - x^\lambda)|^p \\
= {} & \frac{(t_\lambda^+)^2}{2}\int |\nabla u|^2 + u^2 - \frac{(t_\lambda^+)^p}{p}\int Q(\lambda x + a^j - \lambda x^\lambda)|u|^p \\
& + \frac{(t_\lambda^-)^2}{2}\int |\nabla u|^2 + u^2 - \frac{(t_\lambda^-)^p}{p}\int Q(\lambda x + a^j + \lambda x^\lambda)|u|^p \\
& + o(\lambda)
\end{aligned}
$$

where $o(\lambda) \to 0$ as $\lambda \to 0$.

Since $\lambda x^\lambda \to 0$ as $\lambda \to 0$ we see from (2.8) that $I_{Q_\lambda}(w_\lambda) \to 2I_{Q(a^j)}(u) = 2I_{Q_{\text{Max}}}$ as $\lambda \to 0$. This completes the proof of (1).

The proof of (2) is similar. We can simply replace w_λ by w_λ^+ and prove (2).

LEMMA 2.4. – *Assume condition* (Q) *holds. Then there are numbers* $\epsilon, \lambda_\epsilon > 0$ *such that for* $j = 1, \ldots, k$

(1) $\widetilde{J}_\lambda^j > 2I_{Q_{\text{Max}}} + \epsilon$ *for all* $\lambda \in (0, \lambda_\epsilon)$,

(2) $\widetilde{P}_\lambda^j > I_{Q_{\text{Max}}} + \epsilon$ *for all* $\lambda \in (0, \lambda_\epsilon)$.

Proof. – Fix j. Assume to the contrary there is $\lambda_n \to 0$ as $n \to \infty$, such that $\widetilde{J}_{\lambda_n}^j \to c \le 2I_{Q_{\text{Max}}}$. Consequently there exists $\{u_n\} \subset \partial\Lambda_{\lambda_n}^j$ such that

$$(2.9)\qquad \int |\nabla u_n^\pm|^2 + |u_n^\pm|^2 = \int Q(\lambda_n x)|u_n^\pm|^p,$$

$$I_{Q_{\lambda_n}}(u_n) \to c \leq 2I_{Q_{\text{Max}}},$$

and either $g_{\lambda_n}(u_n^+)$, or $g_{\lambda_n}(u_n^-)$ belongs to $\partial C^j_{\ell/\lambda_n}$, as $n \to \infty$.

It then follows that $\{u_n\}$ is bounded in H^1. Let $\rho_n^\pm = |u_n^\pm|^p$, applying the concentration-compactness principle of P.L. Lions [12] to $\rho_n^\pm$, we obtain a subsequence of $\{u_n^\pm\}$ (still denoted by $\{u_n^\pm\}$, and hereafter, we always choose subsequence and denote it by the same sequence if necessary) such that one of the cases (i) Vanishing, (ii) Nonvanishing occurs. If (i) Vanishing occurs, then for $q \in \left(p, \dfrac{2N}{N-2}\right)$ it follows from P.L. Lions [12] that $\int |u_n^\pm|^q \to 0$. By Hölder inequality and the boundedness of $\{u_n^\pm\}$ in $L^2(\mathbb{R}^N)$ we have $\int Q(\lambda_n x)|u_n^\pm|^p \to 0$, which leads to $\int |\nabla u_n^\pm|^2 + |u_n^\pm|^2 \to 0$, a contradiction, since from (2.9) we can find a number $\nu > 0$ such that $\int |\nabla u_n^\pm|^2 + |u_n^\pm|^2 > \nu$ for all n. Hence (ii) Nonvanishing occurs: there are $R > 0, \alpha > 0$ and $\{y_n^\pm\} \subset \mathbb{R}^N$ such that

$$\int_{B_R(y_n^\pm)} |u_n^\pm|^p \geq \alpha \quad \text{for all} \quad n,$$

where $B_r(x_0) = \{x : |x - x_0| < r\}$ for $x_0 \in \mathbb{R}^N,\ r > 0$.

Suppose $g_{\lambda_n}(u_n^+) \in \partial C^j_{\ell/\lambda_n}$ ($g_{\lambda_n}(u_n^-) \in \partial C^j_{\ell/\lambda_n}$ can be considered similarly). Denote y_n^+ by y_n. Let $\tilde{u}_n = u_n^+(x + y_n)$, then

$$\tilde{u}_n \to u_o \quad \text{weakly in} \quad H^1 \quad \text{as} \quad n \to \infty,$$
$$\tilde{u}_n \to u_o \quad \text{a.e. in} \quad \mathbb{R}^N \quad \text{as} \quad n \to \infty,$$

$$\int_{B_R(0)} |\tilde{u}_n|^p \to \int_{B_R(0)} |u_o|^p \geq \alpha > 0, \quad \text{as} \quad n \to \infty.$$

Set $v_n = \tilde{u}_n - u_o$. By Brezis-Lieb lemma [5] we obtain

$$(2.10)\quad \int Q(\lambda_n x + \lambda_n y_n)|\tilde{u}_n|^p = \int Q(\lambda_n x + \lambda_n y_n)|u_o|^p + \int Q(\lambda_n x + \lambda_n y_n)|v_n|^p + o(1)$$

Since $\tilde{u}_n$ converges weakly to u_o, we have

$$(2.11)\quad \int |\nabla \tilde{u}_n|^2 + |\tilde{u}_n|^2 = \int |\nabla u_o|^2 + |u_o|^2 + \int |\nabla v_n|^2 + |v_n|^2 + o(1)$$

It follows from (2.9) that

$$(2.12)\qquad \int |\nabla \tilde{u}_n|^2 + |\tilde{u}_n|^2 = \int Q(\lambda_n x + \lambda_n y_n)|\tilde{u}_n|^p$$

Combining (2.10), (2.11) and (2.12) we have

$$\begin{aligned}(2.13)\quad &\int |\nabla v_n|^2 + |v_n|^2 - \int Q(\lambda_n x + \lambda_n y_n)|v_n|^p \\ &= -\Big(\int |\nabla u_o|^2 + |u_o|^2 - \int Q(\lambda_n x + \lambda_n y_n)|u_o|^p\Big) + o(1)\end{aligned}$$

We consider the following two cases

Case (I). – $\|v_n\| \to 0$ as $n \to \infty$.
By condition (Q) we can choose $\delta > 0$ so that

$$(2.14)\qquad Q(x) < Q_{\text{Max}} \qquad \text{for} \qquad x \in \overline{C}^j_{\ell+\delta}\backslash C^j_{\ell-\delta}.$$

We complete the proof by establishing the contradiction that

$$\lim_{n\to\infty} I_{Q_{\lambda_n}}(u_n) > 2 I_{Q_{\text{Max}}}.$$

Consider the sequence $\{\lambda_n y_n\}$. By passing to a subsequence if necessary, we may assume that one of the following cases occurs:
(a) $\{\lambda_n y_n\} \subset C^j_{\delta+\ell}\backslash \overline{C}^j_{\ell-\delta}$
(b) $\{\lambda_n y_n\} \subset \overline{C}^j_{\ell-\delta}$
(c) $\{\lambda_n y_n\} \subset \mathbb{R}^N\backslash C^j_{\ell+\delta}$, and $\{\lambda_n y_n\}$ is bounded.
(d) $\lambda_n (y_n)_i \to \infty$ as $n \to \infty$, for some $i \in \{1, 2, \dots, N\}$.
Let $\epsilon > 0$ and $R_\epsilon > 0$ be such that

$$(2.15)\qquad \int_{|x|\geq R_\epsilon} |v_n|^p \Big/ \int |v_n|^p \leq \epsilon.$$

In case (a) we may assume $\lambda_n y_n \to \tilde{y} \in \overline{C}^j_{\ell+\delta}\backslash C^j_{\ell-\delta}$, and $Q(\tilde{y}) < Q_{\text{Max}}$. Consequently

$$\begin{aligned}&\lim_{n\to\infty} I_{Q_{\lambda_n}}(u_n^+) \\ &= \lim_{n\to\infty} \Big\{\frac{1}{2}\int |\nabla u_n^+|^2 + |u_n^+|^2 - \frac{1}{p}\int Q(\lambda_n x)|u_n^+|^p\Big\} \\ &= \lim_{n\to\infty} \Big\{\frac{1}{2}\int |\nabla \tilde{u}_n|^2 + |\tilde{u}_n|^2 - \frac{1}{p}\int Q(\lambda_n x + \lambda_n y_n)|\tilde{u}_n|^p\Big\} \\ &= \frac{1}{2}\int |\nabla u_o|^2 + |u_o|^2 - \frac{1}{p}\int Q(\tilde{y})|u_o|^p \\ &> I_{Q_{\text{Max}}},\end{aligned}$$

since we also have

$$\int |\nabla u_o|^2 + u_o^2 = \int Q(\tilde{y})|u_o|^p.$$

In case (b)

$$\begin{aligned} g^i_{\lambda_n}(u_n^+) &= \frac{\int \phi_{\lambda_n}(x_i + (y_n)_i)|\tilde{u}_n|^p}{\int |\tilde{u}_n|^p} \\ &= \frac{\int_{|x_i|\le R_\epsilon} \phi_{\lambda_n}(x_i + (y_n)_i)|\tilde{u}_n|^p + \int_{|x_i|\ge R_\epsilon} \phi_{\lambda_n}(x_i + (y_n)_i)|\tilde{u}_n|^p}{\int |\tilde{u}_n|^p} \end{aligned}$$

In the region $|x_i| \le R_\epsilon$, we have

$$\begin{aligned} x_i + (y_n)_i &\in \left(\frac{a_i^j - (\ell - \delta)}{\lambda_n} - R_\epsilon,\ \frac{a_i^j + (\ell - \delta)}{\lambda_n} + R_\epsilon\right) \\ &\subset \left(-\frac{2K}{\lambda_n}, \frac{2K}{\lambda_n}\right) \quad \text{for sufficiently large } n. \end{aligned}$$

It then follows from (2.15) and the definition of ϕ_{λ_n} that

$$\begin{aligned} g^i_{\lambda_n}(u_n^+) &> \left(\frac{a_i^j - (\ell - \delta)}{\lambda_n} - R_\epsilon\right)(1 - \epsilon) - \frac{2K}{\lambda_n}\epsilon, \\ g^i_{\lambda_n}(u_n^+) &< \left(\frac{a_i^j + (\ell - \delta)}{\lambda_n} + R_\epsilon\right)(1 - \epsilon) + \frac{2K}{\lambda_n}\epsilon, \end{aligned}$$

It is clear from the above inequalities that we can choose $\epsilon > 0$, $\delta > \epsilon$ sufficiently small such that

$$g^i_{\lambda_n}(u_n^+) \in \left(\frac{a_i^j - \ell}{\lambda_n}, \frac{a_i^j + \ell}{\lambda_n}\right) \quad \text{for sufficiently large } n,$$

contradicting $g_{\lambda_n}(u_n^+) \in \partial C^j_{\ell/\lambda_n}$.

In case (c), we may assume that $\lambda_n y_n \to \tilde{y} \in \overline{C}^j_{\ell+\delta}$, as $n \to \infty$, $\tilde{y}_i \ge a_i^j + \ell + \delta$ for some i, and $(y_n)_i > \dfrac{a_i^j + \ell + \delta/2}{\lambda_n}$ for all n. For $|x_i| \le R_\epsilon$ we then have

$$x_i + (y_n)_i > \frac{a_i^j + \ell + \delta/2}{\lambda_n} - R_\epsilon$$

and

$$g_{\lambda_n}^i(u_n^+) > \left(\frac{a_i^j + \ell + \delta/2}{\lambda_n} - R_\epsilon\right)(1-\epsilon) - \frac{2K}{\lambda_n}\epsilon$$
$$> \frac{a_i^j + \ell}{\lambda_n}$$

for sufficiently small $\epsilon, \delta > \epsilon$, and n large enough. This contradicts $g_{\lambda_n}(u_n^+) \in \partial C_{\ell/\lambda_n}^j$.

Case (d) is excluded by assuming $\lambda_n(y_n)_i > 2K$ (or $\lambda_n(g_n)_i < -2K$) for some i and for all n, and using a similar argument to that of case (c).

Case (II). – $\|v_n\| \to L > 0$, as $n \to \infty$.

Set

$$\int |\nabla u_o|^2 + |u_o|^2 - \int Q(\lambda_n x + \lambda_n y_n)|u_o|^p = A + o(1),$$

then by (2.13)

$$\int |\nabla v_n|^2 + |v_n|^2 - \int Q(\lambda_n x + \lambda_n y_n)|v_n|^p = -A + o(1).$$

Suppose $A > 0$ ($A < 0$ can be considered similarly). We can find $t_n \to 1$ such that $w_n = t_n v_n$ satisfies

$$\int |\nabla w_n|^2 + |w_n|^2 - \int Q(\lambda_n x + \lambda_n y_n)|w_n|^p = -A.$$

Since $u_o \in M_{Q(\lambda_n x + \lambda_n y_n)}^{A+o(1)}$, we have

$$\begin{aligned}
I_{Q_{\lambda_n}}(u_n^+) &= \frac{1}{2}\int |\nabla u_o|^2 + u_o^2 - \frac{1}{p}\int Q(\lambda_n x + \lambda_n y_n)|u_o|^p \\
&\quad + \frac{1}{2}\int |\nabla v_n|^2 + |v_n|^2 - \frac{1}{p}\int Q(\lambda_n x + \lambda_n y_n)|v_n|^p + o(1) \\
&\geq \frac{A + o(1)}{2} + \frac{1}{2}\int |\nabla w_n|^2 + |w_n|^2 \\
&\quad - \frac{1}{p}\int Q(\lambda_n x + \lambda_n y_n)|w_n|^p + o(1) \\
&= I_{Q(\lambda_n x + \lambda_n y_n)}^A + I_{Q(\lambda_n x + \lambda_n y_n)}^{-A} + o(1) \\
&> I_{Q(\lambda_n x + \lambda_n y_n)}^0 + \frac{1}{2}\left(\frac{1}{2} - \frac{1}{p}\right)|A| + o(1) \\
&\geq I_{Q_{\mathrm{Max}}} + \frac{1}{2}\left(\frac{1}{2} - \frac{1}{p}\right)|A| + o(1)
\end{aligned}$$

Thus

$$\lim_{n\to\infty} I_{Q_{\lambda_n}}(u_n) = \lim_{n\to\infty}\left(I_{Q_{\lambda_n}}(u_n^+) + I_{Q_{\lambda_n}}(u_n^-)\right)$$
$$\geq 2I_{Q_{\mathrm{Max}}} + \frac{1}{2}\left(\frac{1}{2} - \frac{1}{p}\right)|A|$$

a contradiction.

If $A = 0$, we can find $s_n,\ t_n > 0,\ s_n \to 1,\ t_n \to 1$ as $n \to \infty$ such that $\overline{v}_n = t_n v_n,\ \overline{w}_n = s_n u_o$ satisfy

$$\int |\nabla \overline{v}_n|^2 + |\overline{v}_n|^2 = \int Q(\lambda_n x + \lambda_n y_n)|\overline{v}_n|^p$$
$$\int |\nabla \overline{w}_n|^2 + |\overline{w}_n|^2 = \int Q(\lambda_n x + \lambda_n y_n)|\overline{w}_n|^p.$$

Hence

$$\begin{aligned}\lim_{n\to\infty} I_{Q_{\lambda_n}}(u_n) &= \lim_{n\to\infty} I_{Q_{\lambda_n}}(u_n^+) + \lim_{n\to\infty} I_{Q_{\lambda_n}}(u_n^-)\\ &= \lim_{n\to\infty}\left[\frac{1}{2}\int |\nabla \overline{w}_n|^2 + |\overline{w}_n|^2 - \frac{1}{p}\int Q(\lambda_n x + \lambda_n y_n)|\overline{w}_n|^p\right.\\ &\quad \left. + \frac{1}{2}\int |\nabla \overline{v}_n|^2 + |\overline{v}_n|^2 - \frac{1}{p}\int Q(\lambda_n x + \lambda_n y_n)|\overline{v}_n|^p\right]\\ &\quad + \lim_{n\to\infty} I_{Q_{\lambda_n}}(u_n^-)\\ &\geq 3I_{Q_{\mathrm{Max}}}.\end{aligned}$$

Thus, we have completed our proof of (1) in Lemma 2.4.

The proof of (2) of Lemma 2.4 is similar and is omited here.

LEMMA 2.5. – *For any $u \in \Lambda_\lambda^j$, there exist $\epsilon > 0$ and differentiable functions*

$$t_+ = t_+(w) > 0,$$
$$t_- = t_-(w) > 0$$

defined for $w \in H^1$, $\|w\| < \epsilon$ such that $t_\pm(0) = 1$, the functions $z = t_+(w)(u-w)^+ - t_-(w)(u-w)^-$ belongs to Λ_λ^j and

$$(2.16)\quad \langle t'_\pm(0), v\rangle = \frac{2\int \nabla u^\pm \nabla v + u^\pm v - p\int Q(\lambda x)|u^\pm|^{p-2}u^\pm v}{\int |\nabla u^\pm|^2 + |u^\pm|^2 - (p-1)\int Q(\lambda x)|u^\pm|^p}\quad \textit{for all } v \in H^1.$$

Proof. – The proof is similar to that of Lemma 2.4 in [16].

Define $F^+ : \mathbb{R} \times H^1 \to \mathbb{R}$ by

$$F^+(t,w) = t\int |\nabla(u-w)^+|^2 + |(u-w)^+|^2 - t^{p-1}\int Q(\lambda x)|(u-w)^+|^p.$$

Since $u \in \Lambda_\lambda^j$ we have $F^+(1,0) = 0$ and

$$\frac{d}{dt}F^+(1,0) = \int |\nabla u^+|^2 + |u^+|^2 - (p-1)\int Q(\lambda x)|u^+|^p \neq 0.$$

Therefore we may apply the implicit function theorem to get a function $t_+(w)$ defined for $\|w\| < \epsilon$, $\epsilon > 0$ such that $t_+(0) = 1$, (2.16) holds and $F^+(t_+(w), w) = 0$ which is equivalent to

$$\langle I'_{Q_\lambda}(t_+(w)(u-w)^+),\ t_+(w)(u-w)^+\rangle = 0.$$

Furthermore,

$$g_\lambda(t_+(w)(u-w)^+) \in C^j_{\ell/\lambda}$$

still holds if ϵ is sufficiently small by the continuity of the map g_λ.

Employing the same argument to the functional

$$F^-(t,w) = t\int |\nabla(u-w)^-|^2 + |(u-w)^-|^2 - t^{p-1}\int Q(\lambda x)|(u-w)^-|^p$$

we obtain the second function $t_-(w)$ with analogous properties. Therefore

$$z = t_+(w)(u-w)^+ - t_-(w)(u-w)^- \in \Lambda_\lambda^j$$

for any $w \in H^1$, with sufficiently small norm. This completes our proof of Lemma 2.5.

For the set N_λ^j, by a similar argument, we have

LEMMA 2.5'. – *For any $u \in N_\lambda^j$, there exist $\epsilon > 0$ and a differentiable function $t(w) > 0$ defined for $w \in H^1$, $\|w\| < \epsilon$ such that $t(0) = 1$, the functions $z = t(w)(u-w) \in N_\lambda^j$ and*

(2.16)' $$\langle t'(0), v\rangle = \frac{2\int \nabla u \nabla v + uv - p\int Q(\lambda x)|u|^{p-2}uv}{\int |\nabla u|^2 + |u|^2 - (p-1)\int Q(\lambda x)|u|^p} \quad \textit{for all } v \in H^1.$$

Having established the preliminary lemmas, we are now ready to prove Proposition 2.1.

Proof of Proposition 2.1. – If $\overline{\Lambda}^j_\lambda$ denotes the closure of Λ^j_λ, then we first notice that, $\overline{\Lambda}^j_\lambda = \Lambda^j_\lambda \cup \partial\Lambda^j_\lambda$ and $\partial\Lambda^j_\lambda$ is the boundary of $\overline{\Lambda}^j_\lambda$ for each $j = 1, \dots, k$. This easily follows from the observation that any u with u^+ or u^- equals to zero can't be the limit of a sequence of functions in Λ^j_λ.

Using Lemma 2.3 and Lemma 2.4 we see that there exists $\lambda_0 > 0$ such that

$$J^j_\lambda < \min\{3I_{Q_{\mathrm{Max}}}, \widetilde{J}^j_\lambda\} \text{ for } \lambda \in (0, \lambda_0),\ j = 1, \dots, k. \tag{2.17}$$

It follows that for $\lambda \in (0, \lambda_0)$

$$J^j_\lambda = \inf\{I_{Q_\lambda}(u) : u \in \overline{\Lambda}^j_\lambda\}. \tag{2.18}$$

Applying the variational principle of Ekeland [9] to (2.18) we obtain a minimizing sequence $\{u_n\} \subset \overline{\Lambda}^j_\lambda$ for each fixed $j = 1, \dots, k$, with the properties

$$\begin{aligned} &I_{Q_\lambda}(u_n) < J^j_\lambda + \frac{1}{n} \\ &I_{Q_\lambda}(u_n) < I_{Q_\lambda}(w) + \frac{1}{n}\|w - u\| \quad \text{for any } w \in \overline{\Lambda}^j_\lambda. \end{aligned} \tag{2.19}$$

Using (2.17) we may assume that $u_n \in \Lambda^j_\lambda$ for n sufficiently large. Applying Lemma 2.5 with $u = u_n$ we obtain $\epsilon_n > 0$, two functions $t^n_+(w)$, $t^n_-(w)$ defined for $w \in H^1$, $\|w\| < \epsilon_n$, such that $t^n_+(w)\,(u_n - w)^+ - t^n_-(w)(u - w)^- \in \Lambda^j_\lambda$. Choose $0 < \delta < \epsilon_n$. Let $u \in H^1$, $u \not\equiv 0$ and let $w_\delta = \dfrac{\delta u}{\|u\|}$. Fix n, and let $z_\delta = t^n_+(w_\delta)(u_n - w_\delta)^+ - t^n_-(w_\delta)(u_n - w_\delta)^-$. Since $z_\delta \in \Lambda^j_\lambda$ by Lemma 2.5, using (2.19) we obtain

$$I_{Q_\lambda}(z_\delta) - I_{Q_\lambda}(u_n) \geq -\frac{1}{n}\|z_\delta - u_n\|$$

and by the mean value theorem, we then have

$$\langle I'_{Q_\lambda}(u_n),\ z_\delta - u_n\rangle + o(\|z_\delta - u_n\|) \geq -\frac{1}{n}\|z_\delta - u_n\|.$$

Therefore

$$\begin{aligned} &\langle I'_{Q_\lambda}(u_n), (u_n - w_\delta)^+ - (u_n - w_\delta)^- \\ &\qquad + (t^n_+(w_\delta) - 1)(u_n - w_\delta)^+ - (t^n_-(w_\delta) - 1)(u_n - w_\delta)^- - u_n\rangle \\ &\geq -\frac{1}{n}\|z_\delta - u_n\| + o(\|z_\delta - u_n\|) \end{aligned}$$

Since $(u_n - w_\delta)^+ - (u_n - w_\delta)^- = u_n - w_\delta$, we have

$$
\begin{aligned}
(2.20)\quad \langle I'_{Q_\lambda}(u_n), -w_\delta\rangle &+ (t_+^n(w_\delta) - 1)\langle I'_{Q_\lambda}(u_n),\ (u_n - w_\delta)^+\rangle \\
&- (t_-^n(w_\delta) - 1)\langle I'_{Q_\lambda}(u_n),\ (u_n - w_\delta)^-\rangle \\
&\geq -\frac{1}{n}\|w_\delta - u_n\| + o(\|w_\delta - u_n\|).
\end{aligned}
$$

From $t_\pm^n(w_\delta)(u_n - w_\delta)^\pm \in M_{Q_\lambda}$ we obtain

$$
\langle I'_{Q_\lambda}(z_\delta),\ t_\pm^n(w_\delta)(u_n - w_\delta)^\pm\rangle = 0.
$$

Thus it follows from (2.20) that

$$
\begin{aligned}
-\delta\left\langle I'_{Q_\lambda}(u_n),\ \frac{u}{\|u\|}\right\rangle &+ \frac{(t_+^n(w_\delta) - 1)}{t_+^n(w_\delta)}\langle I'_{Q_\lambda}(z_\delta), t_+^n(w_\delta)(u_n - w_\delta)^+\rangle \\
&- \frac{(t_-^n(w_\delta) - 1)}{t_-^n(w_\delta)}\langle I'_{Q_\lambda}(z_\delta), t_-^n(w_\delta)(u_n - w_\delta)^-\rangle \\
&+ (t_+^n(w_\delta) - 1)\langle I'_{Q_\lambda}(u_n) - I'_{Q_\lambda}(z_\delta),\ (u_n - w_\delta)^+\rangle \\
&- (t_-^n(w_\delta) - 1)\langle I'_{Q_\lambda}(u_n) - I'_{Q_\lambda}(z_\delta),\ (u_n - w_\delta)^-\rangle \\
&\geq -\frac{1}{n}\|z_\delta - u_n\| + o(\|z_\delta - u_n\|)
\end{aligned}
$$

Hence

$$
\begin{aligned}
(2.21)\quad &\left\langle I'_{Q_\lambda}(u_n),\ \frac{u}{\|u\|}\right\rangle \\
&\leq \frac{1}{n}\frac{\|z_\delta - u_n\|}{\delta} + \frac{o(\|z_\delta - u_n\|)}{\delta} \\
&+ \frac{t_+^n(w_\delta) - 1}{\delta}\langle I'_{Q_\lambda}(u_n) - I'_{Q_\lambda}(z_\delta),\ (u_n - w_\delta)^+\rangle \\
&- \frac{t_-^n(w_\delta) - 1}{\delta}\langle I'_{Q_\lambda}(u_n) - I'_{Q_\lambda}(z_\delta),\ (u_n - w_\delta)^-\rangle
\end{aligned}
$$

But

$$
\|z_\delta - u_n\| \leq \delta + [|t_+^n(w_\delta) - 1| + |t_-^n(w_\delta) - 1|]C
$$

for some constant $C > 0$, independent of δ, and

$$
\lim_{\delta\to 0}\frac{|t_\pm^n(w_\delta) - 1|}{\delta} \leq \|t_\pm^{n'}(0)\| \leq C
$$

for some constant $C > 0$, independent of δ, as can be easily verified from (2.16).

For fixed n, let $\delta \to 0$ in (2.22) we obtain

$$\left\langle I'_{Q_\lambda}(u_n), \frac{u}{\|u\|} \right\rangle \leq \frac{C}{n}, \tag{2.22}$$

where we have used the fact that $z_\delta \to u_n$ as $\delta \to 0$, and $I'_{Q_\lambda}(z_\delta) \to I'_{Q_\lambda}(u_n)$ as $\delta \to 0$. (2.21) implies

$$\|I'_{Q_\lambda}(u_n)\|_{H^{-1}} \to 0, \qquad \text{as} \quad n \to \infty.$$

Thus, we have proved (1) of Proposition 2.1.

Similarly, by using (2) of Lemma 2.3, (2) of Lemma 2.4 and Lemma 2.5, we can prove (2) of Proposition 2.1. We will omit the detailed proof here.

3. EXISTENCE OF SOLUTIONS

In this section we establish the existence of at least k positive and k nodal solutions of problem (1.1) for each $\mu \in \left(\frac{1}{\lambda_0^2}, +\infty\right)$, where λ_0 is as in Proposition 2.1.

For fixed j and $\lambda \in (0, \lambda_0)$, we have the following compactness results.

LEMMA 3.1. – *Assume condition (Q) holds, and that $\{u_n^j\} \subset \Lambda_\lambda^j$ is a sequence satisfying*

$$I_{Q_\lambda}(u_n^j) \to J_\lambda^j \qquad as \quad n \to \infty \tag{3.1}$$

$$I'_{Q_\lambda}(u_n^j) \to 0 \text{ in } H^{-1} \qquad as \quad n \to \infty \tag{3.2}$$

$$J_\lambda^j < 3 I_{Q_{\text{Max}}}. \tag{3.3}$$

Then $\{u_n^j\}$ has a subsequence (still denoted by $\{u_n^j\}$) satisfying

$$u_n^j \to u_o \qquad \textit{strongly in} \quad H^1 \quad as \quad n \to \infty,$$

and $u_o^\pm \not\equiv 0$.

Proof. – since $\{u_n^j\}$ is bounded in H^1, we can assume

$$\begin{aligned} &u_n^j \rightharpoonup u_o \quad \text{weakly in} \quad H^1 \quad \text{as} \quad n \to \infty, \\ &u_n^j \to u_o \quad \text{a.e. in} \quad \mathbb{R}^N \quad \text{as} \quad n \to \infty, \end{aligned} \tag{3.4}$$

for some $u_o \in H^1$.

We will show that
a) $u_o^+ \not\equiv 0,\ u_o^- \not\equiv 0$;
b) $u_n^j \to u_o$ strongly in H^1 as $n \to \infty$.
We proceed by contradiction. Assume to the contrary that

$$u_o^+ \equiv 0 \quad \text{or} \quad u_o^- \equiv 0. \tag{3.5}$$

Since $\{u_n^j\}$ is bounded in $L^p(\mathbb{R}^N)$, the concentration-compactness principle [12] implies that $\{u_n^j\}$ satisfies either vanishing or nonvanishing. Vanishing can be ruled out by the same argument used in Lemma 2.4. Therefore, nonvanishing occurs, that is, there are numbers $\alpha > 0,\ R > 0$, and a sequence $\{y_n\} \subset \mathbb{R}^N$ such that

$$\int_{B_R(0)} |u_n^j(x+y_n)|^p \geq \alpha > 0, \quad \text{for } n \text{ large.} \tag{3.6}$$

Set $\tilde{u}_n(x) \equiv u_n^j(x+y_n)$. Since $\{\tilde{u}_n\}$ is bounded in H^1, we may assume

$$\tilde{u}_n \to \tilde{u}_o \quad \text{weakly in} \quad H^1 \quad \text{as} \quad n \to \infty. \tag{3.7}$$

From (3.6) we see that $\tilde{u}_o \not\equiv 0$.

Case 1. – $\tilde{u}_o^+ \not\equiv 0$ and $\tilde{u}_o^- \not\equiv 0$,

Case 2. – either $\tilde{u}_o^+ \equiv 0$ or $\tilde{u}_o^- \equiv 0$.

We show next that each of the Cases 1 and 2 leads to a contradiction to either (3.5) or to (3.3).

Assume Case 1. Set $v_n = \tilde{u}_n - \tilde{u}_o$. If $||v_n|| \to 0$ as $n \to \infty$, and $\{y_n\}$ is unbounded, we employ the argument in Lemma 2.4 to obtain $g_\lambda^i(u_n^+) > a_i^j + \dfrac{3\ell}{2\lambda}$, or $g_\lambda^j(u_n^+) < a_i^j - \dfrac{3\ell}{2\lambda}$ for large n and for some $i \in \{1, 2, \ldots, N\}$, contradicting $u_n \in \Lambda_\lambda^j$.

If $||v_n|| \to 0$ as $n \to \infty$ and $\{y_n\}$ is bounded, so $y_n \to y_o \in \mathbb{R}^N$ as $n \to \infty$, we would have

$$u_n \to \tilde{u}_o(\cdot - y_0) \quad \text{strongly in} \quad H^1 \quad \text{as} \quad n \to \infty$$

which implies that $u_n^\pm \to \tilde{u}^\pm(\cdot - y_o) \not\equiv 0$ (since $||u_n^\pm|| > \gamma > 0$ for some constant $\gamma > 0$), contradicting (3.5).

On the other hand, if $||v_n|| \geq \delta > 0$ for large n and some constant $\delta > 0$, we notice first that (3.2) implies

$$\begin{aligned} &\int |\nabla \tilde{u}_o^\pm|^2 + |\tilde{u}_o^\pm|^2 - \int Q(\lambda x + \lambda y_n)|\tilde{u}_o^\pm|^p = o(1) \\ &\int |\nabla \tilde{u}_n|^2 + |\tilde{u}_n|^2 - \int Q(\lambda x + \lambda y_n)|\tilde{u}_n|^p = 0 \end{aligned} \tag{3.8}$$

By (3.8) and Brezis-Lieb lemma [5] we obtain

$$\int |\nabla v_n|^2 + |v_n|^2 - \int Q(\lambda x + \lambda y_n)|v_n|^p = o(1). \tag{3.9}$$

since $\|v_n\| \geq \delta > 0$ for large n, it is easy to find $s_n > 0$, $s_n \to 1$ as $n \to \infty$ such that $s_n v_n \in M_{Q(\lambda x + \lambda y_n)}$, and to show that

$$\frac{1}{2}\int |\nabla v_n|^2 + |v_n|^2 - \frac{1}{p}\int Q(\lambda x + \lambda y_n)|v_n|^p \geq I_{Q_{\text{Max}}} + o(1). \tag{3.10}$$

Similarly

$$\frac{1}{2}\int |\nabla \tilde{u}_o^{\pm}|^2 + |\tilde{u}_o^{\pm}|^2 - \frac{1}{p}\int Q(\lambda x + \lambda y_n)|\tilde{u}_o^{\pm}|^p \geq I_{Q_{\text{Max}}} + o(1). \tag{3.11}$$

Thus by Brezis-Lieb lemma [5] we obtain

$$\begin{aligned} I_{Q_\lambda}(u_n) &= \frac{1}{2}\int |\nabla \tilde{u}_n|^2 + |\tilde{u}_n|^2 - \frac{1}{p}\int Q(\lambda x + \lambda y_n)|\tilde{u}_n|^p \\ &= \frac{1}{2}\int |\nabla \tilde{u}_o^{+}|^2 + |\tilde{u}_o^{+}|^2 - \frac{1}{p}\int Q(\lambda x + \lambda y_n)|\tilde{u}_o^{+}|^p \\ &+ \frac{1}{2}\int |\nabla \tilde{u}_o^{-}|^2 + |\tilde{u}_o^{-}|^2 - \frac{1}{p}\int Q(\lambda x + \lambda y_n)|\tilde{u}_o^{-}|^p \\ &+ \frac{1}{2}\int |\nabla v_n|^2 + |v_n|^2 - \frac{1}{p}\int Q(\lambda x + \lambda y_n)|v_n|^p + o(1) \\ &\geq 3IQ_{\text{Max}} + o(1) \end{aligned}$$

which implies that

$$\lim_{n\to\infty} I_{Q_\lambda}(u_n) \geq 3I_{Q_{\text{Max}}} \tag{3.12}$$

contradicting (3.3).

In case 2, we may assume, without loss of generality, that $\tilde{u}_o^- \equiv 0$. First notice that we must have $\tilde{u}_n^+ \to \tilde{u}_o^+$ strongly in H^1 as $n \to \infty$, otherwise $\|\tilde{u}_n^+ - \tilde{u}_o^+\| \geq \delta > 0$ would lead to the contradiction (3.12), as above. Next, by the concentration-compactness principle, applied to $\{\tilde{u}_n^-\}$, and by ruling out vanishing as before, we obtain $\alpha > 0$, $R > 0$, and a sequence $\{\tilde{y}_n\}$ such that

$$\int_{B_R(0)} |\tilde{u}^-(x + \tilde{y}_n)|^p \geq \delta > 0 \tag{3.13}$$

Set $\tilde{\tilde{u}}_n(x) \equiv \tilde{u}_n(x+\tilde{y}_n) = u_n(x+y_n+\tilde{y}_n)$, by the boundedness of $\{u_n^-\}$ in H^1, and (3.13) we may assume that

$$\tilde{\tilde{u}}_n \to w_0 \qquad \text{weakly in } H^1 \quad \text{as } n \to \infty$$
$$\tilde{\tilde{u}}_n^- \to w_0^- (\not\equiv 0) \qquad \text{weakly in } H^1 \quad \text{as } n \to \infty.$$

If $w_0^+ \not\equiv 0$, we have a situation similar to Case 1, which is impossible. We may therefore assume $w_0^+ \equiv 0$. In this case we can't have $\|\tilde{\tilde{u}}_n^- - w_o^-\| \geq \delta > 0$ for some $\delta > 0$ and for large n, since otherwise we argue as before to obtain (3.12), contradicting (3.3). Then we must have $\tilde{\tilde{u}}_n^- \to w_0^-$ strongly in H^1 as $n \to \infty$. We are now left with

$$\tilde{u}_n^+ \to \tilde{u}_o^+ \qquad \text{strongly in } H^1 \quad \text{as} \quad n \to \infty \tag{3.14}$$

$$\tilde{\tilde{u}}_n^+ \to w_o^- \qquad \text{strongly in } H^1 \quad \text{as} \quad n \to \infty \tag{3.15}$$

But (3.14) and (3.15) imply, as argued before, that $\{y_n\}, \{y_n + \tilde{y}_n\}$ are bounded. We may assume $y_n \to y_o$, $\tilde{y}_n \to \tilde{y}_o$, as $n \to \infty$. Therefore

$$u_n^+ \to \tilde{u}_o^+(\cdot - y_o), \quad \text{strongly in} \quad H^1 \quad \text{as} \quad n \to \infty \tag{3.16}$$

$$u_n^- \to \tilde{w}_o^-(\cdot - y_o - \tilde{y}_o) \quad \text{strongly in} \quad H^1 \quad \text{as} \quad n \to \infty. \tag{3.17}$$

Hence $u_o^+ = \tilde{u}_o^+(x - y_o) \not\equiv 0$, $u_o^- = w_o^-(x - y_o - \tilde{y}_o) \not\equiv 0$, contradicting (3.5). This proves the conclusion (a).

Using (a) we can show that $u_n \to u_o$ strongly in H^1 as $n \to \infty$, otherwise, we may use a similar argument as above to reach the contradiction (3.12).

This completes the proof of Lemma 3.1.

For the minimizing sequences of P_λ^j, we have

LEMMA 3.2. – *Suppose condition (Q) holds, and $\{v_n^j\} \subset N_\lambda^j$ satisfies*

$$I_{Q_\lambda}(v_n^j) \to P_\lambda^j, \quad \textit{and}$$
$$I'_{Q_\lambda}(v_n^j) \to 0 \qquad \textit{in} \quad H^{-1} \textit{ as } n \to \infty$$
$$P_\lambda^j < 2 I_{Q_{\mathrm{Max}}}.$$

Then $\{v_n^j\}$ has a subsequence converging strongly in H^1.

Since the proof is similar to that of Lemma 3.1, but simpler, we omit it here.

Proof of Theorem A.

It follows from Proposition 2.1 that there exists $\lambda_o > 0$ such that for $\lambda \in (0, \lambda_o)$, fixed $j = 1, \dots, k$ we can find minimizing sequences $\{u_n^j\}$ and $\{v_n^j\}$ of J_λ^j and P_λ^j respectively. $\{u_n^j\}$ satisfies the assumptions in Lemma 3.1, $\{v_n^j\}$ satisfies the assumptions in Lemma 3.2. Therefore we have, as $n \to \infty$,

$$u_n^j \to u_j \qquad \text{strongly in } H^1$$

and $u_j^\pm \not\equiv 0$,

$$v_n^j \to v_j \quad \text{strongly in } H^1.$$

From $I'_{Q_\lambda}(u_n^j) \to 0$, $I'_{Q_\lambda}(v_n^j) \to 0$, as $n \to \infty$, and the strong convergence of $\{u_n^j\}$, $\{v_n^j\}$ we see that u^j is a nodal solution of (2.1), v_j is a nontrivial solution of (2.1). We next show that either $v_j^+ \equiv 0$ or $v_j^- \equiv 0$. Otherwise, suppose $v_j^\pm \not\equiv 0$, then$\langle I'_{Q_\lambda}(v_j^\pm),\ v_j^\pm\rangle = 0$ leads to

$$P_\lambda^j = \lim_{n\to\infty} I_{Q_\lambda}(v_n^j) = I_{Q_\lambda}(v_j) = I_{Q_\lambda}(v_j^+) + I_{Q_\lambda}(v_j^-) > 2I_{Q_{\text{Max}}}$$

a contradiction. So we can assume $v_j \geq 0$ in $\mathbb{R}^N$ (otherwise, $-v_j \geq 0$ in $\mathbb{R}^N$). By a standard regularity argument, we can show that $u_j, v_j \in C^2(\mathbb{R}^N)$ and $v_j > 0$ in $\mathbb{R}^N$ by the maximum principle.

Since $g_\lambda(u_j) \in \overline{C}^j_{\ell/\lambda}$, $g_\lambda(v_j) \in \overline{C}^j_{\ell/\lambda}$, and $\overline{C}^j_{\ell/\lambda}$ are disjoint, v_j, u_j are distinct solutions of (2.1).

Let $\mu_o = \lambda_o^{-2}$, $U_j = \mu^{\frac{1}{2(p-2)}} u_j(\sqrt{\mu}x)$, $V_j = \mu^{\frac{1}{2(p-2)}} v_j(\sqrt{\mu}x)$, then V_j and U_j are k positive and k nodal solutions of problem (1.1). We thus have proved Theorem A.

Remark 3.3. – By Lemma 2.3 and the proof of Theorem A, it is easy to see that for any $\epsilon > 0$, there exists $\lambda_\epsilon > 0$ such that for $\lambda \in (0, \lambda_\epsilon)$, problem (2.1) has at least k positive solutions $v_j (j = 1, \dots, k)$ and k nodal solutions $u_j (j = 1, \dots, k)$ satisfying

$$I_{Q_\lambda}(v_j) \in (I_{Q_{\text{Max}}},\ I_{Q_{\text{Max}}} + \epsilon) \qquad \text{for} \quad j = 1, \dots, k$$
$$I_{Q_\lambda}(u_j) \in (2I_{Q_{\text{Max}}},\ 2I_{Q_{\text{Max}}} + \epsilon) \qquad \text{for} \quad j = 1, \dots, k$$

provided condition (Q) holds.

Remark 3.4. – It is easy to see from the proof of Theorem A that the solutions V_j, U_j $(j = 1, \dots, k)$ satisfy

1) $\|V_j\|_{L^\infty(\mathbb{R}^N)}$, $\|U_j\|_{L^\infty(\mathbb{R}^N)} \to +\infty$ as $\mu \to +\infty$,
2) $\int |\nabla V_j|^2 + \mu|V_j|^2$, $\int |\nabla U_j|^2 + \mu|U_j|^2 \to +\infty$ as $\mu \to +\infty$
3) $\int |V_j|^2$, $\int |U_j|^2 \to +\infty$ as $\mu \to +\infty$ provided $2 < p < 2 + \dfrac{4}{N}$.
4) $\int |V_j|^2$, $\int |U_j|^2 \to 0$ as $\mu \to \infty$ provided $2 + \dfrac{4}{N} < p < 2 + \dfrac{4}{N-2}$

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