BJÖRN BIRNIR
GUSTAVO PONCE
NILS SVANSTEDT

The local ill-posedness of the modified KdV equation


<http://www.numdam.org/item?id=AIHPC_1996__13_4_529_0>
The local ill-posedness of the modified KdV equation

by

Björn BIRNIR*, Gustavo PONCE* and Nils SVANSTEDT†

Department of Mathematics,
University of California,
Santa Barbara, CA 93106.

ABSTRACT. – We find a new method for proving the local ill-posedness of the Cauchy problem for non-linear partial differential equations. The method is used to prove that the Cauchy problem for the Modified KdV equation is ill-posed in Sobolev spaces $H^s(R)$, $s < -1/2$.

Key words: Initial value problem, well-posed, KdV.

1. INTRODUCTION

Consider the initial value problem (IVP) of the Modified KdV equation

$$
\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} + u^2 \frac{\partial u}{\partial x} = 0 \quad t, x \in \mathbb{R},
$$

$$
u(x, 0) = u_0(x).
$$

* Partially supported by the National Science Foundations grants nos. DMS-91-04532 and DMS-93-01351.
† Supported by grants from the Swedish Natural Science Research Council and NUTEK.
It was proven in [6] that this IVP is locally well-posed in Sobolev spaces, $H^s(\mathbb{R}) = (1 - \Delta)^{-s/2} L^2(\mathbb{R})$, for $s \geq 1/4$, and in addition that the IVP has a global solution in $H^s$, $s \geq 1$. Local well-posedness means that there exists a unique solution in $H^s$ for a small time interval $[0,T]$, it is a continuous curve in $H^s$, originating in $u_0$ and the solution depends continuously upon the data.

In this paper we introduce a technique to prove ill-posedness and apply it to the IVP (1.1). A known technique for showing ill-posedness uses a finite-time blow-up of the solution to the IVP, see [4]. If the solution and its lifespan $T^*$ scales with a parameter $\lambda$ then one can scale the solution to get arbitrarily small lifespans $T^*/\lambda$, as $\lambda \to \infty$. This contradicts the existence of a non-zero time interval $[0,T]$ where the solution exists. Our technique is very different from this but it seems to be of general use. In a subsequent publication [2] it will be applied to prove the ill-posedness of the generalized KdV equations, see [3], [6] and (1.2) below, and nonlinear Schrödinger equations in any dimension. The idea is to show that the solution, of (1.1), does not depend continuously on its data in $H^s$, $s < -1/2$, by constructing a sequence converging (strongly) to the data in $H^s$ and then showing that the corresponding sequence of solutions does not converge (strongly) in $H^s$.

The sequence consists of the solitary wave solutions of (1.1), see [5] and [7], and the data is chosen to be the Dirac delta function. Thus we show that the IVP of the generalized KdV equations,

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} + u^n \frac{\partial u}{\partial x} = 0 \quad t, x \in \mathbb{R}, \quad n \in \mathbb{Z}^+, \quad (1.2)$$

$$u(x, 0) = \delta(x),$$

is locally ill-posed in $H^s$, $s < -1/2$, for $n = 2$. In [6] (1.2) was shown to be locally well-posed in $H^s$, $s \geq -3/4$, for $n = 1$, and in [2] (1.2) will be shown to be locally ill-posed in $H^{s_n}$, $s_n < 1/2 - 2/n$, for $n \geq 3$, compare [8]. The Dirac delta function lies in $H^s$, $s < -1/2$, this is sufficiently smooth initial data for KdV, borderline for MKdV and too rough for the (higher order) generalized KdV equations.
2. SOLITARY WAVES

Consider the IVP
\[ \frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} + u^n \frac{\partial u}{\partial x} = 0 \quad t, x \in \mathbb{R}, \quad n \in \mathbb{Z}^+, \]
\[ u(x, 0) = u_0(x). \]

For any \( n \) there exists a one-parameter (\( k \)) family of solutions
\[ u_n(x, t, k) = \left( \frac{n + 2}{2} \right)^{1/4} k^{-2/n} \text{sech}^{2/n} \left( \frac{n}{2} (kx - k^3 t) \right), \quad 0 < k < \infty, \]
which are solitary waves.

The norm of these solutions is finite, with the exception of KdV (\( n = 1 \)), in the Sobolev spaces \( H^{1/2 - \frac{2}{n}} \), and independent of \( k \), as \( k \to \infty \). This is easily shown by use of the Fourier transform,
\[ \hat{u}_n(\xi, 0) = (2n + 2)^{1/n} \frac{1}{n \Gamma(2/n)} \frac{1}{k^{1-\frac{2}{n}}} \Gamma \left( \frac{1}{n} + \frac{i \xi}{nk} \right) \Gamma \left( \frac{1}{n} - \frac{i \xi}{nk} \right), \]
where \( \Gamma \) is the gamma function, see Batemann [1]. The Fourier transform reduces to
\[ \hat{u}_1(\xi, 0) = \frac{3\pi}{2} \xi \text{csch} \left( \frac{\pi \xi}{2k} \right) \]
and
\[ \hat{u}_2(\xi, 0) = \sqrt{2\pi} \text{sech} \left( \frac{\pi \xi}{2k} \right) \]
for \( n = 1 \) and 2, respectively.

**Lemma 2.1.** - The solitary wave solution \( u_n(x, t, k) \) has a finite \( H^s \) norm, \( s = \frac{1}{2} - \frac{2}{n} \), uniformly with respect to \( k > 0 \), for \( n > 2 \).

**Proof.** - By the Plancherel identity,
\[
\|u\|_{H^{1/2 - \frac{2}{n}}}^2 = \frac{(2n + 4)^{2/n}}{4\pi^2} \frac{1}{n^2 \Gamma^2 \left( \frac{2}{n} \right)} \frac{1}{k^{2-\frac{2}{n}}} \times \int_{-\infty}^{\infty} (1 + \xi^2)^{\frac{1}{2} - \frac{2}{n}} \Gamma^2 \left( \frac{1}{n} + \frac{i \xi}{kn} \right) \Gamma^2 \left( \frac{1}{n} - \frac{i \xi}{kn} \right) d\xi
\]
\[
= (2n + 4)^{2/n} \frac{1}{n^2 \Gamma^2 \left( \frac{2}{n} \right)} \int_{-\infty}^{\infty} \left( \frac{1}{k^2 + z^2} \right)^{\frac{1}{2} - \frac{2}{n}} \times \Gamma^2 \left( \frac{1}{n} + \frac{iz}{n} \right) \Gamma^2 \left( \frac{1}{n} - \frac{iz}{n} \right) dz,
\]
where $z = \xi / k$. Then we use Stirling’s formula to obtain

$$
\|u\|^2_{\frac{3}{2} - \frac{2}{n}} = \frac{c(n,k)(2n+4)^{2/n}}{n^2 \Gamma^2(\frac{2}{n})} \int_{-\infty}^{\infty} \frac{(z^2 + \frac{1}{k^2})^{\frac{3}{2} - \frac{2}{n}} e^{-\frac{2\pi|z|}{n}}}{\left((1 - \frac{1}{n})^2 + \frac{z^2}{n}\right)^{1-\frac{2}{n}}} \, dz,
$$

where $c(n,k)$ is a regular function of $k$. The last integral converges uniformly in $k$ and by the dominated convergence theorem we can pass to the limit $k \to \infty$, to get

$$
\|u_n(x,0,\infty)\|^2_{\frac{3}{2} - \frac{2}{n}} = \frac{c(n)(2n+4)^{2/n}}{n^2 \Gamma^2(\frac{2}{n})} \int_{-\infty}^{\infty} \frac{z^{1-\frac{4}{n}} e^{-\frac{2\pi|z|}{n}}}{\left((1 - \frac{1}{n})^2 + \frac{z^2}{n}\right)^{1-\frac{2}{n}}} \, dz.
$$

This integral converges (at the origin) only if $n > 2$. □

**Remark 2.1.** – The limit $k \to \infty$ gives the norm

$$
\|u\|_s = \left(\int_{-\infty}^{\infty} [(-\Delta)^{\frac{s}{2}} u]^2 \, dx\right)^{1/2}
$$

of the homogeneous Sobolev space $\dot{H}^s$. This norm $\|\dot{u}\|_{\frac{3}{2} - \frac{2}{n}}$ is actually independent of $k$. Therefore the index $s = \frac{1}{2} - \frac{2}{n}$ is called the scaling index for $n$.

**Remark 2.2.** – Observe that if $u(x,t)$ solves (2.1), so does $\lambda^{2/n} u(\lambda x, \lambda^3 t)$ for any real-valued $\lambda > 0$. This suggests that one should consider the homogeneous norm $\|\dot{u}\|_{\frac{3}{2} - \frac{2}{n}}$, and this agrees with [6] for $n \geq 4$. However, for $n = 1$ and 2 the formulas (2.4) and (2.5) easily show that the $\dot{H}^s$ norms of $u_n(x,t,k)$ diverge as $k \to \infty$, for $s < -3/2$ and $s < -1/2$, respectively. One must consider the usual inhomogeneous $H^s$ norms to overcome this difficulty.

### 3. THE MODIFIED KdV EQUATION

Consider the IVP for the modified KdV equation (MKdV)

$$
(3.1) \quad \frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} + u_x^2 \frac{\partial u}{\partial x} = 0, \quad x, t, \in \mathbb{R},
$$

$$
u(x,0) = u_0(x).
$$

Annales de l’Institut Henri Poincaré - Analyse non linéaire
We will show that the solution of this IVP cannot depend continuously on its initial data in a Sobolev space $H^s$ of negative index $s < -\frac{1}{2}$. The soliton solution (2.2) of MKdV scales to a constant multiple of the Dirac delta function,

$$\lim_{\epsilon \to 0} \frac{\sqrt{2}}{\epsilon} \text{sech} \left( \frac{x}{\epsilon} \right) = c \delta(x),$$

this is proven in two lemmas below, and we will pose the IVP with this initial data in $H^s$, $s < -1/2$, to prove the theorem.

**Lemma 3.1.** The $H^s$ norm of

$$u_\epsilon(x, 0) = \frac{\sqrt{2}}{\epsilon} \text{sech} \left( \frac{x}{\epsilon} \right)$$

is finite for $s < -1/2$ and

$$\lim_{\epsilon \to 0} \|u_\epsilon(x, 0)\|_s = \left\| \sqrt{2} \pi \delta(x) \right\|_s$$

for $s < -1/2$.

**Proof.** The Fourier transform of $u_\epsilon$, see (2.5), is

$$\hat{u}_\epsilon(\xi, 0) = \frac{\sqrt{2}}{\epsilon} \int_{-\infty}^{\infty} \text{sech} \left( \frac{x}{\epsilon} \right) e^{-i\xi x} \, dx$$

$$= \sqrt{2} \pi \text{sech} \left( \frac{\epsilon \pi \xi}{2} \right).$$

By the Plancherel identity

$$\|u_\epsilon\|_{-s}^2 = \frac{1}{2} \int_{-\infty}^{\infty} \text{sech}^2 (\epsilon \pi \frac{\xi}{2}) \frac{1}{(1 + \xi^2)^s} \, d\xi$$

$$\leq \frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{(1 + \xi^2)^s} \, d\xi < \infty,$$

for $s > 1/2$. Thus the integral converges uniformly in $\epsilon$ and we can bring the limit inside it. Finally one observes that

$$\|\delta\|_{-s}^2 = \frac{1}{2\pi^2} \cdot \frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{(1 + \xi^2)^s} \, d\xi,$$

by the Plancherel identity, since $\hat{\delta} = 1$. \square
Remark 3.1. – The integrals are invariant with respect to translation and this implies that we get the same statements for \( u_\epsilon(x, t) \),

\[
\|u_\epsilon(x, t)\|_s < \infty
\]

and

\[
\lim_{\epsilon \to 0} \|u_\epsilon(x, t)\|_s = \|\sqrt{2\pi \delta}\|_s
\]

for \( t \geq 0 \) and \( s < -1/2 \).

Lemma 3.2. – \( u_\epsilon(x, 0) \) converges weakly to \( \sqrt{2\pi \delta}(x) \), as \( \epsilon \to 0 \), but \( u_\epsilon(x, t) \) converges weakly to zero, for \( t > 0 \), as \( \epsilon \to 0 \), in \( H^s \), \( s < -1/2 \).

Proof. – The \( H^{-s} \) norms are finite

\[
\|u_\epsilon(x, 0)\|_{-s} = \|u_\epsilon(x, t)\|_{-s} < \infty,
\]

by Lemma 3.1 and Remark 3.1, so it suffices to compute the limits

\[
\lim_{\epsilon \to 0} \langle u_\epsilon, v \rangle
\]

for all \( v \) in a strongly dense subset of the dual space \( H^s \), \( s > 1/2 \), where \( \langle , \rangle \) denotes the dual pairing between \( H^{-s} \) and \( H^s \). We choose the smooth compactly supported functions \( C_0^\infty \), they are dense in \( H^s \), \( s > 0 \). Then

\[
\lim_{\epsilon \to 0} \sqrt{2} \int_{-\infty}^{\infty} \frac{\text{sech}\left(\frac{x}{\epsilon}\right)}{\epsilon} \phi(x) \, dx
\]

\[
= \lim_{\epsilon \to 0} \sqrt{2} \int_{-\infty}^{\infty} \text{sech}(y)\phi(\epsilon y) \, dy
\]

\[
= \sqrt{2\pi \phi(0)}, \quad \text{for } \phi \in C_0^\infty,
\]

by the uniform convergence of the integral. This shows that \( \frac{\sqrt{2}}{\epsilon} \text{sech}\left(\frac{x}{\epsilon}\right) \) converges weakly to \( \sqrt{2\pi \delta}(x) \), in \( H^{-s} \). Similarly

\[
\lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \frac{\text{sech}\left(\frac{x}{\epsilon} - \frac{t}{\epsilon^2}\right)}{\epsilon} \phi(x) \, dx
\]

\[
= \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \text{sech}(y)\phi\left(\epsilon y + \frac{t}{\epsilon^2}\right) \, dy = 0,
\]

since \( \phi \in C_0^\infty \) has compact support. This shows that \( \frac{\sqrt{2}}{\epsilon} \text{sech}\left(\frac{x}{\epsilon} - \frac{t}{\epsilon^2}\right) \) converges weakly to 0 in \( H^{-s} \). \( \Box \)

We can now prove the result.
THEOREM 3.1. — The initial value problem (3.1) for the modified KdV equation is locally ill-posed in $H^s$, $s < -1/2$.

Proof. — We will prove that if there exists a local (strong) solution of the IVP (3.1) with $u_0(x) = \sqrt{2\pi} \delta(x)$ in $H^s$, $s < -1/2$, then it does not depend continuously on its initial data.

The $H^s$ norm of $u\epsilon(x,0)$ converges to the norm of $\sqrt{2\pi} \delta(x)$ by Lemma 3.1 and $u\epsilon(x,0)$ converges weakly to $\sqrt{2\pi} \delta(x)$ by Lemma 3.2, therefore $u\epsilon(x,0)$ converges strongly to $\sqrt{2\pi} \delta(x)$. Now we solve the IVP (3.1) with initial data

$$u_0(x, \epsilon) = \frac{\sqrt{2}}{\epsilon} \text{sech} \left( \frac{x}{\epsilon} \right)$$

to get the explicit solutions

$$u\epsilon(x, t) = \frac{\sqrt{2}}{\epsilon} \text{sech} \left( \frac{x - t}{\epsilon^3} \right).$$

By Remark 3.1, the $H^s$ norm of $u\epsilon(x, t)$ converges to the norm of $\sqrt{2\pi} \delta(x)$, which is non-vanishing but, by Lemma 3.2, $u\epsilon(x, t)$ converges weakly to zero. Consequently, $u\epsilon(x, t)$ cannot converge strongly to the solution of the IVP (3.1) with initial data

$$u_0(x) = \sqrt{2\pi} \delta(x),$$
in $H^s$, $s < -1/2$. □

REFERENCES


(Manuscript received November 17, 1994.)

Vol. 13, n° 4-1996.