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# Global strong solutions in Sobolev or Lebesgue spaces to the incompressible Navier-Stokes equations in $\mathbb{R}^3$

by

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**ABSTRACT.** – We construct global strong solutions of the Navier-Stokes equations with sufficiently oscillating initial data. We will show that the condition is for the norm in some Besov space to be small enough.

**RÉSUMÉ.** – Nous construisons des solutions fortes globales des équations de Navier-Stokes, pour des données initiales suffisamment oscillantes. Cette condition se traduit en terme de norme petite dans un certain espace de Besov.

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## INTRODUCTION

We are interested in the following system, for  $x \in \mathbb{R}^3$  and  $t > 0$ ,

$$(1) \quad \begin{cases} \frac{\partial u}{\partial t} + (u \cdot \nabla) u = \nu \Delta u - \nabla p \\ \nabla \cdot u = 0, \end{cases}$$

with initial data  $u(x, 0) = u_0(x)$ . For the sake of simplicity, we suppose that  $\nu = 1$ ; a simple rescaling allows us to obtain any other value. Local

existence and uniqueness in the Sobolev space  $H^s(\mathbb{R}^3)$  and the Lebesgue space  $L^p(\mathbb{R}^3)$  are known, if  $s > 1/2$  and  $p > 3$  (see [4]). We have global solutions for small initial data in  $L^3(\mathbb{R}^3)$  (see [9] or [4]) and  $H^{\frac{1}{2}}(\mathbb{R}^3)$  (see [4] and [5]), or in  $L^2(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$ , with  $p > 3$  (see [1]). We shall extend the results of [4], for  $s > 1/2$  and  $p > 3$ . By adapting the auxiliary spaces used in [4], we shall prove the existence and uniqueness of global solutions in  $H^s(\mathbb{R}^3)$  provided the initial data are small in a sense which will be made precise later, and in  $L^p(\mathbb{R}^3)$  up to additional conditions on  $u_0$ . Let us define the homogeneous Besov spaces  $\dot{B}_{p,q}^\alpha$  :

DEFINITION 1. – Let us choose  $\phi \in \mathcal{S}(\mathbb{R}^n)$  a radial function so that  $\text{Supp } \hat{\phi} \subset \{|\xi| < 1 + \varepsilon\}$ , and  $\hat{\phi}(\xi) = 1$  for  $|\xi| < 1$ . Define  $\phi_j(x) = 2^{nj}\phi(2^jx)$ ,  $S_j$  the convolution operator with  $\phi_j$ , and  $\Delta_j = S_{j+1} - S_j$ . Let  $f \in \mathcal{S}'(\mathbb{R}^n)$ ,  $\alpha \in \mathbb{R}$ ,  $1 < p, q \leq +\infty$ ,  $f \in \dot{B}_{p,q}^\alpha$  if and only if

$$\left[ \sum_{-\infty}^{+\infty} (2^{j\alpha} \|\Delta_j f\|_{L^p})^q \right]^{\frac{1}{q}} < +\infty.$$

The reader should consult [12], [2], or [16] where the properties of Besov spaces are exposed in detail. Let us see how homogeneous Besov spaces arise. If we want to construct a global solution, it is useful to control a norm remaining invariant by the rescaling  $f(x) \rightarrow \lambda f(\lambda x)$ . If this can be achieved in a Besov space with  $\alpha < 0$  and therefore bigger than the usual space where we want to obtain a solution, we will have weaker assumptions on  $u_0$ .

Let us give the results in the case of Sobolev spaces.  $BC$  denotes the class of bounded continuous functions.

THEOREM 1. – There exists an universal constant  $\beta > 0$  such that, if  $s > \frac{1}{2}$ ,  $u_0 \in H^s(\mathbb{R}^3)$ ,  $\nabla \cdot u_0 = 0$  and

$$(2) \quad \|u\|_{\dot{B}_{4,\infty}^{-1/4}} < \beta,$$

then there exists a unique solution  $u$  of (1) such that

$$(3) \quad u \in BC([0, \infty), H^s(\mathbb{R}^3)).$$

Moreover, the following properties hold for  $u$ :

- $\|u(\cdot, t)\|_{L^2}$  is decreasing, and for every  $t > 1$ ,

$$(4) \quad \|u(\cdot, t) - e^{t\Delta}u_0\|_{L^2} \leq \frac{C(\beta, u_0)}{t^{1/4}}.$$

- For every  $t > 1$ ,

$$(5) \quad \| (-\Delta)^{s/2} u(\cdot, t) \|_{L^2} \leq \frac{C(\beta, u_0, s)}{t^{s/2}}.$$

- For every  $t > 0$ ,

$$(6) \quad \| u(\cdot, t) \|_{\infty} \leq \frac{C(\beta, u_0)}{\sqrt{t}}.$$

- If  $s \in (1, 3/2]$ , for every  $t < 1$ ,

$$(7) \quad \| u(\cdot, t) - e^{t\Delta} u_0 \|_{\infty} \leq C(\beta, u_0).$$

Note that the space  $\dot{B}_{4,\infty}^{-1/4}$  is invariant under the scaling  $f(x) \rightarrow \lambda f(\lambda x)$ , and  $\dot{H}^{\frac{1}{2}} \subset \dot{B}_{4,\infty}^{-1/4}$ . It is very interesting that we do not need a small  $H^{\frac{1}{2}}$ -norm to obtain a global solution (see [4]). On the other hand, if we want to include the case  $1/2$ ,  $u$  is unique in the space

$$\begin{cases} u \in BC([0, +\infty), H^{\frac{1}{2}}) \\ t^{1/8} u(\cdot, t) \in BC([0, +\infty), L^4) \\ \lim_{t \rightarrow 0} t^{1/8} \|u\|_{L^4} = 0. \end{cases}$$

which was used in [4], the starting point of the present work. The weak condition (2) is the only remaining obstacle to the problem of existence of global smooth solutions to the Navier-Stokes equations, and we remark that  $\beta$  does not depend on  $s$ . The decay estimates (4) can be found in [8], in a slightly different context. We recall it here as a natural consequence of the construction of  $u$ .

In the Lebesgue spaces, the analogue is

**THEOREM 2.** - Let  $p > 3/2$ , there exists  $\delta(p) > 0$  such that, if  $u_0 \in L^p \cap \dot{B}_{2p,\infty}^{-(1-\frac{3}{2p})}$ ,  $\nabla \cdot u_0 = 0$  and

$$(8) \quad \| u_0 \|_{\dot{B}_{2p,\infty}^{-(1-\frac{3}{2p})}} < \delta(p),$$

then there exists a unique solution  $u$  such that

$$\begin{cases} u \in BC([0, +\infty), L^p) \\ t^{\frac{1}{2}-\frac{3}{4p}} u(\cdot, t) \in BC([0, +\infty), L^{2p}) \\ \lim_{t \rightarrow 0} t^{\frac{1}{2}-\frac{3}{4p}} \|u\|_{L^{2p}} = 0. \end{cases}$$

The restriction  $p > 3/2$  is due to technical considerations, and we could probably obtain 1 instead of  $3/2$ , by slightly modifying the Besov space involved.

PROPOSITION 1. – *The constant  $\delta(p)$  satisfies:*

$$\begin{aligned} \lim_{p \rightarrow +\infty} \delta(p) &= 0, \\ \lim_{p \rightarrow 3/2} \delta(p) &= 0. \end{aligned}$$

PROPOSITION 2. – *In Theorem 2, we can replace  $u_0 \in L^p \cap \dot{B}_{2p,\infty}^{-(1-\frac{3}{2p})}$  by  $u_0 \in L^p \cap L^3$ , and if  $p > 3$  by  $L^2 \cap L^p$ .*

If  $u_0 \in H^s, s \geq 1/2$ , then as  $\dot{H}^{\frac{1}{2}} \subset \dot{B}_{4,\infty}^{-1/4}$ , we have a natural candidate for the useful Besov space. On the contrary, if we take  $L^p$ , we may use two different Besov spaces: the first one is  $\dot{B}_{2p,\infty}^{-\frac{3}{2p}}$ , as  $L^p \subset \dot{B}_{2p,\infty}^{-\frac{3}{2p}}$ . But this space is not invariant by the rescaling. The “right” space is  $\dot{B}_{2p,\infty}^{-(1-\frac{3}{2p})}$ , but unfortunately  $L^p \not\subset \dot{B}_{2p,\infty}^{-(1-\frac{3}{2p})}$ . This explains the additional condition imposed on  $u_0$  in Theorem 2. Both spaces coincide only when  $1 - \frac{3}{2p} = \frac{3}{2p}$ , which means  $p = 3$ . The reader should refer to [9] and [4] for details.

*Proofs.* – We first reformulate the problem in order to obtain an integral equation for  $u$ . This is standard practice, and was first employed by Kato and Fujita (see [10] [11]), and very often used since (see [7] [6] [15]). All these authors use semi-group theory, but in the present case, we do not need this formalism, for the exact expression of the heat kernel in  $\mathbb{R}^3$  allows us to obtain directly the estimates we need (see [9]). Let  $\mathbb{P}$  be the projection operator from  $(L^2(\mathbb{R}^3))^3$  onto the subspace of divergence-free vectors, denoted by  $\mathbb{P}L^2$ , and  $R_j$  the Riesz transform with symbol  $\frac{\xi_j}{|\xi|}$ . We easily see that

$$(9) \quad \mathbb{P} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} - \begin{pmatrix} R_1 \sigma \\ R_2 \sigma \\ R_3 \sigma \end{pmatrix}$$

where  $\sigma = \sum_j R_j u_j$ . It is well-known that  $\mathbb{P}$  can be extended to a bounded operator from  $(L^p)^3$  onto  $\mathbb{P}L^p, 1 < p < +\infty$ , and from  $(H^s)^3$  onto  $\mathbb{P}H^s, s \geq 0$ . Note that  $\mathbb{P}$  commutes with  $S(t) = e^{t\Delta}$ , whereas on an open set  $\Omega$ , we need to introduce the Stokes operator  $-\mathbb{P}\Delta$  and the associated semi-group. Note that

$$\text{Ker } \mathbb{P} = \{u \mid \exists \phi \text{ such that } u = \nabla \phi\}.$$

Using  $\mathbb{P}$ , (1) becomes an evolution equation

$$(10) \quad \begin{cases} \frac{\partial u}{\partial t} = \Delta u - \mathbb{P}\nabla \cdot (u \otimes u), \\ \nabla \cdot u = 0, \\ u(x, 0) = u_0(x). \end{cases}$$

We replace  $(u \cdot \nabla)u$  by  $\nabla \cdot (u \otimes u)$  to avoid problems of definition, and this is possible only because  $\nabla \cdot u = 0$ . It is then standard to study (10) via the corresponding integral equation

$$(11) \quad u(x, t) = S(t)u_0(x) - \int_0^t \mathbb{P}S(t-s)\nabla \cdot (u \otimes u)(x, s)ds$$

in a space of divergence free vectors. The integral should be seen as a Bochner integral. In the general case of evolution equations, a solution of (11) might not be a solution of (10). However, in the case of the Navier-Stokes equations without external forces, it is true without any extra assumptions. Actually, the solutions of (11) are  $C^\infty((0, +\infty) \times \mathbb{R}^3)$  and verify the equations (1) in the classical sense, as we recover easily the pressure up to a constant by

$$(12) \quad -\Delta p = \sum_{i,j} \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i}.$$

The reader should refer to [7] [10] or [13] for proofs. We remark that since a solution of (1) is necessarily a solution of (11), uniqueness for (11) guarantees uniqueness for (1). We aim to solve (11) by successive approximations, with the following lemma:

LEMMA 1. – *Let  $E$  and  $F$  be two Banach functional spaces, endowed with the norms  $\|\cdot\| = \|\cdot\|_E$  and  $|\cdot| = \|\cdot\|_F$ ,  $B$  a continuous bilinear operator from  $F \times F \rightarrow E$  and  $F \times F \rightarrow F$ :*

$$\begin{aligned} \|B(u, v)\| &\leq \eta |u| |v| \\ |B(u, v)| &\leq \gamma |u| |v|, \end{aligned}$$

and define the sequence  $X_0 = 0, X_{n+1} = Y + B(X_n, X_n)$ , where  $Y$  belongs to  $E$  and to  $F$ . If

$$(13) \quad 4\gamma |Y| < 1,$$

then the sequence converges in both spaces  $E$  and  $F$ , and the limit  $X$  satisfies

$$(14) \quad X = Y + B(X, X)$$

and

$$(15) \quad |X| < 2|Y|.$$

The proof is left to the reader. Note that the value of  $\eta$  has no influence on the convergence. Now we have to study the following bilinear operator

$$(16) \quad B(u, v) = \int_0^t \mathbb{P}S(t-s)\nabla \cdot (u \otimes v) ds.$$

In order to simplify the notations, we limit ourselves to the following scalar operator

$$(17) \quad B(f, g) = \int_0^t \frac{1}{(t-s)^2} \theta \left( \frac{\cdot}{(t-s)^2} \right) * fg(s) ds.$$

As  $\mathbb{P}S(t-s)\nabla \cdot$  is a matrix of convolution operators, the components are all operators like (17), with

$$(18) \quad e^{|\xi|^2} \hat{\theta}(\xi) = \frac{\xi_j \xi_k \xi_l}{|\xi|^2} \quad (\text{with a } -\xi_j \text{ on the diagonal}).$$

LEMMA 2. -  $\theta(x) \in C^\infty(\mathbb{R}^3)$  and  $\theta \in L^1 \cap L^\infty$ .

This can be easily seen on the Fourier transform of  $\theta$ .

In what follows,  $C$  denotes a constant which may vary from one line to another.

*Proof of Theorem 1*

PROPOSITION 3. - Let  $1/2 < s \leq 3/4$ , then there exists a solution  $u$  of (11) such that

$$(19) \quad \begin{cases} u \in BC([0, +\infty), \dot{H}^s) = E, \\ \omega(t)u(x, t) \in BC([0, +\infty), L^4) = F, \end{cases}$$

where  $\omega(t) = t^{3/8-s/2}$  if  $0 < t < 1$  and  $\omega(t) = t^{1/8}$  if  $t \geq 1$ .

We want to apply Lemma (1) where E and F are defined by the norms

$$\begin{aligned} \| u \| &= \sup_t \| u \|_{\dot{H}^s} \\ | u | &= \sup_t \omega(t) \| u \|_{L^4}. \end{aligned}$$

If we use Hölder and Young inequalities for  $B(f, g)$ ,  $\Lambda$  being the operator with symbol  $|\xi|$ ,

$$(20) \quad \| \Lambda^s B(f, g)(t) \|_{L^2} \leq \| \theta \|_{L^1} \| f \| \| g \| \int_0^t \frac{1}{(t - \tau)^{1/2+s/2} \omega^2(\tau)} d\tau,$$

$$(21) \quad \| B(f, g)(t) \|_{L^4} \leq \| \theta \|_{L^{4/3}} \| f \| \| g \| \int_0^t \frac{1}{(t - \tau)^{7/8} \omega^2(\tau)} d\tau.$$

We shall then verify that, for all  $t > 0$ ,

$$\begin{aligned} I_1 &= \int_0^t \frac{1}{(t - \tau)^{1/2+s/2} \omega^2(\tau)} d\tau < +\infty, \\ I_2 &= \int_0^t \frac{1}{(t - \tau)^{7/8} \omega^2(\tau)} d\tau < +\infty. \end{aligned}$$

Easy calculations actually show that for  $t < 1$ ,

$$I_i < C t^{s/2-1/4}$$

and for  $t > 1$

$$(22) \quad I_i < C t^{1/4-s/2}.$$

The continuity at  $t = 0$  comes from the estimate when  $t < 1$ . In order to include the case  $s = 1/2$ , we have to impose  $\lim_{t \rightarrow 0} t^{1/8} \| u \|_{L^4} = 0$  (see [4]). Note that the constant  $\gamma$  of Lemma 1 is

$$(23) \quad \gamma = \int_0^1 \frac{\| \theta \|_{L^{4/3}}}{(1 - \tau)^{7/8} \tau^{1/4}} d\tau.$$

Therefore, if  $S(t)u_0$  satisfies condition (13), we obtain  $u \in BC([0, +\infty), \dot{H}^s)$ .

PROPOSITION 4. - We have

$$(24) \quad u \in BC([0, +\infty), L^2).$$



Let  $G = BC([0, +\infty), L^2)$ ;  $B$  is bicontinuous from  $G \times F$  to  $G$ :

$$(25) \quad \| B(f, g)(t) \|_{L^2} \leq |g| \sup_{[0,t]} \| f \|_{L^2} \| \theta \|_{L^{4/3}} \int_0^t \frac{1}{(t-\tau)^{7/8} \omega(\tau)} d\tau.$$

Let

$$I_3 = \int_0^t \frac{\| \theta \|_{L^{4/3}}}{(t-\tau)^{7/8} \omega(\tau)} d\tau,$$

for  $t < 1$ ,  $I_3 < Ct^{s/2-1/4}$ , and for all  $t$

$$I_3 \leq \int_0^1 \frac{\| \theta \|_{L^{4/3}}}{(1-\tau)^{7/8} \tau^{7/8}} d\tau = \rho.$$

$G$  being a Banach space, we can use a contraction argument to show that the sequence defined previously converges in  $G$ . It is sufficient that  $2 |u| \rho < 1$ , which is true as  $\rho \leq \gamma$  and  $u$  verifies (15). Therefore, we proved (24) and hence Proposition 3, and shown that  $\| u(\cdot, t) \|_{L^2}$  is uniformly bounded.

We now show (6): the following estimation is verified by the heat kernel,

$$(26) \quad \sup_{[0,t]} \sqrt{t} \| S(t)u_0 \|_\infty \leq C.$$

We have

$$\| B(f, g)(t) \|_\infty \leq \| \theta \|_{L^{4/3}} \int_0^t \frac{\| f(s) \|_{L^4} \| g(s) \|_\infty}{(t-s)^{7/8}} ds.$$

Let us denote  $W(f, t) = \sup_{[0,t]} \sqrt{t} \| f(\cdot, s) \|_\infty$ , then

$$(27) \quad W(B(f, g), t) \leq |f| W(g, t) \| \theta \|_{L^{4/3}} \int_0^t \frac{\sqrt{t}}{(t-s)^{7/8} s^{5/8}} ds.$$

Let

$$I_4 = \int_0^1 \frac{\| \theta \|_{L^{4/3}}}{(1-\mu)^{7/8} \mu^{5/8}} d\mu$$

then, as  $I_4 \leq 2\gamma$ , we have  $2W(S(t)u_0, t)I_4 < 1$ . Therefore,

$$(28) \quad \sup_{[0,t]} \sqrt{t} \| u(\cdot, t) \|_\infty \leq \frac{C}{1 - 2I_4 \| S(t)u_0 \|}.$$

Now we can prove (4) as follows:

$$\| B(f, g)(t) \|_{L^2} \leq \int_0^t \frac{C}{(t-s)^{2-\frac{3}{2q}}} \| g \|_{L^2} \| f \|_{L^\beta} ds,$$

where

$$\frac{1}{2} = \frac{1}{q} + \frac{1}{2} + \frac{1}{\beta} - 1.$$

If we take  $q$  such that  $\frac{3}{2q} = 1 + \varepsilon$ ,  $\varepsilon > 0$ , using interpolation and (28) we get, for  $t > 1$ ,

$$(29) \quad \|f\|_{L^p} \leq \|f\|_{L^2}^{\frac{2}{p}} \|f\|_{\infty}^{1-\frac{2}{p}}$$

and

$$(30) \quad \begin{aligned} \|B(f, g)(t)\|_{L^2} &\leq \int_0^t \frac{C}{(t-s)^{1-\varepsilon} s^{\frac{1}{6}+\frac{2}{3}\varepsilon}} ds \\ \|B(f, g)(t)\|_{L^2} &\leq \frac{C(\varepsilon)}{t^{\frac{1}{6}-\frac{\varepsilon}{3}}}. \end{aligned}$$

On the other hand, we know by (26) that  $\forall q \geq 2$ ,

$$(31) \quad \sup_{[0,t]} t^{\frac{3}{2}(\frac{1}{2}-\frac{1}{q})} \|S(t)u_0\|_q \leq C.$$

Therefore, as  $u$  satisfies (14), we will improve (30) in the following way: let

$$(32) \quad B_1(f, g) = \int_0^1 \frac{1}{(t-s)^2} \theta\left(\frac{\cdot}{\sqrt{t-s}}\right) * fg(s) ds$$

$$(33) \quad B_2(f, g) = \int_1^t \frac{1}{(t-s)^2} \theta\left(\frac{\cdot}{\sqrt{t-s}}\right) * fg(s) ds.$$

The term  $B_1$  can be handled very easily, so that  $\forall \eta > 0$ ,

$$\|B_1(f, g)(t)\|_{L^2} \leq \frac{C}{t^{1-\eta}}.$$

Now, we split  $B_2(u, u)$  in three parts. By (31) we have

$$\begin{aligned} \|B_2(S(t)u_0, S(t)u_0)\|_{L^2} &\leq \int_1^t \frac{C}{(t-s)^{2-\frac{3}{2q}} s^{\frac{3}{2}(1-\frac{1}{\alpha}-\frac{1}{\beta})}} ds \\ &\leq \frac{C}{t^{1-\frac{3}{2q}+\frac{3}{2}(1-(\frac{3}{2}-\frac{1}{q}))}} \\ &\leq \frac{C}{t^{1/4}}, \end{aligned}$$

as

$$\frac{1}{2} = \frac{1}{\gamma} + \frac{1}{\alpha} + \frac{1}{\beta} - 1.$$

We remark that the exponent  $1/4$  cannot be improved, as it does not depend on  $\gamma, \alpha$  and  $\beta$ .

LEMMA 3. – *Suppose that for  $0 < \mu$*

$$\| B(u, u)(t) \|_{L^2} \leq \frac{C}{t^\mu},$$

then

$$\| B_2(S(t)u_0, B(u, u)) \|_{L^2} \leq \frac{C}{t^{1/4+\mu}},$$

and there exists  $\nu > 0$  such that

$$\| B_2(B(u, u), B(u, u)) \|_{L^2} \leq \frac{C}{t^{\mu+\nu}}.$$

By (31)

$$\begin{aligned} \| B_2(S(t)u_0, B(u, u)) \|_{L^2} &\leq \int_1^t \frac{C}{(t-s)^{2-\frac{3}{2q}} s^{\frac{3}{2}(\frac{1}{2}-\frac{1}{\beta})} s^\mu} ds \\ &\leq \frac{C}{t^{-\varepsilon+\frac{3}{2}(\frac{1}{6}+\frac{2}{3}\varepsilon)+\mu}} \\ &\leq \frac{C}{t^{1/4+\mu}}, \end{aligned}$$

and, by (28) and (29)

$$\begin{aligned} \| B_2(B(u, u), B(u, u)) \|_{L^2} &\leq \int_1^t \frac{C}{(t-s)^{2-\frac{3}{2q}} s^\mu s^{\frac{1}{2}(1-\frac{2}{\beta})+\frac{2}{\beta}\mu}} ds \\ &\leq \frac{C}{t^{-\frac{1}{3}\varepsilon+\frac{1}{6}+(\frac{5}{3}-\frac{4}{3}\varepsilon)\mu}}. \end{aligned}$$

We can start with  $\mu = 1/6 - \varepsilon$ , and obtain any exponent  $\eta > 1/4$ . Thus,

$$(34) \quad \| B_2(u, u) \|_{L^2} \leq \frac{C}{t^{1/4}}.$$

We constructed  $u$  for  $s \leq 3/4$ . Now we will see that if  $s > 3/4$ ,  $u$  as above is actually in  $H^s$ . We limit ourselves to the bilinear form (17), as the term  $S(t)u_0$  satisfies at least the same estimates.

LEMMA 4. – Let  $f, g \in H^s(\mathbb{R}^3), 3/4 \leq s < 3/2,$

$$(35) \quad \|\Lambda^{2s-3/2}(fg)\|_{L^2} \leq C \|\Lambda^s f\|_{L^2} \|\Lambda^s g\|_{L^2} .$$

For a proof see the Appendix. Suppose now that  $s > 3/4,$  and  $u$  is the solution of Proposition 3 for  $s = 3/4.$  Then, if  $\eta \leq 1/4,$  we obtain for  $t < 1$

$$\|\Lambda^{\frac{3}{4}+\eta}B(f, g)(t)\|_{L^2} \leq \int_0^t \frac{C}{(t-\tau)^{\frac{1}{2}+\frac{3/4+\eta}{2}}} d\tau$$

using Lemma 4 and the boundedness of  $f$  in  $H^{3/4},$  so that

$$(36) \quad \|\Lambda^{\frac{3}{4}+\eta}B(f, g)(t)\|_{L^2} \leq Ct^{\frac{1}{8}-\frac{\eta}{2}},$$

which gives the continuity at zero. For  $t > 1,$  we have by (29)

$$\|B(f, g)\|_{L^4} \leq \frac{C}{t^{1/4}}$$

which allows us to improve (22), for  $s \leq 3/4$

$$\begin{aligned} \|\Lambda^s B(f, g)(t)\|_{L^2} &\leq \int_0^1 \frac{\|\theta\|_{L^1}}{(t-\tau)^{1/2+s/2}\omega^2(\tau)} d\tau |f| |g| \\ &+ \int_1^t \frac{C\|\theta\|_{L^1}}{(t-\tau)^{1/2+s/2}\tau^{1/2}} d\tau \\ &\leq \frac{C}{t^{s/2}}. \end{aligned}$$

Then

$$\|\Lambda^{\frac{3}{4}}f(t)\|_{L^2} \leq \frac{C}{t^{\frac{3}{8}}}$$

and,

$$\|\Lambda^{\frac{3}{4}+\eta}B_1(f, g)\|_{L^2} \leq \frac{C}{(t-1)^{\frac{1}{2}+\frac{3/4+\eta}{2}}}$$

and

$$\|\Lambda^{\frac{3}{4}+\eta}B_2(f, g)\|_{L^2} \leq \int_1^t \frac{C}{(t-\tau)^{\frac{1}{2}+\frac{3/4+\eta}{2}}\tau^{\frac{1}{2}}} d\tau$$

then, for all  $t$

$$\|\Lambda^{\frac{3}{4}+\eta}B(f, g)(t)\|_{L^2} \leq \frac{C}{1+t^{\frac{3}{8}+\frac{\eta}{2}}}.$$

We have thus obtained  $u \in H^{\frac{3}{4}+\eta}$ . By applying the same argument we can reach the value  $s > 3/2$ , as

$$\Lambda^{s+\eta} B(f, g)(t) = \int_0^t \frac{1}{(t-\tau)^2} \Lambda^{\eta+3/2-s} \theta * \Lambda^{2s-3/2}(fg)(\tau) d\tau,$$

with  $\eta + 3/2 - s < 1$ . Before dealing with the case  $s > 3/2$ , let us briefly show (7). By Sobolev's injection theorem (see [14]), if  $s < 3/2$  then

$$(37) \quad \|f\|_{L^p} \leq C \|\Lambda^s f\|_{L^2},$$

with  $1/p = 1/2 - s/3$ . If  $s = 1 + \alpha$ ,  $\alpha < 1/2$ , we obtain, for  $t < 1$

$$\|B(f, g)(t)\|_{\infty} \leq \int_0^t \frac{C}{(t-\tau)^{1-\alpha}} d\tau.$$

For small  $t$ ,  $B(f, g)$  is bounded and tends to zero as  $t$  goes to zero. Now we treat the case where  $s > 3/2$ , using the following estimate

LEMMA 5. - Let  $f, g \in H^s(\mathbb{R}^3)$ ,  $s > 3/2$ ,

$$(38) \quad \|\Lambda^s(fg)\|_{L^2} \leq C(s)(\|\Lambda^s f\|_{L^2} \|g\|_{\infty} + \|\Lambda^s g\|_{L^2} \|f\|_{\infty}).$$

For a proof, see the appendix. We will then show

LEMMA 6. - Let  $s > 3/2$ , for all  $t > 0$

$$(39) \quad \|\Lambda^s u(\cdot, t)\|_{L^2} \leq \frac{C(s)}{1+t^{\frac{s}{2}}}.$$

This can be achieved by successive iterations, starting from the previous estimate for  $s = 2$ , and applying Lemma 5. Let us see how it works at each step. Let  $\eta < 1$ , we first treat the case  $t < 1$ .

$$\|\Lambda^{s+\eta} B(f, g)(t)\|_{L^2} \leq \int_0^t \frac{1}{(t-\tau)^2} \|\Lambda^{\eta} \left( \frac{\cdot}{\sqrt{t-\tau}} \right) * \Lambda^s(fg)\|_{L^2} d\tau,$$

and as  $f$  and  $g$  are bounded in  $L^{\infty}$  and in  $H^s$ ,

$$(40) \quad \|\Lambda^{s+\eta} B(f, g)(t)\|_{L^2} \leq C(s)t^{\frac{1}{2}-\frac{\eta}{2}}.$$

For  $t > 1$ ,

$$\begin{aligned} & \|\Lambda^{s+\eta} B_1(f, g)\|_{L^2} \\ & \leq \int_0^1 \frac{1}{(t-\tau)^{\frac{1}{2}+\frac{s+\eta-1/2}{2}}} \|\Lambda^{s+\eta-\frac{1}{2}} \theta\|_{L^1} \|\Lambda f\|_{L^2} \|\Lambda g\|_{L^2} d\tau, \end{aligned}$$

by using Lemma 4. Then

$$\| \Lambda^{s+\eta} B_1(f, g) \|_{L^2} \leq \frac{C(s)}{(t-1)^{\frac{s+\eta+1/2}{2}}}.$$

For  $t > 1$ ,

$$\| \Lambda^{s+\eta} B_2(f, g) \|_{L^2} \leq \int_1^t \frac{1}{(t-\tau)^2} \| \Lambda^\eta \theta \left( \frac{\cdot}{\sqrt{t-\tau}} \right) * \Lambda^s(fg) \|_{L^2} d\tau,$$

and using Lemma 5 we deduce the estimate for  $s+\eta$  from the estimate for  $s$ :

$$(41) \quad \| \Lambda^{s+\eta} B_2(f, g) \|_{L^2} \leq \frac{C(s)}{t^{\frac{s+\eta}{2}}}.$$

This achieves the proof of the existence of  $u \in BC([0, +\infty), H^s)$ . Now, we observe that, as we have local existence and uniqueness for  $s > 1/2$  (see [4]), our solution is unique by applying this theorem on intervals covering  $[0, \infty)$ . In the case  $s = 1/2$ , it is necessary to establish uniqueness directly, (see [4] or [11]). The reader should refer to [11] or [7], in order to see why a solution of (11) is actually a solution in the classical sense. We can nevertheless make a few remarks. By the same process we use to gain the regularity  $s - 3/4$ , we can establish, independently of  $s$ , estimates in  $H^r$ ,  $r > s$ : for all  $t > 0$ , there exists  $\pi(r) > 0$

$$\| \Lambda^r u(\cdot, t) \|_{L^2} \leq \frac{C(r)}{t^{\pi(r)}},$$

and  $\Lambda^r u$  is holderian on every interval  $[t_0, t_1]$ , provided  $t_1 > t_0 > 0$ . This provides the regularity in the space variables. As for regularity in time, it suffices to use the relation, which can be established without knowing (10)

$$(42) \quad u(t) = S(t-\varepsilon)u(\varepsilon) + \int_\varepsilon^t \mathbb{P}S(t-\tau)\nabla \cdot (u \otimes u) d\tau,$$

and the following lemma, (see [11] or [7] for a proof).

LEMMA 7. - Let  $u(t) = \int_0^t e^{-(t-s)\Delta} f(s) ds$ ,  $t \in [0, T]$ ,  $f \in C^\eta([0, T], B)$ ,  $\eta < 1$ ,  $B$  a Banach space. Then  $u \in C^{1+\nu}((0, T], B)$ ,  $Au \in C^\nu((0, T], B)$ , and

$$\partial_t u = -\Delta u + f,$$

for all  $\nu < \eta$ .

We then obtain the  $C^\infty$  regularity of  $u$ , for  $t > 0$ , with a bootstrap argument. Let us see how condition (13) can be expressed on  $u_0$  in terms

of Besov spaces. We set  $|S(t)u_0| < \beta$ , where  $\beta$  has been chosen so that our scheme converges in  $F$ . Remember that

$$|S(t)u_0| = \sup_t \omega(t) \| S(t)u_0 \|_{L^4} .$$

Therefore, as  $3/8 - \inf(s, 3/4)/2 < 1/8$ ,

$$(43) \quad \sup_t t^{3/8 - \inf(s, 3/4)/2} \| S(t)u_0 \|_{L^4} < \beta$$

and

$$(44) \quad \sup_t t^{1/8} \| S(t)u_0 \|_{L^4} < \beta.$$

LEMMA 8. – Let  $u_0 \in \mathcal{S}(\mathbb{R}^3)$ ,  $\alpha > 0$ , and  $\gamma > 1$ ;  $\sup_t t^{\alpha/2} \| S(t)u_0 \|_{L^\gamma}$  is a norm on  $\dot{B}_{\beta, \infty}^{-\alpha}$  which is equivalent to the classical dyadic norm.

We refer to [4] or [12] for a proof. In our case, except for  $s = 3/4$ , the condition on  $u_0$  is equivalent to

$$\begin{cases} \| u_0 \|_{\dot{B}_{4, \infty}^{-1/4}} \leq \beta, \\ \| u_0 \|_{\dot{B}_{4, \infty}^{-(3/4 - \inf(s, 3/4))}} \leq \beta. \end{cases}$$

Thus, as  $\dot{H}^{\frac{1}{2}} \subset \dot{B}_{4, \infty}^{-1/4}$  and  $\dot{H}^{\inf(s, 3/4)} \subset \dot{B}_{4, \infty}^{-(3/4 - \inf(s, 3/4))}$ ,  $u_0$  belongs to both Besov spaces. If  $u$  is a solution with initial condition  $u_0$ ,  $\lambda u(\lambda x, \lambda^2 t)$  is a solution with  $\lambda u_0(\lambda x)$  as initial data. The condition (44) is independent of  $\lambda$  for the norm is invariant by scaling. And (43) can be forced by a suitable choice of  $\lambda$ . For  $s = 3/4$ , we know that  $H^{\frac{3}{4}} \subset L^4$ , and we conclude in the same way. This ends the proof.

*Proof of Theorem 2.* – We introduce as before two Banach spaces  $E = BC([0, +\infty), L^p)$  with the natural norm

$$\| f \| = \sup_t \| f(\cdot, t) \|_{L^p},$$

and  $F = \{f | t^{\frac{1}{2} - \frac{3}{4p}} f \in BC([0, +\infty), L^{2p})\}$  with the norm

$$| f | = \sup_t t^{\frac{1}{2} - \frac{3}{4p}} \| f(\cdot, t) \|_{L^{2p}} .$$

Then, we see that

$$(45) \quad \| B(f, g)(t) \|_{L^p} \leq \| f \| | g | \| \theta \|_{L^q} \int_0^t \frac{1}{(t-s)^{2-\frac{3}{2q}} s^{\frac{1}{2}-\frac{3}{4p}}} ds,$$

$$(46) \quad \| B(f, g)(t) \|_{L^{2p}} \leq | f | | g | \| \theta \|_{L^q} \int_0^t \frac{1}{(t-s)^{2-\frac{3}{2q}} s^{1-\frac{3}{2p}}} ds,$$

$$(47) \quad \text{where } \frac{1}{p} = \frac{1}{q} + \frac{1}{p} + \frac{1}{2p} - 1.$$

which gives the continuity of  $B$  from  $F \times F \rightarrow F$  and  $F \times E \rightarrow E$ , with constants  $\gamma(p)$  and  $\eta(p)$ .

$$\begin{aligned} \gamma(p) &= \|\theta\|_{L^q} t^{\frac{1}{2}-\frac{3}{4p}} \int_0^t \frac{1}{(t-s)^{2-\frac{3}{2q}} s^{1-\frac{3}{2p}}} ds, \\ \eta(p) &= \|\theta\|_{L^q} \int_0^t \frac{1}{(t-s)^{2-\frac{3}{2q}} s^{\frac{1}{2}-\frac{3}{4p}}} ds, \end{aligned}$$

and a simple rescaling shows both quantities are bounded. Then if we use the same sequence as before, Lemma 1 gives us the convergence in  $F$ , and we obtain the convergence in  $E$  by a contraction argument, as  $\eta(p) \leq \gamma(p)$ , we obtain  $2|u|\eta(p) < 1$ . The continuity at  $t = 0$  comes from a slight modification of (45), as we can replace  $|f|$  by  $\sup_{[0,t]} \tau^{\frac{1}{2}-\frac{3}{4p}} \|f(\cdot, \tau)\|_{L^{2p}}$ , which tends to zero with  $t$ . Actually, the value of  $t^{\frac{1}{2}-\frac{3}{4p}} \|f(\cdot, t)\|_{L^{2p}}$  could only be zero: the first term  $u_1 = S(t)u_0$  tends to zero, for if we consider a sequence of  $C_0^\infty$  functions  $(v_j)_j$  which approximate  $u_0$ ,

$$t^{\frac{1}{2}-\frac{3}{4p}} \|S(t)u_0\|_{L^{2p}} = t^{\frac{1}{2}-\frac{3}{4p}} \|S(t)\| \|u_0 - v_j\|_{L^p} + \|S(t)\| t^{\frac{1}{2}-\frac{3}{4p}} \|v_j\|_{L^{2p}}.$$

By Lemma 8 the condition on  $u_0$  becomes,

$$\|u_0\|_{\dot{B}_{2p,\infty}^{-(1-\frac{3}{2p})}} \leq \delta(p),$$

where  $\delta(p) \approx 1/\gamma(p)$ . This proves Proposition 1. Proposition 2 results from the inclusion of  $L^3$  in  $\dot{B}_{2p,\infty}^{-(1-\frac{3}{2p})}$ . Note that for  $p = 2$ , we impose the condition

$$t^{\frac{1}{4}} \|S(t)u_0\|_{L^4} < \delta,$$

which is equivalent to the condition (2). For a general  $u_0 \in L^2$ , we only know

$$t^{\frac{3}{8}} \|S(t)u_0\|_{L^4} < +\infty.$$

In other words, we do not know enough on low frequencies, and a sufficient condition is (2), of which  $u_0 \in L^3$  or  $u_0 \in H^{\frac{1}{2}}$  with small norms are particular cases. We obtained existence and uniqueness in a ball of  $F$  with Lemma 1 and uniqueness in the whole space can be obtained directly as in [11] or [4]. As in the Sobolev case, it is possible to obtain estimates on  $L^q$  norms of  $u(\cdot, t)$ ,  $q > p$ , in order to show the  $C^\infty$  regularity for  $t > 0$ .



**APPENDIX**

We recall that if  $\phi \in \mathcal{S}(\mathbb{R}^n)$  is a radial function so that  $\text{Supp } \hat{\phi} \subset \{|\xi| < 1 + \varepsilon\}$ , and  $\hat{\phi}(\xi) = 1$  for  $|\xi| < 1$ , we define  $\phi_j(x) = 2^{nj}\phi(2^jx)$ ,  $S_j$  the convolution operator with  $\phi_j$ , and  $\Delta_j = S_{j+1} - S_j$ . Then

$$I = \sum_{-\infty}^{+\infty} \Delta_j$$

and  $f(x) \in \dot{H}^s(\mathbb{R}^n)$  if and only if,  $\forall j$ ,

$$(48) \quad \|\Delta_j(f)\|_{L^2} \leq 2^{-js} \|f\|_{\dot{H}^s \varepsilon_j}$$

where  $\sum \varepsilon_j^2 \leq 1$ . We will show the two following inequalities, which are homogeneous variants of well-known inequalities:

for  $s < \frac{n}{2}$ ,

$$(49) \quad \|\Lambda^{2s-\frac{n}{2}}(fg)\|_{L^2} \leq C \|\Lambda^s f\|_{L^2} \|\Lambda^s g\|_{L^2},$$

for  $s > \frac{n}{2}$ ,

$$(50) \quad \|\Lambda^s(fg)\|_{L^2} \leq C(s)(\|\Lambda^s f\|_{L^2} \|g\|_{\infty} + \|\Lambda^s g\|_{L^2} \|f\|_{\infty}).$$

Let us start with the first case: we will use a paraproduct decomposition (see [3]): for  $f, g \in \mathcal{S}$ ,

$$(51) \quad \begin{aligned} f(x)g(x) &= \sum_j \Delta_j(f) \sum_l \Delta_l(g) \\ &= \sum_{|j-l| \leq 1} \Delta_j(f)\Delta_l(g) + \sum_{|j-l| \geq 1} \Delta_j(f)\Delta_l(g). \end{aligned}$$

The second sum is, by reordering the terms, a finite sum of terms like  $S_2 = \sum_j S_{j-1} \Delta_j(g)$ . We will treat only  $S_2$ , as the other ones are of the same kind. The Fourier transform of  $S_2$  is supported in an annulus  $[2^{j-1}(1 - 2\varepsilon), 2^{j+1}(1 + 2\varepsilon)]$ . Using Bernstein's lemma,

$$\begin{aligned} \|\Delta_j(f)\|_{\infty} &\leq C 2^{j\frac{n}{2}} \|\Delta_j f\|_{L^2} \\ &\leq C 2^{j\frac{n}{2}-s} \|f\|_{\dot{H}^s \varepsilon_j}. \end{aligned}$$

Then,

$$\|S_j(f)\|_{\infty} \leq C \sum_{-\infty}^j 2^{q(\frac{n}{2}-s)} \varepsilon_q \|f\|_{\dot{H}^s}.$$

If  $j < 0$ ,

$$\sum_{-\infty}^j 2^{q(\frac{n}{2}-s)} \varepsilon_q = 2^{j(\frac{n}{2}-s)} \tilde{\varepsilon}_j,$$

and

$$\tilde{\varepsilon}_j = \sum_{-\infty}^0 2^{q(\frac{n}{2}-s)} \varepsilon_{j+q}$$

is a convolution product between  $l^1$  and  $l^2$ , therefore in  $l^2$ . For  $j \geq 0$ ,

$$\sum_{-\infty}^j 2^{q(\frac{n}{2}-s)} \varepsilon_q \leq C(1 + \dots + 2^{j(\frac{n}{2})} \varepsilon_j).$$

if

$$(52) \quad 2^{j(\frac{n}{2}-s)} \check{\varepsilon}_j = 1 + \dots + 2^{j(\frac{n}{2}-s)} \varepsilon_j,$$

$(\check{\varepsilon}_j)$  is in  $l^2$  for the same reason as  $\tilde{\varepsilon}_j$ . This gives

$$\|S_j(f)\|_\infty \leq C 2^{j(\frac{n}{2}-s)} \|f\|_{\dot{H}^s} \eta_j.$$

where  $(\eta_j)_j \in l^2$ . Then, if  $(\mu_j)_j$  is associated to  $g$ ,

$$\|S_{j-1}(f)\Delta_j(g)\|_{L^2} \leq 2^{j(\frac{n}{2}-s)} \|f\|_{\dot{H}^s} \|g\|_{\dot{H}^s} \eta_j \mu_j,$$

and as  $(\eta_j \mu_j)_j \in l^1 \subset l^2$ ,  $S_1 \in \dot{H}^{2s-\frac{n}{2}}$ . The terms of the first sum in (51) are like  $S_1 = \sum_j \Delta_j(f)\Delta_j(g)$ , and in this case we only know that the support of the Fourier transform of  $\Delta_j(f)\Delta_j(g)$  is in  $\{|\xi| \leq C2^j\}$ , and

$$\|\Delta_j(f)\Delta_j(g)\|_{L^1} \leq \varepsilon_j \mu_j 2^{-2js} \|f\|_{\dot{H}^s} \|g\|_{\dot{H}^s}.$$

LEMMA 9. - If  $u \in L^1$ ,  $\text{supp } \hat{u} \subset B(0, R)$ , and  $\|\hat{u}(\xi)\|_\infty \leq R^{-2s}$ , then

$$\|\Lambda^{2s-\frac{n}{2}} u\|_{L^2} \leq \int_{S^2} dS.$$

This comes from

$$\begin{aligned} \int_{|\xi| \leq R} (|\xi|^2)^{2s-\frac{n}{2}} |\hat{u}(\xi)|^2 d\xi &\leq R^{-4s} \int_{|\xi| \leq R} |\xi|^{4s-n} d\xi \\ &\leq R^{-4s} \int_{S^2} \int_0^R r^{4s-1} dr dS. \end{aligned}$$

then, applying Lemma 9 to  $\Delta_j(f)\Delta_j(g)$ ,

$$\|S_{j-1}(f)\Delta_j(g)\|_{\dot{H}^{2s-\frac{n}{2}}} \leq C \varepsilon_j \mu_j \|f\|_{\dot{H}^s} \|g\|_{\dot{H}^s}.$$

As  $(\eta_j \mu_j)_j \in l^1$ , this ends the proof. The second inequality can be proved by the same estimates, except that we have a better estimate for  $\|S_j(f)\|_\infty$  and  $\|\Delta_j(f)\|_\infty$ , both bounded by  $\|f\|_\infty$ .

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