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Asymptotic behavior of solutions of a semilinear heat equation with subcritical nonlinearity

by

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ABSTRACT. – We consider the Cauchy problem for $u_t = \Delta u + u^p$ with $1 + 2/N < p$ and $(N - 2)p < N + 2$. We give a complete description of the asymptotic behavior of the positive solution.

RÉSUMÉ. – Nous considérons le problème de Cauchy pour $u_t = \Delta u + u^p$ avec $1 + 2/N < p$ et $(N - 2)p < N + 2$. On donne une description complète de comportement asymptotique de la solution positive.

1. INTRODUCTION AND MAIN RESULT

We study the asymptotic behavior of nonnegative solutions of the following Cauchy problem:

$$(H) \begin{cases} u_t = \Delta u + u^p & \text{in } \mathbf{R}^N \times \mathbf{R}^+, \\ u(x, 0) = u_0 & \text{in } \mathbf{R}^N. \end{cases}$$

We assume $p > 1$ and $u_0 \geq 0$, $\neq 0$ in \mathbf{R}^N . When $u_0 \in L^1 \cap L^\infty$, Problem (H) has a unique local classical solution (see [Kawa, Proposition 2.3]), which we denote by $u(x, t; u_0)$. We set

$$t_{\max}(u_0) := \sup \{T \in \mathbf{R}^+; u(t; u_0) \in L^\infty((0, T); L^\infty)\}.$$

If $t_{\max}(u_0) < \infty$, then we say that $u(t; u_0)$ blows up in finite time. When $p \in (1, 1 + 2/N]$, it is well known (see e.g. [Kavi]) that all solutions of (H) blows up in finite time. In this paper we consider the next subcritical case:

$$(1.1) \quad 1 + 2/N < p \quad \text{and} \quad (N - 2)p < N + 2.$$

In spite of the simple form of Problem (H), we need to transform the equation in order to obtain some important informations on the asymptotic behavior of solutions. Following [Kavi], we set

$$(1.2) \quad v(y, s; u_0) := (t + 1)^{1/(p-1)} u(x, t; u_0),$$

$$(1.3) \quad x = (t + 1)^{1/2} y \quad \text{and} \quad t = e^s - 1.$$

Then $v(y, s; u_0)$ satisfies

$$(TH) \quad \begin{cases} v_s = \Delta v + \frac{y}{2} \cdot \nabla v + \frac{v}{p-1} + v^p & \text{in } Q, \\ v(y, 0) = u_0 & \text{in } \mathbf{R}^N. \end{cases}$$

By studying Problem (TH) Kavian [Kavi] showed

$$(1.4) \quad \|u(t; u_0)\|_{\infty} = O(t^{-1/(p-1)}) \quad \text{as } t \rightarrow \infty,$$

provided $u_0 \in H_{\rho}^1$ and $t_{\max}(u_0) = \infty$. For the definition of H_{ρ}^1 , see Notations just after this section. In this paper we will extend [Kavi] and clarify the structure of space of positive solutions of (H). Let $u_0 \in L_{\rho}^2 \cap L^{\infty}$. Then our main result below shows that $u(t; u_0)$ is classified into one of the next three types:

Type (I): $t_{\max}(u_0) < \infty$, i.e. $u(t; u_0)$ blows up in finite time,

Type (II): $t_{\max}(u_0) = \infty$ and $\|u(t; u_0)\|_{\infty} \sim t^{-N/2}$ as $t \rightarrow \infty$,

Type (III): $t_{\max}(u_0) = \infty$ and $\|u(t; u_0)\|_{\infty} \sim t^{-1/(p-1)}$ as $t \rightarrow \infty$ and that the solution of Type (I) and the solution of Type (II) are stable and the solution of Type (III) is instable.

It is known (see e.g. [Kawa]) that if $E(u_0) < 0$ then $u(t; u_0)$ is of Type (I), where $E(u_0)$ is the ‘energy’ of u_0 defined by

$$(1.5) \quad E(u_0) := \frac{1}{2} \|\nabla u_0\|_2^2 - \frac{1}{p+1} \|u_0\|_{p+1}^{p+1}.$$

Fujita [F] showed that if u_0 is bounded by $\varepsilon \exp(-a|x|^2)$ then $u(t; u_0)$ is of Type (II), where $a > 0$ is a constant and $\varepsilon = \varepsilon(a) > 0$ is some

small constant. In [Kawa] we gave a necessary and sufficient condition for the solution of (H) to be of Type (II) (see Proposition 3 in Section 2), which is one of crucial results to establish our main Theorem. Haraux and Weissler [HW] observed that (H) has a self-similar solution $w(x, t)$ of Type (III) constructed by

$$(1.6) \quad w(x, t) = t^{-1/(p-1)} f\left(\frac{x}{\sqrt{t}}\right),$$

where $f \in S$ and

$$(1.7) \quad S := \left\{ f \in H^1_\rho \cap L^\infty; -\Delta f - \frac{y}{2} \cdot \nabla f = \frac{f}{p-1} + f^p \text{ and } f > 0 \text{ in } \mathbf{R}^N \right\}.$$

Such a solution $w(x, t)$ is invariant by the similarity transformation:

$$(1.8) \quad w_\lambda(x, t) = \lambda^{2/(p-1)} w(\lambda x, \lambda^2 t),$$

namely, we have $w_\lambda(x, t) = w(x, t)$ for $\lambda > 0$.

Now we will state our main result. Let $X := \{f \in L^2_\rho \cap L^\infty; f \geq 0 \text{ in } \mathbf{R}^N\}$ be a closed cone of the Banach space $L^2_\rho \cap L^\infty$ with the norm $\|\cdot\| := \|\cdot\|_2 + \|\cdot\|_\infty$. We set

$$K := \{u_0 \in X; t_{\max}(u_0) = \infty\},$$

$$B := X - K = \{u_0 \in X; t_{\max}(u_0) < \infty\}.$$

We denote by $\text{Int}(K)$ the interior of K in X and by ∂K the boundary of K in X .

THEOREM 1. – *We assume (1.1) Then we obtain the following:*

- (i) *The set K is an unbounded, closed convex set in X and $0 \in \text{Int}(K)$.*
- (ii) *For any $u_0 \in X - \{0\}$ there exists a unique $\tau_0 \in \mathbf{R}^+$ such that*

$$(1.9) \quad \begin{cases} \tau_0 u_0 \in \partial K, \\ \tau u_0 \in \text{Int}(K) & \text{if } \tau \in (0, \tau_0), \\ \tau u_0 \in B & \text{if } \tau \in (\tau_0, \infty). \end{cases}$$

Moreover, $G := \{u_0 \in X; \|u_0\| = 1\}$ and ∂K are homeomorphic by $P|_G$, where $P : X - \{0\} \rightarrow \partial K$ is the well-defined projection: $Pu_0 = \tau_0 u_0 \in \partial K$ in view of (1.9).

(iii) If $u_0 \in \text{Int}(K) - \{0\}$, then we have

$$(1.10) \quad \|u(t; u_0)\|_q \sim t^{-(1-1/q)N/2} \quad \text{for } q \in [1, \infty].$$

More precisely, for $q \in [1, \infty]$

$$(1.11) \quad t^{(1-1/q)N/2} \|u(t; u_0) - m_\infty (4\pi t)^{-N/2} \exp\left(-\frac{|x|^2}{4t}\right)\|_q \rightarrow 0$$

as $t \rightarrow \infty$,

where $m_\infty = \sup_{t \geq 0} \|u(t)\|_1 \in \mathbf{R}^+$.

(iv) If $u_0 \in \partial K$ then we have

$$(1.12) \quad \|u(t; u_0)\|_q \sim t^{N/2q - 1/(p-1)} \quad \text{for } q \in [1, \infty].$$

More precisely, we obtain $\omega(v(s; u_0)) \subset S$, where $\omega(v)$ is ω -limit set of v in $L^2_\rho \cap L^\infty$, i.e.

$$(1.13) \quad \omega(v(s; u_0)) = \bigcap_{t \geq 0} \overline{\{v(s; u_0); s \geq t\}}^{L^2_\rho \cap L^\infty}.$$

(v) If $u_0 \in B$ then we have

$$(1.14) \quad E(u(t; u_0)) \rightarrow -\infty \quad \text{as } t \rightarrow t_{\max}(u_0) - 0.$$

For the Dirichlet problem in bounded domains corresponding to (H) some similar results were established in [Li], [NST], [CL] and [G]. In this case, the solution blows up in finite time or the solution exists time-globally and converges whether to 0 or to nontrivial equilibria in L^∞ (thus in L^q for any $q \in [1, \infty]$) as $t \rightarrow \infty$. We remark that some methods used in their works play important roles in this paper by appropriate modifications. Recently, Lee and Ni [LN] and Wang [W] obtained some interesting necessary conditions and sufficient conditions for the solution of (H) to exist time-globally. In particular, they treat solutions with initial values $u_0(x)$ decaying slowly like $|x|^{-2/(p-1)}$ as $|x| \rightarrow \infty$.

In Section 2 we give some preliminary results in order to establish Theorem 1. In Section 3 we prove Theorem 1 and give some remarks.

Notations. - 1. $\mathbf{R}^+ := (0, \infty)$, $Q := \mathbf{R}^N \times \mathbf{R}^+$, $Q(a, b) := \mathbf{R}^N \times (a, b)$ and $Q[a, b] := \mathbf{R}^N \times [a, b]$.

2. $L^p := L^p(\mathbf{R}^N)$ with the usual norm $\|\cdot\|_p := \left(\int_{\mathbf{R}^N} |\cdot|^p\right)^{1/p}$. We denote $\|\cdot\|_{\infty, Q(a, b)} := \|\cdot\|_{L^\infty(Q(a, b))}$.

3. $\rho(x) := \exp(|x|^2/4)$.
4. $L_\rho^p := \left\{ f \in L^p; \int_{\mathbf{R}^N} |f|^p \rho < \infty \right\}$ is a weighted L^p -space with the norm $|\cdot|_p := \left(\int_{\mathbf{R}^N} |\cdot|^p \rho \right)^{1/p}$.
5. $\|\cdot\|$ denotes the norm of $L_\rho^2 \cap L^\infty$, i.e. $\|\cdot\| := |\cdot|_2 + \|\cdot\|_\infty$.
6. $H_\rho^1 := \{f \in H^1(\mathbf{R}^N); \nabla f \in L_\rho^2\}$ is a Hilbert space with the inner product $(f, g)_\rho := \int_{\mathbf{R}^N} (\nabla f, \nabla g) \rho$ for $f, g \in H_\rho^1$.
7. $f(t) = O(g(t))$ means that $\limsup_{t \rightarrow \infty} |f(t)/g(t)| < \infty$ and $f(t) \sim g(t)$ that $0 < \liminf_{t \rightarrow \infty} |f(t)/g(t)| \leq \limsup_{t \rightarrow \infty} |f(t)/g(t)| < \infty$.

2. PRELIMINARIES

In this section we give some preliminary results to prove Theorem 1.

We defined by (1.5) the energy $E(u)$ for Problem (H). We also define the energy $\hat{E}(v)$ for Problem (TH) by

$$(2.1) \quad \hat{E}(v) := \frac{1}{2} |\nabla v|_2^2 - \frac{1}{2(p-1)} |v|_2^2 - \frac{1}{p+1} |v|_{p+1}^{p+1}.$$

PROPOSITION 1. – (i) Let $u_0 \in X \cap H^1$. If $E(u_0) < 0$ then $u_0 \in B$.

(ii) Let $u_0 \in X \cap H_\rho^1$. If $\hat{E}(u_0) < 0$ then $u_0 \in B$.

Proof. – (i) This is well-known. See e.g. the proof of [Kawa, Proposition 3.1].

(ii) See the proof of [Kavi, Theorem (1.10)]. ■

PROPOSITION 2. – We assume (1.1). Then the following hold.

(i) Let $b \in \mathbf{R}^+$. Then there exists some constant $m \in \mathbf{R}^+$ such that for any $u_0 \in K$ with $\|u_0\|_2 + \|u_0\|_\infty \leq b$ we have $\|u(t; u_0)\|_{\infty, Q} \leq m$.

(ii) The set K is closed in X .

(iii) Let $u_0 \in B$. Then we have (1.14).

Proof. – Using Lemma 1 below, we can prove Proposition 2 essentially by the same argument as in [G]. Therefore, we leave it to the reader. ■

LEMMA 1. – We assume (1.1). Let $t_0 \in \mathbf{R}^+$ and u be a classical solution of (H) on $[0, T)$, $T > t_0$. Assume that

$$(2.2) \quad \int_0^T \|u_t\|_2^2 dt \leq l < \infty,$$

$$(2.3) \quad \|u\|_{\infty, Q(t_0, T)} = \|u\|_{\infty, Q(0, T)}.$$

Then there is some constant $a \in \mathbf{R}^+$ independent of u , u_0 and T (dependant of l and t_0) such that

$$\|u\|_{\infty, Q(0, T)} \leq a.$$

Proof. – Although the proof is similar to that of [G, Lemma], we will describe it for the sake of completeness. We proceed by a contradiction. Suppose that Lemma 1 does not hold. Then there is a sequence of solutions $u_n(x, t)$ of (H) on $[0, T_n)$, $T_n > t_0$ such that

$$(2.4) \quad \int_0^{T_n} \|u_{nt}\|_2^2 dt \leq l,$$

$$(2.5) \quad \|u_n\|_{\infty, Q(t_0, T_n)} = \|u_n\|_{\infty, Q(0, T_n)},$$

and

$$(2.6) \quad \|u_n\|_{\infty, Q(0, T_n)} \rightarrow \infty \quad \text{as } n \rightarrow \infty.$$

Let $(x_n, t_n) \in Q(t_0, T_n)$ be a sequence such that

$$(2.7) \quad |u_n(x_n, t_n)| \geq \frac{1}{2} \|u_n\|_{\infty, Q(0, T_n)}.$$

We choose a sequence $\lambda_n > 0$ such that

$$(2.8) \quad \lambda_n^{2/(p-1)} |u_n(x_n, t_n)| = 1.$$

We remark that λ_n satisfies that $\lambda_n \rightarrow 0$ as $n \rightarrow \infty$. We define the function v_n by

$$v_n(x, t) = \lambda_n^{2/(p-1)} u_n(x_n + \lambda_n x, t_n + \lambda_n^2 t).$$

We can easily verify that v_n is a solution of (H) in $Q_n := Q(-t_n/\lambda_n^2, (T_n - t_n)/\lambda_n^2)$. In view of (2.7) and (2.8) we have

$$(2.9) \quad v_n(0, 0) = 1,$$

$$\|v_n\|_{\infty, Q_n} = \lambda_n^{2/(p-1)} \|u_n\|_{\infty, Q(0, T_n)} \leq 2 \lambda_n^{2/(p-1)} |u_n(x_n, t_n)| = 2.$$

Since $\{v_n\}$ are uniformly bounded, $\{v_n\}$ are equi-continuous on every compact subset of $Q(-\infty, 0]$ (see [D] or [S]). Thus, there is a subsequence (still denoted v_n) and a function $v(x, t)$ such that

$$(2.10) \quad v_n \rightarrow v \quad \text{in } L^\infty(D) \quad \text{as } n \rightarrow \infty,$$

where D is any compact subset of $Q(-\infty, 0]$. The function v is a solution of (H) in the sense of distribution and is bounded in $Q(-\infty, 0]$. Therefore, v is a classical solution of (H). It follows from (H) that

$$(2.11) \quad \int_{-t_0/\lambda_n^2}^0 \|v_{nt}\|_2^2 dt = \lambda_n^{4p/(p-1)-(N+2)} \int_{t_n-t_0}^{t_n} \|u_{nt}\|_2^2 dt \\ \leq l\lambda_n^{4p/(p-1)-(N+2)} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

By (2.9), (2.10) and (2.11), we obtain

$$(2.12) \quad v(0, 0) = 1 \quad \text{and} \quad v_t \equiv 0 \quad \text{in} \quad Q(-\infty, 0].$$

Thus, v is a nontrivial equilibrium solution of (H). This contradicts a Liouville theorem in [GS]. The proof is complete. \blacksquare

PROPOSITION 3. – *Let $p > 1 + 2/N$ and $p_0 := N(p - 1)/2$. Assume that $u_0 \in L^1 \cap L^\infty$, $u_0 \geq 0$, $\neq 0$ in \mathbf{R}^N and $|x|u_0(x) \in L^1$. Then the following (2.13) and (2.14) are equivalent:*

$$(2.13) \quad t_{\max}(u_0) = \infty \quad \text{and} \quad \|u(t; u_0)\|_\infty \sim t^{-N/2},$$

$$(2.14) \quad \inf \{ \|u(t; u_0)\|_{p_0}; t \in [0, t_{\max}(u_0)) \} < \delta_0,$$

where $\delta_0 > 0$ is a constant depending only on N and p . If (2.13) holds then $u(t; u_0)$ satisfies

$$(2.15) \quad t^{(1-1/q)N/2} \|u(t; u_0) - m_\infty(4\pi t)^{-N/2} \exp\left(-\frac{|x|^2}{4t}\right)\|_q \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

for any $q \in [1, \infty]$, where

$$0 < m_\infty := \sup_{t \geq 0} \|u(t)\|_1 = \int_{\mathbf{R}^N} u_0 dx + \int_0^\infty dt \int_{\mathbf{R}^N} u(t)^p dx < \infty.$$

Proof. – The equivalence of (2.13) and (2.14) follows from [Kawa, Corollary 1.1], and (2.15) with $q = \infty$ from [Kawa, Theorem 4.1]. Using [EZ, Lemma 3], we can prove (2.15) with $q = 1$ in the same way as in the proof of (2.15) with $q = \infty$. By linear interpolation we obtain (2.15) for $q \in (1, \infty)$. \blacksquare

PROPOSITION 4. – Assume $p > 1 + 2/N$. We set

$$W := \{u_0 \in X; t_{\max}(u_0) = \infty \text{ and } \|u(t; u_0)\|_\infty \sim t^{-N/2}\}.$$

Then W is open in X .

Proof. – We fix $u_0 \in W$. Let $p_0 = N(p-1)/2 (> 1)$. It suffices to prove

$$(2.16) \quad \exists \delta = \delta(N, p) > 0; u_1 \in X \text{ and } \|u_1 - u_0\|_{p_0} < \delta \implies u_1 \in W.$$

In view of the comparison principle we may assume without loss of generality that $u_1 \geq u_0$ in \mathbf{R}^N . We set $w(x, t) = u(x, t; u_1) - u(x, t; u_0)$ (≥ 0 in \mathbf{R}^N). Then, w satisfies

$$(2.17) \quad \begin{aligned} w_t &= \Delta w + [w + u(t; u_0)]^p - u(t; u_0)^p \\ &= \Delta w + pw \int_0^1 [sw + u(t; u_0)]^{p-1} ds \\ &\leq \Delta w + 2^{p-1}pw[w^{p-1} + \|u(t; u_0)\|_\infty^{p-1}]. \end{aligned}$$

We set $f(t) = 2^{p-1}p\|u(t; u_0)\|_\infty^{p-1}$ and

$$w(x, t) = W(x, t) \exp \left[\int_0^t f(s) ds \right].$$

Then, $f(t) \in L^1(0, \infty)$. The function W satisfies

$$W_t \leq \Delta W + C_1 W^p.$$

Here, $C_1 = 2^{p-1}p \exp \left[(p-1) \int_0^\infty f(s) ds \right] \in \mathbf{R}^+$. By Proposition 3, there exist $\delta = \delta(N, p) > 0$ such that if $\|W(0)\|_{p_0} = \|u_1 - u_0\|_{p_0} < \delta$ then we have

$$t_{\max}(u_1) = \infty \text{ and } \|W(t)\|_\infty = O(t^{-N/2}).$$

Therefore, we obtain $\|u(t; u_1)\|_\infty \sim t^{-N/2}$. Hence, (2.16) holds. \blacksquare

LEMMA 2. – Let $f, g \in S$. If $f \leq g$ in \mathbf{R}^N then $f = g$ in \mathbf{R}^N .

Proof. – Our proof is very close to that of [Li, Lemma 2.2]. By integration by parts we find

$$\int_{\mathbf{R}^N} g^p f \rho = \int (\nabla f \cdot \nabla g) \rho - \frac{1}{p-1} \int f g \rho = \int f^p g \rho,$$

which leads to

$$\int_{\mathbf{R}^N} \rho f g (g^{p-1} - f^{p-1}) = 0.$$

This yields $f \equiv g$ in \mathbf{R}^N . \blacksquare

PROPOSITION 5. – We assume (1.1). Let $u_0 \in X$ and $t_{\max}(u_0) = \infty$. Then $u(t; u_0)$ satisfies whether

$$(2.18) \quad \|u(t; u_0)\|_{\infty} \sim t^{-1/(p-1)}$$

or

$$(2.19) \quad \|u(t; u_0)\|_{\infty} \sim t^{-N/2}.$$

Moreover, if (2.18) holds then we have

$$(2.20) \quad \omega(v(s; u_0)) \subset S,$$

and if (2.19) holds then we have

$$(2.21) \quad \omega(v(s; u_0)) = \{0\}.$$

Proof. – We can verify that

$$(2.22) \quad \|u(t; u_0)\|_q = (t+1)^{(1/q-1/p_0)N/2} \|v(s; u_0)\|_q \quad \text{for } q \in [1, \infty],$$

$$(2.23) \quad \frac{d}{ds} \hat{E}(v(s; u_0)) = -|v_s(s; u_0)|_2^2.$$

Kavian [Kavi, Theorem (1.13)] showed

$$(2.24) \quad \|u(t; u_0)\|_{\infty} = O(t^{-1/(p-1)}) \quad (\iff \|v(s; u_0)\|_{\infty} = O(1))$$

and

$$(2.25) \quad \omega(v(s; u_0)) \subset S \cup \{0\}.$$

He proved (2.24) by using (2.23) and the method in [CL]. Once we obtain (2.24), we can derive (2.25) from the smoothing effect: $v(s; u_0) \in L^{\infty}([\tau, \infty); H_{\rho}^1 \cap C^1(\mathbf{R}^N))$ for $\tau > 0$ and the compactness of the embedding: $H_{\rho}^1 \cap C^1(\mathbf{R}^N) \subset L_{\rho}^2 \cap L^{\infty}$. We remark that the method in [G] is also applicable to deduce (2.24). Indeed, using Lemma 3 below, we can prove (2.24) by the same argument in the proof of Proposition 2, (i).

Next, we will show that if (2.18) does not hold then (2.19) holds. Let $u(t; u_0)$ do not satisfy (2.18). Then we have

$$\liminf_{t \rightarrow \infty} t^{1/(p-1)} \|u(t; u_0)\|_{\infty} = 0,$$

Or equivalently,

$$\liminf_{s \rightarrow \infty} \|v(s; u_0)\|_{\infty} = 0.$$

Therefore, we deduce that

$$(2.26) \quad 0 \in \omega(v(s; u_0)),$$

which leads to

$$(2.27) \quad \liminf_{t \rightarrow \infty} \|u(t; u_0)\|_{p_0} = \liminf_{s \rightarrow \infty} \|v(s; u_0)\|_{p_0} = 0.$$

By Proposition 3 we obtain (2.19). Now, we see that (2.19), (2.21) and (2.26) are equivalent. Thus, (2.18) and (2.20) are also equivalent. ■

LEMMA 3. – We assume (1.1). Let $s_0 \in \mathbf{R}^+$ and v be a classical solution of (TH) on $[0, T)$, $T > s_0$. Assume that

$$\int_0^T |v_s|_2^2 ds \leq l < \infty,$$

$$\|v\|_{\infty, Q(s_0, T)} = \|v\|_{\infty, Q(0, T)}.$$

Then there is some constant $a \in \mathbf{R}^+$ independent of v , u_0 and T (dependant of l and s_0) such that

$$\|v\|_{\infty, Q(0, T)} \leq a.$$

Proof. – Since the proof is essentially the same as that of Lemma 1, we leave it to the reader.

3. PROOF OF THEOREM 1 AND REMARKS

Proof of Theorem 1. – Let W be the open set in X defined in the statement of Proposition 4.

(i) We already proved the closedness of K (see Proposition 2). By the same argument as in [Li], we can verify that K is convex. The

unboundedness of K follows from Proposition 3. Indeed, we can easily find $u_0^n \in X$ such that $\|u_0^n\|_{p_0} < \delta_0$ and $\|u_0^n\|_\infty > n$ for $n \in \mathbf{N}$. By Proposition 3, $\{u_0^n\}$ is an unbounded sequence in K . We can see $0 \in \text{Int}(K)$ also in view of Proposition 3.

(ii) We fix any $u_0 \in X - \{0\}$. We set $L = \{\tau \in \mathbf{R}^+; \tau u_0 \in W\}$ and $M = \{\tau \in \mathbf{R}^+; \tau u_0 \in B\}$. The sets L and M are open connected sets with $L \neq \emptyset$ and $M \neq \emptyset$. Therefore, $\mathbf{R}^+ - (L \cup M) \neq \emptyset$. Set $\tau_0 = \min \{\mathbf{R}^+ - (L \cup M)\}$ and $\tau_1 = \max \{\mathbf{R}^+ - (L \cup M)\}$. By the definition we have $\tau_1 u_0 \in \partial K$, $\tau u_0 \in W$ if $\tau < \tau_0$ and $\tau u_0 \in B$ if $\tau > \tau_1$. We will show that $\tau_0 = \tau_1$. Since $\tau_1 \tau_0^{-1} v(s; \tau_0 u_0)$ is a subsolution of (TH) with the initial value $\tau_1 u_0$, we obtain

$$(3.1) \quad v(y, s; \tau_1 u_0) \geq \frac{\tau_1}{\tau_0} v(y, s; \tau_0 u_0) \quad \text{in } Q.$$

Therefore, there exist $f \in \omega(v(s; \tau_1 u_0))$ and $g \in \omega(v(s; \tau_0 u_0))$ such that

$$(3.2) \quad f \geq \frac{\tau_1}{\tau_0} g \quad \text{in } \mathbf{R}^N.$$

By Proposition 5,

$$(3.3) \quad \omega(v(s; \tau_0 u_0)) \cup \omega(v(s; \tau_1 u_0)) \subset S.$$

It follows from (3.2), (3.3) and Lemma 2 that $f = g$ and $\tau_0 = \tau_1$. Thus we have proved (1.9).

Now we know that the map $P|_G : G \rightarrow \partial K$ is one to one and onto. Clearly, $(P|_G)^{-1}$ is continuous. Thus, it suffices to show that $P|_G$ is continuous. Let $\{x_n\}_{n \in \mathbf{N}} \subset G$ be a sequence in X such that $x_n \rightarrow x_0$ in X as $n \rightarrow \infty$ for some $x_0 \in G$. We will prove

$$(3.4) \quad Px_n \rightarrow Px_0 \quad \text{in } X \quad \text{as } n \rightarrow \infty.$$

We fix a number $\lambda > 1$. We can easily check

$$(3.5) \quad \lambda \|Px_0\| x_n \rightarrow \lambda Px_0 \quad \text{in } X \quad \text{as } n \rightarrow \infty.$$

Since $\lambda Px_0 \in B$ and B is open, $\lambda \|Px_0\| x_n \in B$ for sufficiently large n . Thus we obtain

$$(3.6) \quad \sup_{n \in \mathbf{N}} \|Px_n\| < \infty,$$

By (3.6) there exist a subsequence (still denoted $\{Px_n\}_n$) and a number $a > 0$ such that

$$(3.7) \quad \|Px_n\| \rightarrow a \quad \text{as } n \rightarrow \infty.$$

It follows that

$$(3.8) \quad \begin{aligned} \|Px_n - ax_0\| &\leq \|Px_n - \|Px_n\|x_0\| + \| \|Px_n\|x_0 - ax_0 \| \\ &= \|Px_n\| \|x_n - x_0\| + \| \|Px_n\| - a \| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Therefore, we deduce that $ax_0 \in \partial K$ and $a = \|Px_0\|$. Since a is a unique constant independent of the way to choose a subsequence, we obtain (3.4).

(iii) By the proof of (ii) we can derive that $\text{Int}(K) = W$. Thus we have (1.11) by Proposition 3. The estimate (1.10) follows from (1.11).

(iv) Let $u_0 \in \partial K$. By the proof of Proposition 5 we have

$$(3.9) \quad \|v(s; u_0)\|_q \sim 1$$

for $q = p_0$ and $q = \infty$. Since $v(s; u_0)$ is bounded in X for $s \geq 0$, we have

$$\|v(s; u_0)\|_1 = O(1).$$

Therefore, (3.9) actually holds for any $q \in [1, \infty]$. Combining (2.22) and (3.9), we deduce (1.12). We obtain from (3.9) and Proposition 5 that $\omega(v(s; u_0)) \subset S$.

(v) We have already obtained (1.14) (see Proposition 2). ■

Finally we give two remarks concerning Theorem 1.

Remark 1. – We observe that the Haraux-Weissler self-similar solution $w(t)$ given in (1.6) satisfies $\|w(t)\|_q \rightarrow \infty$ as $t \rightarrow \infty$ for $q \in [1, N(p-1)/2)$. This fact also leads to the unboundedness of K in X .

Remark 2. – With respect to (iv) of Theorem 1 we have the following result:

PROPOSITION. – Assume (1.1) and $p \in \mathbf{N}$. Then for any $u_0 \in \partial K$ the set $\omega(v(s; u_0)) \subset S$ consists of only one element, i.e. $\omega(v) = \{\varphi\}$, where φ is an element of S .

Outline of the proof of Proposition. – We will apply the method by Simon [Si]. Let $\varphi \in \omega(v(s; u_0))$ for $u_0 \in \partial K$. We will derive $\omega(v) = \{\varphi\}$. We set $\mathcal{E}(u) := \hat{E}(u + \varphi)$ (see (2.1) for the definition of \hat{E}) and $w(s) := v(s; u_0) - \varphi$. Then $w(s)$ satisfies

$$(3.10) \quad w_s = \mathcal{M}(w),$$

$$(3.11) \quad \frac{d}{ds} \mathcal{E}(w(s)) = -|w_s|_2^2.$$

where we set

$$(3.12) \quad \begin{aligned} \mathcal{M}(w) &:= \Delta(w + \varphi) + \frac{y}{2} \cdot \nabla(w + \varphi) + \frac{w + \varphi}{p-1} + (w + \varphi)^p \\ &= \Delta w + \frac{y}{2} \cdot \nabla w + \frac{w}{p-1} + (w + \varphi)^p - \varphi^p. \end{aligned}$$

Let $H_\rho^2 := \{f \in H_\rho^1; \nabla f \in H_\rho^1\}$. The space H_ρ^2 is a Hilbert space with the norm $|f|_\rho := \left(\sum_{i,j=1}^N |\partial^2 f / \partial y_i \partial y_j|_2^2 \right)^{1/2}$ for $f \in H_\rho^2$. Since $p \in \mathbf{N}$ and $H_\rho^2 \hookrightarrow L_\rho^{2p}$ (cf. [Kavi, Lemma 2.1]), the map $\mathcal{M} : H_\rho^2 \rightarrow L_\rho^2$ is analytic. We set $L := d\mathcal{M}(0)$. We have

$$(3.13) \quad Lw = \Delta w + \frac{y}{2} \cdot \nabla w + \frac{w}{p-1} + p\varphi^{p-1}w.$$

We define $A : L_\rho^2 \rightarrow L_\rho^2$ by

$$(3.14) \quad Aw := \Delta w + \frac{y}{2} \cdot \nabla w$$

with $D(A) = H_\rho^2$. We know (see [Kavi, Lemma 2.1]) that $-A$ is a positive self-adjoint operator with compact inverse. Since $\varphi \in L^\infty$, there exists a complete ortho-normal system $\{\psi_j\}_{j=1}^\infty$ for L_ρ^2 which consists of eigenfunctions of the operator L . We denote by Π the orthogonal projection of L_ρ^2 onto the (finite-dimensional) subspace $\{\psi \in H_\rho^2; L\psi = 0\}$. It follows that the map $\mathcal{L} : H_\rho^2 \rightarrow L_\rho^2$ defined by

$$(3.15) \quad \mathcal{L}u := \Pi u + Lu$$

is a one to one and onto map. We define $\mathcal{N} : H_\rho^2 \rightarrow L_\rho^2$ by

$$(3.16) \quad \mathcal{N}(u) := \Pi u + \mathcal{M}(u).$$

Then, \mathcal{N} is analytic with $d\mathcal{N}(0) = \mathcal{L}$. Therefore, we obtain from the same argumentation as in [Si, Section 2] that there are constants $\theta \in (0, 1/2)$ and $\sigma \in \mathbf{R}^+$ such that if $u \in H_\rho^2$ with $|u|_\rho \leq \sigma$ then

$$(3.17) \quad |\mathcal{M}(u)|_2 \geq |\mathcal{E}(u) - \mathcal{E}(0)|^{1-\theta}.$$

Let $|w(s)|_\rho < \sigma$ for $s \in [s_1, s_2]$. Then, by (3.11) and (3.17),

$$(3.18) \quad \begin{aligned} \frac{d}{ds} \{\mathcal{E}(w(s)) - \mathcal{E}(0)\}^\theta &= \theta \{\mathcal{E}(w(s)) - \mathcal{E}(0)\}^{\theta-1} \cdot (-|w_s|_2^2) \\ &= -\theta \{\mathcal{E}(w(s)) - \mathcal{E}(0)\}^{\theta-1} \cdot |w_s|_2 |\mathcal{M}(w(s))|_2 \\ &\leq -\theta |w_s|_2 \quad \text{for } s \in [s_1, s_2]. \end{aligned}$$

It follows that

$$(3.19) \quad |w(s_2) - w(s_1)|_2 \leq \theta^{-1} \int_{s_1}^{s_2} |w_s(s)|_2 ds \leq \theta^{-1} \{\mathcal{E}(w(s_1)) - \mathcal{E}(0)\}^\theta.$$

Since $\mathcal{M}(0) = 0$, we can verify that there exist constants $C_j \in \mathbf{R}^+$ ($1 \leq j \leq 3$) such that for $s, \tau > 0$

$$(3.20) \quad |w(s + \tau)|_2 \leq \exp(C_1 \tau) |w(s)|_2,$$

$$(3.21) \quad |w(s + \tau)|_\rho \leq C_2 \left(1 + \frac{1}{\tau}\right) \exp(C_3 \tau) |w(s)|_2.$$

By (3.19), (3.20), (3.21) and the assumption: $0 \in \omega(w(s))$, we obtain that $w(s) \rightarrow 0$ in L_ρ^2 (and also in H_ρ^2) as $s \rightarrow \infty$. Hence, $\omega(v(s)) = \{\varphi\}$. ■

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