ANNALES DE L'I. H. P., SECTION C

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Annales de l'I. H. P., section C, tome 12, nº 3 (1995), p. 339-352
<http://www.numdam.org/item?id=AIHPC_1995__12_3_339_0>

Ann. Inst. Henri Poincaré,

Vol. 12, n° 3, 1995, p. 339-352. *Analyse non linéaire*

Nonlinear instability of double-humped equilibria

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ABSTRACT. – Consider a plasma described by the Vlasov-Poisson system in a cube Q with the specular boundary condition. We prove that an equilibrium $\mu(v)$, which satisfies the Penrose linear instability condition and which decays like $O(|v|^{-3})$, is nonlinearly unstable in the C^1 norm with a weight function in v.

RÉSUMÉ. – On considère un plasma décrit par le système Vlasov-Poisson dans un cube avec la condition aux limites spéculaire. Nous démontrons qu'un équilibre $\mu(v)$ qui satisfait à la condition d'instabilité linéaire de Penrose et qui décroit comme $O(|v|^{-3})$, est instable au sens non linéaire dans la norme de C^1 avec un poids en v.

0. INTRODUCTION

We consider a plasma described by the Vlasov-Poisson system in a cube $Q = [-\pi, \pi]^3$ with the specular boundary condition on the density and the Neumann condition on the potential. We consider an equilibrium $\mu(v)$

This research is supported in part by NSF grant 90-23864 and ARO grant DAAH 04-93-G0198.

which satisfies the Penrose linear instability condition

$$\int [\mu(v_1, v_2, v_3) - \mu(0, v_2, v_3)]/[v_1^2]\, dv > 1. \tag{1}$$

Over thrity years ago Penrose [P] derived his celebrated criterion for linearized instability, which takes the form of (1) for the case of the cube. His criterion is a standard feature in most textbooks on plasmas (e.g. [K]). However, in all the intervening years no one has found a rigorous proof of true nonlinear instability.

Here we prove that $\mu(v)$ is nonlinearly unstable in the C^1 norm with a weight function in v. This is reasonable formulation because it is known that solutions exist globally in this norm for arbitrary initial data. The existence was proved in the present situation by [BR1], following the work of [Pf], [H], [S] and [LP].

Furthermore, it has been known for some time that a monotone decreasing equilibrium $\mu(v)$ is nonlinearly stable. The stability was proved by [BR2], following more formal work of [G], [HM] and [MP]. Still left unresolved is the question of nonlinear stability in the case that $\mu(v)$ is not monotone but satisfies the linear stability criterion of Penrose.

The specular condition in a cube is equivalent to periodicity in each spatial variable. In paragraph 1 we discuss the periodicity and the existence of solutions. In paragraph 2 we explicitly exhibit an unstable eigenfunction of the linearized problem. Then we use a general form of Weyl's Theorem to deduce the discreteness of the spectrum in the unstable half-plane. In paragraph 3 we prove the main instability theorem in the space C^1 with a weight in v. In fact a slightly larger space, denoted Y, suffices. The idea is to show that the eigenvalue with the largest real part dominates. The main estimate is of the L^p norm in x and v with the weight $\langle v\rangle^\alpha$ where $p > 3$ and $3 - p/3 < \alpha < p - 3/p$. The most serious error term involves the derivative $\nabla_v(f - \mu)$, which is estimated separately in Lemma 3.2.

1. VLASOV-POISSON SYSTEM IN A CUBE

We consider the Vlasov-Poisson system in a cube $Q = [-\pi, \pi]^3$, where the particles specularly reflect at the boundary of Q. Let $\overline{\partial}Q$ be the set of points $x \in \partial Q$ which are not corners or edges. Let n_x be any outward normal vector at the boundary point $x \in \overline{\partial} Q$. The Vlasov-Poisson system takes the form

$$(1.1)\qquad \begin{cases} \partial_t f + v \cdot \nabla_x f + \nabla_x \phi \cdot \nabla_v f = 0, \\ f(0, x, v) = f_0(x, v), \\ f(t, x, v) = f(t, x, v - 2(n_x \cdot v) n_x) \quad \text{for} \quad x \in \overline{\partial} Q, \\ \Delta \phi = \rho(t, x) = \int_{R^3} f(t, x, v)\, dv - \rho_0, \\ \left.\dfrac{\partial \phi}{\partial n}\right|_{\partial Q} = 0, \qquad \int_Q \phi dx = 0. \end{cases}$$

Here $f(t, x, v)$ is the density of the electrons, ϕ is the potential, and ρ_0 is a constant background charge density of ions. We must assume that the plasma is initially neutral:

$$(1.2)\qquad \int_{Q \times R^3} f_0(x, v)\, dx dv - \int_Q \rho_0\, dx = 0$$

We first study the global existence of a classical solution for (1.1). The difficulty arises from the complicated particle paths, because particles can bounce repeatedly off the "walls". However, the special geometric structure of (1.1) and the boundary conditions enable us to reduce (1.1) to an easier periodic Cauchy problem, which has been solved by J. Batt and G. Rein in [BR1].

THEOREM 1.1. – *Let $0 \leq f_0 \in C^1(Q \times R^3)$ satisfy (1.2) and $n_x \cdot \nabla_x f_0 = 0$ on $\overline{\partial} Q$. Fix $p > 3$ and assume*

$$\int \langle v \rangle^2 f_0(x, v)\, dv < \infty,$$

$$|f_0(x, v)| + |\nabla_x f_0(x, v)| + |\nabla_v f_0(x, v)| \leq C \langle v \rangle^{-p},$$

where $\langle v \rangle = (1 + |v|^2)^{1/2}$. Then there exists a unique solution (f, ϕ) of (1.1) such that $f \in C^1$ and $\phi \in C^2$.

Proof. – Given a C^1 solution (f, ϕ) in $R^+ \times Q \times R^3$, we extend it to the whole space $R^+ \times R^3 \times R^3$ by the following simple reflection method. We let $x = (x_1, x_2, x_3)$, $x_j = y_j + 2k_j \pi$, $(j = 1, 2, 3)$, for $y \in Q$ and $(k_1, k_2, k_3) \in Z^3$. We define

$$(1.3)\qquad \begin{cases} \overline{f}(t; x; v) = f(t; (-1)^{k_1} y_1, (-1)^{k_2} y_2, (-1)^{k_3} y_3; \\ \qquad\qquad (-1)^{k_1} v_1, (-1)^{k_2} v_2, (-1)^{k_3} v_3), \\ \overline{\phi}(t; x) = \phi(t; (-1)^{k_1} y_1, (-1)^{k_2} y_2, (-1)^{k_3} y_3), \end{cases}$$

for all $y \in Q$ and all $(k_1, k_2, k_3) \in Z^3$. Then $\overline{\phi}$ and $\overline{f}$ are periodic functions of period 4π. Notice that from the boundary conditions in (1.1), $\overline{f}$ and $\overline{\phi}$ are continuous. Thanks to the Neumann and specular boundary conditions, by a direct computation we get

$$\overline{f}_t + v \cdot \nabla_x \overline{f} + \nabla_x \overline{\phi} \cdot \nabla_v \overline{f} = 0 \tag{1.4}$$

in $R^+ \times R^3 \times R^3$. By assumption, $\overline{f}_0 \in C^1 (R^3 \times R^3)$. By [BR1], there exists a unique solution $g \in C^1$ and $\psi \in C^2$ in $R^+ \times R^3 \times R^3$ with $g(0, x, v) = \overline{f}_0$. Now $\overline{f}(t, x, v)$ and $\overline{\phi}(t, x)$ satisfy the same system (1.1) for the same initial data $\overline{f}_0$ by (1.4). Therefore, $\overline{f} = g$ and $\overline{\phi} = \psi$ from the uniqueness. Let (f, ϕ) be the restriction to $\overline{Q}$. Then by (1.3) we recover the specular and Neumann boundary conditions by restricting $x \in \partial Q$.

Q.E.D.

2. LINEARIZED VLASOV-POISSON SYSTEM

Clearly $f = \mu(v)$ is a stationary solution for (1.1) if $\mu(v)$ is a non-negative function even in each coordinate v_1, v_2 and v_3, and

$$\int_{R^3} \mu(v)\, dv = \rho_0.$$

Moreover, we assume the Penrose and finiteness conditions

$$\int_{R^3} [\mu(v) - \mu(0, v_2, v_3)]/v_1^2\, dv > 1, \tag{2.1}$$

$$|\mu(v)| + |\nabla \mu(v)| + |\nabla^2 \mu(v)| \leq C \langle v \rangle^{-p}, \tag{2.2}$$

where $3 < p < \infty$. Condition (2.1) is the form that the Penrose Criterion takes in the cube Q. Here $\|\cdot\|_p$ is the L^p norm of a function. Notice that (2.1) can be written as

$$\begin{aligned} &\int_{R^1} \left[\partial_{v_1} \int_{R^2} \mu(v)\, dv_2 dv_3\right] / v_1 dv_1 \\ &\quad = \int_{R^3} [\mu(v) - \mu(0, v_2, v_3)]/v_1^2\, dv > 1. \end{aligned} \tag{2.3}$$

Therefore $\int_{R^2} \mu(v_1)\, dv_2 dv_3$ cannot be a decreasing function of $|v_1|$. For instance, it may be "double-humped" as when a beam of electrons is injected into the plasma. We are interested in the long-time behavior of solutions of (1.1) near $\mu(v)$. Thus we linearize (1.1) around $\mu(v)$ as:

$$\tag{2.4} \left\{ \begin{aligned} &\partial_t g + v \cdot \nabla_x g + \nabla_x \psi \cdot \nabla_v \mu = 0 \\ &g(t, x, v) = g(t, x, v - 2(n_x \cdot v) n_x), \qquad \forall x \in \partial Q, \\ &\Delta \psi = \int g(t, x, v)\, dv, \quad \frac{\partial \psi}{\partial n} = 0, \quad \int_Q \psi = 0, \end{aligned} \right.$$

where g is a perturbation of $\mu(v)$. In the spirit of Penrose, we establish the following:

LEMMA 2.1. – *If $\mu(v)$ satisfies (2.1) and (2.2), there is a growing mode for the linearized problem (2.4).*

Proof. – By the Penrose condition (2.3), there exists $\lambda > 0$ such that

$$\int \frac{v_1 \partial_{v_1} \mu(v)}{v_1^2 + \lambda^2}\, dv = 1,$$

where we have used the observation that

$$\lim_{\lambda \to \infty} \int \frac{v_1 \partial_{v_1} \mu(v)}{v_1^2 + \lambda^2}\, dv = 0.$$

For this value of λ, we define

$$r(t, x, v) = e^{\lambda t} \frac{v_1 \partial_{v_1} \mu(v)}{v_1^2 + \lambda^2} \cos x_1 - e^{\lambda t} \frac{\lambda \partial_{v_1} \mu(v)}{v_1^2 + \lambda^2} \sin x_1,$$
$$\psi(r) = -e^{\lambda t} \cos x_1.$$

By a direct computation we get

$$\begin{aligned} &\partial_t r + v \cdot \nabla_x r + \nabla_x \psi(r) \cdot \nabla_v \mu \\ &\quad = e^{\lambda t} \left\{ \frac{\partial_{v_1} \mu(v)}{v_1^2 + \lambda^2} [\lambda v_1 \cos x_1 - \lambda^2 \sin x_1 - (v_1^2) \sin x_1 - \lambda v_1 \cos x_1] \right. \\ &\qquad\qquad \left. + \partial_{v_1} \mu(v) \sin x_1 \right\} = 0. \end{aligned}$$

Therefore r is an eigenvector of (2.4) with the eigenvalue $\lambda > 0$ of the operator $-v \cdot \nabla_x - \nabla \mu \cdot \nabla \Delta^{-1} \int dv$.

Q.E.D.

Now we are going to establish a weighted L^p estimate for (2.4). We shall quote a invariant of Weyl's Theorem for a perturbed linear operator. The following lemma is a special case of Shizuta [Sh].

LEMMA 2.2. – *Let Y be a Banach space and A be a linear operator that generates a strongly continuous semigroup on Y such that $\|e^{-tA}\| \leq M$ for all $t \geq 0$. Let K be a compact operator from Y to Y. Then $(A + K)$ generates a strongly continuous semigroup $e^{-t\,(A+K)}$, and $\sigma\,(-A - K)$ consists of a finite number of eigenvalues of finite multiplicity in $\{\mathrm{Re}\,\lambda > \delta\}$ for every $\delta > 0$. These eigenvalues can be labeled by*

$$Re\,\lambda_1 \geq Re\,\lambda_2 \geq \ldots \geq Re\,\lambda_N \geq \delta.$$

Furthermore, for every $\Lambda > Re\,\lambda_1$, there is a constant C_Λ such that

$$\|e^{-t\,(A+K)}\|_{L\,(Y,\,Y)} \leq C_\Lambda e^{\Lambda t}. \tag{2.5}$$

We define

$$\|f\|_Z = \|\langle v\rangle^\alpha\, f\|_{L^p\,(Q\times R^3)} \tag{2.6}$$

for fixed parameters $3 < p < \infty$ and $3 - 3/p < \alpha < p - 3/p$.

LEMMA 2.3. – *Let*

$$\Delta_x\,\psi\,(t,\,x) = \int g\,(t,\,x,\,v)\,dv \qquad \text{for} \quad x \in Q$$
$$\frac{\partial\psi}{\partial n} = 0, \qquad \int_Q \psi dx = 0.$$

Then $\|\nabla_x\,\psi\|_\infty \leq C\|\psi\|_{W^{2,\,p}\,(Q)} \leq C\|\Delta\psi\|_p \leq C\|g\|_Z$.

Proof. – We estimate

$$\begin{aligned}\|\Delta\psi\|_p^p &\leq C \int_Q \left(\int |g|\,dv\right)^p dx \\ &\leq C \int \left(\int |g|^p\,\langle v\rangle^{\alpha p}\,dv\right) \cdot \left(\int \langle v\rangle^{-\alpha p'}\,dv\right)^{p/p'} dx.\end{aligned}$$

Notice that $\langle v\rangle^{-\alpha p'}$ is integrable since $\alpha p' > 3$. Therefore

$$\|\Delta\psi\|_p \leq C\|\langle v\rangle^\alpha\, g\|_p = C\|g\|_Z.$$

The first two inequalities of the lemma come from the Sobolev imbedding and from the standard elliptic estimates applied to the periodic extension.

Q.E.D.

THEOREM 2.4. – *For the linearized problem (2.4), we have*

$$\|g(t)\|_Z \le C_\Lambda e^{\Lambda t}\|f_0\|_Z \tag{2.7}$$

where $\Lambda > \max\{Re\ \lambda : \lambda\ is\ an\ eigenvalue\ of\ (2.4)\}$.

Proof. – We again reduce this boundary value problem (2.4) to the periodic Cauchy problem through the reflection (1.3). Let $\overline{g}$ and $\overline{\psi}$ be as in (1.3). We define the Banach space

$$\mathcal{Z} = \{\overline{g} \in R^3 \times R^3,\ \overline{g} \text{ is periodic in } x_1,\ x_2,\ x_3,\ \|\overline{g}\|_Z < \infty\}.$$

Let $A(f) = v \cdot \nabla_x \overline{f}$ and $K(g) = \nabla_x \overline{\psi} \cdot \nabla_v \overline{\mu}$ where $\Delta\overline{\psi} = \int \overline{g}\, dv$. Then A generates a strongly continuous semigroup on $\mathcal{Z}$ through the formula

$$\overline{g}(t, x, v) = e^{-tA}\,\overline{f}_0 = \overline{f}_0(x - vt, v).$$

By Lemma 2.3

$$\|K(g)\|_Z = \|\nabla\psi\|_p \|\langle v\rangle^\alpha \nabla\mu\|_p \le C\|g\|_Z$$

and K is easily seen to be a compact operator from $\mathcal{Z}$ to $\mathcal{Z}$. By (2.5) with $Y = \mathcal{Z}$, $\overline{g}$ satisfies (2.7). Finally we recover our boundary problem (2.4) by the restriction to $Q \times R^3$.

Q.E.D.

3. NONLINEAR INSTABILITY

Let $p > 3$ and

$$\|f\|_X = \sup_{x,\,v} \langle v\rangle^p (|f| + |\nabla_x f| + |\nabla_v f|). \tag{3.1}$$

Let X be the space of C^1 functions on $Q \times R^3$ for which $\|f\|_X < \infty$.

MAIN THEOREM. – *If $\mu(v)$ is an even function in each coordinate that satisfies (2.1) and (2.2), then $\mu(v)$ is nonlinearly unstable in $\|\cdot\|_X$.*

Explicitly, this means that there exist initial data $f_0^n(x, v)$ and times $t_n \geq 0$ such that $\|f_0^n - \mu\|_X \to 0$, but $\|f^n(t_n) - \mu\|_X$ does *not* go to 0, as $n \to \infty$. We shall prove the Main Theorem through several lemmas.

LEMMA 3.1. – *Let $R(x, v)$ be any eigenvector of the linearized problem (2.4) with a nonzero eigenvalue for which $\|R\|_Z < \infty$. Then $\|R\|_X < \infty$.*

Proof. – By assumption $\|R\|_Z < \infty$. We may consider R and ϕ even and periodic as in paragraph 1. We have

$$
\begin{aligned}
& v \cdot \nabla_x R + \nabla_x \phi \cdot \nabla_v \mu = -\lambda R, \\
& \Delta\phi = \int R dv, \quad \int \phi dx = 0.
\end{aligned} \tag{3.2}
$$

By Lemma 2.3, $\nabla\phi$ is bounded. Let $\Omega(L) = Q \times (-L, L)^3$. Multiplying (3.2) by $\langle v\rangle^{pN}|R|^{N-1}\text{sgn}(R)$ and integrating over $\Omega(L)$, we get

$$
|\lambda| \|\langle v\rangle^p R\|_{N, \Omega(L)} \leq C\|\nabla\phi \langle v\rangle^p \nabla\mu\|_{N, \Omega(L)}.
$$

Letting $N \to \infty$ and then $L \to \infty$, we get $\langle v\rangle^p R \in L^\infty(Q \times R^3)$.

Next, taking the spatial derivatives of (3.2), we get

$$
v \cdot \nabla_x R_{x_j} + \nabla_x \phi_{x_j} \cdot \nabla_v \mu = -\lambda R_{x_j}, \qquad 1 \leq j \leq 3. \tag{3.3}
$$

This process can be justified by a finite difference argument. Multiplying (3.3) by $\langle v\rangle^{\alpha p}|R_{x_j}|^{p-1}\text{sign}(R_{x_j})$ and integrating over $\Omega(L)$, we find

$$
|\lambda| \|\langle v\rangle^\alpha R_{x_j}\|_{p, \Omega(L)} \leq C\| \langle v\rangle^\alpha \nabla_x(\phi_{x_j}) \nabla_v \mu\|_{p, \Omega(L)} \tag{3.4}
$$

where *no* boundary contribution appears of the periodicity, (*i.e.* the boundary condition). By Lemma 2.3, $\nabla^2\phi \in L^p(Q)$. By (2.2), $\langle v\rangle^\alpha \nabla\mu \in L^p(R^3)$. Therefore we deduce from (3.4) that $\langle v\rangle^\alpha R_{x_j} \in L^p(Q \times R^3)$. That is $\nabla_x R \in \mathcal{Z}$. From the equation $\Delta\phi_{x_j} = \int R_{x_j} dv$ and the periodicity, we obtain as in Lemma 2.3

$$
\|\nabla^2\phi\|_\infty \leq C\|\nabla_x R\|_Z < \infty.
$$

Now multiplying (3.3) by $\langle v\rangle^{pN}|R_{x_j}|^{N-1}\mathrm{sgn}\,(R_{x_j})$ and integrating over $\Omega\,(L)$, we get

$$|\lambda|\|\langle v\rangle^p\, R_{x_j}\|_{N(\Omega\,(L))} \leq C\|\nabla^2\phi\|_\infty\|\langle v\rangle^p\,\nabla_v\mu\|_{N(\Omega\,(L))}. \tag{3.5}$$

Letting $N \to \infty$ and then $L \to \infty$, we get $\langle v\rangle^p\,\nabla_x R \in L^\infty(Q \times R^3)$.

Similarly, taking the velocity derivatives of (3.2) yields:

$$v \cdot \nabla_x\, R_{v_j} + R_{x_j} + \nabla_x\,\phi \cdot \nabla_v\,\mu_{v_j} = -\lambda R_{v_j}. \tag{3.6}$$

Multiplying (3.6) by $\langle v\rangle^{pN}|R_{v_j}|^{N-1}\mathrm{sgn}\,(R_{v_j})$ and integrating, we get

$$|\lambda|\|\langle v\rangle^p\, R_{v_j}\|_{N(\Omega\,(L))} \leq C[\|\langle v\rangle^p\, R_{x_j}\|_{N\,(\Omega\,(L))} + \|\langle v\rangle^p\,\nabla\phi\nabla^2\mu\|_{N(\Omega\,(L))}].$$

Again, there is *no* boundary contribution due to the periodicity. Letting $N \to \infty$ and then $L \to \infty$, we deduce $\langle v\rangle^p\, R_{v_j} \in L^\infty\,(Q \times R^3)$.

Q.E.D.

Now let

$$\|f\|_Y^p = \int \left\{\int [\langle v\rangle^\alpha\,(|f| + |\nabla_x f| + |\nabla_v f|]^s\, dx\right\}^{p/s} dv,$$

where we fix $3 < p < \infty$, $3 - 3/p < \alpha < p - 3/p$ and $s = p^2/(p-1) > p$. Since $\alpha < p - 3/p$, we have

$$c\,(\|f\|_Z + \|\nabla f\|_Z)^p \leq \|f\|_Y^p \leq C\|f\|_X^p \int \langle v\rangle^{\alpha p - p^2}\, dv \leq C\|f\|_X^p.$$

LEMMA 3.2. – *If f is a C^1 solution of the nonlinear problem (1.1) with $f\,(t) \in X$, then*

$$\frac{d}{dt}\|\nabla_x f\|_Z^p \leq C\|f - \mu\|_Z\|f\|_Y^p, \tag{3.7}$$

$$\frac{d}{dt}\|\nabla_v(f - \mu)\|_Z \leq C\|f - \mu\|_Z\,(1 + \|f - \mu\|_Y) + C\|\nabla_x f\|_Z. \tag{3.8}$$

Proof. – Let $h = \partial_{x_j} f$. Differentiate the Vlasov equation with respect to x_j to get

$$\{\partial_t + v \cdot \nabla_x + \nabla\phi \cdot \nabla_v\} h = -\partial_{x_j} \nabla\phi \cdot \nabla_v f.$$

Multiply by $p \langle v\rangle^{\alpha p}|h|^{p-1} sgn\,(h)$ and integrate to get:

$$\begin{aligned}\frac{d}{dt} \int \langle v\rangle^{\alpha p}|h|^p \, dxdv \leq & \int |\nabla_v(\langle v\rangle^{\alpha p})||\nabla\phi||h|^p \, dxdv \\ & + p \int \langle v\rangle^{\alpha p}|h|^{p-1}|\nabla^2\phi||\nabla_v f| dxdv = I_1 + I_2.\end{aligned}$$

We estimate the two integrals as

$$\begin{aligned} I_1 &\leq C\|\nabla\phi\|_\infty \int \langle v\rangle^{\alpha p-1}|h|^p \leq C\|\nabla\phi\|_\infty \|\langle v\rangle^\alpha h\|_p^p \\ &\leq C\|f-\mu\|_Z \|f\|_Y^p \end{aligned}$$

by Lemma 2.3 since $s \geq p$, and as

$$\begin{aligned} I_2 &\leq p\|\nabla^2\phi\|_p \int \left\{ \int [(\langle v\rangle^\alpha h)^{p-1} \langle v\rangle^\alpha |\nabla_v f|]^{p'} \, dx \right\}^{1/p'} dv \\ &\leq C\|f-\mu\|_Z \int |\langle v\rangle^\alpha h|_s^{p-1} |\langle v\rangle^\alpha \nabla_v f|_s dv \\ &\leq C\|f-\mu\|_Z \|\langle v\rangle^\alpha \nabla_x f\|_{L_v^p(L_x^s)}^{p-1} \|\langle v\rangle^\alpha \nabla_v f\|_{L_v^p(L_x^s)} \\ &\leq C\|f-\mu\|_Z \|f\|_Y^p. \end{aligned}$$

We thus deduce (3.7).

Next, differentiate the Vlasov equation with respect to v_j. Let $g = \partial_{v_j}(f-\mu)$. Multiply the differentiated equation by $p\langle v\rangle^{\alpha p}|g|^{p-1}\text{sgn}\,(g)$ and integrate to get,

$$\begin{aligned}\frac{d}{dt} \int \langle v\rangle^{\alpha p}|g|^p \, dxdv \leq & \int |\nabla_v(\langle v\rangle^{\alpha p})||\nabla\phi||g|^p \, dxdv \\ & + p \int \langle v\rangle^{\alpha p}|g|^{p-1} \{|\partial_{x_j} f| + |\nabla\phi||\partial_{v_j}\nabla_v \mu|\} \, dxdv \\ & = I_3 + I_4 + I_5.\end{aligned}$$

We estimate these three integrals as

$$|I_3| \le C\|\nabla\phi\|_\infty \int \langle v\rangle^{\alpha p-1}|g|^p\,dxdv \le C\|f-\mu\|_Z\|g\|_Z^p,$$
$$|I_4| \le C\|\langle v\rangle^\alpha g\|_p^{p-1}\|\langle v\rangle^\alpha\nabla_x f\|_p = C\|g\|_Z^{p-1}\|\nabla_x f\|_Z,$$
$$\begin{aligned}|I_5| &\le C\|\langle v\rangle^\alpha g\|_p^{p-1}\|\langle v\rangle^\alpha|\nabla_x\phi\|\nabla_v^2\mu|\|_p\\ &= C\|\langle v\rangle^\alpha g\|_p^{p-1}\|\nabla\phi\|_p\left[\int(\langle v\rangle^\alpha|\nabla^2\mu|)^p\,dv\right]^{1/p}\\ &\le C\|g\|_Z^{p-1}\|f-\mu\|_Z.\end{aligned}$$

The last integral is finite by (2.2). Dividing by $\|g\|_Z^{p-1}$, we deduce (3.8).

Q.E.D.

LEMMA 3.3. – *Assume $f(t, x, v)$ is a C^1 solution of the nonlinear problem (1.1) with $f(t) \in L^\infty([0, T); X)$. Assume there exists $\beta > 0$ such that*

$$\|f(t)-\mu\|_Z \le Ce^{\beta t}\|f(0)-\mu\|_Z$$

in $[0, T)$. Then

$$\|\nabla_v(f(t)-\mu)\|_Z \le Be^{\beta t/p}\|f(0)-\mu\|_Y^{1/p} \tag{3.9}$$

where $0 \le t < T$ and B depends on $\sup_t \|f(t)\|_Y$, on μ and on β.

Proof. – By Lemma 3.2, we have

$$\begin{aligned}\|\nabla_v(f(t)-\mu\|_Z^p &\le C\|\nabla_v(f(0)-\mu)\|_Z^p + C\int_0^t \|f(\tau)-\mu\|_Z^p d\tau\\ &\quad + C\int_0^t \|\nabla_x f(\tau)\|_Z^p\,d\tau\\ &\le C\|f(0)-\mu\|_Y^p + C\int_0^t \|f(\tau)-\mu\|_Z^p\,d\tau\\ &\quad + C\int_0^t\left\{\|\nabla_x f(0)\|_Z^p + \int_0^\tau \|f(\sigma)-\mu\|_Z d\sigma\right\}d\tau\\ &B^p\,N_0\,e^{\beta t}\end{aligned}$$

where $N_0 = \|f(0)-\mu\|_Y \le C\|f(0)-\mu\|_X$.

Q.E.D.

Proof of the main theorem. – We follow the idea of Theorem 6.1 of [GSS]. Suppose the theorem is false, so that μ is nonlinearly stable. Thus for any $\varepsilon > 0$, there exists $\delta > 0$ such that $f \in C([0, \infty); X)$ and

$$\sup_{0 \le t < \infty} \|f(t) - \mu\|_X \le \varepsilon \tag{3.10}$$

provided $\|f(0) - \mu\|_X < \delta$. We choose

$$f(0, x, v) - \mu(v) = \delta R(x, v)$$

where R is an eigenvector in Lemma 3.1 with $\|R\|_X = 1$, such that its eigenvalue $\lambda = \lambda_1$ has the largest real part. By the nonlinear Vlasov equation (1.1),

$$f(t) - \mu = \delta R e^{\lambda t} - \int_0^t e^{-(A+K)(t-\tau)} (\nabla_x \phi \cdot [\nabla_v (f(\tau) - \mu)]\, d\tau \tag{3.11}$$

where $A + K$ is the linearized operator in Theorem 2.4. Let $Re\,\lambda < \Lambda < Re\,\lambda\,(1 + 1/p)$. Let

$$T = \sup \left\{ t' : \|f(t) - \mu - \delta R e^{\lambda t}\|_Z \le \frac{1}{2}\, \delta e^{Re \lambda t} \|R\|_Z \text{ for } 0 \le t \le t' \right\}.$$

Then $0 < T < \infty$. Thus for $0 \le t \le T$ we have

$$\|f(t) - \mu\|_Z \le \frac{3\delta}{2}\, e^{t Re\,\lambda} \|R\|_Z. \tag{3.12}$$

By Lemma 2.3,

$$\|\nabla \phi(t)\|_\infty \le C\, \|f(t) - \mu\|_Z \le C \delta e^{t Re\,\lambda}. \tag{3.13}$$

Thus we may apply Lemma 3.3 with $\beta = Re\,\lambda$ to obtain

$$\begin{aligned} &\|f(t) - \mu - \delta R e^{\lambda t}\|_Z \\ &\le \int_0^t e^{\Lambda(t-\tau)} \|\nabla \phi(\tau)\|_\infty \|\nabla_v (f(\tau) - \mu)\|_Z \, d\tau \\ &\le C \int_0^t e^{\Lambda(t-\tau)}\, \delta e^{\tau Re\,\lambda}\, \delta^{1/p}\, e^{\tau\, Re\,\lambda/p}\, d\tau \\ &\le C \delta^{1+1/p}\, e^{(Re\,\lambda)(1+1/p)\,t}. \end{aligned} \tag{3.14}$$

Thus for $0 \leq t \leq T$ we have

$$\|f(t) - \mu\|_Z \geq \|R\|_Z\, \delta e^{tRe\lambda} - C\,(\delta e^{tRe\,\lambda})^{1+1/p}. \tag{3.15}$$

We choose $t = t_\delta$ so that

$$\delta e^{tRe\lambda} = (\|R\|_Z/(2C))^p.$$

We claim that $0 < t_\delta \leq T$. In fact, if $T < \infty$, then from (3.14) and the definition of T,

$$\frac{1}{2}\|R\|_Z \delta e^{Re\,\lambda T} = \|f(T) - \mu - \delta R e^{\lambda T}\|_Z \leq C\,(\delta e^{TRe\,\lambda})^{1+1/p}.$$

Hence $\delta e^{Re\,\lambda t_\delta} = (\|R\|_Z/2C)^p \leq \delta e^{Re\,\lambda T}$, and so $t_\delta \leq T$. Now that $0 < t_\delta \leq T$, from (3.15) we have

$$\|f(t_\delta) - \mu\|_Z \geq \|R\|_Z^{p+1}/(2^{p+1}\,C^p). \tag{3.16}$$

This contradicts (3.10) since ε is arbitrarily small.

Q.E.D.

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(Manuscript received January 3, 1993.)