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Second order nonpersistence of the sine Gordon breather under an exceptional perturbation

by

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ABSTRACT. – We consider the only nontrivial perturbation of the sine Gordon equation of the type $u_{tt} - u_{xx} + \sin u = \varepsilon\Delta(u) + O(\varepsilon^2)$ under which persistence of the unperturbed breather family cannot be ruled out by first order perturbation theory. We show that in this case, nonpersistence can be proved by second order perturbation theory. A resonant interaction of the second order perturbation function with the first order perturbation of the breathers is responsible for this phenomenon. Number theoretic techniques make the final analysis manageable.

Key words: Sine Gordon, breather, soliton.

RÉSUMÉ. – On discute de la seule perturbation $u_{tt} - u_{xx} + \sin u = \varepsilon\Delta(u) + O(\varepsilon^2)$ non triviale de l'équation sinus-Gordon pour laquelle il est impossible de conclure, sur la base de la seule équation du premier ordre en ε, que l'ensemble des solutions « breather » (« respirateur ») ne soit pas conservé. Dans ce cas, on se sert de l'équation du second ordre en ε pour démontrer que cet ensemble n'est pas conservé. C'est une résonance du second ordre de la perturbation de l'équation avec le premier ordre de la perturbation des solutions « breather », qui est responsable pour ce phénomène. Pour réussir aux évaluations finales, on utilise des résultats de la théorie des nombres.

A.M.S. Classification : 35 B 20, 35 B 10, 35 Q 53.

1. INTRODUCTION

This work considers the perturbed sine Gordon equation

$$u_{tt} - u_{xx} + \sin u = \varepsilon\Delta(u,\varepsilon) = \varepsilon\Delta(u) + \varepsilon^2\Delta^{[2]}(u) + \dots \qquad (1.1)_\varepsilon$$

for an analytic perturbation function $\Delta(\cdot,\cdot)$ such that $\Delta(0,\varepsilon) = 0$ for all ε. For $\varepsilon = 0$, it is well known that the *breathers*

$$u^*(x,t) = u^*(x,t;m) = 4\,\mathrm{atn}\,\frac{m}{\omega}\,\frac{\sin\omega t}{\mathrm{ch}\, mx}, \quad m^2 + \omega^2 = 1 \qquad (1.2)$$

$(m, \omega > 0)$ are a family of solutions to the unperturbed equation $(1.1)_0$. (We use atn for the arc tangent and ch, sh, th for the hyperbolic functions.) Let us here define a breather to be a nontrivial time periodic solution to a wave equation that decays as $|x| \to \infty$ (often, exponential decay is stipulated). Not much is rigorously known about (non-)existence of breathers, but it is generally suspected that their existence is a very rare and singular phenomenon (*see* already [11], or already [9], as the referee kindly suggests).

We are interested whether there exists a similar family of breather solutions to the perturbed equation $(1.1)_\varepsilon$ that reduces to (1.2) for $\varepsilon = 0$, *i.e.* whether the family of breathers *persists* under some perturbation. A necessary condition that the breather $u^*(\cdot,\cdot;m)$ persists is that the variational equation

$$\mathcal{L}v := (\partial_t^2 - \partial_x^2 + \cos u^*(x,t;m))v = \Delta(u^*(x,t;m)) \qquad (1.3)$$

has a solution. Since recently [1], [4], [5], thm. 1, it is known (we omit technical details) that in order for (1.3) to have a solution for infinitely many m, it is necessary and sufficient that $\Delta(\cdot)$ lies in the linear space spanned by the four functions $\sin u$, $u\cos u$, $\cos u$, and a certain $\widetilde{\Delta}(u)$ given below. (The sufficiency part follows from Theorem 12 in [5] and will be worked out explicitly for the present case in section 3.) The first three of those functions can be readily explained: For these perturbations, equation $(1.1)_\varepsilon$ can be reduced to the unperturbed equation $(1.1)_0$ by scaling the variables (x,t) and u and by shifting u by a constant function. We ruled out the latter possibility ($\cos u$) by the *a priori* normalization $\Delta(0,\varepsilon) = 0$, which is obviously necessary for the existence of a breather, whereas $\sin u$ and $u\cos u$ are the leading orders of scaling perturbations, under which breathers trivially persist. By normalizing the scalings, too, we can get rid of the perturbations $\sin u$ and $u\cos u$ as well and are left with $\widetilde{\Delta}(u)$ as the only remaining perturbation under which, according to (1.3), all breathers might persist. There is no simple explanation for this remaining exceptional perturbation

$$\begin{aligned}\widetilde{\Delta}(u) &= \frac{1}{4}\left(1 - 4\cos\frac{u}{2} + 3\cos u + 4\cos u \ln\cos\frac{u}{4}\right)\\ &= -\frac{9}{32}u^2 + O(u^4)\end{aligned} \tag{1.4}$$

It is the purpose of this work to show that there is indeed no analytic perturbation $\Delta(\cdot,\cdot)$ whose leading order is $\widetilde{\Delta}(\cdot)$ and under which infinitely many breathers persist.

An essential ingredient to our proof is the analyticity of the second order perturbation function $\Delta^{[2]}(\cdot)$. Our result uses second order perturbation theory, *i.e.* the order ε^2 of the equation, but no higher orders, so only C^2 in ε is actually needed.

For more background on the persistence problem, we refer to the introduction of [5]. We repeat shortly those of the arguments given there which are basic for the following discussion.

First order perturbation theory for equation $(1.1)_\varepsilon$ is by definition everything that depends only on (1.3). This latter equation can be solved for v if and only if the conditions

$$\int_{-\infty}^{\infty}\int_0^{2\pi/\omega} \Delta(u^*(x,t))\chi_n(x,t)\,dt\,dx = 0 \tag{1.5}$$

are satisfied for $n = 2, 3, 4\ldots$, where $\chi_n(x,t)$ solves $\mathcal{L}\chi_n = 0$ and is asymptotic to $\exp\left(in\omega t + i\sqrt{n^2\omega^2 - 1}\,x\right)$ as $x \to \pm\infty$ [5]. These conditions (which depend on m) have to hold for every value of m for which the corresponding breather is assumed to persist. By an analytic continuation argument (details omitted here), they have to hold identically in m, if they hold for infinitely many m (for details, *see* [1] and [5]).

This determines the four dimensional space spanned by the solutions given above. The two odd ones among the spanning solutions correspond to the scalings and are therefore trivial perturbations. The algebraic calculation involved in determining these solutions (given in [5], but omitted here) is insensitive to parity, and along with the odd ones, it produces two even solutions, namely $\cos u$ and $\widetilde{\Delta}(u)$. As they follow the scalings through all calculations like a shadow, we can call them shadow scalings. However, we restrict this name to $\widetilde{\Delta}(u)$ only, because the geometric meaning of the perturbation $\cos u$ justifies the more precise name "shift perturbation" for it.

The argument by first order perturbation theory in connection with analytic continuation of the obstructions (1.5) with respect to m is due to Birnir, McKean, and Weinstein [1]. In [5], their argument is improved, assuming analyticity of the first order perturbations at 0 only. Moreover, the procedure of solving the first order equations is given there in an accessibly detailed way.

Our approach shows that for the single perturbation $\widetilde{\Delta}$ for which persistence of the breather family cannot be ruled out by first order perturbation arguments, it can be ruled out by second order arguments. This does not explain what makes $\widetilde{\Delta}$ special with respect to the orders of perturbation theory. A recent paper by Birnir [2] gives some numerical evidence that the breather may bifurcate under this perturbation into a quasiperiodic solution.

2. STATEMENT OF THE RESULT AND OUTLINE OF THE PROOF

THEOREM 1. – *Consider the perturbed sine Gordon equation* $(1.1)_\varepsilon$, *where* $\Delta(0,\varepsilon) = 0$ *for all* ε, $\Delta(\cdot,\cdot)$ *is* C^2 *in both variables,* $\Delta(u)$ *is the shadow scaling* (1.4), *and* $\Delta^{[2]}(u)$ *is an arbitrary function that is analytic in some neighbourhood of 0. Let* ρ *be such that* $\left\{u \in \left]-2\pi, 2\pi\right[\;\middle|\; \left|\tan \frac{u}{4}\right| \leq \rho\right\}$ *is contained in this neighbourhood.*

Then, in any interval $[\rho'', \rho'] \subset \left]0, \rho\right[$, *at most finitely many breathers* (1.2) *can persist under the perturbation. The technical definition of persistence is given precisely as one hypothesis of Lemma* 4 *below.*

Note. – With only obvious modifications in Lemmas 5 and 6 below, Theorem 1 holds under the weaker hypothesis that $[\rho'', \rho']$ is contained inside the domain of analyticity of $z \mapsto \Delta^{[2]}(4 \operatorname{atn} z)/(1 + z^2)$ without 0.

From now on, we drop the tilde. Δ will always denote the shadow scaling (1.4).

Before we go into the details, let us give the core of the proof without technicalities: We write a presumed breather to the unperturbed equation in the form $u = u^* + \varepsilon v + \varepsilon^2 v^{[2]} + o(\varepsilon^2)$. The order $O(\varepsilon^0)$ of the equation is satisfied, and we know from [5] that the order $O(\varepsilon)$ of the equation can be solved for v for any m. We want to show that it is impossible that the order $O(\varepsilon^2)$ of the equations can be solved for infinitely many m. In order to achieve this, we first have to calculate v effectively, because it enters into the second order equations. In principle, this calculation is done by a variant of the Fourier transformation and is conceptually easy. But the explicit calculation in our special case is quite a voluminous task and will be accomplished in section 3; *see* Theorem 2 for the result.

We do not obtain a closed formula for the first order solution v. But it turns out that only one qualitative feature of v is important for the conclusion that the second order perturbation equations cannot be satisfied: the fact that v contains terms that are not powers of $1/\operatorname{ch} mx$. More precisely, the asymptotic expansion of $v(x,t)$ as $|x| \to \infty$ involves also a term $e^{-|x|}$ along with terms $e^{-2\ell m|x|}$. The particular term containing $e^{-|x|}$ can be expressed in closed form, and the entire proof relies on this term.

As terms with the asymptotic expansion $e^{-2\ell m|x|}$ appear on the right hand side $\Delta(u^*)$ of the first order equation, it is natural that they appear in the solution v, too. But for the $n = 0$ Fourier component in the time variable and $|x| \to \infty$, the leading term of the operator $\mathcal{L}$ is $-\partial_x^2 + 1$. The crucial terms with the asymptotic behaviour $e^{-|x|}$ are related to the kernel of this operator (on the half lines).

We shall see that the second order equation is

$$\mathcal{L}v^{[2]} = \Delta^{[2]}(u^*) + \Delta'(u^*)v + \frac{1}{2}(\sin u^*)v^2 \tag{2.1}$$

Necessary conditions for this equation to have a solution are obtained, as in first order perturbation theory, by multiplying it with functions $\chi_n \in \ker \mathcal{L}$ and integrating over x and t. The integrals on the right hand side can then be considered as meromorphic functions of the complex variable $m \neq 0$ (but with an essential singularity at $m = 0$). Their sum has to vanish at points where (2.1) can be solved, since $\int \chi_n \mathcal{L}v^{[2]}\, dx\, dt = \int (\mathcal{L}\chi_n)v^{[2]}\, dx\, dt = 0$. We find that the equation

$$-\int \Delta^{[2]}(u^*)\chi_n\, dt\, dx = \int \left(\Delta'(u^*)v + \frac{1}{2}(\sin u^*)v^2\right)\chi_n\, dt\, dx \tag{2.2}$$

holds for infinitely many values of $\frac{m}{\omega} \in [\rho'', \rho']$. Detailed knowledge of the behaviour of the left hand side in dependence of $\frac{m}{\omega}$ (for analytic functions $\Delta^{[2]}$) was already obtained in the discussion of first order perturbation theory. Using this knowledge and an analytic continuation argument, one obtains vanishing pole conditions that are necessary for the persistence of infinitely many breathers: poles that must be expected on the right hand side have to vanish because they do not appear on the left hand side.

As the functions χ_n are asymptotic (for $x \to \pm\infty$) to free Klein Gordon waves $\exp(in\omega t + i\sqrt{n^2\omega^2 - 1}x)$, the integrals over x are simply Fourier integrals; the poles just mentioned correspond to the exponential decay of the terms in the asymptotic expansion of $\Delta'(u^*)v + \frac{1}{2}(\sin u^*)v^2$ as $x \to \pm\infty$, as is well known from the theory of Fourier transformations.

We are in particular interested in those poles which correspond to the decay rates $e^{-2|x|-2\ell m|x|}$, because they can come *only* from $\frac{1}{2}(\sin u^*)v^2$, but not from $\Delta'(u^*)v$ or $\Delta^{[2]}(u^*)$, provided the latter is an analytic function of u^*. Therefore we need not bother with $\Delta^{[2]}$ at all, and also Δ enters only through an explicitly calculated contribution to v. To complete the proof of the theorem, we calculate (some of) these poles and show that they do occur in contradiction to (2.2).

The calculation of the specified poles is explicitly possible, but it has to take into account many single terms and to ensure that they do not cancel altogether by some strange conspiracy. This makes the calculation very lengthy. We stress however that for any given one of these poles, it is a *finite calculation* that can be done *exactly* in some quadratic number field $\mathbb{Q}(\sqrt{D})$, where D depends on the pole chosen.

There is still one serious difficulty: the pole which we check must lie in a domain that corresponds to the domain of analyticity of $\Delta^{[2]}$. Since the latter is allowed to be an arbitrarily small neighbourhood of 0, no single pole is feasible for all possible neighbourhoods. Therefore, we need to calculate that the residues of a whole sequence of possible poles (accumulating in 0 in order to exhaust all neighbourhoods) do not vanish. But the length of the calculations involved grows with the sequence index. Thus instead of mere calculations, one needs an extra argument treating infinitely many poles at once. This difficulty is of the same type as in [5], where the first order of the perturbation was allowed to have arbitrarily small domain of analyticity. In that case, too, one had to work with poles accumulating at 0, and a "locally, but not uniformly" finite calculation had to be replaced by a hopefully clever argument. But there, an explicit formula was available, which we do not have (nor expect to have) here.

In order to treat infinitely many poles with one argument, we reduce the calculations in the fields $\mathbb{Q}(\sqrt{D})$ to calculations in the rings $\mathbb{Z}[\sqrt{D}]$ by multiplying with the common denominator. This leaves us with showing that certain expressions (in fact multiples of the residues) do not vanish in this ring. We intend to show that they do not even vanish considered as elements in an appropriate quotient ring $\mathbb{Z}[\sqrt{D}]/I$ modulo some ideal I. It turns out that it is possible to choose I in dependence of the pole in such a way that most (but not all) terms are killed by dividing out the ideal, and only a uniformly finite number of terms remains, whose sum can then be shown not to vanish in the quotient ring. In these calculations, one uses elementary number theory: binomial coefficients modulo a prime, Fermat's theorem, and quadratic reciprocity.

We give the calculation of v in section 3, then explain how to obtain the vanishing pole conditions of second order perturbation theory in section 4. At this stage, one can work out the easiest of these conditions explicitly (still 3×24 terms), thus proving the theorem for a sufficiently large analyticity domain of $\Delta^{[2]}$. This is done in section 4.4. We finally give the reduction to a quotient ring in section 5, which completes the proof without this assumption on the domain.

3. CALCULATING FIRST ORDER PERTURBATION THEORY

In this section, we solve the equation

$$\mathcal{L}v(x,t) = (\partial_t^2 - \partial_x^2 + \cos u^*(x,t))v(x,t) = \Delta(u^*(x,t)) \tag{3.1}$$

with respect to the boundary conditions $v(x, t + 2\pi/\omega) = v(x,t)$, $|v(x,t)| \to 0$ as $|x| \to \pm\infty$, where Δ is the shadow scaling (1.4). We use that for Δ, the necessary conditions $\int \chi_n \Delta(u^*)\, dx\, dt = 0$ are satisfied, where the χ_n satisfy $\mathcal{L}\chi_n = 0$ and are $2\pi/\omega$-periodic in t. These conditions are also sufficient for the solvability of (3.1). *See* [5] for the arguments. The calculation applies the scheme of eigenfunction expansions given in [7].

3.1. The general scheme

In order to calculate v, we need a modified scheme of the sufficiency proof given in [5]. Let $\vec{v} = \begin{bmatrix} v \\ v_t \end{bmatrix}$, $\vec{\Delta} = \begin{bmatrix} 0 \\ \Delta \end{bmatrix}$, $A = \begin{bmatrix} 0 & 1 \\ \partial_x^2 - \cos u^* & 0 \end{bmatrix}$, $A^* = \begin{bmatrix} 0 & \partial_x^2 - \cos u^* \\ 1 & 0 \end{bmatrix}$. In order to solve the equation $\partial_t \vec{v} = A\vec{v} + \vec{\Delta}$, which is equivalent to (3.1), we fix t and make a generalized Fourier transform of the right hand side with respect to x. Any 2-component function $\vec{f}(\cdot, t_0)$, in particular $\vec{\Delta}$ and $\vec{v}$, can be written as

$$\vec{f}(x, t_0) = \int_\Gamma \hat{f}(\lambda, t_0)\, e^{-i\Omega(\lambda)t_0}\, \Phi(x, t_0, \lambda)\, d\lambda \tag{3.2}$$

where the functions Φ solve a homogeneous version of (3.1), namely

$$\partial_t(e^{-i\Omega(\lambda)t}\Phi(x,t,\lambda)) = A\,(e^{-i\Omega(\lambda)t}\Phi(x,t,\lambda))$$

and $\Phi \sim \frac{i}{\lambda}\begin{bmatrix} -1 \\ i\Omega(\lambda)\end{bmatrix} e^{-ik(\lambda)x}$ for $x \to -\infty$, and where

$$\Omega(\lambda) = 2\lambda + \frac{1}{8\lambda} \qquad k(\lambda) = 2\lambda - \frac{1}{8\lambda} \qquad \Omega(\lambda)^2 - k(\lambda)^2 = 1$$

The path of integration Γ will be discussed a little later. The possibility of a formula like (3.2) is due to a completeness relation [in $L^2(\mathbb{R})$] for solutions to the linearized sine Gordon equation given already by McLaughlin and Scott in [7]. *See* [5] for a sketch of a proof. For real λ, the coefficients $\hat{f}(\lambda, t_0)$ can be calculated as

$$\hat{f}(\lambda, t_0) = \frac{\lambda}{4\pi i a^2(\lambda)} \int_{-\infty}^{\infty} e^{i\Omega(\lambda)t_0} \Phi^{\mathrm{A\,T}}(x, t_0, \lambda)\vec{f}(x, t_0)\, dx \tag{3.3}$$

where

$$\begin{aligned} a(\lambda) &= \frac{k(\lambda) - im}{k(\lambda) + im} = \frac{\bar{n}(\lambda)}{n(\lambda)} \\ n(\lambda) &= \frac{\lambda}{2}(k(\lambda) + im)\,, \quad \bar{n}(\lambda) = \frac{\lambda}{2}(k(\lambda) - im) \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} \Phi_2^{\mathrm{A}}(x, t, \lambda) &= \Phi_1(-x, -t, \lambda) \\ \Phi_1^{\mathrm{A}}(x, t, \lambda)e^{i\Omega(\lambda)t} &= \partial_t(\Phi_2^{\mathrm{A}}(x, t, \lambda)e^{i\Omega(\lambda)t}) \end{aligned} \tag{3.5}$$

As a function of x, $\Phi(\cdot, t_0, \lambda)$ is bounded for real $\lambda \neq 0$, and it even decays as $x \to \pm\infty$ for the zeros $\lambda_\pm = (im \pm \omega)/4$ of the function $a(\lambda)$. For other non-real values of λ, Φ grows exponentially as either $x \to \infty$ or $x \to -\infty$. We cite the explicit formulas for Φ form [7]; they are derived in the scattering theory for the sine Gordon equation, but can be checked without this theory in a straightforward, though lengthy way:

$$\begin{aligned} \Phi_1(x, t, \lambda)\, e^{-i\Omega(\lambda)t} &= \frac{\lambda}{n(\lambda)^2} e^{-ik(\lambda)x - i\Omega(\lambda)t}\left(1 + \frac{m^2 s^2}{\omega^2 C^2}\right)^{-1} m^2 \times \\ &\left\{-\frac{k(\lambda)}{2m}\frac{S}{C} - \frac{\Omega(\lambda)}{2\omega}\frac{cs}{C^2} - i\left[\left(\frac{k(\lambda)}{2m}\right)^2 - \frac{1}{4}\right] - i\left[\left(\frac{\Omega(\lambda)}{2\omega}\right)^2 + \frac{1}{4}\right]\frac{s^2}{C^2}\right\} \\ \Phi_2(x, t, \lambda)\, e^{-i\Omega(\lambda)t} &= \partial_t(\Phi_1(x, t, \lambda)\, e^{-i\Omega(\lambda)t}) \end{aligned} \tag{3.6}$$

where we have set

$$C = \operatorname{ch} mx \qquad S = \operatorname{sh} mx \qquad c = \cos \omega t \qquad s = \sin \omega t$$

Following [7], the path of integration Γ given in (3.2) is a path in the complex λ-plane that coincides with the real axis except that it leaves it twice in order to go around $\lambda_\pm$. *See* figure 1. The contributions from $\lambda_\pm$ are quite fundamental for the theory, because the basis necessarily includes functions whose time evolution is not periodic ($\dot{\widetilde{\Phi}}_\pm$ below). The small circles in the figure are not important now, but will be explained after (3.15).

For fixed x, the integral in (3.2) is well-defined and reduces to an integral over *real* λ plus the residual contributions from $\lambda_\pm$, but as soon as we have to rely on growth assumptions in dependence of x, we interpret (3.2) only as a shorthand notation for the latter, because no growth assumptions hold on Γ off the real axis. Formula (3.3) holds for real $\lambda \neq 0$ only and should be complemented by similar formulas for the Fourier coefficients corresponding to the residual contributions. Instead, we are going to argue that $\hat{f}$ can be continued analytically, and that inserting the continued function into (3.2) accounts correctly for the residual contributions.

We represent both $\vec{v}$ and $\vec{\Delta}$ in the form (3.2) with generalized Fourier transforms $\hat{v}$ and $\widehat{\Delta}$.

Letting $\widetilde{\Phi} := \Phi\, e^{-i\Omega(\lambda)t}$ and $\widetilde{\Phi}^{\mathrm{A}} := \Phi^{\mathrm{A}}\, e^{+i\Omega(\lambda)t}$, the integral (3.2) reads, according to our interpretation,

$$\vec{v} = \int_{-\infty}^{\infty} \frac{\lambda}{4\pi i a^2(\lambda)} <\widetilde{\Phi}^{\mathrm{A}}|\vec{v}> \widetilde{\Phi}(\lambda)\, d\lambda \tag{3.7}$$

$$- 2\pi i \sum_{\pm} <\operatorname*{res}_{\lambda_\pm} \frac{\lambda}{4\pi i a^2(\lambda)}\, \widetilde{\Phi}^{\mathrm{A}}|\vec{v}> \dot{\widetilde{\Phi}}_\pm - 2\pi i \sum_{\pm} <\operatorname*{res}_{\lambda_\pm} \frac{\lambda(\lambda-\lambda_\pm)}{4\pi i a^2(\lambda)}\, \widetilde{\Phi}^{\mathrm{A}}|\vec{v}> \widetilde{\Phi}_\pm$$

$$=: \int_{-\infty}^{\infty} \hat{v}(\lambda)\widetilde{\Phi}(\lambda)\, d\lambda + \sum_{\pm} \hat{\dot{v}}_\pm \dot{\widetilde{\Phi}}_\pm + \sum_{\pm} \hat{v}_\pm \widetilde{\Phi}_\pm \tag{3.8}$$

where $<\cdot|\cdot>$ denotes the $L^2(\mathbb{R} \to \mathbb{R}^2)$ scalar product with respect to the variable x.

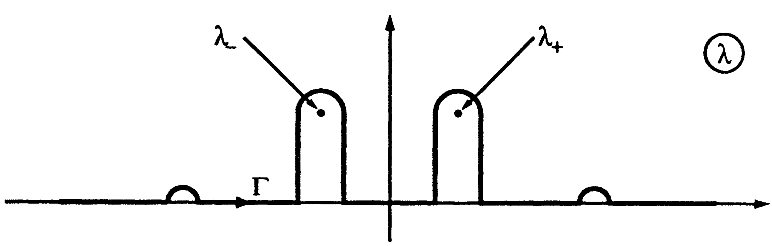

FIG. 1. – The path of integration Γ.

These formulas follow from (3.2) by calculating the residues of the second order poles of the λ-integral in $\lambda_\pm$ and introducing the abbreviations $\hat{\hat{v}}_\pm$ and $\hat{v}_\pm$. The differential equation $\partial_t \vec{v} = A\vec{v} + \vec{\Delta}$ now reduces to differential equations for $\hat{v}(\lambda)$, $\hat{v}_\pm$, $\hat{\hat{v}}_\pm$, the first of which is:

$$\partial_t \hat{v}(\lambda, t) = \widehat{\Delta}(\lambda, t) = \int_{-\infty}^{\infty} \frac{\lambda\, \widetilde{\Phi}_2^{\mathrm{A}}(x, t, \lambda)\, \Delta(u^*(x, t))}{4\pi i a^2(\lambda)}\, dx \tag{3.9}$$

for *real* λ. But the right hand side of this equation is well-defined for $|\operatorname{Im} k(\lambda)| < 2m$ and represents a meromorphic function of $\lambda \neq 0$ there [with double poles at $\lambda_\pm$ coming from $a^2(\lambda)$]. We *define* $\widehat{\Delta}$ for complex λ to be the analytic continuation of the right hand side of this equation.

Starting with the definitions introduced in (3.8) and using the differential equations for $\vec{v}$ and $\widetilde{\Phi}^{\mathrm{A}}$, straightforward calculations show that the differential equations for $\hat{v}_\pm$ and $\hat{\hat{v}}_\pm$ are

$$\begin{aligned} \partial_t \hat{\hat{v}}_\pm &= -2\pi i \langle \operatorname*{res}_{\lambda_\pm} \frac{\lambda(\lambda - \lambda_\pm)}{4\pi i a^2(\lambda)} \widetilde{\Phi}^{\mathrm{A}} | \vec{\Delta} \rangle = -2\pi i \operatorname*{res}_{\lambda_\pm} (\lambda - \lambda_\pm) \widehat{\Delta}(\lambda) \\ \partial_t \hat{v}_\pm &= -2\pi i \langle \operatorname*{res}_{\lambda_\pm} \frac{\lambda}{4\pi i a^2(\lambda)} \widetilde{\Phi}^{\mathrm{A}} | \vec{\Delta} \rangle = -2\pi i \operatorname*{res}_{\lambda_\pm} \widehat{\Delta}(\lambda) \end{aligned} \tag{3.10}$$

In these calculations, one can pull the residue in $\lambda_\pm$ in front of the x-integral only *after* the differential equations for $\vec{v}$ and $\widetilde{\Phi}^{\mathrm{A}}$ have been used to replace $\vec{v}$ by $\vec{\Delta}$, because otherwise undefined intermediate terms would formally arise.

From (3.9) and the boundary conditions $v(x, t) = v(x, t + 2\pi/\omega)$, *i.e.* $\hat{v}(\lambda, t + 2\pi/\omega) = e^{2\pi i \Omega(\lambda)/\omega}\, \hat{v}(\lambda, t)$, one gets

$$\begin{aligned} \hat{v}(\lambda, t) =& (e^{2\pi i \Omega(\lambda)/\omega} - 1)^{-1} \frac{\lambda}{4\pi i a^2(\lambda)} \\ & \times \int_t^{t+2\pi/\omega} \int_{-\infty}^{\infty} e^{i\Omega(\lambda)t} \Phi_2^{\mathrm{A}}(x, t, \lambda) \Delta(u^*(x, t))\, dx\, dt \end{aligned} \tag{3.11}$$

Up to nonvanishing scaling factors, $e^{i\Omega(\lambda_n)t}\Phi_2^{\mathrm{A}}(x, t, \lambda_n)$ equals the function χ_n mentioned in the introduction, where λ_n satisfies by the condition $\Omega(\lambda_n) = n\omega$. Therefore, the fact that the necessary conditions (1.5) are satisfied for the shadow scaling means that the poles of $\left(e^{2\pi i \Omega(\lambda)/\omega} - 1\right)^{-1}$ on the real λ-axis are all compensated by zeros of the integral in (3.11) and $\hat{v}(\lambda)$ has no singularities there. Hence, after the Fourier synthesis, v will indeed be a decaying solution.

As was explained in [5], the periodicity conditions for $\hat{\dot{v}}_\pm$ give necessary conditions that are however automatically satisfied for any Δ, and the constants of integration that appear when one calculates $\hat{\dot{v}}_\pm$ from (3.10) are uniquely determined by the periodicity conditions for $\hat{v}_\pm$. The constants of integration from the calculation of $\hat{v}_\pm$ correspond to the 2-dimensional kernel of the operator $\partial_t - A$.

According to (3.9) and (3.11), $e^{-i\Omega(\lambda)t}\widehat{\Delta}(\lambda, t)$ and $e^{-i\Omega(\lambda)t}\hat{v}(\lambda, t)$ involve only even time harmonics (*i.e.* $e^{in\omega t}$ with even integer n), because Δ is an even function and Φ_2^A also involves only even time harmonics. This property of $\hat{v}(\lambda)$ survives the analytic continuation in λ, and we see that $v_0(x, t)$, which will be defined and calculated by

$$v_0(x,t) := \int_\Gamma \hat{v}(\lambda, t) e^{-i\Omega(\lambda)t} \Phi_1(x, t, \lambda)\, d\lambda \tag{3.12}$$

also involves only even time harmonics. But *a priori,* we have to calculate v from the first component of (3.8), not from (3.12), so let us see why both is equivalent (*i.e.* $v = v_0$):

Using (3.9) and integrating (3.10) shows that

$$\begin{aligned} \hat{\dot{v}}_\pm &= -2\pi i \operatorname*{res}_{\lambda_\pm} (\lambda - \lambda_\pm)\hat{v}(\lambda) + \dot{c}_\pm \\ \hat{v}_\pm &= -2\pi i \operatorname*{res}_{\lambda_\pm} \hat{v}(\lambda) + c_\pm \end{aligned} \tag{3.13}$$

for appropriate constants $\dot{c}_\pm$ and $c_\pm$. The residues on the right hand side involve only odd time harmonics [note that $\Omega(\lambda_\pm) = \pm\omega$]. Repeating the discussion of the constants of integration in [5], this parity result implies that $\hat{v}_\pm$ satisfies the periodicity conditions, if and only if $\dot{c}_\pm = 0$. In our calculation, we shall furthermore *choose* $c_\pm = 0$, thus obtaining the unique solution v_0 to equation (3.1) subject to periodicity and decay boundary conditions that contains only even time hamonics. Thus we have shown that v_0 is indeed a solution and that the general solution is $v_0 + \ker \mathcal{L}$, where $\ker \mathcal{L}$ is spanned by $\widetilde{\Phi}_+$ and $\widetilde{\Phi}_-$.

Concluding, let us emphasize that $\mathcal{L}$ [say as an operator defined in $L^2(\mathbb{R} \times S^1 \to \mathbb{R})$] is neither injective nor surjective. Its range has infinite codimension [characterized by the vanishing of the integrals in (3.11)]. The above procedure applies only to this range. The basis used in the above argument is for *fixed t* and consists of generalized (not in L^2) eigenfunctions of the operator A going from $\mathbb{R} \ni x$ to $\mathbb{R}^2$, some of which have non-periodic time evolution.

We give the detailed evaluation of (3.11), (3.12) below; the author has also done the much messier calculation according to (3.8) with $\hat{\dot{v}}_\pm$ and $\hat{v}_\pm$

evaluated independently [6]. Doing both calculations is a means of checking against calculational errors.

3.2. Effective calculation

We now prove

THEOREM 2. – *The equation* (3.1) *with boundary conditions* $v \to 0$ *as* $x \to \pm\infty$ *and* v *of period* $2\pi/\omega$ *in* t, *where* $u^*(x,t;m) = 4 \operatorname{atn} \dfrac{m \sin \omega t}{\omega \operatorname{ch} mx}$ *is the breather solution and* Δ *is the shadow scaling given in* (1.4), *has a* 2-*dimensional space* $v_0 + \ker \mathcal{L}$ *of solutions, where* $\ker \mathcal{L}$ *is spanned by the time- and space derivatives of* u^*, *and* v_0 *is even in* x *and* t *and* $\dfrac{\pi}{\omega}$ *-periodic in* t.

$v_0(x,t;m)$ *is an analytic function of* x, t, *and* m $\left(or\ \dfrac{m}{\omega}\right)$, *in the domain* $|\operatorname{Im} mx| < \dfrac{\pi}{2}$, $t \in \mathbb{C}$, $\left|\dfrac{m}{\omega}\right| < 1$. *Moreover, it satisfies the following decomposition for real* x:

$$\begin{aligned}\left(1 + \frac{m^2 \sin^2 \omega t}{\omega^2 \operatorname{ch}^2 mx}\right) v_0(x,t;m) &= \tilde{v}(e^{-2m|x|}, \omega t; m)\\ &+ \frac{-\pi e^{-|x|}}{2\omega^2 \sin \dfrac{\pi}{2m}} \left((1+m)^2 - 2m \frac{e^{-m|x|}}{\operatorname{ch} mx} - \frac{m^2 \sin^2 \omega t}{\operatorname{ch}^2 mx}\right) \end{aligned} \tag{3.14}$$

where $\tilde{v}(\zeta, \omega t; m)$ *is analytic for* $|\zeta| < 1$ *and meromorphic in* $m \neq 0$ *with poles at* $\pm m = \dfrac{1}{2}, \dfrac{1}{4}, \dfrac{1}{6}, \ldots$ *just compensating the ones introduced by the first term. Also, the nondifferentiability at* $x = 0$ *of the first term is compensated by a similar singularity of the second.*

Proof. – Most of the first part has just been explained. We have seen in the previous section that the solution v_0, selected by choosing $c_\pm = 0$, contains only even time harmonics, hence it is π/ω-periodic. The integrand in (3.11) is analytic for $|\operatorname{Im} mx| < \dfrac{\pi}{2}$, therefore $\hat{v}(\lambda, t)$ can be estimated by a constant times $e^{-c|\operatorname{Re} k(\lambda)|}$ for all $c < \pi/2m$. Therefore, according to (3.12), v is analytic in the strip specified in the theorem.

It is an easy calculation that span $\langle \widetilde{\Phi}_+, \widetilde{\Phi}_- \rangle =$ span $\langle \partial_t u^*, \partial_x u^* \rangle$.

Among the functions in $\ker \mathcal{L}$, none except the zero function contains only even harmonics, therefore v_0 is determined uniquely by this property. Now, for any solution $v(x,t)$ to $\mathcal{L} v(x,t) = \Delta(u^*)$, $v(x,-t)$ and $v(-x,t)$

are also solutions with the same harmonics, and so, v_0 (being unique) is even in x and t.

A standard calculation shows the following series expansion as a consequence of (1.4) :

$$\frac{\Delta(4\,\mathrm{atn}\,z)}{1+z^2} = \sum_{\substack{p=2 \\ p \text{ even}}}^{\infty} \Delta_p z^p\,, \qquad \Delta_p = (-1)^{p/2}(p+1)^2 \sum_{\substack{j=2 \\ j \text{ even}}}^{p} \frac{1}{j} \tag{3.15}$$

(In fact, the shadow scaling was constructed through this expansion in [5], and (1.4) is a consequence. We have reversed this order only for the exposition.) Evaluation of $\hat{v}$ proceeds by inserting this series expansion into (3.11). A side effect of this is that $\hat{v}$, which is known to be regular on the real axis as a consequence of the shadow scaling's satisfying the necessary conditions of first order perturbation theory, is written as a series whose single terms have poles on the real axis. This is the reason, why the path Γ used in (3.2) contains small half circles around the points where these poles will arise intermediately in the calculation. We know from the beginning that the poles will cancel in the end, so it is irrelevant on which side these small half circles pass.

Inserting (3.15) and the formula for Φ_2^{A}, *see* (3.5), (3.6), into (3.11), we have to evaluate integrals of the type $\int e^{ikx}\,\mathrm{ch}^{-p}\,mx\,dx$ and $\int e^{i\Omega t}\cos^p \omega t\,dt$. The integrals with $\sin\omega t$ and $\mathrm{sh}\,mx$ that also appear can be reduced to the former ones by integration by parts. We finally find that

$$\begin{aligned}
\hat{v}(\lambda,t) = {}& \frac{\lambda^2}{16ia^2(\lambda)n^2(\lambda)} \sum_p \frac{\Delta_p(2i\omega)^{-p}}{(p+1)!} \frac{k}{\mathrm{sh}\,\frac{\pi k}{2m}} \prod_{r=2}^{p-2}\left[k^2+r^2m^2\right] \\
& \times \Bigg\{ (p+1)\left(2k^2 - p(k^2-m^2)\right) \sum_{j=0}^{p} (-1)^j \binom{p}{j} \frac{e^{i(\Omega+(p-2j)\omega)t}}{\Omega+(p-2j)\omega} + \\
& + (k^2+m^2p^2)\frac{\Omega^2+\omega^2}{4\omega^2} \sum_{j=0}^{p+2}(-1)^j \binom{p+2}{j} \frac{e^{i(\Omega+(p+2-2j)\omega)t}}{\Omega+(p+2-2j)\omega} - \\
& - \frac{(k^2+m^2p^2)\Omega}{2\omega} \sum_{j=0}^{p+1} (-1)^j \binom{p+1}{j} \\
& \qquad \times \left(\frac{e^{i(\Omega+(p-2j)\omega)t}}{\Omega+(p-2j)\omega} + \frac{e^{i(\Omega+(p+2-2j)\omega)t}}{\Omega+(p+2-2j)\omega} \right) \Bigg\}
\end{aligned} \tag{3.16}$$

Here, p and r run over even positive integers only. We have suppressed the argument λ from k and Ω. The series converges for $\left|\frac{m}{\omega}\right| < 1$, because the series $\sum_p \Delta_p z^p$ converges for $|z| < 1$.

In this formula, the terms of the type $e^{i(\Omega+q\omega)t}/(\Omega + q\omega)$ carry the poles, which we know have to cancel after the summation over p. The term $k/\mathrm{sh}\,\frac{\pi k}{2m}$ carries the decay properties with respect to λ [remember $k = k(\lambda)$]. Everything else is just Nature's Camouflage: mess we have to struggle through but that should not divert the readers' attention.

In order to calculate $v(x,t)$ from (3.12) and (3.16), we intend to change the variable of integration from λ to $k = k(\lambda) = 2\lambda - 1/8\lambda$. Since $\lambda \mapsto k(\lambda)$ is two-to-one on $\mathbb{R} \setminus \{0\}$, but one-to-one on each half-line, we split the integral in halves. Φ_1 contains a factor $\lambda/4n^2(\lambda)$, which goes together with the first factor in the formula for $\hat{v}(\lambda, t)$. We use that $a = \bar{n}/n$ according to (3.4) and that

$$\frac{\lambda^3\, d\lambda}{64in^2(\lambda)\bar{n}^2(\lambda)} = \frac{dk/\Omega}{4i(k^2+m^2)^2}$$

where $\Omega = 2\lambda + 1/8\lambda = \pm\sqrt{k^2+1}$ with the $\pm$ sign for $\lambda \gtrless 0$. So, (k, Ω) lives on a 2-sheeted cover of the complex k-plane (branched at $k = \pm i$), and the two halves of Γ (in the λ-plane) map into two paths $\Gamma_\pm$ on this 2-sheeted cover, both of which lie above the same path Γ_0, which goes from $-\infty$ to $+\infty$ in the complex k-plane.

We do another manipulation: the binomial coefficients $\binom{p}{j}$ in (3.16) are defined to be 0 for $j > p$ or $j < 0$, which acquits us of bothering about the limits of the sum. We substitute $j = \frac{p}{2} - q$ or $j = \frac{p}{2} - q + 1$ in these sums in order to collect terms with the same denominator $\Omega + 2q\omega$. This gives us:

$$\begin{aligned} v(x,t) =& \left(1 + \frac{m^2s^2}{\omega^2C^2}\right)^{-1} \sum_p \frac{\Delta_p}{(p+1)!}\left(\frac{1}{\omega}\right)^p \frac{1}{(2i)^p} \sum_q (-1)^{p/2-q} \\ &\times \int_{\Gamma_-+\Gamma_+} \frac{dk/\Omega}{4i(k^2+m^2)^2} \frac{e^{-ikx}k}{\mathrm{sh}\,\dfrac{\pi k}{2m}} \prod_{r=2}^{p-2} \left[k^2 + r^2m^2\right] \\ &\times \left(-2mk\frac{S}{C} - \frac{2m^2cs}{C^2}\frac{\Omega}{\omega} - i(k^2-m^2) - i\frac{m^2s^2}{\omega^2C^2}(\Omega^2+\omega^2)\right) \end{aligned}$$

$$\times \left\{ (p+1)\Big(2k^2 - p(k^2-m^2)\Big)\binom{p}{p/2-q}\frac{e^{2i\omega qt}}{\Omega+2q\omega} - \right.$$
$$-\frac{1}{4\omega^2}(k^2+m^2p^2)(\Omega^2+\omega^2)\binom{p+2}{p/2-q+1}\frac{e^{2i\omega qt}}{\Omega+2q\omega} -$$
$$-\frac{1}{2}(k^2+m^2p^2)\frac{\Omega}{\omega}\binom{p+1}{p/2-q}\frac{e^{2i\omega qt}}{\Omega+2q\omega} +$$
$$\left. + \frac{1}{2}(k^2+m^2p^2)\frac{\Omega}{\omega}\binom{p+1}{p/2-q+1}\frac{e^{2i\omega qt}}{\Omega+2q\omega} \right\}$$

Here again, we have set $S = \operatorname{sh} mx$, $C = \operatorname{ch} mx$, $s = \sin\omega t$, $c = \cos\omega t$. Now add the contributions from Γ_+ and Γ_- under the integral. By this, the term under the integral will become symmetric under the change $\Omega \to -\Omega$, *i.e.* it will depend only on $\Omega^2 = k^2+1$, but not on Ω itself. Then, the integral in the 2-sheeted cover of the k-plane reduces to one over Γ_0 in the k-plane.

Moreover, since q runs from $-\infty$ to $+\infty$, we can replace each term in the sum by its even part in q. We get:

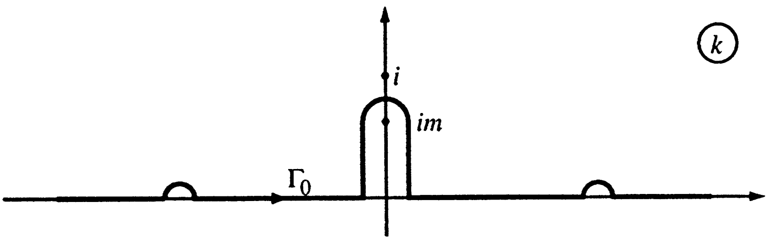

FIG. 2. – The path of integration Γ_0.

$$v(x,t) = \left(1+\frac{m^2s^2}{\omega^2C^2}\right)^{-1}\sum_p\sum_q \frac{(-1)^{p/2-q}\Delta_p\,(2i\omega)^{-p}}{4i\left(\frac{p}{2}-q+1\right)!\left(\frac{p}{2}+q+1\right)!}$$
$$\int_{\Gamma_0}\frac{e^{-ikx}k\,dk}{(k^2+m^2)^2\operatorname{sh}\dfrac{\pi k}{2m}}\prod_{r=2}^{p-2}[k^2+r^2m^2]\frac{1}{\Omega^2-4q^2\omega^2}$$
$$\times\left\{\frac{m^2}{\omega C^2}\left[\Big(2k^2-p(k^2-m^2)\Big)\left(\frac{p}{2}-q+1\right)\left(\frac{p}{2}+q+1\right)-\right.\right.$$
$$-\frac{1}{4\omega^2}(k^2+m^2p^2)(\Omega^2+\omega^2)(p+2)-$$
$$\left.-(k^2+m^2p^2)\frac{\Omega^2}{2\omega^2}\right]4i\omega q\sin 2\omega qt\sin 2\omega t+$$

$$
\begin{aligned}
&+ \left(-2mk\frac{S}{C} - i(k^2 - m^2) - i\frac{m^2 \sin^2 \omega t}{\omega^2 C^2}(\Omega^2 + \omega^2) \right) \\
&\times \Bigg[(2k^2 - p(k^2 - m^2))\Big(\frac{p}{2} - q + 1\Big)\Big(\frac{p}{2} + q + 1\Big) - \\
&\qquad - \frac{1}{4\omega^2}(k^2 + m^2 p^2)(\Omega^2 + \omega^2)(p+2) - \\
&\qquad - 2q^2(k^2 + m^2p^2) \Bigg] 2 \cos 2\omega q t \Bigg\}
\end{aligned}
\tag{3.17}
$$

where $\Omega^2 = k^2 + 1$.

Here, only the line containing the integral sign displays important qualitative features, the rest being Nature's Camouflage again. For $x < 0$, we shall close the path of integration Γ_0 in the upper half plane. This is possible, because the integrand is a meromorphic function of k in the whole plane. For $|\operatorname{Re} k| \geq |\operatorname{Im} k|$, the sh $\dfrac{\pi k}{2m}$ term in the denominator ensures a fast decay of the integrand as $k \to \infty$. For $|\operatorname{Re} k| \leq \operatorname{Im} k$ and $x < 0$, the e^{-ikx} term ensures a similar decay provided the path keeps outside neighbourhoods of the poles on the imaginary axis. Therefore, for $x < 0$, we can evaluate the integral into a sum of residues in the upper half plane. For $x > 0$, we could close the path in the lower half plane, but we simply make use of the fact that v is even in x. There are three types of contributions to be discussed:

• The poles on the real axis (or between i and $-i$ on the imaginary axis, if qm is small), due to $1/(k^2 + 1 - 4q^2m^2)$ for $q \neq 0$, still cancel after the summation over p, or in other words they do not bother anyway, since Γ_0 can be chosen not to include them.

• But for $q = 0$, the same term gives a pole in $k = \pm i$: its residue yields a term containing $e^{-|x|}$. It is this term that will play the basic role in the later discussion, and it will be calculated explicitly.

• There are also poles from $\dfrac{k}{\operatorname{sh} \pi k/2m}$: they give terms containing $e^{-2\ell m|x|}$, which add up to $\tilde{v}$ in the theorem, and we shall not calculate them.

One of these latter uninteresting poles might coalesce with the pole at i to a second order pole. Then, the decomposition just explained becomes singular, but the integral itself does not. This happens for $1 \in 2m\mathbb{Z}$. Moreover, closing the path introduces the distinction according to the sign of x. The single terms are non-differentiable at $x = 0$. This non-differentiability will also cancel when the single residual contributions are added, it is only our method of evaluation that hides this fact.

Let us calculate the contribution from the pole at $k = i$. To this end, simply choose the $q = 0$ term above and calculate the residue; for $x < 0$, we get the contribution:

$$\left(1+\frac{m^2s^2}{\omega^2C^2}\right)^{-1}\frac{e^x\pi}{\omega^4\sin\dfrac{\pi}{2m}}\sum_p\frac{\Delta_p\,(-1)^{p/2}}{2(2i\omega)^p\left(\frac{p}{2}+1\right)!^2}\prod_{r=2}^{p-2}\left[r^2m^2-1\right]$$
$$\times\left(1+m^2-2m\frac{S}{C}-\frac{m^2s^2}{C^2}\right)$$
$$\times\left[(-2+p(m^2+1))\left(\frac{p}{2}+1\right)^2-\frac{1}{4}(m^2p^2-1)(p+2)\right]$$

We claim that this simplifies to

$$-\frac{\pi e^x}{2\omega^2\sin\dfrac{\pi}{2m}}\left(1+\frac{m^2s^2}{\omega^2C^2}\right)^{-1}\left(1+m^2-2m\frac{S}{C}-\frac{m^2s^2}{C^2}\right)$$

In fact, one evaluates the sum over p explicitly in the following way: Set $p = 2n$, $r = 2l$, $m^2 = z$ for simplification and use from (3.15) that

$$\Delta_{2n} = \frac{(-1)^n}{2}(2n+1)^2\sum_{j=1}^{n}\frac{1}{j} =: \frac{(-1)^n}{2}(2n+1)^2h_n$$

Pulling in front of the sum all the terms that do not depend on p, we come to use the left hand side of the formula in the following lemma. With it, the claimed equality follows immediately. The lemma is proved in the next section.

LEMMA 3. – *Let* $h_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$. *Then, for* $\left|\frac{z}{1-z}\right| < 1$, *it holds*

$$\sum_{n=1}^{\infty}\frac{(-1)^n(2n+1)^2\,(4n^2+4nz-3)\,h_n}{n!\,(n+1)!\,(1-z)^{n+1}}\prod_{l=1}^{n-1}\left[l^2z-\frac{1}{4}\right] = -16 \quad (3.18)$$

This last calculation was valid for $x < 0$; we know already, that v is even in x. So simply replace x by $-|x|$ and get (3.14). This proves the theorem relative to the last lemma. ■

3.3. Proof of Lemma 3

We start by stating an auxiliary formula:

$$\sum_{n=0}^{N} \frac{(-1)^n (2n+1)^2\,(4n^2+4nz-3)}{n!\,(n+1)!\,(1-z)^n} \prod_{l=0}^{n-1} \left[l^2 z - \frac{1}{4}\right] = \frac{(-1)^{N+1}(2N+1)(2N+3)}{N!\,(N+1)!\,(1-z)^N} \prod_{l=1}^{N} \left[l^2 z - \frac{1}{4}\right] \tag{3.19}$$

It is straightforward to prove this by induction over N.

We need (3.19) in order to do a partial summation in (3.18). The partial summation yields

$$\sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)^2\,(4n^2+4nz-3)\,h_n}{n!\,(n+1)!\,(1-z)^{n+1}} \prod_{l=0}^{n-1} \left[l^2 z - \frac{1}{4}\right] = -\sum_{n=1}^{\infty} \frac{1}{n}\,\frac{(-1)^n(2n-1)(2n+1)}{(n-1)!\,n!\,(1-z)^n} \prod_{l=1}^{n-1} \left[l^2 z - \frac{1}{4}\right]$$

and we have to prove that this equals 4. (Note the different lower limits in the products over l.) To this end, we claim, similarly to (3.19), that

$$\sum_{n=0}^{N} \frac{(-1)^n(2n-1)(2n+1)}{n!^2\,(1-z)^n} \prod_{l=0}^{n-1} \left[l^2 z - \frac{1}{4}\right] = \frac{(-1)^{N+1}}{N!^2(1-z)^N} \prod_{l=1}^{N} \left[l^2 z - \frac{1}{4}\right]$$

which again is proved by straight-forward induction; letting $N \to \infty$ and isolating the $n = 0$ term proves the lemma. ■

4. CONDITIONS FROM SECOND ORDER PERTURBATION THEORY

Having isolated the important contribution of the solutions to $\mathcal{L}v = \Delta(u^*)$ in Theorem 2, we can now insert v into the equations of second order. We derive these equations in section 4.1, then derive from them conditions that do not depend on the "uninteresting" part $\tilde{v}$ of v in section 4.2, and state an algorithm for checking these conditions in section 4.4.

4.1. The second order equations and basic conventions

Just to be definite about the periodicity and decay assumptions, let

$$\mathbf{X} = \left\{ u(\cdot,\cdot) \in C^2(\mathbb{R} \times \mathbb{R}/2\pi\mathbb{Z}) \;\middle|\; |u(x,t)| + |u_x(x,t)| + |u_t(x,t)| \le \frac{c}{1+x^2} \right\}$$

The smallest possible c can be taken as a norm, making $\mathbf{X}$ a (dense) subspace of a Banach space (of weighted C^1-functions). The particular choice of the function space is not really important for the following arguments, however. We say sloppily that a T-periodic (in the second argument) function u is in $\mathbf{X}$, if the 2π-periodic function $(x,\tau) \mapsto u\left(x, \tau\frac{T}{2\pi}\right)$ is in $\mathbf{X}$.

LEMMA 4 (Second order perturbation theory). – *Consider* $(1.1)_\varepsilon$, *and assume that* $(u,\varepsilon) \mapsto \varepsilon\Delta(u,\varepsilon)$ *is* C^2 *with respect to both variables. Write*

$$\varepsilon\Delta(u,\varepsilon) = \varepsilon\Delta(u) + \varepsilon^2\Delta^{[2]}(u) + o(\varepsilon^2)$$

Assume that the breather for $m = m_0$ *persists under the perturbation, in other words, that for* $0 \le \varepsilon < \varepsilon_{\max}$, *there exists a* $T(\varepsilon)$*–periodic (in* t*) classical solution* $u(x,t,\varepsilon)$ *to* $(1.1)_\varepsilon$ *satisfying* $u(x,t,0) = u^*(x,t;m_0)$, *such that* $T(\cdot)$ *is continuous and the map from* ε *to the function* $(x,t) \mapsto u\left(x, t\frac{2\pi}{T}, \varepsilon\right)$ *is in* $C^2([0,\varepsilon_{\max}[\to \mathbf{X})$.

Moreover, assume that one can solve (3.1) *for all* m *in a whole real neighbourhood of* m_0, i.e. *that we have* $T(\varepsilon)$*-periodic solutions* $v(x,t;m)$ *in* $\mathbf{X}$ *to*

$$(\partial_t^2 - \partial_x^2 + \cos u^*(x,t;m))\, v(x,t;m) = \Delta(u^*(x,t;m))$$

Then, for $m(\varepsilon) = \sqrt{1 - \left(2\pi/T(\varepsilon)\right)^2}$,

$$v^{[2]}(x,t) := \partial_\varepsilon^2|_{\varepsilon=0}\left(u(x,t,\varepsilon) - u^*(x,t;m(\varepsilon)) - \varepsilon v(x,t;m(\varepsilon))\right)$$

solves the equation (2.1), i.e.

$$\mathcal{L}v^{[2]} = \Delta^{[2]}(u^*) + \Delta'(u^*)v + \frac{1}{2}(\sin u^*)v^2 \tag{4.1}$$

where u^* *and* v *stand for the corresponding functions at* $m = m_0$.

Proof. – We can write

$$u(x,t,\varepsilon) = u^*(x,t,m(\varepsilon)) + \varepsilon v(x,t,m(\varepsilon)) + \varepsilon^2 v^{[2]}(x,\tau) + o(\varepsilon^2) \quad (4.2)$$

Inserting this into $(1.1)_\varepsilon$, the $O(\varepsilon^2)$-contribution of the resulting equation is just the claimed condition. ■

Remember that for $\left|\frac{m}{\omega}\right| < 1$, there is a bijective correspondence between any two of $\frac{m}{\omega}$, m, and ω. We shall refer to points in the (unit disc of the) $\frac{m}{\omega}$ -plane by giving either of these coordinates without further comment. Also remember that $k(\lambda_n) = \sqrt{n^2\omega^2 - 1}$ (either of both values of the root).

In Theorem 1, we have assumed that $\Delta^{[2]}$ is analytic in a neighbourhood of $\left\{u \,\middle|\, \left|\tan\frac{u}{4}\right| \leq \rho \leq 1\right\}$. Let χ_n be the bounded $2\pi/\omega$-periodic solutions to $\mathcal{L}\chi = 0$ mentioned in the introduction and given exactly by

$$\chi_n(x,t) = e^{i\Omega(\lambda_n)t}\Phi_2^{\mathrm{A}}(x,t,\lambda_n) \times \left(\frac{\lambda_n m^2}{n(\lambda_n)^2}\right)^{-1} \quad (4.3)$$

where λ_n satisfies $\Omega(\lambda_n) = n\omega$ and $\Phi_2^{\mathrm{A}}(x,t,\lambda_n)$ is given by (3.5), (3.6), and the last factor is chosen just for convenience. This leaves two choices for λ_n, which can be distinguished by the sign of $k(\lambda_n)$, but nothing of the following will depend on this sign. Then, we can test equation (4.1) with χ_n to obtain

$$-\int \Delta^{[2]}(u^*)\chi_n\,dx\,dt = \int\left(\Delta'(u^*)v + \frac{1}{2}(\sin u^*)v^2\right)\chi_n\,dx\,dt \quad (4.4)$$

The right hand side of this equation reduces to 0 for even n because of the t-integration; so we choose n odd in all of the following. Suppose that equation (4.4) holds for infinitely many values of $\frac{m}{\omega}$ that accumulate in $]0,\rho[$. It is known [1], [5] that for odd n, $\operatorname{ch}\pi k(\lambda_n)/2m$ times the left hand side of this equation can be continued to an analytic function of $\frac{m}{\omega}$ in the disc $\left|\frac{m}{\omega}\right| < \rho$.

Therefore so can the right hand side. Let us state this formally:

LEMMA 5. – *If* (4.1) *has a* $2\pi/\omega$*-periodic decaying solution for infinitely many values of* $\frac{m}{\omega}$ *in some interval* $0 < \rho'' \leq \frac{m}{\omega} \leq \rho' < \rho$, *then the function*

$$\mathcal{V}_n\left(\frac{m}{\omega}\right) := \int\left(\Delta'(u^*)v + \frac{1}{2}(\sin u^*)v^2\right)\chi_n\,dx\,dt$$

originally defined on $\frac{m}{\omega} \in \,]0, \rho[$, *can be continued to an analytic function in the disc* $\left|\frac{m}{\omega}\right| < \rho$, *except for poles at the zeros of the function* $\operatorname{ch} \frac{\pi\sqrt{n^2\omega^2 - 1}}{2m}$ *and an essential singularity at* $\frac{m}{\omega} = 0$.

To completely prove Lemma 5, consider the analyticity properties of the right hand side of (4.4). The term in parentheses decays exponentially, namely at least as fast as $e^{-(1+|\operatorname{Re} m|)|x|}$, so the right hand side is analytic as long as $|\operatorname{Im}\sqrt{n^2\omega^2 - 1}| < 1 + |\operatorname{Re} m|$, in particular in a small complex neighbourhood of the real interval $0 < \frac{m}{\omega} < \infty$. This function on the right agrees with the one on the left, which is analytic in a neighbourhood of $0 < \frac{m}{\omega} < \rho$ according to [1], [5], in a set of points that accumulates in the interior of the common domain of analyticity. This allows to do the analytic continuation and thus concludes the proof of Lemma 5. ■

Note. – For the more general theorem stated in the note after Theorem 1, a corresponding result on the domain of analyticity of the left hand side of (4.4) has to be cited from first order perturbation theory. *See* section 4 of [5].

There is a subtlety in the assumptions here that is not needed in first order perturbation theory: The point $\frac{m}{\omega} = 0$ is in the interior of the domain of analyticity of the function

$$\operatorname{ch} \frac{\pi k(\lambda_n)}{2m} \int \Delta^{[2]}(u^*)\, \chi_n \, dx\, dt$$

by the above-mentioned non-obvious result of first order perturbation theory. But 0 must be expected to be on the boundary of the domain of analyticity of $\operatorname{ch} \frac{\pi k(\lambda_n)}{2m}$ times the right hand side of (4.4). Therefore, 0 is not sufficient as a point of accumulation for the analytic continuation argument. It is for this purpose that ρ'' has been introduced in the hypotheses. This detail has erroneously been omitted in Theorem 2.5 and Lemmas 5.4, 5, 8 of [6]. This extra hypothesis may well be an artefact of our method of proof; but we simply cannot get rid of it.

4.2. Isolating the relevant vanishing pole conditions

We use the abbreviation $k_n := k(\lambda_n) = \pm\sqrt{n^2\omega^2 - 1}$. Occasionally, we will find it convenient to reinstate the dependence $k_n = k_n\left(\frac{m}{\omega}\right)$ that was suppressed until now.

Before continuing, let us note that the poles when $\pm ik_n = pm$ (discussed in the last lemma) appear only with $p \geq n$: in all terms that enter under the integral $\int \Delta(u^*)\chi_n \, dx \, dt$, every power of $1/\operatorname{ch} mx$ is multiplied by time harmonics up to an order no bigger than the power to which $1/\operatorname{ch} mx$ is raised. The factor $e^{in\omega t}$ hidden in χ_n has the effect that during the t-integration it kills all the terms containing the factor $1/\operatorname{ch}^p mx$ with $p < n$, because their time dependent coefficient does not contain a Fourier component $e^{-in\omega t}$. Therefore, the integral over t and x will be an analytic function of $\frac{m}{\omega}$ in the domain defined by $|\operatorname{Im} k_n| < nm$. This argument will be used repeatedly below.

Remember that n is chosen to be odd. The following sets of points will play a role:

- $D(\rho) := \left\{ \frac{m}{\omega} \,\middle|\, \left| \frac{m}{\omega} \right| < \rho, \ \frac{m}{\omega} \neq 0 \right\}$,
the punctured disc.
- $P_n := \left\{ \frac{m}{\omega} \,\middle|\, \pm ik_n = pm \text{ for some odd } p \geq n \right\}$,
the points where poles appear in first order perturbation theory.
- $P'_n := \left\{ \frac{m}{\omega} \,\middle|\, \pm ik_n = 1 + pm \text{ for some odd } p \geq n \right\}$,
the points where poles may appear due to the $e^{-|x|}$-terms.
- $P''_n := \left\{ \frac{m}{\omega} \,\middle|\, \pm ik_n = 2 + pm \text{ for some odd } p \geq n \right\}$,
the points where poles may appear due to the $e^{-2|x|}$-terms.
- $P^* := \left\{ \frac{m}{\omega} \,\middle|\, 1/m \in 2\mathbb{Z} \right\}$,
the points where artificial singularities are introduced by the splitting into interesting and non-interesting terms according to Theorem 2.

Lemma 5 implies that under its hypotheses, $\mathcal{V}_n$ is analytic in $D(\rho) \setminus P_n$. Here, the above argument for the time harmonics (and a parity argument of the same type) are responsible that only odd $p \geq n$ lead to points that must be taken out from $D(\rho)$.

Our goal will be to eliminate from $\mathcal{V}_n$ those terms of v that have not been calculated in Theorem 2. After this procedure, only the term that contains $e^{-2|x|}$ will be left over. Some of the eliminated terms contain a factor $e^{-|x|}$ and may therefore be responsible for singularities in points of P'_n. The detailed argument gives us the following

LEMMA 6. – *If* (4.1) *has a* $2\pi/\omega$*-periodic decaying solution for infinitely many values of* $\frac{m}{\omega}$ *in some interval* $0 < \rho'' \leq \frac{m}{\omega} \leq \rho' < \rho$*, then the function*

$$\mathcal{W}_n\left(\frac{m}{\omega}\right) := \int \left(1 + \frac{m^2 \sin^2 \omega t}{\omega^2 \operatorname{ch}^2 mx}\right)^{-2} (\sin u^*) w^2 e^{-2|x|} \chi_n \, dx \, dt \quad (4.5a)$$

with

$$w(x,t;m) = (1+m)^2 - 2m \frac{e^{-m|x|}}{\operatorname{ch} mx} - \frac{m^2 \sin^2 \omega t}{\operatorname{ch}^2 mx} \quad (4.5b)$$

can be continued to an analytic function in $D(\rho) \setminus (P_n \cup P_n' \cup P^*)$.

On the other hand, the singularities of $\mathcal{W}_n$ can immediately be discussed from its formula. The result is

LEMMA 7. – *The function* $\mathcal{W}_n$ *defined in Lemma 6 can be continued to an analytic function in* $D(1) \setminus P_n''$. $\mathcal{W}_n$ *has either a simple pole or no singularity at all in each of the points of* P_n''.

Similarly, the function $\kappa \mapsto \mathcal{W}_n\left(\frac{m}{\omega}, \kappa\right)$ *defined exactly as* $\mathcal{W}_n\left(\frac{m}{\omega}\right)$ *in* (4.5a) *except that the term* $e^{ik_n(m/\omega)\,x}$*, which is implicit in* χ_n*, is replaced by* $e^{i\kappa x}$ *with an independent variable* κ*, is meromorphic for* $\kappa \in \mathbb{C}$ *with at most simple poles at the points* $\kappa = \pm i(2 + \ell m)$ *with* ℓ *odd and positive. When* $\frac{m}{\omega}$ *is a solution to* $k_n\left(\frac{m}{\omega}\right) = \pm i(2 + \ell m)$*, these poles with respect to* κ *have non-vanishing residues, if and only if the pole of* $\mathcal{W}_n(\cdot)$ *at that value of* $\frac{m}{\omega}$ *has non-vanishing residue.*

All lemmas will be proved in the next section.

According to Lemma 8 below, the sets $P_n \cup P_n'$ and P_n'' are disjoint. Therefore we conclude, combining Lemmas 6 and 7, that under the stated persistence assumption the residues of the simple poles of $\mathcal{W}_n$ in each of those points in P_n'' for which $\left|\frac{m}{\omega}\right| < \rho$ have to vanish (*i.e.* there are no poles at all). These are the vanishing pole conditions we are going to check. Showing that some of them are violated will prove Theorem 1.

In particular, note that these vanishing pole conditions do not contain the second order perturbation function $\Delta^{[2]}$ any more. The only trace that is left of it is the parameter ρ that describes its domain of analyticity. Only this qualitative feature enters in our argument.

Let us now state the lemma on the point sets P_n' and P_n'' in detail.

LEMMA 8. – *Given any* $n \geq 3$ odd. *The points in* P'_n, i.e. *the solutions to* $\pm ik_n = 1 + pm$ *for some* odd $p \geq n$ *are given by*

$$m'_\pm(p,n) = \frac{-p \pm \sqrt{p^2 - (p^2 - n^2)n^2}}{p^2 - n^2} = \frac{-p \pm i\sqrt{(n^2-1)(p^2-n^2-1) - 1}}{p^2 - n^2} \tag{4.6}$$

if $p > n$, *and* $m'(n,n) = -n/2$.

The points in P''_n, i.e. *the solutions to* $\pm ik_n = 2 + \ell m$ *for some* odd $\ell \geq n$ *are given by*

$$m''_\pm(\ell,n) = \frac{-2\ell \pm \sqrt{4\ell^2 - (\ell^2 - n^2)(n^2+3)}}{\ell^2 - n^2} = \frac{-2\ell \pm 2i\sqrt{(n^2-1)\left(\dfrac{\ell^2-n^2}{4} - 1\right) - 1}}{\ell^2 - n^2} \tag{4.7}$$

for $\ell > n$, *and* $m''(n,n) = -(n^2+3)/4n$.

In particular, $m'(n,n)$ *and* $m''(n,n)$ *lie outside the disc* $\left|\dfrac{m}{\omega}\right| < 1$, *and the other points lie neither on the real nor on the imaginary axis. Moreover, the families* P'_n *and* P''_n *are disjoint for every fixed odd* n.

For $m = m''_\pm(\ell,n)$ *defined by* (4.7), *it holds*

$$\frac{m^2}{\omega^2} = \frac{8\ell^2 - (\ell^2+3)(n^2+3) \mp 4\ell\sqrt{4\ell^2 - (\ell^2-n^2)(n^2+3)}}{(\ell^2-1)(\ell^2-9)} \tag{4.8a}$$

$$\left|\frac{m}{\omega}\right|^4 = \frac{(n^2+3)^2}{(\ell^2-1)(\ell^2-9)} \tag{4.8b}$$

4.3. Proof of technical Lemmas

Proof of Lemma 6. – For convenience, we introduce the abbreviation $Q := \dfrac{m \sin \omega t}{\omega \operatorname{ch} mx}$.

Let us first get rid of the contribution $\tilde{v}$ to v, which has not been calculated in Theorem 2. To this end, write (according to this theorem)

$$v(x,t;m) = (1+Q^2)^{-1}(e^{-|x|}w(x,t;m) + \tilde{v}(e^{-2m|x|},t;m))$$

Remember that the important term in χ_n is $e^{ik_n x}$. The powers of $1/\operatorname{ch} mx$ that appear in χ_n and u^* can be expressed in terms of $e^{-|mx|}$ according to the formula

$$\frac{1}{\operatorname{ch}^s mx} = 2^s \sum_j \binom{-s}{j} e^{-(s+2j)|mx|} \qquad (mx \neq 0 \text{ real}) \tag{4.9}$$

In u^*, χ_n and $\tilde{v}$, x enters always in the combination $mx =: \xi$, except in the oscillating term $e^{ik_n x}$, and t enters always in the combination $\omega t =: \tau$. With this, one has an expansion of the type

$$\begin{aligned}\Big((1+Q^2)^{-1}\Delta'(u^*)\tilde{v} + \left(1+Q^2\right)^{-2}\frac{1}{2}(\sin u^*)\tilde{v}^2\Big)\,\chi_n \\ = \sum_{p\,\mathrm{odd}} (a + b \operatorname{th}\xi)\, e^{-p|\xi|} e^{i(k_n/m)\xi}\end{aligned} \tag{4.10}$$

where a and b are analytic functions of τ and $\frac{m}{\omega}$ (except for poles at the points where $1/m \in 2\mathbb{Z}$). The series converges for real $\xi \neq 0$ provided $1/m \notin 2\mathbb{Z}$ and $\frac{m}{\omega} < 1$. Its convergence for $\xi = 0$ is dubious, but we need not care: all that is needed is that the series is asymptotic as $\xi \to \pm\infty$. Consequently, cut off the series at p_0. The remainder is the oscillating term $e^{i(k_n/m)\xi}$ times a function that decays like $e^{-p_0|\xi|}$. By a standard estimate, its integral over ξ is an analytic function of $\frac{m}{\omega}$ in the domain

$$D_{p_0} := \left\{ \frac{m}{\omega} \in D(1) \setminus P^* \;\middle|\; |\operatorname{Im} k_n/m| < p_0 \right\}$$

The same property still holds after integrating this over τ from 0 to 2π.

What analyticity properties do the integrals of the single terms in the sum have? When integrating over ξ from $-\infty$ to $+\infty$, only the even part of the integrand survives. One can integrate twice this even term from 0 to ∞ instead. But for $\xi > 0$, it holds

$$\operatorname{th}\xi = 1 - 2e^{-2\xi} + 2e^{-4\xi} - 2e^{-6\xi} + - \ldots \tag{4.11}$$

So, all that remains are terms $e^{-p\xi}e^{\pm i(k_n/m)\xi}$. Using

$$\int_0^\infty e^{-s\xi} e^{\pm i\kappa\xi}\, d\xi = \frac{1}{s \mp i\kappa} \tag{4.12}$$

one sees that the integrals of the single terms are analytic functions of $\frac{m}{\omega}$ except for simple poles in points $\frac{m}{\omega} \in P_n$. Therefore, for every p_0, it holds that the integral of (4.10) over ξ and τ (which is $1/m\omega$ times the analytic continuation of the integral over x and t) is analytic in the domain $D_{p_0} \setminus P_n$. One can therefore subtract this term from the right hand side of (4.4) and gets that

$$\int \Big(\left(1+Q^2\right)^{-2} \frac{1}{2} \sin u^* (2e^{-|x|} w\tilde{v} + e^{-2|x|} w^2) + \left(1+Q^2\right)^{-1} \Delta'(u^*) e^{-|x|} w \Big) \chi_n \, dx \, dt$$

can be continued to an analytic function in $D(\rho) \setminus (P_n \cup P^*)$.

There is still one term left that contains $\tilde{v}$. This time, the presence of $e^{-|x|}$ in front of it makes it useless (but also unnecessary) to scale from (t, x) to (τ, ξ) for the purpose of analytic continuation. The other term containing $e^{-|x|}$, namely $\Delta'(u^*) w e^{-|x|}$, is responsible for the same qualitative features as $(\sin u^*) e^{-|x|} w\tilde{v}$, and these features are therefore hidden by the unknown term: We have to get rid of both terms containing $e^{-|x|}$. Expanding as above, one finds

$$\begin{aligned} &\{\left(1+Q^2\right)^{-1} \Delta'(u^*) w + \left(1+Q^2\right)^{-2} (\sin u^*) \tilde{v} w\} e^{-|x|} \chi_n \\ &\quad = \sum_{p \,\mathrm{odd}} (c + d \,\mathrm{th}\, mx) \, e^{-(1+pm)|x|} e^{ik_n x} \end{aligned} \tag{4.13}$$

where c and d are analytic functions in the same domain as a and b above. Again, the series converges for real $x \neq 0$, but we only use that it is asymptotic. The poles that come from the single terms in the sum are at those points where $\pm i k_n = (1 + pm)$ for some odd $p \geq n$, and the domain of analyticity of the remainder term for cutoff at $p = p_0$ is

$$D'_{p_0} := \left\{ \frac{m}{\omega} \in D(1) \setminus (P'_n \cup P^*) \;\middle|\; |\operatorname{Im} k_n| < (1 + p_0 |\operatorname{Re} m|) \right\}$$

Note that without the 1 in the term $1 + p_0 |\operatorname{Re} m|$, this would exclude the whole imaginary axis. This is why we had to introduce ξ in the case of the "pure" $\tilde{v}$-terms before continuing analytically. (This scaling corresponds to turning the path of x-integration by minus the argument of the complex number m.) Here, the imaginary axis is not excluded, because for $\dfrac{m}{\omega} = iz$ with $-1 < z < 1$, one has $\omega^2 = 1/(1 - z^2) > 0$, so $|\operatorname{Im} k_n| < 1$. We stress this here, because we need to cross the imaginary axis later. The union of all D'_{p_0} is the full disc $D(1) \setminus (P'_n \cup P^*)$.

This is enough information to subtract the $e^{-|x|}$-terms from $\mathcal{V}_n$, and Lemma 6 follows.

Proof of Lemma 7. – The very same argument that was used to discuss the $e^{-|x|}$-terms immediately shows Lemma 7 without any assumptions on the solvability of (4.1), but directly from formula (4.5a). Formula (4.12) can be used as well for the variable κ as for the variable $\frac{m}{\omega}$. Although the residues are different with respect to both variables, they both vanish or both don't. We use here that the poles are simple in either case. This in turn follows from calculating them explicitly. We do not write down these straightforward calculations; they result in the formulas in Lemma 8.

Proof of Lemma 8. – Beyond these calculations, we only have to show that the families are disjoint. Suppose, they were not, *i.e.* that for some triple (ℓ, p, n) one has

$$\frac{-2\ell \pm \sqrt{4\ell^2 - (\ell^2 - n^2)(n^2+3)}}{\ell^2 - n^2} = \frac{-p \pm \sqrt{p^2 - (p^2 - n^2)n^2}}{p^2 - n^2}$$

Then the real parts and the arguments of the complex numbers on either side have to coincide, *i.e.*

$$\frac{p}{2\ell} = \frac{p^2 - n^2}{\ell^2 - n^2} \qquad \text{and} \qquad n^2\left(\frac{p^2 - n^2}{p^2}\right) - 1 = \frac{n^2+3}{4}\left(\frac{\ell^2 - n^2}{\ell^2}\right) - 1$$

The second condition can be written in the form

$$\left(\frac{p}{2\ell}\right)^2 = \frac{n^2}{n^2+3}\,\frac{p^2 - n^2}{\ell^2 - n^2}$$

Dividing this by the first condition gives

$$p(n^2+3) = 2\ell n^2$$

But for odd ℓ and n, the left hand side of this last equation is divisible by 4, whereas the right hand side is not. ■

4.4. Evaluation of the vanishing pole conditions

The calculation of the residues is elementary, but very involved. So, let us collect what we have to do. According to the above discussion and using (4.5 a, b), (4.3), (3.5) and (3.6) together with the series expansion

$$\frac{\sin(4\,\mathrm{atn}\,z)}{(1+z^2)^3} = \frac{1}{3}\sum_p (-1)^{(p-1)/2}\underbrace{\frac{(p+1)(p+3)^2(p+5)}{16}}_{=:\,\eta_p} z^p \tag{4.14}$$

we want to check whether

$$
\begin{aligned}
0 = \mathcal{W}_n\left(\frac{m}{\omega}, \kappa\right) &= \frac{m}{3\omega} \sum_p (-1)^{(p-1)/2} \eta_p \left(\frac{m^2}{\omega^2}\right)^{(p-1)/2} \\
&\times \operatorname*{res}_{\kappa=i(2+\ell m)} \int_{-\infty}^{\infty} \frac{e^{-2|x|} e^{i\kappa x}}{\operatorname{ch}^p mx} \int_0^{2\pi/\omega} B_n w^2 \sin^p \omega t \, e^{in\omega t} \, dt \, dx \qquad (4.15)
\end{aligned}
$$

where

$$
\begin{aligned}
B_n(x,t;m) = &\frac{k_n}{2m} \operatorname{th} mx + \frac{n}{4} \frac{\sin 2\omega t}{\operatorname{ch}^2 mx} \\
&- i\left[\left(\frac{k_n}{2m}\right)^2 - \frac{1}{4}\right] - i\frac{n^2+1}{4} \frac{\sin^2 \omega t}{\operatorname{ch}^2 mx} \qquad (4.16)
\end{aligned}
$$

and

$$
w(x,t;m) = (1+m)^2 - 2m \frac{e^{-m|x|}}{\operatorname{ch} mx} - \frac{m^2 \sin^2 \omega t}{\operatorname{ch}^2 mx} \qquad (4.17)
$$

Expanding $B_n w^2$ gives $4 \times 6 = 24$ separate terms. For the t-integration of each of them, we use one of

$$
\begin{aligned}
\frac{\omega}{2\pi} \int_0^{2\pi/\omega} \sin^s \omega t e^{in\omega t} \, dt &= \frac{(-1)^{(s+n)/2}}{(2i)^s} \binom{s}{(s+n)/2} \\
\frac{\omega}{2\pi} \int_0^{2\pi/\omega} \sin 2\omega t \sin^s \omega t e^{in\omega t} \, dt &= \frac{(-1)^{(s+n)/2}}{(2i)^{s+1}} \binom{s+2}{(s+n)/2+1} \frac{n}{s+2} \qquad (4.18)
\end{aligned}
$$

We shall use later that $\frac{n}{s+2}\binom{s+2}{(s+n)/2+1}$ is an integer; in fact, it is equal to $\binom{s}{(s+n)/2-1} - \binom{s}{(s+n)/2+1}$.

The x-integration of each term is not carried out completely, but only the residues are calculated with the help of (4.9) and (4.11). We find that

$$
\begin{aligned}
i \operatorname*{res}_{\kappa=i(2+\ell m)} \int_{-\infty}^{\infty} \frac{e^{-2|x|} e^{i\kappa x}}{\mathrm{ch}^s\, mx}\, dx &= (-1)^{(\ell+s)/2}\, 2^s\, L_s^\ell \\
i \operatorname*{res}_{\kappa=i(2+\ell m)} \int_{-\infty}^{\infty} \frac{e^{-(2+m)|x|} e^{i\kappa x}}{\mathrm{ch}^s\, mx}\, dx &= (-1)^{(\ell-1+s)/2}\, 2^s\, L_s^{\ell-1} \\
i \operatorname*{res}_{\kappa=i(2+\ell m)} \int_{-\infty}^{\infty} \frac{e^{-(2+2m)|x|} e^{i\kappa x}}{\mathrm{ch}^s\, mx}\, dx &= (-1)^{(\ell-2+s)/2}\, 2^s\, L_s^{\ell-2} \\
i \operatorname*{res}_{\kappa=i(2+\ell m)} \int_{-\infty}^{\infty} \frac{e^{-2|x|}\, \mathrm{th}\, mx\, e^{i\kappa x}}{\mathrm{ch}^s\, mx}\, dx &= (-1)^{(\ell+s)/2}\, 2^s\, G_s^\ell \\
i \operatorname*{res}_{\kappa=i(2+\ell m)} \int_{-\infty}^{\infty} \frac{e^{-(2+m)|x|}\, \mathrm{th}\, mx\, e^{i\kappa x}}{\mathrm{ch}^s\, mx}\, dx &= (-1)^{(\ell-1+s)/2}\, 2^s\, G_s^{\ell-1} \\
i \operatorname*{res}_{\kappa=i(2+\ell m)} \int_{-\infty}^{\infty} \frac{e^{-(2+2m)|x|}\, \mathrm{th}\, mx\, e^{i\kappa x}}{\mathrm{ch}^s\, mx}\, dx &= (-1)^{(\ell-2+s)/2}\, 2^s\, G_s^{\ell-2}
\end{aligned}
\tag{4.19}
$$

Here, we have introduced

$$
\begin{aligned}
L_s^\ell &:= \frac{s(s+1)\ldots\left(\dfrac{\ell+s}{2}-1\right)}{\left(\dfrac{\ell-s}{2}\right)!} = \binom{(\ell+s)/2-1}{(\ell-s)/2} \\
G_s^\ell &:= L_s^\ell + 2\sum_{j=1}^{(\ell-s)/2} L_s^{\ell-2j} = 2\binom{(\ell+s)/2}{(\ell-s)/2} - \binom{(\ell+s)/2-1}{(\ell-s)/2}
\end{aligned}
\tag{4.20}
$$

provided $\ell - s$ is even and non-negative; else $L_s^\ell := G_s^\ell := 0$. In particular, the vanishing of L and G for negative $\ell - s$ guarantees that the sum over p ends with $p = \ell$. This is also clear from the very beginning by the qualitative discussion of the analyticity domains of the single terms.

For every p in the sum (4.15), we isolate $(-1)^{(p+n)/2}/(2i)^p$ from the t-integration and $(-1)^{(p+\ell)/2}\, 2^p$ from the x-integration and get the condition

$$
\begin{aligned}
&\operatorname*{res}_{\kappa=i(2+\ell m)} \mathcal{W}_n\left(\frac{m}{\omega}, \kappa\right) = -\frac{2\pi i\, m}{3\omega^2}(-1)^{(n+\ell)/2} W(n,\ell) = 0 \\
&\text{with} \quad W(n,\ell) := \sum_{p=1}^{\ell} \eta_p \left(\frac{m^2}{\omega^2}\right)^{(p-1)/2} R_p(n,\ell)
\end{aligned}
\tag{4.21}
$$

where $R_p(n,\ell)$ are the remaining terms coming from the above integration formulas and displayed in Table 1. Here, and in Table 1, one should consider m as determined by $m := m''_+(n,\ell)$ according to (4.7).

w^2 \ $-iB_n$	$\frac{(2+\ell m)^2+m^2}{4m^2}$ $\times 1$	$\frac{(2+\ell m)}{2m}$ $\times \operatorname{th} mx$	$-(n^2+1)/4 \times$ $\frac{\sin^2 \omega t}{\operatorname{ch}^2 mx}$	$n/4 \times$ $\frac{-i \sin 2\omega t}{\operatorname{ch}^2 mx}$
$4m^2 \times$ $e^{-2m\|x\|}/\operatorname{ch}^2 mx$	$4\binom{p}{\frac{n+p}{2}} L_{p+2}^{\ell-2}$	$4\binom{p}{\frac{n+p}{2}} G_{p+2}^{\ell-2}$	$-4\binom{p+2}{\frac{n+p+2}{2}} L_{p+4}^{\ell-2}$	$\frac{8n}{p+2}\binom{p+2}{\frac{n+p+2}{2}} L_{p+4}^{\ell-2}$
$-4m(1+m)^2 \times$ $e^{-m\|x\|}/\operatorname{ch} mx$	$2\binom{p}{\frac{n+p}{2}} L_{p+1}^{\ell-1}$	$2\binom{p}{\frac{n+p}{2}} G_{p+1}^{\ell-1}$	$-2\binom{p+2}{\frac{n+p+2}{2}} L_{p+3}^{\ell-1}$	$\frac{4n}{p+2}\binom{p+2}{\frac{n+p+2}{2}} L_{p+3}^{\ell-1}$
$(1+m)^4 \times$ 1	$\binom{p}{\frac{n+p}{2}} L_{p}^{\ell}$	$\binom{p}{\frac{n+p}{2}} G_{p}^{\ell}$	$-\binom{p+2}{\frac{n+p+2}{2}} L_{p+2}^{\ell}$	$\frac{2n}{p+2}\binom{p+2}{\frac{n+p+2}{2}} L_{p+2}^{\ell}$
$4m^3 \times \quad e^{-m\|x\|}$ $\sin^2 \omega t/\operatorname{ch}^3 mx$	$-2\binom{p+2}{\frac{n+p+2}{2}} L_{p+3}^{\ell-1}$	$-2\binom{p+2}{\frac{n+p+2}{2}} G_{p+3}^{\ell-1}$	$2\binom{p+4}{\frac{n+p+4}{2}} L_{p+5}^{\ell-1}$	$\frac{-4n}{p+4}\binom{p+4}{\frac{n+p+4}{2}} L_{p+5}^{\ell-1}$
$-2m^2(1+m)^2 \times$ $\sin^2 \omega t/\operatorname{ch}^2 mx$	$-\binom{p+2}{\frac{n+p+2}{2}} L_{p+2}^{\ell}$	$-\binom{p+2}{\frac{n+p+2}{2}} G_{p+2}^{\ell}$	$\binom{p+4}{\frac{n+p+4}{2}} L_{p+4}^{\ell}$	$\frac{-2n}{p+4}\binom{p+4}{\frac{n+p+4}{2}} L_{p+4}^{\ell}$
$m^4 \times$ $\sin^4 \omega t/\operatorname{ch}^4 mx$	$\binom{p+4}{\frac{n+p+4}{2}} L_{p+4}^{\ell}$	$\binom{p+4}{\frac{n+p+4}{2}} G_{p+4}^{\ell}$	$-\binom{p+6}{\frac{n+p+6}{2}} L_{p+6}^{\ell}$	$\frac{2n}{p+6}\binom{p+6}{\frac{n+p+6}{2}} L_{p+6}^{\ell}$

TABLE 1. – The left column displays the six terms of w^2, the top row the four terms of B_n. m has the value $m''_+(n,\ell)$ as in (4.7). — Each of the 6×4 entries arises from the integration of the (x,t)-dependent terms (after the $\times$ sign) given in the corresponding left column and first row entries and therefore has still to be multiplied with the coefficients (in front of the $\times$ sign) of these very entries. Then, the sum of these 24 products equals $R_p(n,\ell)$ in (4.21).

As we have to evaluate the terms in Table 1 for all odd $p = 1, 3, \dots, \ell$, the shortest useful calculation is for $\ell = 5$, $n = 3$, and it involves 3×24 terms. In this case, according to Lemma 8, one has $m = (-5 + \sqrt{-23})/8$, and $\left|\frac{m}{\omega}\right|^4 = \frac{3}{8}$. It is straightforward, though lengthy, to check that $W(3,5) = -(195635 + 104993\sqrt{-23})/384 \neq 0$. This already proves Theorem 1 for the case where $\rho > \sqrt[4]{3/8}$.

5. NUMBER THEORETIC REDUCTION OF SECOND ORDER CONDITIONS

5.1. Outline of the argument

In this section, our goal is to exhibit a set of pairs (n, ℓ) such that the corresponding values of $m''_+(n, \ell)$ accumulate at 0, and such that $W(n, \ell) \neq 0$. When we succed in this, Theorem 1 is proved completely, *i.e.* for arbitrarily small ρ. According to (4.8*b*), $m''_+(n, \ell) \to 0$ if and only if $n/\ell \to 0$, in particular ℓ must tend to infinity. This has the consequence that the number of terms in (4.21) also tends to infinity. The basic trick by which we overcome this difficulty is the following:

Instead of showing $W(n, \ell) \neq 0$ by a calculation in the algebraic number field $\mathbb{Q}(\sqrt{D})$ [where D can be read off from (4.7)], it suffices to show that $\widetilde{W}(n, \ell) \neq 0$, where $\widetilde{W}$ arises from W by multiplication with common denominators ($\neq 0$), such that the calculation now takes place in the integer subring $\mathbb{Z}[\sqrt{D}]$. Now, if for some ideal I in this ring, $\widetilde{W}$ maps to $\widehat{W} \neq 0$ under the quotient mapping $\mathbb{Z}[\sqrt{D}] \longrightarrow \mathbb{Z}[\sqrt{D}]/I$, then of course $\widetilde{W} \neq 0$, and it is the calculation of $\widehat{W}$ in the quotient ring that we shall actually carry out.

It turns out that one can choose the ideal in such a way that among the originally $\frac{\ell + 1}{2} \times 24$ terms, only a finite number (*independent* of ℓ) survive the quotient mapping. Their sum can then be evaluated by brute force.

Before carrying out this program, we collect all the choices that will be made (different choices may work as well).

(1) We let $n = 3$ and $\ell = 2\lambda - 5$, where $\lambda \geq \lambda_0$ is a prime number. It then follows that $D \equiv -23 \bmod \lambda$. The ideal will be taken to be $I := (\lambda)$, the principal ideal generated by $\lambda \in \mathbb{Z}[\sqrt{D}]$.

(2) We assume that -23 is a square modulo λ (*i.e.* a square in the finite field $\mathbb{Z}/\lambda\mathbb{Z}$), which, according to Gauss' law on quadratic

reciprocity (e.g. [8]) is equivalent to λ being a square modulo 23. There are infinitely many such primes, because there are infinitely many primes in every prime residue class according to a classical theorem by Dirichlet ([10], [3]). [As a consequence of this condition, our quotient ring $\mathbb{Z}[\sqrt{D}]/(\lambda)$ is *not* the Galois field with λ^2 elements, but a ring that has zero divisors.]

Condition (1) will guarantee that only 42 of the $\dfrac{\ell+1}{2} \times 24$ terms survive in the quotient ring. Condition (2) will help to evaluate them.

5.2. Evaluation of the vanishing pole conditions in some quotient ring

All the 24 entries in Table 1 are integers; for the last column, this was discussed after (4.18). So are the numbers η_p. There enter numbers in $\mathbb{Q}(\sqrt{D})$ through m and $\dfrac{m^2}{\omega^2}$, where

$$D = -(n^2-1)\left(\frac{\ell^2-n^2}{4}-1\right)+1 \tag{5.1}$$

according to (4.7). In particular, $D<0$ and $D \equiv 1 \bmod 8$. Recall that we have to set

$$m = \frac{-\ell+\sqrt{D}}{(\ell^2-n^2)/2}, \qquad \frac{m^2}{\omega^2} = \frac{\ell^2-(\ell^2+3)(n^2+3)/8-\ell\sqrt{D}}{(\ell^2-1)(\ell^2-9)/8}$$

We let

$$\begin{aligned} \widetilde{W}(n,\ell) :=& 4m^2\, W(n,\ell)\left(\frac{\ell^2-n^2}{2}\right)^6\left(\frac{(\ell^2-1)(\ell^2-9)}{8}\right)^{(\ell-1)/2} \\ =& \sum_{p=1}^{\ell} C_p(n,\ell)\,\boldsymbol{a}(n,\ell)\boldsymbol{T}_p(n,\ell)\boldsymbol{b}(n,\ell) \end{aligned} \tag{5.2}$$

where

$$\begin{aligned} C_p(n,\ell) =& \eta_p\left(\ell^2-\frac{(\ell^2+3)(n^2+3)}{8}-\ell\sqrt{D}\right)^{(p-1)/2} \\ &\times\left(\frac{(\ell^2-1)(\ell^2-9)}{8}\right)^{(\ell-p)/2} \end{aligned}$$

and $\boldsymbol{T}_p(n,\ell)$ is the 6×4-matrix displayed in Table 1, $\boldsymbol{a}(n,\ell)$ is the row 6-vector

$$\boldsymbol{a}(n,\ell) = \left(\frac{\ell^2 - n^2}{2}\right)^4 \big(4m^2,\ -4m(1+m)^2,\\ (1+m)^4,\ 4m^3,\ -2m^2(1+m)^2,\ m^4\big) \tag{5.3}$$

and $\boldsymbol{b}(n,\ell)$ is the column 4-vector

$$\boldsymbol{b}(n,\ell) = \left(\frac{\ell^2 - n^2}{2}\right)^2 \begin{pmatrix} (2+\ell m)^2 + m^2 \\ 2m(2+\ell m) \\ -(n^2+1)m^2 \\ nm^2 \end{pmatrix} \tag{5.4}$$

$\widetilde{W}(n,\ell)$ is manifestly an element of $\mathbb{Z}[\sqrt{D}]$, the ring of all numbers of the form $\alpha + \beta\sqrt{D}$ with integers α, β. [Algebraic number theorists will note that this is not the ring of algebraic integers in $\mathbb{Q}(\sqrt{D})$, but a strict subring of it, because $D \equiv 1 \bmod 4$, but this distinction is not important for our purpose.] We have to show that for some sequence of (n,ℓ), such that $m_+''(n,\ell) \to 0$, $\widetilde{W}(n,\ell)$ does not vanish.

We choose $n = 3$, $\ell = 2\lambda - 5$, λ prime, and calculate modulo the ideal (λ). Then

$$\begin{aligned} D &\equiv -23 \mod \lambda \\ \frac{(\ell^2-1)(\ell^2-9)}{8} &\equiv 48 \mod \lambda \\ \ell^2 - \frac{(\ell^2+3)(n^2+3)}{8} - \ell\sqrt{D} &\equiv -17 + 5\sqrt{D} \mod (\lambda) \end{aligned}$$

$$\begin{aligned} \boldsymbol{a}(n,\ell) \equiv{}& 8\,(64\,(1+5\sqrt{D}),\ -48\,(11+23\sqrt{D}),\ 721+949\sqrt{D}, \\ & 16\,(-55+13\sqrt{D}),\ 18\,(79-21\sqrt{D}),\ -287+5\sqrt{D}) \\ \boldsymbol{b}(n,\ell) \equiv{}& (4\,(-123+25\sqrt{D}),\ 4\,(35-17\sqrt{D}), \\ & -20\,(1+5\sqrt{D}),\ 6\,(1+5\sqrt{D}))^T \end{aligned}$$

mod (λ). – But what about the matrices $\boldsymbol{T}_p(n,\ell)$ for $p = 1, 3, 5, \ldots, \ell$? We claim that

LEMMA 9. – $\eta_p \boldsymbol{T}_p(n,\ell) \equiv 0 \bmod \lambda$, *provided either* $7 \le p \le \lambda - 6$ *or* $\lambda \le p \le 2\lambda - 5$.

Proof. – (1) Let $\lambda \le p \le 2\lambda - 11$. Here, the binomial coefficients $\binom{\alpha}{\beta}$ that appear explicitly are all divisible by λ. Namely

(*a*) $\alpha \ge p \ge \lambda$, hence $\lambda \mid \alpha!$. In the last column, $p + 6 \ge \alpha \ge p + 2$, hence $\lambda < \alpha < 2\lambda$ and thus $\lambda \Big| \frac{n}{\alpha}\alpha!$.

(*b*) $\beta \le \dfrac{n+p+6}{2} \le \lambda - 1$, hence $\lambda \nmid \beta!$.

(*c*) $\alpha - \beta \le \dfrac{-n+p+6}{2} \le \lambda - 4$, hence $\lambda \nmid (\alpha - \beta)!$.

(2) The same holds, if $p = 2\lambda - 9$, except for (1*b*): $\lambda \mid \beta!$, if $\beta = \dfrac{n+p+6}{2}$, but in this case $L^{\ell}_{p+6} = L^{2\lambda-5}_{2\lambda-3} = 0$.

The same holds also, if $p = 2\lambda - 7$, except for (1*b*): But again in those cases, in which $\lambda \mid \beta!$, the L or G terms vanish.

(3) If $p = 2\lambda - 5$, then $\eta_p \equiv 0 \bmod \lambda$.

(4) Let $7 \le p \le \lambda - 6$. Here, the binomial coefficients that are implicit in the terms L^{α}_{β} and G^{α}_{β} are divisible by λ, namely $\binom{(\alpha+\beta)/2}{(\alpha-\beta)/2}$ and $\binom{(\alpha+\beta)/2-1}{(\alpha-\beta)/2}$, where the former appears only for G^{α}_{β}.

(*a*) $\dfrac{\alpha+\beta}{2} - 1 \ge \dfrac{p+\ell}{2} - 1 \ge \lambda$, hence $\lambda \Big| \left(\dfrac{\alpha+\beta}{2} - 1\right)!$ and $\lambda \Big| \left(\dfrac{\alpha+\beta}{2}\right)!$.

(*b*) $\dfrac{\alpha-\beta}{2} \le \dfrac{\ell-p}{2} \le \lambda - 6$, hence $\lambda \nmid \left(\dfrac{\alpha-\beta}{2}\right)!$.

(*c*) $\beta \le p + 4 \le \lambda - 2$ for the G^{α}_{β} terms, and $\beta - 1 \le p + 5 \le \lambda - 1$ for all terms, hence hence $\lambda \nmid \beta!, (\beta - 1)!$. ■

This lemma leaves us with five values for p, namely $p \in \{1, 3, 5, \lambda-4, \lambda-2\}$. But also for these values, there are several zero entries modulo λ due to arguments of the the same type. Taking into account the vanishing of binomial coefficients modulo λ and the formulas (4.20), one obtains the remaining 6×4-matrices:

$$\boldsymbol{T}_1(3,2\lambda-5)$$

$$\equiv \begin{bmatrix} 0 & 0 & -4\binom{\lambda-2}{4} & 8\binom{\lambda-2}{4} \\ 0 & 0 & -2\binom{\lambda-2}{3} & 4\binom{\lambda-2}{3} \\ 0 & 0 & -\binom{\lambda-2}{2} & 2\binom{\lambda-2}{2} \\ -2\binom{\lambda-2}{3} & 2\binom{\lambda-2}{3}-4\binom{\lambda-1}{4} & 10\binom{\lambda-1}{5} & 12\binom{\lambda-1}{5} \\ -\binom{\lambda-2}{2} & \binom{\lambda-2}{2}-2\binom{\lambda-1}{3} & 5\binom{\lambda-1}{4} & -6\binom{\lambda-1}{4} \\ 5\binom{\lambda-1}{4} & -5\binom{\lambda-1}{4} & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{T}_3(3,2\lambda-5)$$

$$\equiv \begin{bmatrix} 4\binom{\lambda-2}{4} & -4\binom{\lambda-2}{4}+8\binom{\lambda-1}{5} & -20\binom{\lambda-1}{6} & 24\binom{\lambda-1}{6} \\ 2\binom{\lambda-2}{3} & -2\binom{\lambda-2}{3}+4\binom{\lambda-1}{4} & -10\binom{\lambda-1}{5} & 12\binom{\lambda-1}{5} \\ \binom{\lambda-2}{2} & -\binom{\lambda-2}{2}+2\binom{\lambda-1}{3} & -5\binom{\lambda-1}{4} & 6\binom{\lambda-1}{4} \\ -10\binom{\lambda-1}{5} & 10\binom{\lambda-1}{5} & 0 & 0 \\ -5\binom{\lambda-1}{4} & 5\binom{\lambda-1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{T}_5(3,2\lambda-5) \equiv \begin{bmatrix} 20\binom{\lambda-1}{6} & -20\binom{\lambda-1}{6} & 0 & 0 \\ 10\binom{\lambda-1}{5} & -10\binom{\lambda-1}{5} & 0 & 0 \\ 5\binom{\lambda-1}{4} & -5\binom{\lambda-1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{T}_{\lambda-4}(3,2\lambda-5) \equiv \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{-12}{\lambda}\begin{pmatrix} \lambda \\ \dfrac{\lambda+3}{2} \end{pmatrix}\begin{pmatrix} \dfrac{3\lambda-7}{2} \\ \dfrac{\lambda-7}{2} \end{pmatrix} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$
\boldsymbol{T}_{\lambda-2}(3,2\lambda-5) \equiv \begin{bmatrix}
0 & 8\binom{\lambda-2}{\frac{\lambda+1}{2}}\binom{\frac{3\lambda-7}{2}}{\frac{\lambda-7}{2}} & 0 & \frac{24}{\lambda}\binom{\lambda}{\frac{\lambda+3}{2}}\binom{\frac{3\lambda-7}{2}}{\frac{\lambda-9}{2}} \\
0 & 0 & 0 & \frac{12}{\lambda}\binom{\lambda}{\frac{\lambda+3}{2}}\binom{\frac{3\lambda-7}{2}}{\frac{\lambda-7}{2}} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
$$

The coefficients are

$$
\begin{aligned}
C_1(3,2\lambda-5) &\equiv 12\cdot 48^{\lambda-3}\\
C_3(3,2\lambda-5) &\equiv 72\big(-17+5\sqrt{D}\big)\,48^{\lambda-2}\\
C_5(3,2\lambda-5) &\equiv 240\big(-17+5\sqrt{D}\big)^2\,48^{\lambda-1}\\
C_{\lambda-4}(3,2\lambda-5) &\equiv \frac{(\lambda-3)(\lambda-1)^2(\lambda+1)}{16}\big(-17+5\sqrt{D}\big)^{(\lambda-5)/2}\,48^{(\lambda-1)/2}\\
C_{\lambda-2}(3,2\lambda-5) &\equiv \frac{(\lambda-1)(\lambda+1)^2(\lambda+3)}{16}\big(-17+5\sqrt{D}\big)^{(\lambda-3)/2}\,48^{(\lambda-3)/2}
\end{aligned}
$$

For the further evaluation of the binomial coefficients in the matrices $\boldsymbol{T}_p(n,\ell)$, we can use

LEMMA 10. – *Let λ be any prime; then the following congruences hold* $\bmod \lambda$:

(1) $\binom{\lambda-\alpha}{\beta} \equiv (-1)^\beta\binom{\alpha+\beta-1}{\beta}$, *provided* $\lambda > \beta$.

(2) $\binom{\lambda-2}{(\lambda+1)/2} \equiv (-1)^{(\lambda+1)/2}\,\frac{\lambda+3}{2}$

(3) $\frac{3}{\lambda}\binom{\lambda}{(\lambda+3)/2} \equiv (-1)^{(\lambda+1)/2}\,2$, *provided* $\lambda > 3$

(4) $\binom{(3\lambda-7)/2}{(\lambda-7)/2} \equiv 1$

(5) $\binom{(3\lambda-7)/2}{(\lambda-9)/2} \equiv \frac{\lambda-7}{2}$

Proof. – We show the third one; all others are proved by the same method.

$$\left(\frac{\lambda+3}{2}\right)! \frac{1}{\lambda} \binom{\lambda}{(\lambda+3)/2} = (\lambda-1)(\lambda-2)\dots\left(\lambda-\frac{\lambda+1}{2}\right)$$
$$\equiv (-1)^{(\lambda+1)/2}\left(\frac{\lambda+1}{2}\right)!$$

We may cancel $\left(\frac{\lambda+1}{2}\right)!$ from this congruence, because it is nonzero in the field $\mathbb{Z}/\lambda\mathbb{Z}$. Multiplying by 2 gives the claimed result. ■

This lemma can be used to evaluate the terms completely. A symbolic manipulation package is very helpful (we used *Mathematica*), but patience and a pocket calculator may also suffice. We get

$$\sum_{p=1}^{5} C_p \boldsymbol{a} \boldsymbol{T}_p \boldsymbol{b} \equiv A := 12 \times 48^{\lambda-3}$$
$$\times 2176(-3686888226461 + 3249824828879\sqrt{D})$$

and

$$\sum_{p=\lambda-4}^{\lambda-2} C_p \boldsymbol{a} \boldsymbol{T}_p \boldsymbol{b} \equiv B := (-1)^{(\lambda+1)/2} \times 48^{(\lambda-3)/2}$$
$$\times \left(-17+5\sqrt{D}\right)^{(\lambda-5)/2} \times 2064384\left(269-95\sqrt{D}\right)$$

Therefore we have to show that $A + B \neq 0$ in the ring $\mathbb{Z}[\sqrt{D}]/(\lambda)$. For this, it is sufficient to show that $((3A)^2 - (3B)^2)(-17+5\sqrt{D})^4 \neq 0$ in this ring. For this latter expression, we can use Fermat's little theorem $a^{\lambda-1} \equiv 1 \bmod \lambda$ for $a = 48$, but according to the next lemma also for $a = -17 + 5\sqrt{D}$, provided -23 is a square modulo λ. The result is quite ugly, but different from zero; we get in fact

$$2^{19}\left(412900291171260646060599121129\right.$$
$$\left. - 18367854246388929869140462291\sqrt{D}\right)$$

This is not congruent to 0 modulo any $\lambda \neq 2$ (because the two long integers are relatively prime).

Now, here is why the use of Fermat's theorem was allowed for $(-17 + 5\sqrt{D})$ as well.

LEMMA 11. – *Let a, b, and D be integers and $\lambda \neq 2$ a prime that does not divide $a^2 - Db^2$. Suppose that D is a square modulo λ (but not a square in $\mathbb{Z}$), and $\lambda \nmid D$. Then in the ring $\mathbb{Z}[\sqrt{D}]$, it holds $(a + b\sqrt{D})^{\lambda-1} \equiv 1 \bmod (\lambda)$.*

Proof. – Let $s \in \mathbb{Z}$ such that $s^2 \equiv D \bmod \lambda$. Applying the binomial formula to both $(a + b\sqrt{D})^{\lambda-1}$ and $(a \pm bs)^{\lambda-1}$, one gets

$$\begin{aligned}(a + b\sqrt{D})^{\lambda-1} = & \frac{(a+bs)^{\lambda-1} + (a-bs)^{\lambda-1}}{2} \\ & + \sqrt{D}\,\frac{(a+bs)^{\lambda-1} - (a-bs)^{\lambda-1}}{2s}\end{aligned} \tag{5.5}$$

where the quotients on the right hand side can be evaluated in $\mathbb{Z}$.

Since $(a+bs)(a-bs) \equiv a^2 - Db^2 \not\equiv 0 \bmod \lambda$, neither factor is divisible by λ. According to Fermat's little theorem, it holds $(a \pm bs)^{\lambda-1} \equiv 1 \bmod \lambda$. Therefore,

$$(a+bs)^{\lambda-1} \pm (a-bs)^{\lambda-1} \equiv \begin{cases} 2 \\ 0 \end{cases} \bmod \lambda$$

Neither 2 nor $2s$ vanishes in the field $\mathbb{Z}/\lambda\mathbb{Z}$, thus the right hand side of (5.5) is congruent to $1 + 0\sqrt{D} \bmod (\lambda)$. ■

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