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The Cauchy problem for semilinear weakly hyperbolic equations in Hilbert spaces

by

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ABSTRACT. – We consider an abstract nonlinear second order equation, whose principal part satisfies some conditions inspired by Oleinik's conditions on degenerate hyperbolic equations. We prove local existence and regularity results in suitable scales of abstract Sobolev spaces. Several applications to concrete equations are given.

Key words: Nonlinear abstract equations, nonlinear hyperbolic equations, Oleinik conditions.

RÉSUMÉ. – On considère une équation abstraite du second ordre, nonlinéaire, dont la partie principale satisfait à des conditions inspirées par celles de Mme Oleinik pour les équations hyperboliques dégénérées. On obtient des résultats d'existence locale et de régularité dans des échelles d'espaces de Sobolev abstraits. Plusieurs applications aux équations concrètes sont données.

1. INTRODUCTION

We investigate here the local solvability of a second order abstract Cauchy problem of the following form:

(1) $$u'' + \mathcal{A}(t)u = f(t, u)$$

(2) $$u(0) = u_0, \quad u'(0) = u_1$$

Classification A.M.S.: 34 G 20, 35 L 70.

(a prime denotes a time derivative) in the framework of Banach spaces.

More precisely, let V, V' be two Hilbert spaces coupled by the (sesquilinear) pairing $\langle , \rangle$; denote with H the Hilbert space completion of V with respect to the scalar product $(,)$, restriction of $\langle , \rangle$ to $V \times V$, and identify H with its dual space. This gives the *Hilbert triple* $V \subseteq H \subseteq V'$(see [L1], [LM]).

Consider now Eq. (1). We shall assume that $\mathcal{A}(t)$ can be decomposed as

$$\mathcal{A}(t) = A(t) + B(t) + C(t) \tag{3}$$

where the operator $A(t) : V \to V'$ satisfies, for all $v, w \in V$, $t \in [0, T]$, $T > 0$,

$$\langle A(t)v, w \rangle = \overline{\langle A(t)w, v \rangle} \tag{4}$$

$$\langle A(t)v, v \rangle \geq 0, \tag{5}$$

while A, B, C satisfy

$$\begin{gathered} A \in C^1(I; \mathcal{L}(V, V')), \quad B \in C^0(I; \mathcal{L}(V, H)) \cap C^0(I; \mathcal{L}(H, V')), \\ C \in C^0(I; \mathcal{L}(H, H)), \quad M \in C^1(I; \mathcal{L}(H, H)), \end{gathered} \tag{6}$$

(thus we can think of A, B, C as operators of order 2,1,0 respectively) and

$$f : I \times V \times H \to H \text{ is continuous.} \tag{7}$$

Under the above assumptions, Eq. (1) is a quasilinear *weakly hyperbolic* equation. As it is well known from concrete counter examples (see [CS]), even in the linear case $f \equiv 0$ Pb. (1), (2) is not in general locally solvable; stronger assumptions on the operator $\mathcal{A}$ are needed.

When ass. (5) is strengthened to

$$\langle A(t)v, v \rangle \geq \nu_0 \|v\|_V^2, \qquad \nu_0 > 0, \tag{8}$$

so that Eq. (1) becomes *strictly hyperbolic*, then various methods are available, in order to prove the local existence for (1), (2). Among the most important, we mention the energy method of [LM] and the theory developed in [K], based on semigroup methods; see also [L2]. Indeed, it is also possible to handle more general equations of the form

$$u'' = \mathcal{F}(u) \tag{9}$$

(see [K]). Of course in general the local solutions thus obtained cannot be extended to global ones, since, as it is well known from concrete examples, a blow up may occur in a finite time.

Another approach to the study of (1),(2) is to impose restrictions on the data, assuming that they belong to suitable Banach scales of subspaces of H, while the coefficients have the right order in the scale. From this point of view, hyperbolicity is inessential, and what we get are abstract versions of the nonlinear Cauchy Kowalewski theorem (see e.g. [BG], [Ni], [Ov], [Y]).

Here we follow a different path. Starting from some sufficient conditions devised by O. Oleinik [O] for the *global* solvability in C^∞ of weakly hyperbolic equations of second order, in [D] a corresponding theory was developed in order to study weakly hyperbolic abstract equations of the form

$$u'' + \mathcal{A}(t)u + M(t)u' = f(t). \tag{10}$$

Our aim here is to apply the theory to the study of the local solvability for Pb.(1),(2).

We begin by recalling the framework of [D]. We assume that there exists an n-tuple of bounded commuting operators

$$\mathbf{d} = (d_1, \dots, d_n), \qquad d_k \in \mathcal{L}(V, H)$$

which generate the norm of V:

$$\|v\|_V^2 = \sum_{k=1}^{n} |d_k v|_H^2 + |v|_H^2 \qquad \forall v \in V \tag{11}$$

(in the following we shall write shortly $\|v\| \equiv \|v\|_V, |v| \equiv |v|_H$) and enjoy the property

(12) H *has a countable basis of common eigenvectors of* $d_1, \dots, d_n$.

Moreover, we define for $j \geq 1$ the Banach spaces

$$\begin{aligned} &H_j = \{v \in V \ : \ \mathbf{d}^\alpha v \in V \ \forall |\alpha| < j\}, \\ &\quad \text{with norm} \ \ |v|_j^2 = \sum_{|\alpha| \leq j} |\mathbf{d}^\alpha v|^2 \end{aligned} \tag{13}$$

where, for any multiindex $\alpha = (\alpha_1, \dots, \alpha_n), \mathbf{d}^\alpha = d_1^{\alpha_1} \circ \dots \circ d_n^{\alpha_n}$. We have thus the *Banach scale generated by* $\mathbf{d}$ (see [AS])

$$H_s \subseteq \cdots \subseteq H_1 \equiv V \subseteq H_0 \equiv H \subseteq H_{-1} \equiv V'.$$

We remark that H_j, endowed with the product

$$(v, w)_j \equiv \sum_{|\alpha| \leq j} (\mathbf{d}^\alpha v, \mathbf{d}^\alpha w),$$

has a natural Hilbert structure, which however will not be used here. For sake of simplicity we shall make the assumption

$$H_{j+1} \quad \textit{is dense in} \quad H_j, \quad 0 \le j \le s-1. \tag{14}$$

Then we consider the Cauchy problem

$$u'' + \mathcal{A}(t)u + M(t)u' = f(t,u) \tag{15}$$

$$u(0) = u_0, \quad u'(0) = u_1 \tag{16}$$

under the following assumptions.

As to the operator $\mathcal{A}(t)$, we shall assume that it has the form (3) and that for some nonnegative constant α, for some functions $\beta(t), \gamma(t)$ in $L^1(0,T)$ such that $\beta > 0$ a.e., for almost any t the following inequality holds:

$$A'(t) + \alpha A(t) + \gamma(t) \cdot i - \frac{1}{\beta(t)} B(t) \cdot {}^t B(t) \ge 0 \tag{17}$$

where i is the inclusion $V \subseteq V'$, and (17) means that the left hand side is a nonnegative operator for the product $\langle , \rangle$. Note that, thanks to the requirement that $B(t) \in \mathcal{L}(H, V')$ (see (6)), both ${}^tB \cdot B$ and $B \cdot {}^tB$ are defined and belong to $\mathcal{L}(V, V')$.

Moreover, we shall need the following assumptions on the commutators of A, B, C, M with $\mathbf{d}$: for some $s \ge 1$, for $j = 1, \dots, s, |\alpha| = j$,

$$\begin{gathered}[A(t), \mathbf{d}^\alpha] \in C^0(I; \mathcal{L}(H_{j+1}, H)), \quad [B(t), \mathbf{d}^\alpha] \in C^0(I; \mathcal{L}(H_j, H)), \\ [C(t), \mathbf{d}^\alpha], [M(t), \mathbf{d}^\alpha] \in C^0(I; \mathcal{L}(H_{j-1}, H)),\end{gathered} \tag{18}$$

with norms in these spaces not greater than μ_j. Further, we assume that for $j = 1, \dots, s$, for all α with $|\alpha| = j$ the following decomposition holds:

$$[A, \mathbf{d}^\alpha] = \sum_{|\beta| = j-1} K_{\alpha,\beta} \mathbf{d}^\beta + R_\alpha,$$

where the operators $K_{\alpha,\beta}, R_\alpha$ are such that: $K_{\alpha,\beta} \in C^0(I; \mathcal{L}(H_2, H)) \cap C^0(I; \mathcal{L}(V, V'))$,$R_\alpha \in C^0(I; \mathcal{L}(H_j, H))$,$[K_{\alpha,\beta}, d_h] \in C^0(I; \mathcal{L}(V, H))$ $(1 \le h \le n)$ with norms not greater than μ_j, and, for any $v \in H_2$,

$${}^tK_{\alpha,\beta} = K_{\beta,\alpha}, \qquad |K_{\alpha,\beta} v| \le \mu_j \sum_{h=1}^{n} \langle A d_h v, d_h v \rangle^{1/2} + \mu_j \|v\|. \tag{19}$$

Evidently we can suppose that $\mu_{j+1} \ge \mu_j$. In concrete examples, (18) is trivially satisfied, and only (19) needs verification (see Section 4).

As to the regularity of the coefficients, we shall assume that, for some integer $s \ge 2$,

$$\begin{gathered}\partial_t^r \mathcal{A} \in C^0(I; \mathcal{L}(H_{j+2+r}, H_j)), \quad \partial_t^r M \in C^0(I; \mathcal{L}(H_{j+1+r}, H_j)), \\ 0 \le r \le s-2, \ 0 \le j \le s-r-2.\end{gathered} \tag{20}$$

We have then

THEOREM 1. – *Assume that, for some integer $s \geq 1$, (3)-(6), (12), (14), (17)-(19) hold, and that*

$$f \in C^0(I; C^1(H_s; H_s)). \tag{21}$$

Then, for all $u_0, u_1 \in H_s$, there exists an interval $I' = [0, \tau], 0 < \tau \leq T$, such that Pb.(15), (16) has a unique solution

$$u(t) \in C^0(I'; H_s) \cap C^1(I'; H_{s-1}) \tag{22}$$

Note that assumption (21) implies an estimate of the form

$$|f(t, v)|_s \leq \phi_s(t; |v|_s) \tag{23}$$

for a suitable continuous function $\phi_s(t; r)$, increasing in each argument. If we assume instead that a stronger estimate holds (see (25) below), we can prove a result of C^∞ existence for solutions of (15), (16). To this end, we define the graded Fréchet space

$$H_\infty = \bigcap_{s \geq 0} H_s$$

endowed with the grading defined in (13). We have then

THEOREM 2. – *Let $s_0 \geq 0$. Assume (3)-(6), (12), (14), (17)-(19) hold for all $s \geq 0$, that (21) holds for all $s \geq s_0$, and that the following estimate holds for $s \geq s_0$, $v \in H_s$:*

$$|f(t, v)|_s \leq \psi_s(t; |v|_{s_0}) \cdot (|v|_s + 1) \tag{24}$$

where $\psi_s(t; r)$ is a continuous function, increasing in each argument.

Then, for all $u_0, u_1 \in H_\infty$, there exists an interval $I' = [0, \tau]$, $0 < \tau \leq T$, such that Pb.(15), (16) has a unique solution belonging to

$$u \in C^1(I'; H_\infty). \tag{25}$$

Moreover, if for all $r, h \geq 0$ the derivative $\partial_t^r D_v^h f(t, v)$ (in the sense of Fréchet) can be extended to an element of

$$\partial_t^r D_v^h f(t, v)\{v_1, \ldots, v_h\} \in C^0(I \times H_s, H_s \times \ldots \times H_s; H_s) \tag{26}$$

for all $s \geq s_0$, and if the regularity assumptions (20) are satisfied for all $s \geq 0$, then we have

$$u \in \bigcap_{k \geq 0} C^k(I'; H_\infty). \tag{27}$$

In the last section, we shall give some applications of Thms. 1,2 to the mixed problem for weakly hyperbolic partial differential equations of the form

$$u_{tt} = \sum_{i,j}^{n} \left(a_{ij}(t,x)u_{x_j}\right)_{x_i} + \sum_{j=1}^{n} b_j(t,x)u_{x_j} + b_0(t,x)u + c(t,x)u_t + f(t,x,u(t,x)),$$

both with Dirichlet and with periodic boundary conditions (see also [DM] for related results in the concrete case). We shall also consider applications to equations on manifolds and of higher order (of non-kowalewskian type).

2. PRELIMINARY RESULTS

A. Existence and regularity for the linear equation

We recall here in the following lemmas some results from [D], concerning abstract linear equations, that we shall need in the proof of Thms. 1,2.

We consider the *linear* Cauchy problem

$$u'' + \mathcal{A}(t)u + M(t)u' = f(t) \tag{28}$$

$$u(0) = u_0, \quad u'(0) = u_1. \tag{29}$$

We have the following a priori estimates for the solutions of (28), (29):

LEMMA 1. – *Assume (3)-(6), (17)-(19) hold for* $j = 1, \dots, s$, *and let* $u \in C^0(I; H_s) \cap C^1(I; H_{s-1})$ *be a solution to Pb.(28), (29). Then, for* $t \in I$,

$$|u(t)|_s^2 + |u'(t)|_{s-1}^2 \le C(|u_0|_s^2 + |u_1|_s^2 + \int_0^t |f(\sigma)|_s^2 d\sigma) \tag{30}$$

where the constant C *depends on* $\alpha, \beta, \gamma, T, \mu_s$, *the norms of* $B(t), C(t), M(t), M'(t)$ *in* $C^0(I; \mathcal{L}(H,H))$ *and the* $C^1(I; \mathcal{L}(V,V'))$ *norm of* $A(t)$.

Moreover, the following existence and regularity result holds:

LEMMA 2. – *Let $s \geq 2$; assume that (3)-(6), (12), (14), (17)-(19) hold. Then for all $u_0, u_1 \in H_s$, $f \in C^0(I; H_s)$,Pb.(29), (30) has a unique solution $u(t)$ in $C^0(I; H_s) \cap C^1(I; H_{s-1})$.*

Moreover, if the regularity assumptions (20) hold, and $f \in \cap_{k=0}^{s-2} C^k(I; H_{s-k})$, then $u(t)$ belongs to

$$u \in \bigcap_{k=0}^{s} C^k(I; H_{s-k}). \tag{31}$$

B. The strictly hyperbolic case

In this subsection we recall some results on strictly hyperbolic linear equations (see also [D]), which are proved by combining classical arguments with suitable energy estimates.

We consider again Pb.(28),(29); instead of condition (5), we shall assume that the strict condition (8) holds. Then we have:

LEMMA 3. – *Let $s \geq 0$. Assume that (3), (4), (6), (8) hold, that, for $j = 0, \ldots, s$, for all $v \in H_{j+1}$,*

$$\sum_{|\alpha|=j} |[A(t), \mathbf{d}^\alpha]v| \leq \mu_j |v|_{j+1} \tag{32}$$

and that

$$\begin{gathered} A(t) \in C^0(I; \mathcal{L}(H_{j+1}, H_{j-1})), \quad B(t) \in C^0(I; \mathcal{L}(H_{j+1}, H_j)), \\ C(t), M(t) \in C^0(I; \mathcal{L}(H_j, H_j)) \end{gathered} \tag{33}$$

for $j = 0, \ldots, s$.

Then for all $u_0 \in H_{s+1}$, $u_1 \in H_s$, $f \in C^0(I; H_s)$ Pb.(28), (29) has a unique solution $u(t)$ belonging to

$$u(t) \in C^0(I; H_{s+1}) \cap C^1(I; H_s) \cap C^2(I; H_{s-1}). \tag{34}$$

Moreover, the following estimate holds:

$$|u(t)|_{s+1}^2 + |u(t)|_s^2 \leq C\left(|u_0|_{s+1}^2 + \int_0^t |f(\sigma)|_s^2 d\sigma\right) \tag{35}$$

where the constant C depends on the norms of $A(t), B(t), C(t), M(t)$ in the spaces listed in (33) and on the $C^0(I; \mathcal{L}(V, V'))$ norm of $A'(t)$.

SKETCH OF THE PROOF. – The basic tool in the proof is the method of a priori estimates of higher order, which now we recall briefly. Define for $j = 1, \dots, s$ the *energy of order* j of the solution u to (28), (29) as

$$E_j(t) = |u'|_j^2 + \sum_{|\alpha| \leq j} \langle A(t)\mathbf{d}^\alpha u, \mathbf{d}^\alpha u \rangle. \tag{36}$$

By differentiating (36) with respect to time we obtain

$$E_j'(t) = \sum_{|al| \leq j} (2\Re\langle \mathbf{d}^\alpha u'' + A(t)\mathbf{d}^\alpha u, \mathbf{d}^\alpha u' \rangle + \langle A'(t)\mathbf{d}^\alpha u, \mathbf{d}^\alpha u \rangle). \tag{37}$$

Now, if we apply $\mathbf{d}^\alpha$ to Eq. (31) for $|\alpha| \leq j$, we can write

$$\mathbf{d}^\alpha u'' + A(t)\mathbf{d}^\alpha u = [A(t), \mathbf{d}^\alpha]u - \mathbf{d}^\alpha(B + C)u - \mathbf{d}^\alpha M u' + \mathbf{d}^\alpha f(t). \tag{38}$$

Since by (32) and (8)

$$|[A(t), \mathbf{d}^\alpha]u| \leq \mu_j |u(t)|_{j+1} \leq \frac{\mu_j}{\nu_0^{1/2}} \sqrt{E_j}, \tag{39}$$

and also

$$|\mathbf{d}^\alpha B u + \mathbf{d}^\alpha C u + \mathbf{d}^\alpha M u'| \leq c(|u|_{j+1} + |u'|_j) \leq C(\mu_j, \nu_0)\sqrt{E_j}, \tag{40}$$

we arrive at an estimate of the form ($j = 1, \dots, s$)

$$E_j'(t) \leq c E_j + |f(t)|_{j-1}^2. \tag{41}$$

An application of Gronwall's lemma gives the a priori estimate (35), after some easy passages.

Now it is not difficult to conclude the proof of Lemma 3, by the following standard argument (Faedo-Galerkin finite dimensional approximation). Assumption (12) implies the existence of a sequence P_N of projections on H, with finite dimensional image V_N, commuting with $d_1, \dots, d_n$ and strongly converging to the identity of H. We consider then the Cauchy problems in V_N

$$\begin{aligned} &v'' + P_N A(t) v + P_N M(t) v' = P_N f(t) \\ &v(0) = P_N u_0, \quad v'(0) = P_N u_1 \end{aligned}$$

which have a global solution $u_N(t)$, owing to the finite dimension of V_N. Moreover, estimate (35) clearly holds for the functions $u_N(t)$, with constants independent of N, since $P_N \mathcal{A}$, $P_N M$ and $P_N f$ satisfy exactly the same assumptions as $\mathcal{A}, M, F$. Note in particular that, since $V_N = P_N(H)$ is spanned by a finite set of eigenvectors for $d_1, \ldots, d_n$, then $u_N(t)$ is H_s-valued for all $s \geq 0$.

Thanks to the boundedness of the sequence $\{u_N\}$ in $L^2(0,T;H_s)$ and of $\{u'_N\}$ in $L^2(0,T;H_{s-1})$, by extracting suitable subsequences we can assume that

$$\begin{aligned} u_N &\rightharpoonup w_1 \quad \text{in} \quad L^2(0,T;H_s), \\ u'_N &\rightharpoonup w_2 \quad \text{in} \quad L^2(0,T;H_{s-1}). \end{aligned}$$

This implies evidently $w'_2 = w_1$; moreover, by standard arguments (see e.g. [LM]) it is easy to see that $w_1 = u$ is the required solution to (28), (29), and that it satisfies (34), (35). □

3. PROOF OF THE THEOREMS

PROOF OF THEOREM 1. – We begin by observing that estimate (23) easily follows from ass. (21) with the choice

$$\phi_s(t;r) = \sup\{|f(\sigma,v)|_s \,:\, 0 \leq \sigma \leq t,\ |v|_s \leq r\}. \tag{42}$$

Theorem 1 will be proved using the contraction mapping principle. To this end, consider the Banach space

$$X_\tau \equiv C^0(I';H_s) \cap C^1(I';H_{s-1}), \qquad I' = [0,\tau] \tag{43}$$

where $\tau \in]0,T]$ is to be chosen, with the natural norm. We define a map $F(v)$ on X_τ as follows: for $v \in X_\tau$, $u = F(v)$ is the solution to the *linear* problem

$$u'' + \mathcal{A}(t)u + M(t)u' = f(t,v(t)) \tag{44}$$

$$u(0) = u_0, \quad u'(0) = u_1 \tag{45}$$

which exists and belongs to X_τ, by Lemma 2.

Next we will show that F maps the closed, bounded set

$$Y = \{v \in X_\tau \,:\, |v'(t)|^2_{s-1} + |v(t)|^2_s \leq R\} \tag{46}$$

into itself, provided τ is small enough and R large enough. Indeed, by estimate (30) of Lemma 1 we have, for $v \in Y$and $u = F(v)$

$$\begin{aligned} |u(t)|^2_s + |u'(t)|^2_{s-1} &\leq c_1(|u_0|^2_s + |u_1|^2_s + \int_0^t |f(\sigma,v(\sigma))|^2_s d\sigma) \\ &\leq c_1(|u_0|^2_s + |u_1|^2_s + \tau \cdot \phi_s(\tau;R)) \end{aligned} \tag{47}$$

and choosing e.g.

$$R = 2c_1(|u_0|_s^2 + |u_1|_s^2), \tag{48}$$

and τ so small that

$$\tau \cdot \phi_s(\tau; R) \leq c_1(|u_0|_s^2 + |u_1|_s^2) \tag{49}$$

we obtain that $u = F(v) \in Y$.

Now we show that F is a contraction on Y, provided the value of τ is (possibly) reduced. Let $u, v \in Y$; the difference $F(u) - F(v)$ will solve a linear problem like (44), (45) with null initial data and $f(t, u(t)) - f(t, v(t))$ as right hand member, hence applying again (30) we have for $w(t) = F(u)(t) - F(v)(t)$

(50)
$$\begin{aligned} |w(t)|_s^2 + |w'(t)|_{s-1}^2 \leq & c_1 \int_0^t |f(\sigma, u(\sigma)) - f(\sigma, v(\sigma))|_s^2 d\sigma \\ \leq & c_1(|u_0|_s^2 + |u_1|_s^2 + \tau \cdot M^2 \cdot \sup_{0 \leq \sigma \leq \tau} |u(\sigma) - v(\sigma)|_s^2) \end{aligned}$$

where we have used Taylor's formula and

$$M = \sup_{0 \leq \sigma \leq \tau} \sup_{|u|_s \leq R} ||D_u f(\sigma, u)||. \tag{51}$$

Thus, if

$$\tau \leq (2c_1 M^2)^{-1} \tag{52}$$

we conclude that F is a contraction on Y, whose unique fixed point is the desired local solution to Pb.(15),(16). □

PROOF OF THEOREM 2. – Assumption (21) does not imply (26), in general; indeed, the lifespan τ of the solution, determined in the preceding proof, may depend on s, and in particular it may happen that $\tau_s \to 0$ as $s \to \infty$.

On the other hand, the stronger ass. (25) implies that $\tau_s = \tau_{s_0}$ for all $s \geq s_0$.

To prove this, we begin by observing that the result is true for the following strictly hyperbolic Cauchy problem ($\epsilon > 0$):

$$v'' + \left(\mathcal{A}(t) + \epsilon \cdot \sum_{k=1}^{n} {}^t d_k d_k + \epsilon \right) v = f(t, u(t)) \tag{53}$$

$$v(0) = u_0, \quad v'(0) = u_1 \tag{54}$$

under the assumptions of Thm. 2. Indeed, if we apply Thm. 1 we obtain that Pb.(53), (54) has a local solution $u_\epsilon(t)$ on some interval $I' = [0, \tau_{s_0}]$, $\tau_{s_0} > 0$, such that

$$u_\epsilon(t) \in C^0(I'; H_{s_0}) \cap C^1(I'; H_{s_0-1}) \tag{55}$$

and, since $u_0, u_1 \in H_\infty$, while $f(t, u(t)) \in C^0(I'; H_{s_0})$, by Lemma 3 we obtain also

$$u_\epsilon(t) \in C^0(I'; H_{s_0+1}) \cap C^1(I'; H_{s_0}). \tag{56}$$

The argument can be repeated, using (21) for $s = s_0 + 1$. By induction, we obtain (26) for $u_\epsilon(t)$.

Now we observe that Pb.(53), (54) satisfies the assumptions of Lemma 1 with constants independent of ϵ, thus we can apply estimate (30),and by (25) we have, for each $s \geq s_0$,

$$\begin{aligned} |u_\epsilon(t)|_s^2 + |u_\epsilon'(t)|_{s-1}^2 &\leq c(s)(|u_0|_s^2 + |u_1|_s^2 + \int_0^t |f(\sigma, u_\epsilon(\sigma))|_s^2 d\sigma) \\ &\leq c(s)\left(|u_0|_s^2 + |u_1|_s^2 + \psi_s(\tau; \sup_{I'} |u_\epsilon(\sigma)|_{s_0}) \cdot \int_0^t |u_\epsilon(\sigma)|_s^2 d\sigma\right). \end{aligned} \tag{57}$$

By (55) we have

$$\sup_{I'} |u_\epsilon(\sigma)|_{s_0} \leq M < +\infty,$$

hence (57) gives

$$|u_\epsilon(t)|_s^2 + |u_\epsilon'(t)|_{s-1}^2 \leq c(s, M)(|u_0|_s^2 + |u_1|_s^2 + \int_0^t |u_\epsilon(\sigma)|_s^2 d\sigma). \tag{58}$$

Now defining

$$E_s^\epsilon(t) = \int_0^t \left(|u_\epsilon(\sigma)|_s^2 + |u_\epsilon'(\sigma)|_{s-1}^2\right) d\sigma \tag{59}$$

we have by (58)

$$E_s^{\epsilon\prime}(t) \leq c(s, M) \cdot (E_s^\epsilon + |u_0|_s^2 + |u_1|_s^2); \tag{60}$$

by Gronwall's lemma this implies that

$$E_s^\epsilon(t) \leq c(s, M, \tau_{s_0}) \cdot E_s^\epsilon(0) \tag{61}$$

and again by (60), (59),

$$|u_\epsilon(t)|_s^2 + |u'_\epsilon(t)|_{s-1}^2 \le c(s, M, \tau_{s_0}) \cdot (|u_0|_s^2 + |u_1|_s^2) \quad \text{on} \quad I' \tag{62}$$

where the constant at the right hand member does not depend on ϵ.

Hence we see that, for each $s \ge s_0$, $\{u_\epsilon(t)\}$ is a bounded sequence in $C^1(I'; H_{s-1})$, and by Eq. (15) itself, also in $C^2(I'; H_{s-2})$. We can thus extract a subsequence which converges in $C^1(I'; H_s)$ for all s, whose limit is the desired solution to Pb.(15), (16), and satisfies (26).

The final remark on the regularity of the solution (see (27)) follows by differentiating Eq. (15) with respect to time:

$$\partial_t^{j+2} u = -\sum_{r=0}^{j} \binom{j}{r} \partial_t^r \mathcal{A}(t) \partial_t^{j-r} u - \sum_{r=0}^{j} \binom{j}{r} \partial_t^r M(t) \partial_t^{j+1-r} u$$
$$-\sum_{r=0}^{j} \sum_{h=0}^{j} \sum_{\substack{\nu_1+\dots+\nu_h=j-r \\ \nu_i \ge 1}} \frac{j!}{\nu_1! \cdots \nu_h! r!} \partial_t^r D_u^h f(t, u(t)) \{\partial_t^{(\nu_1)} u, \dots, \partial_t^{(\nu_h)} u\}.$$

Then, starting from (25), and using (26) inductively, we easily obtain (27). □

4. APPLICATIONS

We list here some applications of Thms. 1,2 to concrete examples of Cauchy problems for partial differential equations. We give only sketchy proofs for the following results, since most verifications are straightforward.

1) *Weakly hyperbolic equations with periodic boundary conditions.* Consider the following Cauchy problem on $[0, T] \times \mathbf{T}^n$ (where $\mathbf{T}^n = \mathbf{R}^n / 2\pi \mathbf{Z}^n$ is the n-dimensional torus)

$$u_{tt} = \sum_{i,j}^{n} \left(a_{ij}(t,x) u_{x_j} \right)_{x_i} + \sum_{j=1}^{n} b_j(t,x) u_{x_j} + b_0(t,x) u + c(t,x) u_t + f(t, x, u(t,x)) \tag{63}$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x) \tag{64}$$

where a_{ij}, b_j, c, u_0, u_1 are $C^\infty(\mathbf{T}^n)$ functions, while $f(u)$ is a C^∞ function.

We shall apply Thm. 2 with the choices $H = L^2(\mathbf{T}^n)$, $V = H^1(\mathbf{T}^n)$, $\mathbf{d} = \nabla$ so that $H_s = H^s(\mathbf{T}^n)$. To this end we recall the following result (due essentially to O. Oleinik, [O]; see [D] for details) concerning commutators of second order degenerate elliptic operators:

LEMMA. – *Let* Ω *be an open nonempty subset of* $\mathbf{R}^n$, *and* a_{ij}, $i, j = 1, \cdots, n$ *functions in* $C^\infty(\Omega)$ *such that*

$$a_{ij} = \overline{a_{ji}}, \qquad \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \geq 0, \quad \xi \in \mathbf{R}^n. \tag{65}$$

and such that, for some integer $s \geq 1$

$$\sum_{i,j=1}^{n} \sum_{|\alpha| \leq s} \|\partial^\alpha a_{ij}\|_{L^\infty(\Omega)} \equiv \nu_s < \infty \tag{66}$$

where as usual $\partial^\alpha = \partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n}$. *Denote by* A *the operator*

$$Av = -\sum_{i,j=1}^{n} \partial_{x_i}(a_{ij}(x)\partial_{x_j} v).$$

Then for every α *of length* s *we can write the commutator* $[A, \partial^\alpha]$ *in the following form*

$$[A, \partial^\alpha] = \sum_{|\beta| = s-1} K_{\alpha,\beta}\partial^\beta + R_\alpha$$

where $K_{\alpha,\beta}$ *are second order selfadjoint operators, while* R_α *contains the lower order terms and has order* s. *In particular, the operators* $K_{\alpha,\beta}$ *satisfy, for all* $v \in H^2(\Omega)$, *the estimate*

$$\|K_{\alpha,\beta}v\| \leq C(\nu_2)\sum_{h=1}^{n}(A\partial_{x_h}v, \partial_{x_h}v)^{1/2} + C(\nu_2)\sum_{|\gamma|\leq 1} \|\partial^\gamma v\| \tag{67}$$

while the operators R_α *satisfy, for all* $v \in H^s(\Omega)$,

$$\|R_\alpha v\| \leq C(\nu_s)\sum_{|\beta| \leq s} \|\partial^\beta v\|;$$

here $\|\cdot\|$ *and* $(\ ,\ )$ *are the norm and the scalar product in* $L^2(\Omega)$.

This Lemma implies (19) for the operator $A(t)$ ($\Omega = \mathbf{R}^n$). The verification of the other assumptions of Thm. 2 is straightforward; in particular (24) is a consequence of the inequality

$$\|f(t, x, u(t, x)))\|_{H^k} \leq \phi_k(t; \|u\|_\infty) \cdot \|u\|_{H^k} \tag{68}$$

which follows from the chain rule and the Gagliardo-Nirenberg inequalities, and of the Sobolev immersion theorems ($s_0 = [n/2] + 1$).

Hence we obtain (see also [DM])

PROPOSITION 1. – *Assume a_{ij} satisfy the weak hyperbolicity condition (65), and that there exist a positive constant α and two functions β, γ in $L^1(0,T)$ with $\beta > 0$ a.e., such that for almost any $t \in [0,T]$, for all x, ξ*

$$\sum \partial_t a_{ij}(t,x)\xi_i\xi_j + \alpha \cdot \sum a_{ij}(t,x)\xi_i\xi_j + \gamma(t) \geq \frac{1}{\beta(t)} \cdot \left(\sum b_j(t,x)\xi_j\right)^2. \tag{69}$$

Then for all C^∞periodic initial data $u_0(x), u_1(x)$, Pb.(63), (64) has a unique local solution

$$u(t,x) \in C^\infty([0,T'] \times \mathbf{T}^n)$$

for some $T' > 0$.

The above result can be generalized without difficulty to the case of a vector valued u, i.e. to the case of systems of weakly hyperbolic differential equations of second order.

2) *Mixed problem with Dirichlet boundary conditions.*Let Ω be a bounded open subset of $\mathbf{R}^n$ with smooth boundary, and consider Pb.(63), (64) on $[0,T] \times \Omega$. We can choose now $\mathbf{d} = \nabla$, $H = L^2(\Omega)$, $V = H_0^1(\Omega)$, and we obtain $H_s = H_0^s(\Omega)$ (completion of $C_0^\infty(\Omega)$ in $H^s(\Omega)$). Note that $V' \equiv H_{-1} = H^{-1}(\Omega)$. Then, applying Thms. 1,2, we have

PROPOSITION 2. – *Let $s \geq [n/2] + 1 \equiv s_0$. Assume a_{ij}, b_j, c are in $C^\infty([0,T] \times \overline{\Omega})$, that $f(t,x,u) \in C^\infty([0,T] \times \overline{\Omega} \times \mathbf{C})$ such that $\partial_x^\alpha f(t,x,0) \equiv 0$ for $x \in \partial\Omega, |\alpha| \leq s$, and that conditions (65) and (69) hold. Then, for all initial data $u_0, u_1 \in H_0^s(\Omega)$, Pb.(63), (64) has a unique local solution*

$$u(t,x) \in C^0([0,T']; H_0^s(\Omega)) \cap C^1([0,T']; H_0^{s-1}(\Omega))$$

for some $T' \in]0,T]$. Moreover, if u_0, u_1, f are C^∞functions vanishing with all their space and time derivatives at the boundary of Ω, then the same holds for $u(t)$.

3) *Weakly hyperbolic equations on manifolds.* Let M be a smooth compact Riemannian manifold without boundary and let Δ_M be the associated Laplace-Beltrami operator. We choose $\mathbf{d} = (1 - \Delta_M)^{1/2}$ ($\mathbf{d}$ is composed

of one single operator), $V = H^1(M)$, we have thus $H_s = H^s(M)$ (see e.g. [A]), and we consider the Cauchy problem ($k \geq 0$ integer) on $[0,T] \times M$:

$$u_{tt} - t^k \Delta_M u = f(u(t,x)) \tag{70}$$

$$u(0,x) = u_0(x), \quad u_t(x) = u_1(x). \tag{71}$$

We shall assume that $u_0, u_1 \in H^s(M)$ and $f \in C^\infty(\mathbf{C})$.

Then we have:

PROPOSITION 3. – *Let $s \geq s_0 \equiv [n/2] + 1$. For all $u_0, u_1 \in H^s(M)$ Pb.(70), (71) has a unique local solution*

$$u(t,x) \in C^0([0,T']; H^s(M)) \cap C^1([0,T']; H^{s-1}(M))$$

for some $T' \in]0,T]$. Moreover, if the initial data are in $C^\infty(M)$, then $u \in C^\infty([0,T'] \times M)$.

A similar result holds for the equations of the form

$$u_{tt} - t^{2k} \Delta_M u + t^{k-\lambda} X u = f(t,x,u)$$

where $\lambda < 1$, while X is a smooth vector field on M.

5) *A non-kowalewskian degenerate equation.* We consider now the following space periodic fourth order Cauchy problem on $[0,T] \times \mathbf{T}^n$:

$$u_{tt} + a(t)\Delta^2 u + b(t)\Delta u = f(u) \tag{72}$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x). \tag{73}$$

We shall assume that $a(t), b(t)$ are C^∞ function satisfying

$$a(t) \geq 0, \qquad a'(t) + \alpha \cdot a(t) \geq \frac{b(t)^2}{\beta(t)} \tag{74}$$

for some positive constant α and some function $\beta \in L^1$,$\beta > 0$ a.e., while $f(u)$ is a C^∞ function. We choose $\mathbf{d} = \Delta$, $H = L^2(\mathbf{T}^n)$,$V = H^2(\mathbf{T}^n)$; this gives $H_s = H^{2s}(\mathbf{T}^n)$. Then, applying Thms. 1,2 we have

PROPOSITION 4. – *Let $s_0 = [n/4] + 1$, and assume (74) holds. Then, for $s \geq s_0$, for all $u_0, u_1 \in H^{2s}(\mathbf{T}^n)$, Pb.(72), (73) has a unique local solution*

$$u(t,x) \in C^0([0,T']; H_0^{2s}(\mathbf{T}^n)) \cap C^1([0,T']; H_0^{2s-2}(\mathbf{T}^n))$$

for some $T' \in]0,T]$. Moreover, if the initial data are in $C^\infty(\mathbf{T}^n)$, then $u \in C^\infty([0,T'] \times \mathbf{T}^n)$.

We can also consider the mixed problem on $[0,T] \times \Omega$ for (72), (73) with Dirichlet boundary conditions. We obtain a local existence result in the subspaces of $H^s(\Omega)$ defined by conditions of the form

$$\Delta^j v = 0 \quad on\ \partial\Omega, \quad 0 \leq j \leq s. \tag{75}$$