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Multiple solutions of a semilinear elliptic equation in $\mathbb{R}^N$

by

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ABSTRACT. – In this paper, we are concerned with the existence of multiple solutions of

$$-\Delta u + u = \lambda b(x) |u|^{p-1} u + c(x) |u|^{q-1} u$$

where $1 < p, q < \frac{N+2}{N-2}$ if $N \geq 3$, $1 < p, q < +\infty$ if $N = 2$, $\lambda > 0$.

We obtain the existence of multiple solutions by using concentrations-compactness method and dual variational principle to establish the corresponding existence of critical points.

Key words: Semilinear elliptic equations, variation, critical point, concentration-compactness.

RÉSUMÉ. – Nous obtenons dans cet article un résultat d’existence et de multiplicité de solutions de

$$-\Delta u + u = \lambda b(x) |u|^{p-1} u + c(x) |u|^{q-1} u$$

où $1 < p, q < \frac{N+2}{N-2}$, $N \geq 3$, $1 < p, q < +\infty$ si $N = 2$, $\lambda > 0$.

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1. INTRODUCTION

We consider the existence of multiple solutions of the following semi-linear elliptic equation

\[
\begin{cases}
-\Delta u + u = \lambda b(x)|u|^p - 1 u + c(x)|u|^q - 1 u & \text{in } \mathbb{R}^N \\
u \in H^1(\mathbb{R}^N)
\end{cases}
\]

where \(1 < p, q < \frac{N+2}{N-2}\) if \(N \geq 3\), \(1 < p, q < +\infty\) if \(N = 2\), \(\lambda > 0\) is a real number, \(b(x)\) and \(c(x)\) satisfy

\[
\begin{cases}
b(x) \in C(\mathbb{R}^N), & b(x) \geq 0 \text{ in } \mathbb{R}^N \\
\lim_{|x| \to \infty} b(x) = b_\infty > 0,
\end{cases}
\]

\[
\begin{cases}
c(x) \in C(\mathbb{R}^N), & c(x) \geq 0 \text{ in } \mathbb{R}^N \\
\lim_{|x| \to \infty} c(x) = 0.
\end{cases}
\]

Existence of nontrivial solutions (positive solutions, for example) concerning (1.1) has been extensively studied even for more general nonlinearity—see, for instance, W. Strauss [12], H. Berestycki and P. L. Lions [4], W. Y. Ding and W. M. Ni [5], P. L. Lions [9], [10], A. Bahri and P. L. Lions [2] and the references therein. For the multiplicity of solutions we refer to H. Berestycki and P. L. Lions [4], X. P. Zhu [13] and Y. Y. Li [8].

It is known to some extent that the equation

\[
-\Delta u + u = c(x)|u|^{q-1} u \quad \text{in } \mathbb{R}^N
\]

may have infinitely many solutions because (1.3) ensures that the corresponding variational functional

\[
I^*(u) = \frac{1}{2} \int \nabla u^2 + u^2 - \frac{1}{q+1} \int c(x)|u|^{q+1}
\]
satisfies the (PS) (Palais-Smale) condition and the dual variational principle of A. Ambrosetti and P. Rabinowitz [1] may be applied. When $\lambda$ is small, (1.1) can be taken as a small perturbation of (1.4) and thus it seems reasonable to hope that (1.1) has more and more solutions as $\lambda$ tends to 0.

As mentioned in P. L. Lions ([9], [10]) that the variational functional corresponding to (1.1) defined by

$$I_\lambda(u) = \frac{1}{2} \int |\nabla u|^2 + u^2 - \frac{\lambda}{p+1} \int b(x) |u|^{p+1} - \frac{1}{q+1} \int c(x) |u|^{q+1}$$

fails to satisfy the (PS) condition because of the lack of compactness of the Sobolev embedding $H^1(\mathbb{R}^N) \subset L^2(\mathbb{R}^N)$.

Such a failure creates difficulties for the application of standard variational techniques. In section 2, arguing as P. L. Lions [10], we show by using the concentration-compactness principle that $I_\lambda(u)$ satisfies (PS)$_c$ condition if $c$ belongs to an interval depending on $\lambda$ which becomes large as $\lambda$ tends to 0. In section 3, using a variant of the dual variational principle (dealing with unbounded even functionals) of A. Ambrosetti and P. Rabinowitz [1] we obtain the existence of multiple solutions by establishing the corresponding existence of critical points of $I_\lambda(u)$ with critical values in the interval in which $I_\lambda(u)$ satisfies (PS)$_c$ condition.

We conclude this introduction by remarking that some more general nonlinearities can be considered and similar existence results can be obtained by the arguments in this paper.

2. EXISTENCE OF A POSITIVE SOLUTION

In this section, we are concerned with the existence of a positive solution of (1.1). As preparations and for the discussion of next section, we first give some notations, definitions and auxiliary results.

Define

$$M_\lambda = \{ u \in H^1(\mathbb{R}^N) \mid u \neq 0, I'_\lambda (u) u = 0 \}$$

$$M^\infty_\lambda = \{ u \in H^1(\mathbb{R}^N) \mid u \neq 0, I'^\infty_\lambda (u) u = 0 \}$$

where $I_\lambda(u)$ is defined by (1.6), $I^\infty_\lambda(u)$ is defined by

$$I^\infty_\lambda (u) = \frac{1}{2} \int |\nabla u|^2 + u^2 - \frac{\lambda}{p+1} \int b_{\infty} |u|^{p+1}$$

Let

$$I_\lambda = \inf \{ I_\lambda(u) \mid u \in M_\lambda \}$$

$$I^\infty_\lambda = \inf \{ I^\infty_\lambda(u) \mid u \in M^\infty_\lambda \}$$
We have

**PROPOSITION 2.1.** — For each \( \lambda > 0 \), \( I_\lambda \leq I^* \).

**Proof.** — If \( c(x) \equiv 0 \), then \( I^* = +\infty \), thus \( I_\lambda \leq I^* \). In what follows, we assume \( c(x) \neq 0 \).

Suppose \( u \in H^1(\mathbb{R}^N) \), \( u \neq 0 \) such that

\[
(2.8) \quad \int \nabla u \cdot u + u^2 = \int c(x) |u|^{q+1}.
\]

Let \( v = \sigma u \) such that \( v \in M_\lambda \), i. e.,

\[
(2.9) \quad \int \nabla u \cdot u + u^2 = \sigma^{p-1} \int \lambda b(x) |u|^{p+1} + \sigma^{q-1} \int c(x) |u|^{q+1}
\]

Comparing (2.8) and (2.9) we deduce that such \( \sigma \) exists and \( \sigma \in (0, 1) \).

Letting \( h(\sigma) = \frac{\sigma^2}{2} \int \nabla u \cdot u + u^2 - \frac{\sigma^{q+1}}{q+1} \int c(x) |u|^{q+1} \), we have

\[
(2.10) \quad I_\lambda(v) = \sigma^2 \int \nabla u \cdot u + u^2 - \frac{\sigma^{q+1}}{q+1} \int \lambda b(x) |u|^{p+1}
\]

Thus \( I_\lambda \leq I^* \) and we have proved Proposition 2.1.

**PROPOSITION 2.2.** — We have

\[
(2.11) \quad I_\lambda^\circ = \frac{p-1}{2(p+1)} S^{(p+1)/(p-1)}(\lambda, b_\infty)^{-(2/(p-1))}.
\]
Proof. – We can easily find that
\[
S = \inf \left\{ \int |\nabla u|^2 + u^2 \mid u \in H^1(\mathbb{R}^N), \int |u|^{p+1} = 1 \right\}
\]
which has a positive minimum \( \tilde{u} \in H^1(\mathbb{R}^N) \cap C^2(\mathbb{R}^N) \) satisfying
\[
-\Delta u + u = S |u|^{p-1} u \quad \text{in } \mathbb{R}^N
\]
(see W. Strauss [12], P. L. Lions ([9], [10]) for examples). By Gidas, Ni and Nirenberg [7] we may assume \( \tilde{u} \) is radial.

On the other hand, there exists a positive radial function \( \tilde{u} \in H^1(\mathbb{R}^N) \cap C^2(\mathbb{R}^N) \) achieving \( I_\lambda^\infty \) such that \( \tilde{u} \) satisfying
\[
-\Delta u + u = \lambda b_\infty |u|^{p-1} u \quad \text{in } \mathbb{R}^N
\]
(see also W. Strauss [12], P. L. Lions ([9], [10]) for examples).

Let \( \tilde{u} = \left( \frac{S}{\lambda b_\infty} \right)^{1/(p-1)} v \), then \( v > 0 \) in \( \mathbb{R}^N \) and solves (2.13). By the uniqueness of radial positive solution due to M. K. Kwong [11] we deduce \( v \equiv \tilde{u} \) and thus
\[
I_\lambda^\infty = I_\lambda^\infty (\tilde{u}) = \frac{p-1}{2(p+1)} \int |\nabla \tilde{u}|^2 + \tilde{u}^2 - \frac{p-1}{2(p+1)} S^{(p+1)/(p-1)} (\lambda b_\infty)^{-(2/(p-1))}
\]
proving Proposition 2.2.

Lemma 2.3. – \( I_\lambda(u) \) satisfies \( (PS)_c \) condition if
\[
c \in (-\infty, I_\lambda^\infty).
\]
Proof. – Suppose \( \{ u_n \} \subset H^1(\mathbb{R}^N) \) such that
\[
I_\lambda(u_n) \to c \in (-\infty, I_\lambda^\infty)
\]
\[
I'_\lambda(u_n) \to 0 \quad \text{in } H^1(\mathbb{R}^N)
\]

It is easy to deduce from (2.16) and (2.17) that \( \{ u_n \} \) is bounded in \( \tilde{H}^1(\mathbb{R}^N) \). By choosing subsequence if necessary we assume
\[
u_0 \to u_0 \quad \text{weakly in } H^1(\mathbb{R}^N).
\]

By the method of concentration-compactness, as in A. Bahri and P. L. Lions [2], P. L. Lions [10], V. Benci and G. Cerami [3] we deduce that there exist a nonnegative integer \( k \), \( \{ x_i \} (1 \leq i \leq k) \) in \( \mathbb{R}^N \), solutions \( \tilde{u}_i \in H^1(\mathbb{R}^N) \) (1 \( \leq i \leq k \)) of (2.14) such that (extracting subsequence if necessary)
\[
\left\| u_n - u_0 - \sum_{i=1}^{k} \tilde{u}_i(x-x_i) \right\| \to 0
\]
(2.20) \[ c = I_\lambda (u_0) + \sum_{i=1}^{\lambda} I^{\infty}_\lambda (\tilde{u}_i). \]

Since \( I^{\infty}_\lambda (\tilde{u}_i) = \frac{p-1}{2(p+1)} \int |\nabla \tilde{u}_i|^2 + \tilde{u}_i^2 \geq 0 \) for \( i = 1, \ldots, k \) if for some \( i \), \( \tilde{u}_i \neq 0 \), then \( I^{\infty}_\lambda (\tilde{u}_i) \geq I^{\infty}_\lambda \) which implies \( c \geq I^{\infty}_\lambda \) because \( I_\lambda (u_0) \geq 0 \). Thus \( \tilde{u}_i \equiv 0 \) for \( 1 \leq i \leq k \). Hence \( u_n \) converges to \( u_0 \) strongly and therefore Lemma 2.3 has been proved.

We are now going to use the preceding result to obtain the existence of a positive solution.

**Theorem 2.4.** Suppose \( I_\lambda < I^{\infty}_\lambda \). Then (1.1) has a positive solution.

**Proof.** By Ekeland's variational principle [6] and the definition of \( I_\lambda \), there exists a minimizing sequence \( \{ u_n \} \) such that \( \{ u_n \} \subset M_\lambda \)

\[
\begin{align*}
(2.21) & \quad I_\lambda (u_n) \to I_\lambda \\
(2.22) & \quad I'_{\lambda | M_\lambda} (u_n) \to 0 \quad \text{in} \quad H^{-1} (\mathbb{R}^N).
\end{align*}
\]

\[
(2.23) \quad I'_\lambda (u_n) \to 0 \quad \text{in} \quad H^{-1} (\mathbb{R}^N).
\]

Indeed, from (2.21), \( u_n \in M_\lambda \), using Sobolev inequality we can find \( C_1, C_2 > 0 \) such that

\[
(2.24) \quad C_1 < \int |\nabla u_n|^2 + u_n^2 < C_2 \quad \text{for all} \quad n = 1, 2, \ldots
\]

Letting \( J_\lambda (u) = \int |\nabla u|^2 + u^2 - \int \lambda \beta (x) |u|^{p+1} - \int c(x) |u|^{q+1} \), we have

\[
(2.25) \quad M_\lambda = \{ u \in H^1 (\mathbb{R}^N) \setminus \{ 0 \} \mid J_\lambda (u) = 0 \}.
\]

Thus

\[
(2.26) \quad I'_{\lambda | M_\lambda} (u_n) = I'_\lambda (u_n) - \theta_n J'_{\lambda | M_\lambda} (u_n)
\]

for some \( \theta_n \in \mathbb{R} \).

Since \( u_n \in M_\lambda \), we have from (2.26)

\[
(2.27) \quad I'_{\lambda | M_\lambda} (u_n) u_n - \theta_n J'_{\lambda | M_\lambda} (u_n) u_n = I'_\lambda (u_n) u_n = 0
\]

\[
(2.28) \quad J'_\lambda (u_n) u_n = 2 \int |\nabla u_n|^2 + u_n^2 - (p+1) \int \lambda \beta (x) |u_n|^{p+1} - (q+1) \int c(x) |u_n|^{q+1}
\]

\[= -(p+1) \int \lambda \beta (x) |u_n|^{p+1} - (q+1) \int c(x) |u_n|^{q+1}.
\]
Thus from (2.24), (2.28) and $u_n \in M_\lambda$ we have

$$-C_3 < J'(u_n) u_n < -C_4$$

for some constants $C_3, C_4 > 0$ independent of $n$.

From $I'_{\lambda n} (u_n) \to 0$, we obtain by (2.27) and (2.29) that $\theta_n \to 0$ which combined with (2.26) deduces $I'(u_n) \to 0$ in $H^{-1} (\mathbb{R}^N)$. Thus (2.23) holds.

Following Lemma 2.3, we can assume (by choosing subsequence if necessary)

$$u_n \to u_0 \quad \text{strongly in} \quad H^1 (\mathbb{R}^N).$$

By Sobolev inequality, we have $I_\lambda > 0$. Thus $u_0$ is a nontrivial solution of (1.1). Letting $u_0 = u_0^+ + u_0^-$, where $u_0^+ = \max \{ u_0, 0 \}$, $u_0^- = u_0 - u_0^+$, we have $I_\lambda (u_0) = I_\lambda (u_0^+) + I_\lambda (u_0^-)$. Since $I_\lambda (u_0^+) u_0^+ = 0$, i.e., $u_0^+ \in M_\lambda$ if $u_0^+ \neq 0$ we have $I_\lambda (u_0^-) \geq I_\lambda$ if $u_0^- \neq 0$. Therefore $u_0^+ \equiv 0$ or $u_0^- \equiv 0$. Without loss of generality, assume $u_0^- \equiv 0$. Thus $u_0 \geq 0$ in $\mathbb{R}^N$. It follows from standard regularity method and maximum principle that $u_0 \in C^2 (\mathbb{R}^N)$, $u_0 > 0$ in $\mathbb{R}^N$. Thus, we conclude the proof of Theorem 2.4.

**Corollary 2.5.** Suppose (1.2) holds, $c (x)$ satisfies

$$\begin{cases}
  c (x) \in C (\mathbb{R}^N), & c (x) \geq 0 \quad \text{in} \quad \mathbb{R}^N, \\
  c (x) \to 0, & c (x) \neq 0 \quad \text{in} \quad \mathbb{R}^N.
\end{cases}$$

Then (1.1) has a positive solution provided

$$\lambda \in \left( 0, \left[ \frac{p-1}{2 (p+1) I^*} \right]^{(p-1)/2} S^{(p+1)/2} b^{-1} \right).$$

**Proof.** From (2.31) we have

$$I^* < \frac{p-1}{2 (p+1)} S^{(p+1)/(p-1)} (\lambda b) - (2/(p-1)) = I_\lambda^\infty$$

which combined with Proposition 2.1 implies

$$I_\lambda < I_\lambda^\infty.$$  

Thus, by Theorem 2.4 we know (1.1) has a positive solution.

We end this section by a few remarks.

**Remark 2.6.** The fact that if $I_\lambda < I_\lambda^\infty$ then $I_\lambda$ has a minimum has been proved in P. L. Lions ([9], [10]). We reprove this fact for the sake of completeness.

**Remark 2.7.** Consider the following equation

$$-\Delta u + u = Q (x) |u|^{p-1} u \quad \text{in} \quad \mathbb{R}^N$$

where $Q (x) \in C (\mathbb{R}^N)$, $Q (x) \geq 0$ in $\mathbb{R}^N$, $Q (x) \to Q > 0$ as $|x| \to \infty$. 

(2.35) can be obtained by taking $\lambda = 1$, $Q(x) \equiv b(x)$, $c(x) \equiv 0$ in (1.1). From Theorem 2.4 we can deduce the corresponding results concerning the existence of positive solution of (2.35) in section 3 of W. Y. Ding and W. M. Ni [5] [for the case $Q(x) \to \bar{Q}$ as $|x| \to \infty$]. Corollary 2.5 gives a type of precise condition under which $I_\lambda < I_\infty$.

Suppose $Q(x) = \lambda b(x) + c(x)$, where $b(x)$ satisfies (1.2) and
c(x) satisfies (2.30) with $\text{supp } c(x)$ bounded. Corollary 2.5 ensures the existence of positive solution if $\lambda$ is properly small. It should be pointed out that in this case $Q(x)$ does not satisfy the condition proposed by A. Bahri and P. L. Lions in [2].

3. EXISTENCE OF MULTIPLE SOLUTIONS

First of all, let us state a variant of the dual variational principle of A. Ambrosetti and P. Rabinowitz [1] dealing with unbounded even functionals.

Let $E$ be a Banach space, $B_r$ be the ball in $E$ centered at 0 with radius $r$, $\partial B_r$ be the boundary of $B_r$. $A \subset E$ is called symmetric if $u \in A$ implies $-u \in A$. Let

$$\Sigma = \{ A | A \subset E \setminus \{0\}, A \text{ is closed and symmetric} \}$$

For $A \subset \Sigma$, $v(A)$ denotes the genus of $A$. We set for $f \in C^1(E, \mathbb{R})$

$$E_+ = \{ u \in E | f(u) \geq 0 \}$$

$$H_n = \{ h | h \in C(E, E), h \text{ is odd homeomorphism } h(B_1) \subset E_+ \}$$

$$\Gamma_n = \{ A \subset \Sigma | A \text{ is compact, } v(A \cap h(\partial B_1)) \geq n \text{ for any } h \in H \}$$

Replacing (PS) by condition, we have the following lemma proved exactly as in [1].

**Lemma 3.1.** Suppose $f \in C^1(E, \mathbb{R})$ satisfies

(C1) $f(0) = 0$ and there exist $\rho, \alpha > 0$ such that $f(u) > 0$ for any $u \in B_\rho \setminus \{0\}$, $f(u) \geq \alpha$ for all $u \in \partial B_\rho$;

(C2) for any finite dimensional subspace $E^n \subset E$, $E^n \cap E_+$ is bounded;

(C3) $f(u) = f(-u)$.

Set

$$b_n = \inf_{A \in \Gamma_n} \sup_{A \in \Gamma_n} \{ f(u) | u \in A \}, \quad n = 1, 2, \ldots$$

Then

(i) $\Gamma_n \neq 0$ for $n = 1, 2, \ldots$, $b_n \geq \alpha$;

(ii) $b_n$ is a critical level if $f$ satisfies (PS)$_c$ condition for $c = b_n$.  

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Furthermore, if \( b = b_n = \ldots = b_{n+m} \), then \( v(K_b) \geq m+1 \), where \( K_b = \{ u \in E \mid f(u) = b, f'(u) = 0 \} \).

In what follows, we always take \( E = H^1(\mathbb{R}^N) \) and use the same notations \( \Sigma, B_\ast, \partial B_\ast \) and \( v(A) \). Let
\[
E_\lambda = \{ u \in H^1(\mathbb{R}^N) \mid I_\lambda(u) \geq 0 \}
\]
\[
E_\ast = \{ u \in H^1(\mathbb{R}^N) \mid I^*(u) \geq 0 \}
\]
\[
H_\lambda = \{ h \in C(H^1(\mathbb{R}^N), H^1(\mathbb{R}^N)) \mid h \text{ is odd homeomorphism}, h(B_\lambda) \subset E_\lambda \}
\]
\[
H_\ast = \{ h \in C(H^1(\mathbb{R}^N), H^1(\mathbb{R}^N)) \mid h \text{ is odd homeomorphism}, h(B_\lambda) \subset E_\ast \}
\]

Obviously \( E_\lambda \subset E_\ast \), \( H_\lambda \subset H_\ast \).

**Proposition 3.2.** — If \( b(x) \) satisfies (1.2), \( c(x) \) satisfies
\[
\begin{align*}
&c(x) \in C(\mathbb{R}^N), \quad c(x) \geq 0 \quad \text{in } \mathbb{R}^N, \\
&\text{meas} \{ x \in \mathbb{R}^N \mid c(x) = 0 \} = 0,
\end{align*}
\]
\[
c(x) \to 0 \quad \text{as} \quad |x| \to \infty
\]

Then \( I_\lambda(u) \) and \( I^*(u) \) satisfy (C1), (C2) and (C3) in the previous lemma.

**Proof.** — The verification of (C1) and (C3) is trivial. We only show that (C2) holds for \( I_\lambda(u) \) [resp. \( I^*(u) \)]. We argue by way of contradiction. Suppose there exists a \( m \) dimensional subspace \( E^m \subset H^1(\mathbb{R}^N) \), a sequence \( \{ u_n \} \subset E^m \cap E_\lambda \) (resp. \( \{ u_n \} \subset E^m \cap E_\ast \)) such that \( \| u_n \| \to +\infty \). Let \( e_1, e_2, \ldots, e_m \) be the basis of \( E_m \). Then
\[
u_{n}=t_{11}e_{1}+\ldots+t_{m}e_{m}
\]
for some \( t_n = (t_{11}, \ldots, t_{m}) \in \mathbb{R}^m \).

Set \( |t_n| = \max_{1 \leq i \leq m} |t_{ii}| \), we have \( |t_n| \to +\infty \).

\[
\int |\nabla u_n|^2 + u_n^2 = 0 (|t_n|^2)
\]
\[
\int b(x)|u_n|^{p+1} \geq 0
\]
\[
\int c(x)|u_n|^{q+1} \geq C_5 |t_n|^{q+1} \quad \text{for } n \text{ large enough}
\]

where \( C_5 > 0 \) is some constant.

(3.14), (3.15) and (3.16) deduce \( I_\lambda(u_n) < 0 \) for \( n \) large enough [resp. \( I^*(u_n) < 0 \) for \( n \) large enough], which contradicts \( u_n \in E_\lambda \) (resp. \( u_n \in E_\ast \)).
Define

(3.17) \[ \Gamma^n_k = \{ A \subset \Sigma \mid A \text{ is compact and } \nu(A \cap h(\partial B_i)) \geq n \} \text{ for any } h \in \mathcal{H}_k, \quad n = 1, 2, \ldots, \]

(3.18) \[ \Gamma^n_* = \{ A \subset \Sigma \mid A \text{ is compact and } \nu(A \cap h(\partial B_i)) \geq n \} \text{ for any } h \in \mathcal{H}_*, \quad n = 1, 2, \ldots, \]

(3.19) \[ c^n_k = \inf_{A \in \Gamma^n_k} \max_{u \in A} \{ I_k(u) \}, \quad n = 1, 2, \ldots, \]

(3.20) \[ c^n_* = \inf_{A \in \Gamma^n_*} \max_{u \in A} \{ I^*(u) \}, \quad n = 1, 2, \ldots, \]

By the definitions we have

(3.21) \[ \Gamma^n_1 = \Gamma^n_* \quad \text{for } n = 1, 2, \ldots. \]

Suppose (3.10) holds then by Proposition 3.2 and Lemma 3.1, \( \Gamma^n_* \neq \emptyset \) for each \( n = 1, 2, \ldots, \) and consequently \( c^n_* < +\infty. \)

Let

\[ \lambda_k = \left[ \frac{p-1}{2(p+1)c^n_k} \right]^{(p-1)/2} S^{(p+1)/2} b^{-1}_\infty, \quad k = 1, 2, \ldots. \]

We have

**Theorem 3.3.** Suppose (1.2) and (3.10) hold. Then for each \( n = 1, 2, \ldots, \) (1.1) has \( n \) pair of solutions \( \{-u_i, u_i\}, \quad i = 1, \ldots, n \) if \( \lambda \in (0, \lambda_n). \)

**Proof.** By the definition of \( c^n_k, c^n_* \), \( n = 1, 2, \ldots \) we have

\[ c^n_k = \inf_{A \in \Gamma^n_k} \max_{u \in A} \{ I_k(u) \} \]

\[ \leq \inf_{A \in \Gamma^n_*} \max_{u \in A} \{ I_k(u) \} \]

\[ \leq \inf_{A \in \Gamma^n_*} \max_{u \in A} \{ I^*(u) \} \]

\[ = c^n_. \]

Thus

(3.23) \[ c^n_k \leq c^n_* \quad \text{for } n = 1, 2, \ldots. \]

Next we claim that for each \( c^n_k, k = 1, \ldots, n, \) \( I_k(u) \) satisfies (PS)c condition.

Indeed, \( \lambda < \lambda_n \) implies

\[ \lambda \left[ \frac{p-1}{2(p+1)c^n_k} \right]^{(p-1)/2} S^{(p+1)/2} b^{-1}_\infty. \]
Thus
\[ c_n^* < \frac{p-1}{2(p+1)} S^{(p+1)/(p-1)}(h, b_\infty)^{-2/(p-1)} = I_k^\infty \]

which combining with (3.23) deduces
\[ (3.24) \quad c_n^* < I_k^\infty. \]

On the other hand, obviously we have
\[ (3.25) \quad c^1_k \leq \ldots \leq c^n_k. \]

Thus, by Lemma 2.3, \( I_k(u) \) satisfies (PS)\(_c\) condition for \( c^k, k = 1, 2, \ldots, n \). Following Lemma 3.1, has at least \( n \) different critical points \( u_i \in H^1(\mathbb{R}^n) (1 \leq i \leq n) \) such that \( I_k(u_i) = c^i_k (1 \leq i \leq n) \). Since \( I_k(u) \) is an even functional \( -u_i \) is critical point either \( (1 \leq i \leq n) \), \( \{ -u_i, u_i \} \) are the solutions we are looking for. Hence we have proved Theorem 3.3.

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