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by

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ABSTRACT. – We prove a result on the topological entropy of a large class of Hamiltonian systems. This result is obtained variationally by constructing “multibump” homoclinic solutions.

Key words : Hamiltonian systems, convexity, dual variational methods, concentration-compactness, homoclinic orbits, Bernoulli shift, topological entropy, chaos.

RÉSUMÉ. – On démontre un résultat sur l’entropie topologique d’une grande classe de systèmes hamiltoniens. Ce résultat est obtenu par une méthode variationnelle qui permet de construire des solutions homoclines « multi-bosses ».

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1. INTRODUCTION

1. Some history

Homoclinic orbits were first introduced by H. Poincaré (see [M] for a modern exposition). Considering a hyperbolic fixed point \( p \) of a diffeomorphism \( \varphi \) in \( \mathbb{R}^{2N} \), we say that a point is homoclinic if it belongs to the intersection of the unstable and stable manifolds \( W^u, W^s \) associated to \( (p, \varphi) \); the orbit of \( r \) is called a homoclinic orbit. Assuming that \( W^u, W^s \) intersect transversally at \( r \), and that \( \varphi \) is symplectic, Poincaré proved that there are infinitely many homoclinic orbits, geometrically distinct in the following sense:

\[
(\text{the orbits of } r, r' \text{ are geometrically distinct}) \iff (\forall n \in \mathbb{Z} : \varphi^n(r) \neq r').
\]

Birkhoff, Smale and other authors also studied homoclinic orbits, and their relation with Bernoulli shifts. We state here a result of Smale on homoclinics (see [M]): if \( r \neq p \) is a point of transverse intersection of \( W^u, W^s \), then there are \( \ell \in \mathbb{N}^* \) and a homeomorphism \( \tau : \{0, 1\}^\mathbb{Z} \to I \), where \( I \) is an invariant set for \( \varphi^\ell \), such that \( \varphi^\ell \circ \tau = \tau \circ \sigma \). Here, \( \sigma ((a_n)) = (b_n) \) with \( b_n = a_{n+1} \) and \( \{0, 1\}^\mathbb{Z} \) is endowed with the standard metric

\[
d(a, b) = \frac{1}{3} \sum_{n \in \mathbb{Z}} \frac{|b_n - a_n|}{2^|n|}.
\]

This structure is called a Bernoulli shift.

Bernoulli shifts are an important tool in the study of chaotic behavior. For instance, Smale’s result given above implies that the topological entropy of \( \varphi \), \( h_{\text{top}}(\varphi) \), is greater than \( \frac{\ln 2}{\ell} \). This is a direct consequence of the following definition (see [O], p. 182-183):

\[
h_{\text{top}}(\varphi) = \sup_{R > 0} \lim_{\epsilon \to 0} \left( \lim_{n \to \infty} \frac{\log s(n, \epsilon, R)}{n} \right),
\]

where

\[
s(n, \epsilon, R) = \max \{ \text{Card}(E) : E \subset B(0, R), \forall x \neq y \in E \exists k \in [0, n]) : |\varphi^k(x) - \varphi^k(y)| \geq \epsilon \}.
\]

2. Variational approach

The results described in the preceding section were proved by dynamical systems methods, with a transversality assumption on \( W^u, W^s \). The question examined in this paper is the following one:

We assume that \( \varphi \) is the time-one map of a Hamiltonian system \( x' = J \nabla_x H(t, x), H \) being one-periodic in time. Is it possible to say some-
thing about Bernoulli shifts and topological entropy, using a variational method? We will see that this approach has several advantages:

- The existence of a homoclinic point \( r \) is not an assumption any more, but follows from global hypotheses on \( H \) that we call \((hA), (hR)\).
- The classical transversality hypothesis can be replaced by a weaker condition, denoted \( (\mathcal{H}) \).

3. Main results

We work with the same Hamiltonian system as in the paper \([CZ-E-S]\):

\[
x' = JA x + J \nabla x R(t, x), \quad x \in \mathbb{R}^{2N}, \quad t \in \mathbb{R}, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

We are looking for non-zero solutions satisfying \( x(\pm \infty) = 0 \), i.e. solutions homoclinic to 0.

We make the following assumptions on \( A, R \):

- \( A^* = A \), and \( JA = E \) is a constant matrix, all eigenvalues of which have a non-zero real part. \( (hA) \)
- \( R(. + 1, .) = R(. , .) \), and \( R \) is \( C^2 \).
- \( (\forall \ t \in \mathbb{R}), \ R(t, .) \) is strictly convex.
- for some \( \alpha > 2 \), \( 0 < k_1 < k_2 < + \infty \), we have

\[
\forall (t, x) \in \mathbb{R} \times \mathbb{R}^{2N}, \quad R(t, x) \leq \frac{1}{\alpha} \left( \nabla_x R, x \right),
\]

\[
k_1 \ |x|^\alpha \leq R(t, x) \leq k_2 \ |x|^\alpha.
\]

In \([CZ-E-S]\), it was proved under these assumptions that there are at least two homoclinic orbits \( x, y \), geometrically distinct, i.e. such that \( \forall n \in \mathbb{Z} : n \ast x \neq y \), where \( n \ast x(t) = x(t-n) \). One of them was obtained by a mountain-pass argument on a dual action functional. This paper has motivated some related work.

Concerning the existence of at least one homoclinic solution, the convexity assumption was relaxed in \([H-W]\) and \([T]\), by two different methods.

Concerning multiplicity, a novel variational argument was introduced in \([S]\), and the following result was proved:

**Theorem I.** — Assume \((hA), (hR)\) are true. Then there are infinitely many orbits homoclinic to 0, geometrically distinct in the sense

\[
x_1 \neq x_2 \iff (\forall n : n \ast x_1 \neq x_2).
\]

The idea in \([S]\) was to look for solutions near \((-n) \ast x + n \ast x \), where \( x \) is the homoclinic orbit found in \([CZ-E-S]\) by mountain-pass, and \( n \) is large enough. We call them "solutions with two bumps distant of \( 2n \)."
The existence of such solutions is a well-known fact of classical dynamical systems theory, in many particular situations. Let describe briefly one of them (see [W]):

Consider the autonomous system associated to the Hamiltonian

$$H(p, q) = p^2 - q^2 + p^4 + q^4, \quad (p, q) \in \mathbb{R}^2.$$ 

It is integrable, and does not have any solution with two (or more) bumps. But in the autonomous case, we have a continuum of solutions which are the translates of one of them in time, and Theorem I is not contradicted.

By Melnikov's theory, it is possible to find small non-autonomous perturbations $H(p, q) + \varepsilon K(t, p, q)$ of the Hamiltonian such that $W^u, W^s$ intersect transversally. Then, using the implicit function theorem, multi-bump homoclinic solutions can be constructed.

To give more detailed comments on Theorem I, we need some notations:

$f$ is the dual action functional introduced in [CZ-E-S]. It is defined on the space $L^p(\mathbb{R}, \mathbb{R}^{2N})$, with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ (the exact form of $f$ will be given in section II). $f^g = \{ x : f(x) \leq g \}$, $\mathcal{C}$ is the set of non-zero critical points, and $\mathbb{Z}$ acts by integer translations in time.

$L : L^p \rightarrow W^{1, p}$ is an isomorphism such that, if $u \in \mathcal{C}$, then $L u$ is a homoclinic orbit (see §II).

$c$ is the mountain-pass level, let us define it precisely:

$0$ is a strict local minimum for $f$, and $f(0) = 0$. Moreover, $f$ is not bounded from below (see [CZ-E-S]. So we consider $r$ is non-empty, and we choose $c = \inf \max_{\gamma \in \Gamma} f(\gamma) > 0$ as mountain-pass level.

In [S], the variational gluing of two bumps was possible under the following assumption:

(*) There is some $c' > c$ such that $(\mathcal{C} \cap f^{-c'})/\mathbb{Z}$ is finite.

The following result, which is a more precise version of Theorem I, is an immediate consequence of the arguments given in [S]:

**Theorem I'**. Assume that (hA), (hR) and (*) are true. Then there are two critical points $u, v$ such that for any $r, h > 0$ and $n \geq N(r, h)$, exists a critical point $u_n$, with

$$\| u_n - [(-n) \star u + n \star v] \|_{L^p} < r \quad \text{and} \quad f(u_n) \in [2c - h, 2c + h].$$

$u, v$, possibly equal, satisfy $f(u) = f(v) = c$. The homoclinic orbit $y_n = L u_n$ is called a solution with two bumps distant of $2n$. It satisfies

$$\| y_n - [(-n) \star L u + n \star L v] \|_{W^{1, p}} < \| L \| \cdot r.$$
Theorem I is trivial when (*) is not satisfied ("degenerate" situation), and Theorem I' implies Theorem I when (*) is satisfied ("non-degenerate" situation).

In the later work [CZ-R]$^1$, Coti Zelati and Rabinowitz apply the ideas of [S] to the case of second order systems, and construct, under assumption (*), solutions with $m$ bumps, i.e. located in a ball of center $p^1 \ast x_1 + \ldots + p^m \ast x_m$ and radius $\epsilon$, for the norm of the functional space $E=W^{1,2}(\mathbb{R}, \mathbb{R}^N)$. The $x_i$ are in a fixed finite set of critical points of the action functional $\int \frac{x^2}{2} - V$ defined on $E$. They are found thanks to a mountain-pass. Moreover, for any $i$, $(p^{i+1} - p^i) \geq K(\epsilon, m)$. In the construction of [CZ-R]$^1$, the minimal distance $K$ between bumps goes to infinity as $m$ goes to infinity, for $\epsilon$ fixed.

Other applications, in the domain of partial differential equations, are given in [CZ-R]$^2$, [LI]$^1$, [LI]$^2$.

In the paper [C-L] of Chang and Liu, the assumption (*) is replaced by (**) : $\mathcal{C} \cap f^c$ contains only isolated points.

In the present work, (**) is replaced by the weaker assumption ($\mathcal{H}$) : $\mathcal{C} \cap f^c$ is at most countable.

Moreover, multibump solutions are constructed for a minimal distance $K$ between bumps independent of $m$. This last point, whose proof requires many modifications in the arguments of [S], [CZ-R]$^1$, allows to study the topological entropy of the Hamiltonian system. The main theorem that we will prove can be stated as follows:

**Theorem II.** - Assume (hA), (hR) and ($\mathcal{H}$) are true. Then there exists a homoclinic orbit $x$ such that, for any $\epsilon > 0$, and any finite sequence of integers $\bar{p}=(p^1, \ldots, p^m)$, satisfying

$$ (\forall i) : \quad (p^{i+1} - p^i) \geq K(\epsilon), $$

there is a homoclinic orbit $y_{\bar{p}}$, with

$$ (\forall t \in \mathbb{R}) : \quad \left| y_{\bar{p}}(t) - \sum_{i=1}^{m} x(t-p^i) \right| \leq \epsilon. $$

Here, $K$ is a constant independent of $m$.

**Remark 1.** - The assumption ($\mathcal{H}$) cannot be satisfied in the autonomous situation, where the translates of $x$ in time form a continuum. Now, if $W^u$, $W^s$ intersect transversally, then their intersection is at most countable, and so is the set of homoclinic solutions; but the converse is false.

**Remark 2.** - The estimate on $y_{\bar{p}} - \sum_{i=1}^{m} x(t-p^i)$ is given in $L^\infty$ norm. In [S] and [CZ-R]$^1$, it was given in global $W^{1,q}(\mathbb{R})$ norm. Without this change,
it seems impossible, or at least very difficult, to choose K independently of m.

Since K does not depend on m, we can study the limit \( m \to \infty \), and get solutions with infinitely many bumps (those are not homoclinic orbits any more). We have

**Corollary II.1.** With the hypotheses and notations of Theorem II, for any interval \( 1 \subset \mathbb{Z} \), finite or infinite, and any sequence of integers \( \bar{p} = (p_i)_{i \in \mathbb{N}} \) such that \( \forall i : (p_{i+1} - p_i) \geq K(\varepsilon) \), there is a solution \( y_{\bar{p}} \) of (1) satisfying

\[
(\forall t \in \mathbb{R}) : |y_{\bar{p}}(t) - \sum_{i \in \mathbb{N}} x(t - p_i)| \leq \varepsilon.
\]

If I is infinite, we say that \( y \) has infinitely many bumps.

As a consequence, we have an "approximate" Bernoulli shift structure:

**Corollary II.2.** Under the hypotheses of Theorem II, there is \( x_0 \in \mathbb{R}^{2N} \setminus \{0\} \) such that, for any \( \varepsilon > 0 \), exist K = K(\varepsilon) > 0 and

\[
\tau = \tau(\varepsilon) : (\{0, 1\}^\mathbb{Z}, d) \to (\mathbb{R}^{2N}, |.|),
\]

with:

- \( \tau \) is injective, and \( \tau^{-1} \) is uniformly continuous.
- \( (\forall n \in \mathbb{Z}) \|\tau \circ \sigma^n - \varphi^{K_n} \circ \tau\|_\infty < 2 \varepsilon \).
- \( s_0 = 1 \Rightarrow |\tau(s) - x_0| < \varepsilon \)
- \( s_0 = 0 \Rightarrow |\tau(s)| < \varepsilon \).

Here, \( \varphi \) is the time-one flow of (1), and \( \sigma(s) = s_{n+1} \). Note that we cannot say that \( \tau \) is continuous. We call \( (\tau(\{0, 1\}^\mathbb{Z}), \varphi^K) \) an approximate Bernoulli shift structure.

Corollary II.2 will be proved in section VI.

Now, we are in a position to state the result on topological entropy. Choose \( \varepsilon \leq \frac{|x_0|}{3} \). If two sequences \( s, s' \) are such that for some \( k \), then

\[
|\Phi^K(\varepsilon)^k \circ \tau(s) - \Phi^K(\varepsilon)^k \circ \tau(s')| \geq \frac{|x_0|}{3}.
\]

So, for \( \varepsilon < \frac{|x_0|}{3} \) and \( R > |x_0| + \varepsilon \), we get \( s(K_n, \varepsilon, R) \geq 2^n \), and

\[
h_{top}(\varphi) \geq \frac{\text{Ln} 2}{K(\varepsilon)}.
\]

So Corollary II.2 implies

**Corollary II.3.** With the hypotheses of Theorem I, the flow of (1) has a positive topological entropy.

Note: Independently of the present paper, Bessi in [B] constructs variationally an approximate Bernoulli shift for the one-dimensional pendulum,
by a method inspired of [S]. He replaces assumption (*) by a weakening of the classical Melnikov condition, and his result is given for small perturbations of an autonomous system.

II. VARIATIONAL FRAMEWORK

AND SKETCH OF PROOF OF THEOREM II

We use a variational formulation based on Clarke's dual action principle (see [CZ-E-S], [E]). Define $G(t, y) = \max \{ (z, y) - R(t, z) | z \in \mathbb{R}^{2N} \}$. $G$ is 1-periodic in time, strictly convex in $y$, and satisfies, for $\alpha + \frac{1}{\beta} = 1$:

$$0 \leq \frac{1}{\beta} (\nabla_y G, y) \leq G(t, y) \leq (\nabla_y G, y),$$

$$(\exists c_1, c_2 > 0) (\forall y_0, \iota) \quad c_1 |y|^\beta \leq \frac{G(t, y)}{y |y|^\beta} \leq c_2 |y|^\beta - 1.$$ 

We define

$$D : W^{1, \beta}(\mathbb{R}, \mathbb{R}^{2N}) \rightarrow L^\beta(\mathbb{R}, \mathbb{R}^{2N})$$

$$z \mapsto \left( -J \frac{d}{dt} - A \right) z,$$

$L = D^{-1}$.

We call $\mathcal{C}$ the set of non-zero critical points of the following functional $f$:

$$f(u) = \int G(t, u) dt - \frac{1}{2} \int (u, Lu) dt, \quad u \in L^\beta(\mathbb{R}, \mathbb{R}^{2N}).$$

We have (see [CZ-E-S])

**Lemma 1.** - If $u \in \mathcal{C}$, then $x = Lu$ is a non-zero solution of (1) such that $x(\pm \infty) = 0$, i.e. an orbit homoclinic to 0.

Our task will be to find a large class of elements of $\mathcal{C}$.

For this purpose, we need some compactness properties of $f$. Unfortunately, $f$ does not satisfy the Palais-Smale (PS) condition, because it is invariant for the action of the non-compact group $\mathbb{Z} : n \circledast u = u (+ n)$. To deal with this problem, we use the concentration-compactness theory of P. L. Lions (see [LS]).

We have (see [CZ-E-S])

**Lemma 2.** - Suppose (hA), (hR) are true. Then $f$ satisfies the following compactness property:

Let $(u_n)_{n \geq 0}$ be a sequence such that

$$f(u_n) \rightarrow a > 0, \quad f'(u_n) \rightarrow 0.$$
Then there exist $m > 0$, a subsequence $(n_p)_{p \geq 0}$, and $u^1, \ldots, u^m$ in $\mathcal{G}$, not necessarily distinct, such that

$$\left\| u_{n_p} - \sum_{i=1}^{m} k^i_p \ast u^i \right\|_{p \to +\infty} \to 0,$$

where $k^i_p \in \mathbb{Z}$, $(k^i_p - k^j_p) \to +\infty$ as $p \to +\infty$ if $i < j$.

To simplify notations, we will write

$$\bar{k}_p = (k^1_p \ldots k^m_p) \in \mathbb{Z}^m, \quad \bar{u} = (u^1 \ldots u^m) \in \mathcal{G}^m,$$

$$\bar{k}_p \ast \bar{u} = \sum_{i=1}^{m} k^i_p \ast u^i.$$ 

Moreover, \((\lim_{k \to \infty} (k^i_p - k^j_p) = +\infty \text{ if } i < j)\)

will be summarized by

$$\bar{k}_p \to \Omega \text{ as } p \to +\infty.)$$

Now, what is special here is that the splittings $\bar{k} \ast \bar{u}$ do not vary continuously when $\bar{k}$ varies. This leads to introduce a new compactness condition (see [CZ-E-S], [S]).

**CONDITION \( \overline{PS} \) (a).** Let \((u_n)\) be a sequence such that $f(u_n) \leq a \in \mathbb{R}$, $f'(u_n) \to 0$, $(u_{n+1} - u_n) \to 0$. Then \((u_n)\) is convergent.

We have:

**LEMMA 3.** Assume \((hA), (hR)\) and \((\mathcal{H})\) are true. Then \( \overline{PS} (c') \) holds.

Lemma 3 will be proved in section III, and will be used in the proof of Lemma 7, section IV.

The interest of \( \overline{PS} \) is that, if $f$ is bounded on a pseudo-gradient line, then one can find a \( \overline{PS} \) sequence on this line. So \( \overline{PS} \) can give the same kind of deformation lemmas as the Palais-Smale condition. If \( \overline{PS} \) is satisfied under level $c'$, by deforming a particular curve in $\Gamma$, one finds at least one critical point $u$ between levels $c$ and $c'$. When (*) holds, one can impose $f(u) = c$. When only $(\mathcal{H})$ holds, the best that can be done is to take $u$ with $(f(u) - c)$ arbitrarily small.

In [S], under assumption (*), a “product min-max” is constructed at level $2c$, for the “split” functional $\bar{f}(x) = f(x \chi_{\mathbb{R}_-}) + f(x \chi_{\mathbb{R}_+})$, where $\chi_t$ is the characteristic function of $I$. Theorems I and I’ are then proved by contradiction, thanks to a deformation argument. This argument works because the differentials $f'$ and $\bar{f}'$ “look the same” near $(-n) \ast u + n \ast v$, where $u, v$ are critical points associated to the mountain-pass, possibly equal.

The proof of Theorem II is based on the same ideas, but contains several technical improvements.

We first construct, for any $r, h > 0$, a non-trivial homology class in $H_1(f^c+ h, f^c)$, containing a chain included in $B(u, r)$, thanks to assumption
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(\mathcal{H}) Here, \tilde{c} = f(u) \in \langle c, c' \rangle, and u \in \mathcal{C}, found thanks to the mountain-pass, is independent of r, h (see § IV).

Then, roughly speaking, we consider a product of m "copies" of this homology class, and find a "product min-max" in a neighborhood of \( \sum_{i=1}^{m} p^i \ast u \). This is done in section IV thanks to Künnew's formula,

\[
H_\ast(X \times Y, (Z \times Y) \cup (X \times T)) = H_\ast(X, Z) \otimes H_\ast(Y, T).
\]

Note that in [S], [CZ-R]1, a more elementary procedure (without homology) is used to construct the product min-max. It would be possible to use this procedure in the proof of Theorem II. But the method involving homology seems easier to generalize to situations where the min-max is not of mountain-pass type.

Finally, we find a critical point \( u_p \) in a neighborhood of \( \sum_{i=1}^{m} p^i \ast u \), provided \((p^{i+1} - p^i) \geq K\), K depending only on r, not on m. To do this, we assume that \( u_p \) does not exist, construct a more precise version of the deformation used in [S], and apply it to the "product min-max" to obtain a contradiction (see § V).

In the proof of Theorem II, a crucial point is to make a suitable choice of the neighborhood of \( \sum_{i=1}^{m} p^i \ast u \) in which we want to find \( u_p \); this choice allows to control K as \( m \) increases. The correct neighborhood will be defined in the statement of Theorem III (see the end of section V), after the introduction of some technical notations. Theorem II will be a direct consequence of Theorem III.

III. COMPACTNESS PROPERTIES OF f

We first prove the following result:

**Lemma 4.** Suppose (hA), (hR) and (\mathcal{H}) are true. Then there is an at most countable compact set \( D \) such that:

If \( (u_n)_{n \geq 0} \) satisfies \( f(u_n) \leq c', f'(u_n) \to 0 \), then

\[
(v \ast u) \leq c', f(u_n) \to 0 \text{ on } D.
\]

Here, \( B(D, r) = \{ x \in [0, +\infty]/d(x, D) < r \} \).

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Proof. - Consider the set
\[ D = \left\{ x \in [0, + \infty) / x = \sum_{i=1}^{m} \| u_i - v_i \|, \ m \geq 1, \ u_i, v_i \in \mathcal{C} \cup \{0\}, \ \sum_{i=1}^{m} f(u_i) \leq c', \ \sum_{i=1}^{m} f(v_i) \leq c' \right\}. \]

From (\mathcal{H}), D is at most countable.

Let us prove that D is compact. We know (see \cite{CZ-E-S}) that there is \( \Lambda > 0 \) such that
\[ (\forall u \in \mathcal{C}) \quad f(u) \geq \Lambda. \]

Consider a sequence \((d^n)\) in D, with
\[ d^n = \sum_{i=1}^{M_n} \| u^n_i - v^n_i \|, \quad u^n_i, v^n_i \in \mathcal{C} \cup \{0\}, \quad \sum_{i=1}^{M_n} f(u^n_i) \leq c', \]
\[ \sum_{i=1}^{M_n} f(v^n_i) \leq c', \quad (u^n_i = 0 \Rightarrow v^n_i \neq 0). \]

We have \( M_n \leq 2c' / \Lambda. \)

So, after extraction, we may assume that \( M_n = M \) is constant and, by Lemma 2, that, \( \forall i \in [1, M]: \)
\[ \| u^n_i - k^n_i \ast U_i \| \to 0, \quad U_i \in \mathcal{C}'^{m(i)}, \quad k^n_i \to \Omega, \]
\[ \| v^n_i - p^n_i \ast V_i \| \to 0, \quad V_i \in \mathcal{C}'^{m'(i)}, \quad p^n_i \to \Omega. \]

One easily sees that
\[ d_n \to \sum_{k=1}^{m'} \| \mathcal{U}_k - \mathcal{V}_k \| = d_\infty \]
where \( \mathcal{U}_k, \) resp. \( \mathcal{V}_k, \) if non-zero, are of the form \( n \ast U_l, \) resp. \( n \ast V_l, \) and \( d_\infty \in D. \)

We have thus proved that D is compact. The last step is to study \((u_n)\) such that
\[ f(u_n) \leq c', \quad f'(u_n) \to 0. \]

Assume there are two subsequences \((u_{pm})_{m \geq 0}, (u_{qm})_{m \leq 0}\) satisfying \( \| u_{pm} - u_{qm} \| \notin B(D, \rho) \) for some \( \rho > 0. \) After extraction, we may impose
\[ \| u_{pm} - \bar{\kappa}_m \ast \bar{\mu} \| \to 0, \quad \bar{\mu} = (\mu^1, \ldots, \mu^{r'}), \quad \bar{\kappa}_m \to \Omega, \]
\[ \| u_{qm} - \bar{\kappa}_m \ast \bar{\nu} \| \to 0, \quad \bar{\nu} = (\nu^1, \ldots, \nu^{s'}), \quad \bar{\kappa}_m \to \Omega, \]
\[ \sum f(\mu^i) \leq c', \quad \sum f(\nu^i) \leq c'. \]
After a new extraction, each sequence \((\kappa_m^i - \lambda_m^j)\) has a limit \(l_{i,j}\) in \(\mathbb{Z} \cup \{-\infty, +\infty\}\). Moreover, for each \(i\), \(\text{Card}\{j : |l_{i,j}| < +\infty\} \leq 1\).

Hence
\[
\|u_{pm} - u_{qm}\| \to \sum_{k=1}^{t} \|l_k \ast w_k - w_k'\|,
\]

where \((w_k)_{1 \leq k \leq t}\) is a reindexing of
\[\mu^1, \ldots, \mu^r, 0, \ldots, 0,\]

\((w_k')_{1 \leq k \leq t}\) is a reindexing of
\[\nu^1, \ldots, \nu^s, 0, \ldots, 0,\]

and \(l_k \in \mathbb{Z}\).

Clearly, \(\sum f(w_k) = \sum f(\mu) \leq c', \sum f(w_k') = \sum f(\nu) \leq c'. \) So \(\sum_{k=1}^{t} \|w_k - w_k'\| \in \mathcal{D},\)

which contradicts the assumption \(\|u_{pm} - u_{qm}\| \notin \mathcal{B}(\mathcal{D}, \rho)\). The last assertion of Lemma 4 is thus proved by contradiction. \(\square\)

We now give another lemma, that will be used in section V.

**Lemma 5.** Suppose that \(f\) satisfies \((h_A), (h_R)\) and \((\mathcal{M})\). Then the set
\[F = \left\{ x = \sum_{k=1}^{m} f(u_k)/m \geq 1, (u_1, \ldots, u_m) \in \mathcal{C}^m, (\forall k), f(u_k) \leq c' \right\}\]
is closed and a most countable.

The proof of Lemma 5 is analogous to that of Lemma 4, so we won't give it. Now, we prove Lemma 3 as a consequence of Lemma 4.

**Proof.** Consider a sequence \((u_n)\) such that
\[f(u_n) \leq c', \quad f'(u_n) \to 0, \quad (u_{n+1} - u_n) \to 0.\]

we want to prove by contradiction that \((u_n)\) is a Cauchy sequence.

Assume the contrary, \(i.e., \|u_{n} - u_{p_n}\| \to \delta > 0, p_n < q_n < p_{n+1}.\)

The open set \(\delta\setminus\mathcal{D}\) contains an interval \([d_1 - d_2, d_1 + d_2]\). And there is \(P\) such that
\[\left( p > P \Rightarrow \|u_{p+1} - u_p\| \leq \frac{d_2}{2} \right) .\]

So, if \(p_n > P,\)
\[\|u_{r_n} - u_{p_n}\| \in \left[ d_1 - \frac{d_2}{2}, d_1 + \frac{d_2}{2} \right] \text{ for some } r_n \in [p_n, q_n].\]

But this implies \(\|u_{r_n} - u_{p_n}\| \notin \mathcal{B}(\mathcal{D}, d_2/2),\) which is impossible by Lemma 4.

So \((u_n)\) is Cauchy, hence convergent. Lemma 3 is thus proved. \(\square\)
We now study the local compactness of $\mathcal{C}$. We prove

**Lemma 6.** — Assume (hA) and (hR) are true. There is $r_0 > 0$ such that, if a sequence $(u_n)$ satisfies

$$\begin{cases} f'(u_n) \to 0 \\ (\exists R > 0), \ (\forall p, q), \ \| (u_p - u_q) \chi_{R \setminus [-R, R]} \| \leq 2r_0 \end{cases}$$

then $(u_n)$ is precompact.

**Proof.** — We remark (see [CZ-E-S]) that there is $r_0 > 0$ such that

$$\frac{3r_0}{2} < \| u \| \quad (\forall u \in \mathcal{C})$$

We now apply Lemma 2 to the sequence $(u_n)$. If $m \geq 2$ or if $(m = 1)$ and

$$\lim_{p \to \infty} | k_p | = +\infty,$$

then for any $P > 0$, there are $p > q > P$ such that

$$\| (k_p \ast u - k_q \ast u) \chi_{R \setminus [-R, R]} \| \geq 3r_0.$$ 

This contradicts $\| (u_p - u_q) \chi_{R \setminus [-R, R]} \| \leq 2r_0$, for $P$ large enough. So $m = 1$, and we may extract a subsequence $u_{n(p)}$ such that $k_{n(p)} = k$ is constant, and $u_{n(p)} \rightharpoonup k \ast u \in \mathcal{C}$. Lemma 6 is thus proved. □

Lemma 6 will be used in the proof of Lemma 12, section V.

**IV. THE PRODUCT MIN-MAX**

We want to find a min-max at each level $k \sigma, \ k \geq 2$. This will be done thanks to singular homology over $\mathbb{Z}$. We first need to “localize” the min-max

$$\inf_{\gamma \in \Gamma} (\max f \circ \gamma) = c.$$ 

This will be done thanks to $(\mathcal{H})$.

We recall some notations:

$$f^l = \{ x / f(x) \leq l \}, \quad f^{< l} = \{ x / f(x) < l \},$$

$$f = (-f)^{-1}, \quad f^a = f \cap f^b,$$

$$B(x, \rho) = \{ y / \| y - x \| < \rho \}, \quad S(x, \rho) = \{ y / \| y - x \| = \rho \}.$$ 

We have

**Lemma 7.** — Assume (hA), (hR) and $(\mathcal{H})$ are true. Choose $r \in \mathbb{R}_+ \setminus \mathbb{D}$, with the notation of Lemma 4.
Then for any \( h > 0 \), exist \( p = p(h, r) \in \mathbb{N}^* \), \( (u^1, \ldots, u^p) \in (\mathcal{C} \cap f_{c+h}^\infty)^p \), and \( \gamma \in \Gamma \), with:

\[
\begin{align*}
(i) & \quad \text{Im}(\gamma) \cap f_c = \bigcup_{i=1}^p B(u^i, r) \\
(ii) & \quad \text{Im}(\gamma) \cap f_{c+h} = \emptyset \\
(iii) & \quad \text{Im}(\gamma) \cap f_c \cap \left[ \bigcup_{i=1}^p S(u^i, r) \right] = \emptyset
\end{align*}
\]

Proof. - Given \( r > 0 \), we just have to prove the result for \( h \) small enough. We take \( \gamma^h \in \Gamma \) such that \( f^* \gamma^h < c + h \).

We are going to take \( \gamma \) as a deformation of \( \gamma^h \). We choose \( \varepsilon > 0 \) such that \([r - 2\varepsilon, r + 2\varepsilon] \cap D = \emptyset \). For \( d \geq 0 \), we define

\[
\begin{align*}
U^d &= \{ x \in f_{c+h}^\infty(\forall y \in \mathcal{C} \cap f_{c+h}^\infty) \mid x - y \geq r + d \} \\
V^d &= \{ x \in f_{c+h}^\infty(\exists y \in \mathcal{C} \cap f_{c+h}^\infty) \mid x - y \in [r - d, r + d] \} \\
K^d &= \{ x \in f_{c+h}^\infty(\exists y \in \mathcal{C} \cap f_{c+h}^\infty) \mid x - y < r - d \} \\
&\quad \cup \{ x \in f_{c+h}^\infty(\exists y \in \mathcal{C} \cap f_{c+h}^\infty) \mid x - y > r + d \} \setminus V^d
\end{align*}
\]

We assume \( c + h < c' \). From Lemma 4, there is \( \mu > 0 \), independent of \( h \), and such that \( \inf \{ \| f'(x) \| / x \in V^{2\varepsilon} \} \geq \mu \). We assume, moreover, that \( h < \varepsilon e/2 \). We build a locally Lipschitz vector field \( V \) on \( f_{c+h}^\infty \), such that:

\[
\begin{align*}
(i) & \quad x \in K^{2\varepsilon} \cup f_{c-h}^\infty \Rightarrow V(x) = 0 \\
(ii) & \quad (\forall x) \quad f'(x). V(x) \leq 0, \quad | V(x) | \leq 2 | f'(x) |^{-1} \\
(iii) & \quad x \in U^{e} \cup V^{e} \Rightarrow f'(x). V(x) \leq -1
\end{align*}
\]

Consider the flow \( \phi_t \) defined by

\[
(\forall (t, x) \in \mathbb{R}_+ \times f_{c+h}^\infty) \quad \begin{cases} 
\phi_0(x) = x \\
\frac{\partial}{\partial t} \phi_t(x) = V \circ \phi_t(x).
\end{cases}
\]

Assume that for some \( x \in f_{c+h}^\infty \), the maximal interval of definition of \( t \mapsto \phi_t(x) \) is \([0, L[\, L < +\infty \). Then \( \int_0^L \| V \circ \phi_t(x) \| \, dt = +\infty \). So we can define a sequence \( (t_n) \) by

\[
t_0 = 0 \\
\int_{t_n}^{t_{n+1}} \| V \circ \phi_t(x) \| \, dt = \sqrt{L - t_n}
\]

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So we get
\[ (\forall (u, v) \in [t_n, t_{n+1}])^2: \| \phi_u(x) - \phi_v(x) \| \leq \sqrt{L - t_n} \]
\[ (\beta) \exists \gamma_n \in [t_n, t_{n+1}]: \left\{ \begin{array}{l}
|f' \circ \phi_{\gamma_n}(x)| \leq 2 \| V \circ \phi_{\gamma_n}(x) \|^{-1} \leq 2 \sqrt{L - t_n} \\
\phi_{\gamma_n}(x) \in f^{c + h} \mathcal{K}^{2e} \end{array} \right. \]
\[ (\gamma) \int_0^l \| V \circ \phi_t(x) \| dt = \sum_{n=0}^{+\infty} \sqrt{L - t_n}, \text{ where } l = \lim_{n \to \infty} t_n. \]

If \( l < L \), the left term of \((\gamma)\) is finite, and the right one infinite. So we have
\[ l = L, \text{ and} \]
\[ (\phi_{\gamma_{n+1}}(x) - \phi_{\gamma_n}(x)) \to 0, \quad f' \circ \phi_{\gamma_n}(x) \to 0. \]

Since \( f \) satisfies property \( \overline{PS}(c') \), we get
\[ u_{\infty} = \lim_{n \to \infty} \phi_{\gamma_n}(x) \in (f^{c + h} \mathcal{K}^{2e}) \cap \mathcal{G}. \]

But this intersection is empty. So we have proved that \( \phi_t \) is defined on \( \mathbb{R}_+ \times f^{c + h} \).

Now, suppose that \( f(x) < c + h \), and that \( \phi_h(x) \in \mathcal{U}^0 \cup \mathcal{V}^0 \). Then three situations may occur:

- \((\forall t \in [0, h]), \quad \phi_t \in \mathcal{U}^e \cup \mathcal{V}^e \) apply \((\text{iii})\), and conclude \( f' \circ \phi_h(x) < c \); contradiction.

- \((\exists y \in \mathcal{G} \cap f^{c + h} \quad (\exists \alpha, \beta) \subset [0, h]), \quad \| \phi_a(x) - y \| = r - e, \quad \| \phi_b(x) - y \| = r, \quad (\forall t \in [\alpha, \beta]), \quad \| \phi_t(x) - y \| \in [r - e, r]. \)

- \((\exists y \in \mathcal{G} \cap f^{c + h} \quad (\exists \alpha, \beta) \subset [0, h]), \quad \| \phi_a(x) - y \| = r + e, \quad \| \phi_b(x) - y \| = r, \quad (\forall t \in [\alpha, \beta]), \quad \| \phi_t(x) - y \| \in [r, r + e]. \)

In the second and third situations, we have \( \| \phi_b(x) - \phi_a(x) \| \geq e \), and from \((\text{ii}), (\text{iii})\), \( f' \cdot \mathcal{V}_y \leq -\frac{1}{2} \| f' \| \cdot \| \mathcal{V}_y \| \leq -\frac{\mu}{2} \| \mathcal{V}_y \| \) if \( y \in \mathcal{Q}[x, \beta](c) \cap f^{c - h} \).

Since \( h < \mu e/2 \), we also conclude \( f' \circ \phi_h(x) < c \); contradiction.

So we have proved that if \( f(x) < c + h \), then either \( f' \circ \phi_h(x) < c \), or \( \phi_h(x) \in \mathcal{K}^0 \).

Finally, \( \gamma = \phi_h \circ \gamma^h \) is such that
\[ \text{Im} \gamma \cap \bigcup_{y \in \mathcal{G} \cap f^{c + h}} \mathcal{S}(y, r) \cap f^{c} = \emptyset, \]
\[ (\text{Im} \gamma \cap f^{c}) \subset \bigcup_{y \in \mathcal{G} \cap f^{c + h}} \mathcal{B}(y, r). \]
Since \( \text{Im} \gamma \cap f_c \) is compact, we can extract a finite subcovering:

\[
(\text{Im} \gamma \cap f_c) \subseteq \bigcup_{i=1}^{p} B(u_i, r).
\]

Lemma 7 is thus proved. \( \square \)

Lemma 7 has a direct consequence:

**Corollary 7.1.** Assume \((\mathscr{H})\) is true. Choose \( r > 0, h > 0 \). Then there is \( u = u(r, h) \in C \cap f_c \) such that \( i_* \neq 0 \), where

\[
i_* : H_1(f^{<c+h} \cap B(u, r), f^{<c} \cap B(u, r)) \rightarrow H_1(f^{<c+h}, f^{<c})
\]

is the morphism induced by the canonical injection

\[
i : B(u, r) \rightarrow L^p.
\]

**Proof.** We just have to prove the result when \( r \in \mathbb{R}^+ \setminus D \): it will then be true for any \( r' \geq r \).

Let \( p_0 \) be the minimal value of \( p \) such that there are \((u^1, \ldots, u^p) \in C \cap f_c^p\) satisfying the conclusion of Lemma 7. \( \text{Im} \gamma \cap B(u^{p_0}, r) \) is the image of a 1-dimensional complex \( \omega \in C_1(f^{<c+h}) \), with \( \omega \in \tilde{\omega} \), for some \( \tilde{\omega} \in H_1(f^{<c+h} \cap B(u^{p_0}, r), f^{<c} \cap B(u^{p_0}, r)) \).

If \( i_* \tilde{\omega} = 0 \), then there is a singular 2-dimensional complex \( \Omega \in C_2(f^{<c+h}) \) such that \( \partial \Omega = \omega - \alpha \), with \( \alpha \in C_1(f^{<c}) \). So, replacing the curves of \( \omega \) by curves of \( \alpha \) in \( \gamma \), we get \( \tilde{\gamma} \) satisfying the conclusion of Lemma 7 with \( u^1, \ldots, u^{p_0-1} \). This contradicts the minimality of \( p_0 \). So \( i_* \tilde{\omega} \neq 0 \). Corollary 7.1 is thus proved, with \( u = u^{p_0} \). \( \square \)

Corollary 7.1 gives the existence of at least one critical point \( u \neq 0 \). The hypothesis \((\mathscr{H})\) seems too weak to get \( u \) independent of \( r, h \), and we cannot say that \( f(u) = c \). The fundamental reason for this is that the Palais-Smale condition is not satisfied. To overcome this difficulty, we shall make use of Lemma 6 which gives a local Palais-Smale condition.

We first choose \( p_0 \in ]0, r_0] \), \( d_0 > 0 \), such that \([p_0 - d_0, p_0 + d_0] \cap D = \emptyset \), \( r_0 \) being defined in Lemma 6.

We define

\[
\mu^0 = \frac{1}{2} \inf \{ ||f^c(x)||/x \in f_c, (\exists y \in C \cap f_c) : ||x - y|| \in [p_0, p_0 + d_0] \}.
\]

We take \( 0 < h < \min(\mu^0 d_0, c' - c) \). By Corollary 7.1, there are

\[
u^0 \in C \cap f_c, \quad \tilde{\omega} \in H_1(B(u^0, p_0) \cap f^{<c+h}, B(u^0, p_0) \cap f^{<c}),
\]

such that \( i_* \tilde{\omega} \neq 0 \), where

\[
i_* : H_1(f^{<c+h} \cap B(u^0, p_0), f^{<c} \cap B(u^0, p_0)) \rightarrow H_1(f^{<c+h}, f^{<c})
\]

is the morphism induced by the canonical injection

\[
i : B(u^0, p_0) \rightarrow L^p.
\]
We define
\[ X = (f^{c+h} \cap B(u^0, \rho^0)) \]
\[ \cup \left\{ x \in L^p \mid \|x - u^0\| \in [\rho^0, \rho^0 + d^0], f(x) < c + h \left( 1 - \frac{\|x - u^0\| - \rho^0}{d^0} \right) \right\}, \]
\[ Y = f^c \cap B(u^0, \rho^0 + d^0). \]

We call
\[ j_* : H_1 (f^{<c+h} \cap B(u^0, \rho^0), f^{<c} \cap B(u^0, \rho^0)) \rightarrow H_1 (X, Y) \]
the morphism induced by the canonical injections
\[ j_+ : f^{<c+h} \cap B(u^0, \rho^0) \rightarrow X, \]
\[ j_- : f^{<c} \cap B(u^0, \rho^0) \rightarrow Y. \]

Clearly, we have \( j_* \tilde{\omega} \neq 0. \)

We define \( \tilde{c} = \inf_{z \in I, \tilde{a}} (\max f(z)) \in [c, c + h]. \)

By arguments similar to those proving Lemma 7 and Corollary 7.1, we find, for any \( n \in \mathbb{N}^*, \) a critical point \( u^n \in \mathcal{C} \cap f^{<\tilde{c}+(1/n)} \cap B(u^0, \rho^0 - d^0), \) such that \( i^n_* \neq 0, \) where
\[ i_*^n : H_1 \left( f^{<\tilde{c}+(1/n)} \cap B\left(u^n, \frac{d^0}{n}\right), f^{<\tilde{c}} \cap B\left(u^n, \frac{d^0}{n}\right) \right) \rightarrow H_1 \left( f^{<\tilde{c}+(1/n)} \cap B(u^n, d^0), f^{<\tilde{c}} \cap B(u^n, d^0) \right) \]
is the morphism induced by the canonical injection
\[ i^n_* : B\left(u^n, \frac{d^0}{n}\right) \rightarrow B(u^n, d^0). \]

By Lemma 6, the sequence \( (u^n) \) is precompact (recall that \( \rho^0 < r_0). \) Considering one of its limit points, and taking \( r_1 = d^0/2, \) we get

**Lemma 8.** Assume that \((hA), (hR)\) and \((\mathcal{C})\) are true.

Then there are \( u \in \mathcal{C} \) with \( f(u) = \tilde{c} \in [c, c^\prime] \) and \( r_1 > 0, \) such that, for any \( r \in [0, r_1] \) and \( h > 0, \) we have \( i_* \neq 0 \) where
\[ i_* : H_1 \left( f^{<(\tilde{c}+h)} \cap B(u, r), f^{<\tilde{c}} \cap B(u, r) \right) \rightarrow H_1 \left( f^{<(\tilde{c}+h)} \cap B(u, r_1), f^{<\tilde{c}} \cap B(u, r_1) \right) \]
is the morphism induced by the canonical injection
\[ i : B(u, r) \rightarrow B(u, r_1). \]

The great difference with Corollary 7.1 is that \( u \) does not depend on \( r, \) \( h \) any more.

Lemma 8 gives a min-max localized around \( u. \) To get our multiplicity result, we are going to make products of several “copies” of this min-max. At each product will be associated a new critical point. We first
**Corollary 8.1.** Assume that (hA), (hR) and (H) are true. Choose $r \in ]0, 1[,$ $h > 0.$

Then there is $N = N(r, h)$ such that

$$(\forall (a, b) \in [N, + \infty)^2): \quad I_* \neq 0,$$

where

$$I_* : H_1 (f^{-<\varepsilon + h}) \cap B(u, r) \cap L^\beta_{(-a, b)}, f^{-<\varepsilon} \cap B(u, r) \cap L^\beta_{(-a, b)}) \to H_1 (f^{-<\varepsilon + h}) \cap B(u, r_1) \cap L^\beta_{(-a, b)}, f^{-<\varepsilon} \cap B(u, r_1) \cap L^\beta_{(-a, b)}$$

is the morphism induced by

$$I : B(u, r) \cap L^\beta_{(-a, b)} \to B(u, r_1) \cap L^\beta_{(-a, b)}.$$

and

$$L^\beta_{(-a, b)} = \{ x \in L^\beta / \text{supp}(x) \subset [-a, b] \}.$$

**Proof.** We choose $\tilde{\omega} \in H_1 (f^{-<\varepsilon + h}) \cap B(u, r), f^{-<\varepsilon} \cap B(u, r))$ such that $i_* (\tilde{\omega}) \neq 0,$

with the notations of Lemma 8.

The class $\tilde{\omega}$ has an element of the form $\sum_{i=1}^{r} \lambda_i \sigma_i,$ satisfying

$$(P) \quad [\lambda_i \in \mathbb{R}, \sigma_i : S^1 \to L^\beta \text{ continuous or } \sigma_i : [0, 1] \to L^\beta \text{ continuous, with } \sigma_i(0), \sigma_i(1) \in f^{-<\varepsilon}, \text{ and } \text{Im} (\sigma_i) \subset f^{-<\varepsilon + h}) \cap B(u, r) \text{ in both cases}].$$

For $t_1, t_2 \in \mathbb{R},$ we define

$$K_{t_1, t_2} : L^\beta (\mathbb{R}, \mathbb{R}^{2N}) \to L^\beta (\mathbb{R}, \mathbb{R}^{2N})$$

$$x(t) \mapsto \chi_{[t_1, t_2]}(t) x(t)$$

We note that $\bigcup_{i=1}^{r} \text{Im} \sigma_i$ is compact, so that

$$\lim_{(t_1, t_2) \to (-\infty, +\infty)} \left( \sup_{i=1}^{r} \left\{ \| x - K_{t_1, t_2}(x) \| ; x \in \bigcup_{i=1}^{r} \text{Im} \sigma_i \right\} \right) = 0.$$ 

Moreover, $f^{-<\varepsilon + h}) \cap B(u, r)$ and $f^{-<\varepsilon} \cap B(u, r)$ are open.

So there is $N = N(r, e, h) \in \mathbb{N}$ such that, if $(a, b) \in [N, + \infty)^2,$ then

$$\sum_{i=1}^{r} \lambda_i (K_{-a, b} \circ \sigma_i) \in \tilde{\omega}.$$

As a consequence, there is

$$\tilde{\omega} \in H_1 (f^{-<\varepsilon + h}) \cap B(u, r) \cap L^\beta_{(-a, b)}, f^{-<\varepsilon} \cap B(u, r) \cap L^\beta_{(-a, b)})$$

such that $\sum_{i=1}^{r} \lambda_i (K_{-a, b} \circ \sigma_i) \in \tilde{\omega},$ and $i_* (\tilde{\omega}) \neq 0$ implies $I_* (\tilde{\omega}) \neq 0.$ So $I_*$ cannot be zero.
Corollary 8.1 is thus proved. □

We now have to introduce some notations.
Take $x_i \in l^E$, $p = (p^1, \ldots, p^m) \in \mathbb{Z}^m$, $m \geq 1$, $p^i < p^{i+1}$. Denote

$$x_i = x_i f_i((p^i - 1 + p^{i+1})/2), \quad f_i(x) = f(x_i),$$

with $x_i$ the characteristic function of $I$, $p^0 = -\infty$, $p^{m+1} = +\infty$.

We have $x = \sum_{i=1}^m x_i$, but $f \neq \sum_{i=1}^m f_i$.

Consider the sets

$$\mathcal{L}^+ (h) = \bigcap_{i=1}^m (f_i)^{<\delta + h}, \quad \mathcal{L}^- (h) = \bigcup_{i=1}^m (f_i)^{<\delta - h},$$

and the “product” ball

$$
B^u_{\bar{p}, \rho} = \{ x \in L^p ((\forall i) \| (x - p^i * u)_i \|_{l^p} < \rho ) \}
$$

for $\rho > 0$, $u \in \mathbb{C}$.

From Künneth’s formula,

$$H_* (X \times Y, (Z \times Y) \cup (X \times T)) = H_* (X, Z) \otimes H_* (Y, T),$$

immediately follows

**Lemma 9.** Assume that (hA), (hR) and (Jf) are true. $u, r_1$ are the same as in Lemma 8. Choose $r \in [0, r_1]$, $h > 0$.

Then there is $N = N(r, h)$ such that, if $m \geq 1$ and $\bar{p} = (p^1, \ldots, p^m)$ satisfy $p^{i+1} - p^i \geq N$ for $1 \leq i \leq m - 1$, then

$$J_* \neq 0,$$

where

$$J_* : H_m (\mathcal{L}^+ (h) \cap B^u_{\bar{p}, r_1}, \mathcal{L}^- (0) \cap \mathcal{L}^+ (h) \cap B^u_{\bar{p}, r}) \rightarrow H_m (\mathcal{L}^+ (h) \cap B^u_{\bar{p}, r_1}, \mathcal{L}^- (0) \cap \mathcal{L}^+ (h) \cap B^u_{\bar{p}, r_1})$$

is the morphism associated to the canonical injection

$$J : B_{\bar{p}, r} \rightarrow B_{\bar{p}, r_1}.$$

Lemma 9 gives the desired product min-max.

**V. A DEFORMATION ARGUMENT**

In what follows, we assume once again that (hA), (hR) and (Jf) are true. $D, F$ are the same as in Lemmas 4, 5, $r_0$ is the same as in Lemma 6, $u, \tilde{c}, r_1$ are the same as in Lemmas 8, 9.
5.1. Construction of a vector field

From (hA) (hR), we know that ($\exists \theta, C_1 > 0$) ($\forall (X, Y) \in (L^2)^2$):

$$\left| \int (X, LY) \right| \leq C_1 \exp (-\theta \delta (X, Y)) \|X\|_\theta \|Y\|_\theta,$$

for $\delta (X, Y) = \text{dist} (\text{supp} X, \text{supp} Y)$.

From (hR), we know that

$$\left( \exists c_1 > 0 \right) \left( \forall (y, t) \in \mathbb{R}^{2N} \times \mathbb{R} \right), \quad c_1 |y|^\theta \leq G (y, t) \leq (V G (y, t), y),$$

$$\left( \exists c_2 > 0 \right) \left( \forall (y, t) \in \mathbb{R}^{2N} \times \mathbb{R} \right), \quad |V G (y, t)| \leq c_2 |y|^{\theta - 1}.$$

We choose $0 < r_2 < \min (1, r_1)$ such that

$$\frac{c_1}{2} (r_2)^\theta > 6 C_1 (r_2)^2, \quad \text{and} \quad B (u, r_2) \subset f^-.$$

We are going to use these technical conditions in the proof of the following Lemma:

**Lemma 10.** Assume that (hA), (hR) and ($\mathcal{X}$) are true, and to $0 < r < \frac{r_2}{2}$, associate $\epsilon = \epsilon (r)$ such that

$$r + 2 \epsilon \leq \frac{r_2}{2} \quad \text{and} \quad [r - 2 \epsilon, r + 2 \epsilon] \cap D = \emptyset.$$

There are $\mu = \mu (r) > 0$, $A = A (r) > 0$ such that:

If $m \geq 2$, and if $p \in \mathbb{Z}^m$ satisfies ($\forall i$): $p^{i+1} - p^i > A$, then:

$$\left( \forall x \in B_{p_\theta, r + \epsilon} \setminus B_{p_\theta, r - \epsilon} \right) \left( \exists \nu \in B_0^0 \right):$$

1) $f (x). V_x > \mu$;

2) ($\forall i$): $(f y) (x). V_x \geq 0$;

3) $\|y_i\| \geq r - \epsilon \Rightarrow (f y) (x). V_x > \mu$,

with the notation $y_i = (x - p^i * u)$.

**Proof.** Define

$$\bar{\mu} = \frac{1}{2} \inf \{ \|f^* (x)\|_{\theta, x \in B (u, r + 2 \epsilon (r)) \setminus B (u, r - \epsilon (r))} \}.$$

$\bar{\mu}$ depends only on $r$, and $\bar{\mu} > 0$ by Lemma 4. Let $x \in B_{p_\theta, r + \epsilon} \setminus B_{p_\theta, r - \epsilon}$, $i \in [1, m]$, and $y_i = (x - p^i * u)$. Impose $A > 64$.

We always have $\|x_i\| \leq \|u\| + r_2$. So there is $\tau^i \in [2 \sqrt{A}, A/2 - 2 \sqrt{A}]$ such that

$$\|x_i \tau^i \sqrt{\bar{\mu}} \|_{\theta, 1 - \|p^i\| \leq \tau^i \sqrt{\bar{\mu}}} \leq \frac{C_2}{A^{1/2^\theta}}.$$

Here, $C_2$ is a constant, but $\tau^i$ depends on $x$, $i$, $A$, $\bar{p}$.

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Now, impose \( \| u \chi_{|t| > \sqrt{\lambda}} \| \leq \frac{e}{3} \), and \( \frac{C_2}{A^{1/2}} \leq \frac{e}{3} \), which is possible for \( A \geq A^0(e) \).

Then, three possibilities may occur:

**First case:**

\[
\| x_i \chi_{|t| - p^t \leq \sqrt{\lambda}} \| \leq \frac{e}{3}.
\]

We take

\[
V_{x, i} = x_i (h - \chi_{|t| - p^t - \sqrt{\lambda}} + h + \chi_{|t| - p^t + \sqrt{\lambda}})
\]

with

\[
h_+ = 1 \quad \text{if} \quad \| x_i \chi_{|t| - p^t + \sqrt{\lambda}} \| \leq \frac{e}{6}, \quad h_+ = 0 \quad \text{otherwise},
\]

\[
h_- = 1 \quad \text{if} \quad \| x_i \chi_{|t| - p^t - \sqrt{\lambda}} \| \leq \frac{e}{6}, \quad h_- = 0 \quad \text{otherwise}.
\]

We have

\[
(f_3)'(x). V_{x, i} \geq c_1 \| V_{x, i} \|_{\beta} - C_1 \| V_{x, i} \|_{\beta} \frac{e}{3} - C_1 \| x \chi_{|t| - p^t - \sqrt{\lambda}} \|_{\beta} \| V_{x, i} \|_{\beta} - C_1 \| x \chi_{|t| - p^t + \sqrt{\lambda}} \|_{\beta} \| V_{x, i} \|_{\beta} \exp (-2 \theta \sqrt{\lambda})
\]

\[
\geq \frac{3c_1}{4} \| V_{x, i} \|_{\beta} - C_1 \frac{e}{3} \| V_{x, i} \|_{\beta} - C_1 (\| u \|_{\beta} + r_2) \| V_{x, i} \|_{\beta} \exp (-2 \theta \sqrt{\lambda})
\]

\[
\geq \frac{3c_1}{4} \| V_{x, i} \|_{\beta} - C_1 e \| V_{x, i} \|_{\beta} \quad \text{for} \quad A \geq A^1(e)
\]

\[
\geq \frac{3c_1}{4} \| V_{x, i} \|_{\beta} - 6 C_1 \| V_{x, i} \|_{\beta}^2
\]

\[
\geq \frac{c_1}{4} \| V_{x, i} \|_{\beta} \geq \frac{c_1}{4} \left( \frac{e}{6} \right)^{\beta}.
\]

\[
\left[ \text{We recall that} \quad \frac{e}{6} \leq \| V_{x, i} \|_{\beta} \leq \| u \chi_{|t| \geq \sqrt{\lambda}} \| + (r + e) \leq r_2 < 1, \quad \text{and that} \quad \frac{c_1}{2} (r_2)^{\beta} > 6 C_1 (r_2)^2 \right].
\]

**Second case:** \( \| x_i \chi_{|t| - p^t \leq \sqrt{\lambda}} \| < \frac{e}{3}, \) and \( \| y_i \| < r - e \). Then we take \( V_{x, i} = 0 \).
Finally,

\[
\begin{align*}
\| x \chi_{\{1 - p \leq t - \sqrt{A}\}} - p \ast u \| & \geq \| y_i \| - \| x \chi_{\{1 - p \leq t - \sqrt{A}\}} \| \\
& \geq r - e - \frac{e}{3} - \frac{e}{3} = r - 2e.
\end{align*}
\]

So there is \( W_{x,i} \in L^\beta \) such that \( \| W_{x,i} \| \leq 1 \), and

\[
f'(x) = f'(x_i \chi_{\{1 - p \leq t - \sqrt{A}\}}) + f'(x \chi_{\{1 - p \leq t + \sqrt{A}\}}) + \sum_{j \neq i} f'(x_j).
\]

But \( \| x^\beta \| \leq \frac{C_2}{A^{1/2\beta}} \), and \( \max \{ \| x^\alpha \|, \| x^\beta \|, \| x_j \| (j \neq i) \} \leq \| u \| + r_2 \).

We choose \( V_{x,i} = W_{x,i} \chi_{\{1 - p \leq t\}} \). Clearly, \( \| V_i \| \leq 1 \). Moreover, we have:

\[
f'(x) \cdot V_{x,i} \geq f'(x^\alpha) \cdot W_{x,i} - f'(x^\beta) \cdot V_{x,i} - \sum_{j \neq i} f'(x_j) \cdot V_{x,i} \\
\geq \bar{\mu} - C_1 (\| u \| + r_2) \exp(-\theta \sqrt{A}) \\
- c_2 \left( \frac{C_2}{A^{1/2\beta}} \right)^{\beta - 1} - C_1 \frac{C_2}{A^{1/2\beta}} - C_1 (\| u \| + r_2) \exp(-\theta \sqrt{A}) \\
- \sum_{j \neq i} C_1 (\| u \| + r_2) \exp(-\theta \sqrt{A}) \exp[-\theta (|i-j| - 1) A] \\
\geq \bar{\mu} - c_2 \left( \frac{C_2}{A^{1/2\beta}} \right)^{\beta - 1} - C_1 \frac{C_2}{A^{1/2\beta}} \\
- C_1 (\| u \| + r_2) \left( 2 + \frac{2}{1 - \exp(-\theta \sqrt{A})} \right) \exp(-\theta \sqrt{A}) \\
\geq \bar{\mu}/2 \quad \text{for} \quad A \geq A^2(r).
\]
Identically,

\((f') (x) \cdot V_{x,i} = f' (x^a + x^b + x^c) \cdot V_{x,i} \)

\[\geq \tilde{\mu} - c_2 \left( \frac{C_2}{A^{1/2\beta}} \right)^{\beta-1} - C_1 \frac{C_2}{A^{1/2\beta}} - 2 C_1 \left( \|u\| + r_2 \right) \exp \left( -\theta \sqrt{A} \right) \]

\[\geq \frac{\tilde{\mu}}{2} \quad \text{for } A \geq A^2.\]

**Conclusion.** - We now take \(V_x = \sum_i V_{x,i}.\) By construction, \(V_x \in B^\infty_{p,1}.\)

Denote by \(I^1, I^2, I^3\) the sets of indices \(i\) corresponding to Cases 1, 2, 3 respectively. We write

\[f' (x) \cdot V_x = \sum_{i \in I^1} f' (x) \cdot V_{x,i} + \sum_{i \in I^3} f' (x) \cdot V_{x,i} \]

\[\geq \sum_{i \in I^1} f' (x) \cdot V_{x,i} + \frac{\tilde{\mu}}{2} \text{ card } (I^3).\]

Now, there is a family \(J^1 \subset [0, m]\) such that

\[\sum_{i \in I^1} V_{x,i} = \sum_{j \in J^1} X^j,\]

where

\[X^j = (\xi^j_+ X^j_+ + \xi^j_- X^j_-) \chi_{p^j + 1 - t^j + 1 - \sqrt{\lambda}} \chi_{p^j + 1 - t^j + 1 + \sqrt{\lambda}} \chi_{((p^j + 1)/2, p^j + 1 - t^j + 1 - \sqrt{\lambda})} \chi_{((p^j + 1)/2, p^j + 1 - t^j + 1 + \sqrt{\lambda})} X \]

with \(\xi^j_\pm \in \{0, 1\}\), and

\[(\forall s \in \{+, -\}) \quad (\forall j \in [0, m]) \]

\[\left( \xi^j_s = 1 \Rightarrow \|X^j_x\| \geq \frac{e}{6}, \xi^j_s = 0 \Rightarrow \|X^j_x\| < \frac{e}{3}. \right)\]

So there are three possible situations

\[\xi^j_- = 0 \quad \text{and } \xi^j_+ = 1, \quad (\xi^j_- = 1 \quad \text{and } \xi^j_+ = 0).\]

**First situation:** \(\xi^j_- = 0 \quad \text{and } \xi^j_+ = 1.\)

Denote

\[Y^j = X^j_+ X^j_+ \chi_{p^j + 1 - t^j + 1 - \sqrt{\lambda}} \chi_{p^j + 1 - t^j + 1 + \sqrt{\lambda}} \cup \chi_{p^j + 1 - t^j + 1 - \sqrt{\lambda}} \chi_{p^j + 1 - t^j + 1 + \sqrt{\lambda}} \]

\[Z^j = x^j_+ + x^j_{j+1} - X^j - Y^j.\]

We have

\[f' (x) \cdot X^j = f' (X^j) \cdot X^j + f' (Y^j) \cdot X^j + f' (Z^j) \cdot X^j\]

\[\geq \frac{3 c_1}{4} \left( \|X^j\| - \frac{2}{3} \|X^j\| - 2 C_1 \|X^j\| \left( \|u\| + r_2 \right) \exp \left( -2 \theta \sqrt{A} \right) \right.\]

\[\left. - 2 \|X^j\| \left( \|u\| + r_2 \right) \sum_{i \geq 0} \exp \left( -2 \theta \sqrt{A} \right) \exp \left( -\theta i A \right) \right)\]

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Second situation: \( \xi^- = 0, \xi^+ = 1. \)

We now take

\[ Y_j = x (x_{1, j+1} + x_{j+1, j+2}) + x_{1, j+2} - x_{j+1, j+2}, \]
\[ Z_j = x_j + x_{j+1} - Y_j. \]

We have \( \| Y_j \| \leq \frac{e}{3} + \frac{e}{3} = \frac{2e}{3}, \) dist (supp \( Z_j \), Supp \( X_j \)) \( \geq \sqrt{A}. \) As in the first situation, we get

\[ f' (x) . X_j \geq \frac{3c_1}{4} \| X_j \| - C_1 e \| X_j \|, \]
\[ \geq \frac{3c_1}{4} \| X_j \| - 6C_1 \| X_j \|^2, \]
\[ \geq \frac{c_1}{4} \| X_j \| \geq \frac{c_1 e^\beta}{4 \cdot 6^\beta}. \]

The third situation is identical to the second one. Since \( I^1 \cup I^3 \) is non-empty, we take

\[ A(r) = \max (A^0, A^1, A^2, A^3, A^4) \quad \text{and} \quad \mu (r) = \min \left( -\mu, \frac{c_1}{4 \cdot 6^\beta} \right), \]

and Lemma 10 is proved. \( \Box \)

Lemma 11. - Suppose \( f \) satisfies (hA), (hR) and (\( A \)). To \( l < c' \), associate \( \eta = \eta (l) > 0 \) such that \( l + 2 \eta \leq c' \), and \( [l - 2 \eta, l + 2 \eta] \) \( \cap F = \emptyset. \)

Then there are \( \mathcal{A} = \mathcal{A} (l) \) and \( v = v (l) \) such that for any \( m \geq 2, \bar{p} \in \mathbb{Z}^m \), with \( (\forall \, i) \, p^{i+1} > p^i > \mathcal{A} \), we have:

\[ \left( \forall \, x \in B_{\bar{p}, (r/2)} \cap \bigcup_{i=1}^m (f_i)^{l+\eta}_{l-\eta} \right) (\exists \, \mathcal{V}_x \in B_{\bar{p}, 1}^0) : \]

- \( f' (x) . \mathcal{V}_x > v; \)
- \( (\forall \, i \in [1, m]) : (x \in (f_i)^{l+\eta}_{l-\eta} \Rightarrow (f_i)' (x) . \mathcal{V}_x > v); \)
- \( (\forall \, i) : (f_i)' (x) . \mathcal{V}_x > 0. \)
Proof. - We know that $f$ is uniformly continuous on any bounded part of $L^b$. So there is $\varepsilon (\eta) > 0$ such that, if $X, Y \in B(0, \|u\| + r_2)$, then

$$\|X - Y\| \leq \varepsilon \Rightarrow \|f(x) - f(y)\| \leq \eta.$$ 

Now, consider $\tilde{v} = \frac{1}{2} \inf \{ \|f'(x)\|; x \in f_i^{l+2\eta}\}$. From Lemma 5, $\tilde{v} > 0$. The proof of Lemma 11 is similar to that of Lemma 10, replacing $V$ by $\mathcal{V}$, $\mu$ by $\tilde{v}$, $A$ by $\mathcal{A}$, $e$ by $\mathcal{E}$. So we just sketch it. The three possibilities are:

First case: $\|x_i \chi_{\{t - p^i \leq t + \sqrt{\mathcal{A}}\}}\| \geq \frac{\mathcal{E}}{3}$, then

$$\mathcal{V}_{x, i} = x_i (h - \chi_{\{t - p^i \leq t - \sqrt{\mathcal{A}}\}} + h + \chi_{\{t - p^i \leq t + \sqrt{\mathcal{A}}\}}),$$

$$(f_i')(x_i) \mathcal{V}_{x, i} \geq \frac{c_1}{2} \mathcal{E}^b$$

for $\mathcal{A} \geq \max (\mathcal{A}_0, \mathcal{A}^1)$.

Second case: $\|x_i \chi_{\{t - p^i \leq t + \sqrt{\mathcal{A}}\}}\| < \frac{\mathcal{E}}{3}$, and $f_i(x) \notin [l - \eta, l + \eta]$, then

$$\mathcal{V}_{x, i} = 0.$$

Third case: $\|x_i \chi_{\{t - p^i \leq t + \sqrt{\mathcal{A}}\}}\| < \frac{\mathcal{E}}{3}$, and $f_i(x) \in [l - \eta, l + \eta]$, then

$$f(x \chi_{\{t - p^i \leq t + \sqrt{\mathcal{A}}\}}) \in [l - 2\eta, l + 2\eta]$$

for $\mathcal{A} \geq \mathcal{A}_0$, hence $f'(x \chi_{\{t - p^i \leq t + \sqrt{\mathcal{A}}\}}). \mathcal{W}_{x, i} > \tilde{v}$,

$$\|\mathcal{W}_{x, i}\| \leq 1, \quad \mathcal{V}_{x, i} = \mathcal{W}_{x, i} \chi_{\{t - p^i \leq t\}}$$

$$f'(x). \mathcal{V}_{x, i} \geq \tilde{v}/2, \quad (f_i')(x). \mathcal{V}_{x, i} \geq \tilde{v}/2,$$

for $\mathcal{A} \geq \mathcal{A}^2$.

The final study of $f'(x). \mathcal{V}_x$ is the same as in Lemma 10, and 11 is proved with $\mathcal{A} = \max (\mathcal{A}_0, \ldots, \mathcal{A}^4)$, $\nu = \min \left( \frac{\tilde{v}}{2}, \frac{c_1}{2} \mathcal{E}^b \right)$.

Lemma 12. - Suppose $f$ satisfies (hA), (hR) and (H).

$r, e(r), A(r), \mu(r)$ are the same as in Lemma 10. We impose, moreover, $r < r_0$, with the notation of Lemma 6.

Choose $\lambda > 0$ such that $\bar{c} + \lambda < c'$

and

$$\left\{ \begin{array}{l}
\bar{c} + \lambda \notin \mathcal{F} \\
\bar{c} - \lambda \notin \mathcal{F}.
\end{array} \right.$$ 

Suppose $m \geq 2, \bar{p} \in \mathbb{Z}^m$,

$$(p^{i+1} - p^i) \geq \max (A(r), \mathcal{A} (\bar{c} - \lambda), \mathcal{A} (\bar{c} + \lambda))$$

$$= \mathcal{B} (r, \lambda)$$

($\mathcal{A}$ has been defined in Lemma 11).
If \( \mathcal{C} \cap B_{p,r}^u \cap \mathcal{L}^+ (\lambda) \mathcal{L}^- (\lambda) = \emptyset \), then there are \( \xi = \xi (p, r, \lambda) > 0 \) and a locally Lipschitz vector field \( V(x) \) such that:

(i) \( (\forall x) : V(x) \in B_{p,1}^0 \) and \( x \notin B_{p,(r/2)}^u \Rightarrow V(x) = 0 \);

(ii) \( \forall x \in B_{p, r}^u \setminus B_{p, (r-e)}^u, \forall i \in \{1, m\}, \quad \left\| y_i \right\| \in [r-e, r] \Rightarrow (f_i)'(x) \cdot V(x) > \frac{\mu(r)}{3} \).

(iii) \( \forall x \in B_{p, r}^u \cap \mathcal{L}^+ \mathcal{L}^- : f'(x) \cdot V(x) > \xi \).

(iv) \( \forall x \in B_{p, (r/2)}^u (\forall i \in \{1, m\}) : \)

\( (f_i(x)) \in \{ \tilde{c} + \lambda, \tilde{c} - \lambda \} \Rightarrow (f_i)'(x) \cdot V(x) > 0 \).

**Proof.** — In Lemma 6, take \( R = \max (|p^1|, |p^m|) \). Consider a sequence \( (u_n) \in B_{p, r}^u \mathcal{L}^+ (\lambda - \eta (c + \lambda)) \mathcal{L}^- (\lambda - \eta (c - \lambda)) \).

\( (u_n) \) satisfies

\( (\forall p, q) : \left\| (u_p - u_q) \chi_{\mathbb{R} \setminus [-r, r]} \right\| < 2r_2 < 2r_0. \)

So, if \( \mathcal{C} \cap B_{p, r}^u \cap \mathcal{L}^+ (\lambda) \mathcal{L}^- (\lambda) = \emptyset \), we cannot have \( f'(u_n) \to 0 \), and there is \( \alpha (p, u, r, \lambda) > 0 \) such that

\[ \forall x \in B_{p, r}^u \cap \mathcal{L}^+ (\lambda - \eta (c + \lambda)) \mathcal{L}^- (\lambda - \eta (c - \lambda)) : \| f'(x) \| \geq 2 \alpha. \]

Now, if \( x \in [B_{p, (r + e)}^u \setminus B_{p, (r-e)}^u] \), we find \( V_x \) satisfying the conclusion of Lemma 10, and we choose \( V_x = 0 \) otherwise.

For \( s \in \{-, +\} \). if \( x \in B_{p, (r/2)}^u \cap \mathcal{L}^+ (\lambda + s \kappa) \mathcal{L}^- (\lambda - \eta (c + s \kappa)) \), we find \( V_x \) satisfying the conclusion of Lemma 11 with \( i = c + s \kappa \), and we choose \( V_x^s = 0 \) otherwise.

If \( x \in B_{p, r}^u \cap \mathcal{L}^+ (\lambda) \mathcal{L}^- (\lambda) \) and if \( V_x = V_x^+ = V_x^- = 0 \), we find \( V_x \in B_{p, 1}^0 \) such that \( f'(x) \cdot V_x > \alpha \), and we choose \( V_x = \frac{1}{3} (V_x^++V_x^-) \) otherwise.

We take \( \xi = \min \left\{ \frac{1}{3} (\mu(r) + \nu(c + \lambda) + \nu(c - \lambda)) \right\} \).

\( V_x \) satisfies:

(I) \( (\forall x) : V_x \in B_{p, 1}^0 \) and \( x \notin B_{p, (r/2)}^u \Rightarrow V_x = 0 \).

(II) \( (\forall x) : V_x \in [B_{p, r+e}^u \setminus B_{p, (r-e)}^u], \forall i \in \{1, m\}, \quad \left\| y_i \right\| \in [r-e, r+e] \Rightarrow (f_i)'(x) \cdot V_x > \frac{\mu(r)}{3} \).

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We suppose (hA), (hR) and (H) are true. r, e (r), μ (r), λ are the same as in Lemma 12. On λ, we impose one more condition:

\[ \lambda \leq \frac{\mu (r) e (r)}{6}. \]

As in Lemma 12, we suppose that

\[ \mathcal{C} \cap \mathbb{B}^e_{\bar{p}, r} \cap (\mathcal{L} \setminus \mathcal{L}_-)(\lambda) = \emptyset, \]

and we take \( m \geq 2, \bar{p} \in \mathbb{Z}^m \) with

\[ (\forall i) \quad (p^{i+1} - p^i) \geq \mathcal{B} (r, \lambda). \]

We define \( \varphi (t, x) \) for \( (t, x) \in \mathbb{R} \times \mathbb{L}^b \) by

\[ \varphi (0, x) = x \]

\[ \frac{\partial \varphi}{\partial t} (t, x) = -V^* \varphi (t, x), \]

where \( V (x) \) is the vector field of Lemma 12.

We have

**Lemma 13.** - With the notations and hypotheses above, there is \( \mathcal{F} = \mathcal{F} (r, \lambda, \bar{p}) \) such that

\[ \varphi (\mathcal{F}, \cdot) [\mathbb{B}^e_{\bar{p}, r-e} \cap \mathcal{L} \setminus (\lambda)] = \mathcal{L} \setminus (\lambda) \cap \mathcal{L} + (\lambda). \]

**Proof.** - Take \( x \in \mathbb{B}^e_{\bar{p}, r-e} \cap \mathcal{L} + (\lambda) \). Then

\[ (\forall t \geq 0), \quad \varphi (t, x) \in \mathbb{B}^e_{\bar{p}, (r/2)} \cap \mathcal{L} + (\lambda), \]

by (i) and (iv) of Lemma 12. Moreover, if \( \varphi (t, x) \in \mathcal{L} - (\lambda) \), then for any \( t' \geq t \), \( \varphi (t', x) \in \mathcal{L} - (\lambda) \), by (iv). Now, define

\[ S = S (\bar{p}) = \sup \{ |f (X) - f (Y)|; (X, Y) \in (\mathbb{B}^e_{\bar{p}, (r/2)})^2 \}. \]

Define

\[ \mathcal{F} = \frac{2 S (\bar{p})}{\xi (\bar{p}, r, \lambda)}. \]
By (iii) of Lemma 12, there is \( t_x \in [0, \mathcal{F}] \) such that
\[
\varphi(t_x, x) \notin B_{\tilde{p} r} \cap (\mathcal{L}^+ (\lambda) \setminus \mathcal{L}^- (\lambda)).
\]
By (i), (ii) of Lemma 12, this implies \( \varphi(\mathcal{F}, x) \in \mathcal{L}^- (\lambda) \) (we recall that \( 2 \lambda \leq \mu(r) e(r)/3 \)).

Lemma 13 is thus proved. \( \square \)

Now, we impose
\[
(\forall i) \quad (p_i^{i+1} - p_i) \geq N(r - e(r), \lambda),
\]
with the notations of Lemma 9.

The conclusion of Lemma 13 clearly implies \( J_* = 0 \), which contradicts the conclusion of Lemma 9.

Now, for any \( h > 0 \), we may choose \( \lambda < h \) satisfying all the conditions above.

So, by contradiction, we have proved the following result:

**THEOREM III.** - Assume that \((hA), (hR)\) and \((\mathcal{H})\) are true.

Then there is \( u \in \mathcal{C} \), with \( f(u) = \tilde{c} \in [c', c] \), and such that for any \( r, h > 0 \), for all \( m \geq 1 \) and \( \tilde{p} = (p_1, \ldots, p_m) \in \mathbb{Z}^m \):
\[
[ (\forall i) : (p_i^{i+1} - p_i) \geq M(r, h) ] \Rightarrow [ \mathcal{C} \cap U_{\tilde{p}, r, h} \neq \emptyset ].
\]

\( M(r, h) \) is a constant independent of \( m \), and \( U_{\tilde{p}, r, h} \) is a neighborhood of
\[
\sum_{i=1}^m p_i \ast u \text{ defined as follows:}
\]
\[
U_{\tilde{p}, r, h} = B_{\tilde{p} r} \cap (\mathcal{L}^+ (h) \setminus \mathcal{L}^- (h)), \text{ with the notations of Lemma 9.}
\]

We now prove Theorem II:

We take a fixed value of \( h \), and we write \( M(r) \) instead of \( M(r, h) \). We may choose \( K > M(r) \) large enough to get \( \| u_{\chi_{\{1 \leq K/2 \}}} \| \leq r \), which implies
\[
\sum_{i=1}^m p_i \ast u \in B_{\tilde{p} r}, \text{ for any } m \geq 2, \text{ and } \tilde{p} \in \mathbb{Z}^m \text{ such that } (\forall i) (p_i^{i+1} - p_i) \geq K. \text{ So, from Theorem III, there is } u_{\tilde{p}} \in \mathcal{C} \text{ such that}
\]
\[
(\forall i \in \mathbb{Z}) : \quad \left\| \left( u_{\tilde{p}} - \sum_{i=1}^m p_i \ast u \right) \chi_{\{ ((p_i^{i-1} + p_i)/2), ((p_i + p_i^{i+1}))/2 \}} \right\|_\beta \leq 2r.
\]
So, defining \( y_{\tilde{p}} = L u_{\tilde{p}} \):
\[
\left\| y_{\tilde{p}} - \sum_{i=1}^m p_i \ast x \right\|_\infty \leq 3 C_3 \sum_{n \geq 0} 2r \exp [-2 \theta' n M(r)]
\]
\[
= \frac{6 C_3 r}{1 - \exp (-2 \theta' K)} \leq \varepsilon,
\]
for \( K (\varepsilon) \) large enough. So Theorem II is a direct consequence of Theorem III. \( \square \)

We are now going to study the limit \( (m \to + \infty) \).

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Our first task here is to prove Corollary II.1 of Theorem II. We consider a sequence \( \bar{p} = (p^i)_{i \in \mathbb{Z}} \) of integers with \( I \subset \mathbb{Z} \) a finite or infinite interval, and \( p^{i+1} - p^i \geq K(\varepsilon) \).

The case \( 0 \leq \text{Card}(I) < \infty \) is clear. So we just consider the case of an infinite \( I \). We may write \( I = \bigcup_{k \geq 0} I^k \), each \( I^k \) being finite. From Theorem II, we get an orbit \( y^k \) such that
\[
\left\| y^k - \sum_{i \in I^k} p^i \ast x \right\|_{\infty} \leq \varepsilon.
\]
The \( y^k \)'s being orbits, \( \left\| y^k \right\|_{\infty} + \left\| \frac{d}{dt} y^k \right\|_{\infty} \) is a bounded sequence. So, after extraction, by Ascoli’s theorem, \( y^k \) converges to some orbit \( y_p \) in the \( C^0_{\text{loc}} \) topology, and Corollary II.1 is proved.

Now, we take \( s \in \{0, 1\}^\mathbb{Z} \) arbitrary (i.e. with possibly infinitely many \( 1 \)'s). There are an interval \( I \) of integers and a sequence \( (q^i)_{i \in \mathbb{Z}} \subset \mathbb{Z} \), with \( (\forall i) q^{i+1} > q^i \), and \( s_n = \chi_{[q^i, q^i+1)}(n) \).

We denote \( p^i = K(\varepsilon)q^i \), and we define \( \mathcal{T}(s) = y_p \), using Corollary II.1.

We recall that \( \{0, 1\}^\mathbb{Z} \) may be given the topology associated to the metric \( d(s, s') = \frac{1}{3} \sum_{n \in \mathbb{Z}} \frac{|s_n - s'_n|}{2^n} \).

We define
\[
\bar{\tau} : \{0, 1\}^\mathbb{Z} \to \mathbb{R}^{2N}
\]
\[
s \mapsto \mathcal{T}(s)(0).
\]
Since
\[
\left\| \mathcal{T}(s) - \sum_n s_n (K n \ast x) \right\|_{\infty} \leq \varepsilon,
\]
we have \( \limsup_{d(s, s') \to 0} |\bar{\tau}(s') - \bar{\tau}(s)| \leq 2\varepsilon \).

Now, we take \( \delta > 0 \). There is \( I(\delta) > 0 \) such that if \( d(s, s') \geq \delta \), then \( s' \neq (s')' \).

So, taking \( K(\varepsilon) \) large enough in Corollary II.1, there is \( \rho > 0 \) independent of \( s, s', \varepsilon \), with
\[
\left\| \left( \sum_n s_n (K n \ast x) - \sum_n s'_n (K n \ast x) \right) \chi_{[-2I, 2I]} \right\|_{\infty} \geq 2 \rho.
\]
So
\[
\left\| \left( \mathcal{T}(s) - \mathcal{T}(s') \right) \chi_{[-2I, 2I]} \right\|_{\infty} \geq \rho
\]
for \( \varepsilon < \frac{\rho}{2} \).
Now, define
\[ \mathcal{O} : \mathbb{R}^{2N} \to C^0 \left( [-2I, 2I], \mathbb{R}^{2N} \right) \]
\[ x \mapsto \mathcal{O}(x) \]
where
\[ \frac{d}{dt} \mathcal{O} - J A \mathcal{O} = J \nabla R(t, \mathcal{O}) \]
\[ \mathcal{O}(x)(0) = x. \]

By the classical continuity results on the Cauchy problem, \( \mathcal{O} \) is uniformly continuous on any bounded part of \( \mathbb{R}^{2N} \). So there is \( \rho' (\delta) > 0 \), independent of \( s, s', r \), such that
\[ \| \mathcal{O}(s) - \mathcal{O}(s') \| \leq \rho'. \]

So \( \tau \) is injective, and \( \tau^{-1} \) is uniformly continuous. The other assertions of Corollary II.2 are easy to check, if we choose \( x_0 = x(0) \). Corollary II.2 is thus proved. One would like \( \tau \) to give a Bernoulli shift structure, i.e. \( \tau \) homeomorphism, and \( \tau * \sigma = \varphi^k * \tilde{\tau} \) (see [M], [W]). Unfortunately, this is not the case. We only have the estimate \( \| F(s) - \sum_n n \cdot x_n \|_\infty \leq \varepsilon \). The points \( s \) such that \( s_n = 0 \) except for a finite number of \( n \)'s correspond to homoclinic orbits passing through \( \tilde{\tau}(s) \) at time \( 0 \): there are infinitely many of them.

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