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VLADIMIR ŠVERÁK

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## On Tartar's conjecture

by

**Vladimír ŠVERÁK**

Department of Mathematics,  
Heriot-Watt University, Edinburgh EH14 4AS, UK.  
(On leave from MFF UK, Sokolovská 83, Praha)

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**ABSTRACT.** — We prove that the only probability measures supported at connected subsets of  $2 \times 2$  matrices without rank-one connections and commuting with the determinant are Dirac masses. We also prove some regularity results for fully nonlinear  $2 \times 2$  elliptic systems of the first order.

*Key words :* Young measures, compactness, regularity.

**RÉSUMÉ.** — Soit  $K$  un sous-ensemble connexe de matrices deux par deux sans connexion de rang un et soit  $\nu$  une mesure de probabilité concentrée sur  $K$  qui commute avec le déterminant. On démontre que  $\nu$  est une masse de Dirac. On démontre aussi quelques résultats de régularité pour des systèmes elliptiques deux par deux du premier ordre.

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### 1. INTRODUCTION

Let  $\Omega \subset \mathbf{R}^2$  be open and bounded. For functions  $v : \Omega \rightarrow \mathbf{R}^2$  we consider nonlinear systems given by  $Dv(x) \in K$ , where  $K$  is a submanifold of the

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set  $M^{2 \times 2}$  of all  $2 \times 2$  matrices. We shall be interested in regularity of solutions of these systems and also in the following question: if  $v_j: \Omega \rightarrow \mathbf{R}^2$  is a sequence of functions such that  $|Dv_j| \leq c$  and  $\text{dist}(Dv_j(\cdot), K) \rightarrow 0$  in  $L^p$ , what can be said about compactness of the sequence  $Dv_j$  in  $L^p$ ? Since for every  $A, B \in M^{2 \times 2}$  with  $\text{rank}(A - B) = 1$  we can construct a sequence of piecewise linear functions whose gradients oscillate between  $A$  and  $B$ , a necessary condition to get some positive results is that  $\text{rank}(A - B) \geq 2$  for any two distinct matrices  $A, B \in K$ . Tartar's conjecture (see [14]) in our special situation says that this condition should be also sufficient for the compactness of the sequences above. Here we prove that this holds true under the additional assumption that  $K$  is connected. (Without additional assumptions the conjecture fails. For a counterexample with  $K$  consisting of four matrices see [7]. Counterexamples in higher dimensions can be found in [2].) We also give a simple proof of the fact that if  $K$  is connected,  $\text{rank}(A - B) \geq 2$  for each  $A, B \in K$  distinct, and the system  $Dv(x) \in K$  is elliptic (i. e. planes tangent to  $K$  do not contain rank-one directions), then the solutions which are Lipschitzian belong to  $C^{1, \alpha}$  for some  $\alpha > 0$ . If, moreover,  $K$  is smooth, then the solutions are smooth. *A priori* estimates for the  $C^{1, \alpha}$ -norm of twice differentiable solutions of the systems considered here are well-known. (See, for example, [8], Chapter 12.) I am not aware of any previous regularity results for Lipschitzian solutions, with the exception of the Monge-Ampère equation, which, of course, can be considered as a first-order elliptic system. In general, if  $K$  is two dimensional and is contained in symmetric matrices, then the equation  $Dv(x) \in K$  can be viewed as a fully nonlinear scalar equation of the second order for the potential of the vector field  $v$ . *A priori* estimates for solutions of such equations in arbitrary dimensions have been obtained in [5]. See also [8], Chapter 17.

## 2. PRELIMINARIES

Throughout this paper  $\Omega$  denotes a nonempty, bounded, open subset of  $\mathbf{R}^2$ . The Lebesgue spaces  $L^p$ , the Sobolev spaces  $W^{k, p}$  and the spaces  $C^{k, \alpha}$  of Hölder continuous functions are defined in the usual way.

Let us briefly recall basic facts concerning Young measures. (We refer the reader to [1] or [14] for more details.) Let  $z_j: \Omega \rightarrow \mathbf{R}^n$  be a sequence of functions bounded in  $L^\infty(\Omega)$ . It is possible to prove that there exists a subsequence  $z_\mu$  of  $z_j$  such that for any continuous function  $f: \mathbf{R}^n \rightarrow \mathbf{R}$  the sequence  $f \circ z_\mu$  converges weakly\* in  $L^\infty(\Omega)$  to some function  $h_f$ . Moreover, it is also possible to prove that there is a subset  $S$  of  $\Omega$  of measure zero and a family  $\{\nu_x, x \in \Omega \setminus S\}$  of probability measures on  $\mathbf{R}^n$  such that for

each continuous  $f: \mathbf{R}^n \rightarrow \mathbf{R}$  we have  $h_f(x) = \int_{\mathbf{R}^n} f(\lambda) dv_x(\lambda)$  for almost every  $x \in \Omega$ . We shall use the notation  $\int_{\mathbf{R}^n} f(\lambda) dv_x(\lambda) = \langle v_x, f \rangle$ . If almost all of the measures  $v_x$  are Dirac masses, then the sequence  $z_\mu$  is compact in  $L^r(\Omega)$  for any  $r < \infty$  and *vice versa*. The measures  $v_x$  are called Young measures.

We shall use the following lemma.

LEMMA 1. — *Let  $K$  be a connected topological space and let  $g: K \times K \rightarrow \mathbf{R}$  be a continuous function such that  $g(x, y) = g(y, x) \neq 0$  for every  $x, y \in K$ ,  $x \neq y$  and  $g(x, x) = 0$  for every  $x \in K$ . Then either  $g(x, y) \geq 0$  for every  $x, y \in K$  or  $g(x, y) \leq 0$  for every  $x, y \in K$ .*

*Proof.* — We notice that if  $g$  changes sign on  $K \times K$ , then there exists  $y \in K$  such that  $g(\cdot, y)$  changes sign on  $K$ . Indeed, supposing this is not the case, we consider the sets  $K^+ = \{y \in K, g(\cdot, y) \geq 0 \text{ on } K\}$  and  $K^- = \{y \in K, g(\cdot, y) \leq 0 \text{ on } K\}$ . These sets are clearly closed and  $K^+ \cap K^- = \emptyset$ . Since  $K$  is connected, we cannot have  $K^+ \cup K^- = K$ . Therefore the lemma will be proved if we show that under our assumptions the function  $g(\cdot, y)$  does not change sign for any  $y \in K$ . Suppose this is not true and let  $y_0 \in K$  be such that  $g(\cdot, y_0)$  changes sign. Let  $K_+ = \{x \in K, g(x, y_0) \geq 0\}$  and  $K_- = \{x \in K, g(x, y_0) \leq 0\}$ . We claim that  $K_+$  and  $K_-$  are connected. To see this, suppose that  $K_+ = U \cup V$ , where  $U, V$  are nonempty disjoint closed subsets of  $K_+$ . We can suppose  $y_0 \notin V$ . We now consider the sets  $\tilde{U} = K_- \cup U$  and  $\tilde{V} = V$ . These are closed sets covering  $K$ , i. e.  $\tilde{U} \cup \tilde{V} = K$ . We have

$$\tilde{U} \cap \tilde{V} = (K_- \cap V) \cup (U \cap V) \subset (K_- \cap K_+) \setminus \{y_0\}.$$

Since  $g$  does not vanish outside the diagonal, the last set is empty. Since  $K$  is connected and  $\tilde{U}$  is nonempty, the set  $V = \tilde{V}$  must be empty. This shows that  $K_+$  is connected. The proof for  $K_-$  is the same. Let  $x_+ \in K_+ \setminus \{y_0\}$  and  $x_- \in K_- \setminus \{y_0\}$ . The function  $g(x_+, \cdot)$  is positive at  $y_0$  and does not vanish on the connected set  $K_-$  containing  $y_0$ . Therefore it is positive on  $K_-$  and in particular  $g(x_+, x_-) > 0$ . On the other hand, the function  $g(x_-, \cdot)$  is negative at  $y_0$  and does not vanish on the connected set  $K_+$  containing  $y_0$  and therefore  $g(x_-, x_+) = g(x_+, x_-) < 0$ , a contradiction. The proof is finished.

## 3. COMPACTNESS

LEMMA 2. — Let  $K$  be a connected subset of  $M^{2 \times 2}$  and suppose that  $\text{rank}(X - Y) \geq 2$  for every two distinct matrices  $X, Y \in K$ . Then either  $\det(X - Y) \geq 0$  for all  $X, Y \in K$  or  $\det(X - Y) \leq 0$  for all  $X, Y \in K$ .

*Proof.* — This is an obvious consequence of Lemma 1.

LEMMA 3. — Let  $K$  be a bounded Borel measurable subset of  $M^{2 \times 2}$  such that  $\text{rank}(X - Y) \geq 2$  for any two distinct  $X, Y \in K$  and suppose that  $\det(X - Y)$  does not change sign on  $K \times K$ . Let  $\nu$  be a probability measure on  $M^{2 \times 2}$  carried by  $K$  (i.e.  $\nu(M^{2 \times 2} \setminus K) = 0$ ) and satisfying  $\langle \nu, \det \rangle = \det \langle \nu, \text{identity} \rangle$ . Then  $\nu$  is a Dirac mass, i.e.  $\nu = \delta_A$  for some  $A \in K$ .

*Proof.* — Let  $A = \langle \nu, \text{identity} \rangle$  be the centre of mass of  $\nu$ . Let  $b$  be the symmetric bilinear form on  $M^{2 \times 2}$  determined by  $\det X = \frac{1}{2}b(X, X)$ . We can write

$$\begin{aligned} & \int_{M^{2 \times 2}} d\nu(X) \int_{M^{2 \times 2}} d\nu(Y) \det(X - Y) \\ &= \int_{M^{2 \times 2}} d\nu(X) \int_{M^{2 \times 2}} d\nu(Y) (\det X + \det Y - b(X, Y)) \\ &= \int_{M^{2 \times 2}} d\nu(X) (\det X + \det A - b(X, A)) \\ &= \det A + \det A - b(A, A) = 0. \end{aligned}$$

Since  $\det(X - Y)$  does not change sign and vanishes only at the diagonal of  $K \times K$ , we see that the measure  $\nu \otimes \nu$  is supported at the diagonal of  $K \times K$  and therefore it must be a Dirac mass. The proof is finished.

THEOREM 1. — Let  $U^{(j)} = \begin{pmatrix} u_1^{(j)} & u_2^{(j)} \\ v_1^{(j)} & v_2^{(j)} \end{pmatrix}$  be a uniformly bounded sequence of matrix-valued functions on  $\Omega$  and suppose that the sequences  $\text{curl} u^{(j)}$  and  $\text{curl} v^{(j)}$  are compact in  $H^{-1}(\Omega)$ . Let  $K$  be a closed connected subset of  $M^{2 \times 2}$  such that  $\text{rank}(X - Y) \geq 2$  for any two distinct  $X, Y \in K$  and suppose that  $\text{dist}(U^{(j)}(x), K) \rightarrow 0$  for a.e.  $x \in \Omega$ . Then the sequence  $U^{(j)}$  is compact in  $L^p(\Omega)$  for every  $1 \leq p < \infty$ .

*Proof.* — Following L. Tartar [14] we consider a family of Young measures  $\nu_x$  associated to a subsequence of the sequence  $U^{(j)}$  we and use the div-curl lemma (see [14]) to infer that  $\langle \nu_x, \det \rangle = \det \langle \nu_x, \text{identity} \rangle$  for almost every  $x \in \Omega$ . Our assumptions clearly imply that  $\nu_x$  is supported

on a bounded subset of  $K$  for a.e.  $x \in \Omega$ . From Lemma 2 and Lemma 3 we see that  $\nu_x$  is a Dirac mass for almost every  $x \in \Omega$ . The proof is finished.

#### 4. RANK-ONE CONNECTIONS IN SETS OF GRADIENTS

The results of Section 3 can be used to generalize some results of [2], Section 5.

**THEOREM 2.** — *Let  $u: \Omega \rightarrow \mathbf{R}^2$  be a Lipschitzian function which coincides with an affine function  $A$  at the boundary of  $\Omega$  and suppose that  $Du$  is continuous in  $\Omega$ . If  $u$  is not affine, then there exist  $x, y \in \Omega$  such that  $\text{rank}(Du(x) - Du(y)) = 1$ .*

*Proof.* — Let us first assume that  $\Omega$  is connected and  $A = 0$ . Let  $K = \{Du(x), x \in \Omega\}$  and let  $\nu$  be the probability measure on  $M^{2 \times 2}$  given by

$$\langle \nu, f \rangle = \frac{1}{\text{meas } \Omega} \int_{\Omega} f(Du(x)) dx$$

for every continuous function  $f: M^{2 \times 2} \rightarrow \mathbf{R}$ . Under our assumptions the set  $K$  is bounded and connected. The measure  $\nu$  is carried by  $K$ . We claim that  $\langle \nu, \det \rangle = \det \langle \nu, \text{identity} \rangle$ . For this it is enough to prove that under our assumptions we have  $\int_{\Omega} Du(x) dx = 0$  and  $\int_{\Omega} \det Du(x) dx = 0$ . This is well known if  $u$  is Lipschitzian and compactly supported in  $\Omega$ . (See, for example, [11].) The general case can be brought to this case by extending  $u$  by 0 outside  $\Omega$  and integrating over a sufficiently large ball in which  $\Omega$  is compactly contained. We can now apply Lemma 2 and Lemma 3 and we see that if  $Du$  is not constant, then there must be rank-one connections in  $K$ . The proof in the case when  $\Omega$  is connected and  $A = 0$  is finished. The general case follows easily, since clearly  $u = A$  on the boundary of every connected component of  $\Omega$  and since we can replace  $u$  by  $u - A$ , if necessary.

*Remarks.* — 1. For any open set  $\Omega \subset \mathbf{R}^2$  it is possible to construct a Lipschitzian function  $u: \Omega \rightarrow \mathbf{R}^2$  vanishing at the boundary of  $\Omega$  and a bounded countable set  $S \subset M^{2 \times 2}$  such that there are no rank-one connections in the closure  $K$  of  $S$ ,  $0 \notin K$ , and  $Du \in S$  a.e. in  $\Omega$ . See [13].

2. For examples showing that Theorem 2 fails in higher dimensions (except, perhaps, for mappings from  $\Omega \subset \mathbf{R}^2$  to  $\mathbf{R}^3$ ) see [2].

## 5. REGULARITY

**THEOREM 3.** — *Let  $K$  be a bounded subset of  $M^{2 \times 2}$  and suppose that there is  $\lambda > 0$  such that either  $\det(X - Y) \geq \lambda |X - Y|^2$  for each  $X, Y \in K$  or  $\det(X - Y) \leq -\lambda |X - Y|^2$  for each  $X, Y \in K$ . Let  $v: \Omega \rightarrow \mathbf{R}^2$  be a Lipschitzian function satisfying  $Dv(x) \in K$  for almost every  $x \in K$ . Then there is  $p > 2$  such that  $v$  belongs to  $W_{loc}^{2,p}(\Omega)$ . In particular, the gradient  $Dv$  of  $v$  is Hölder continuous.*

*Proof.* — We will consider only the case  $\det(X - Y) \geq \lambda |X - Y|^2$ . For the proof in the case  $\det(X - Y) \leq -\lambda |X - Y|^2$  it is enough to replace  $\det$  by  $-\det$  in the formulae below. Let  $a \in \mathbf{R}^2$  and for  $h > 0$  let  $v_h(x) = (v(x + ha) - v(x))/h$ . (We can extend  $v$  by zero outside  $\Omega$ , for example.) Let  $\eta$  be a smooth nonnegative function compactly supported in  $\Omega$ . Let  $b \in \mathbf{R}^2$ . For sufficiently small  $h$  we have

$$\begin{aligned} 0 &= \int_{\Omega} \det D(\eta(v_h - b)) \, dx \\ &\geq \int_{\Omega} (-\eta |Dv_h| |D\eta| |v_h - b| + \eta^2 \det Dv_h) \, dx \\ &\geq \int_{\Omega} (-\eta |Dv_h| |D\eta| |v_h - b| + \lambda \eta^2 |Dv_h|^2) \, dx \\ &\geq -\frac{1}{2\lambda} \int_{\Omega} |D\eta|^2 |v_h - b|^2 \, dx + \frac{\lambda}{2} \int_{\Omega} \eta^2 |Dv_h|^2 \, dx. \end{aligned}$$

We see that the  $L^2$ -norm of  $Dv_h$  on compact subsets of  $\Omega$  is estimated by the  $L^2$ -norm of  $v_h$ . We can now use the well-known Nirenberg's Lemma to infer that  $Dv \in W_{loc}^{1,2}(\Omega)$ . It is well-known that if there exists  $C > 0$  such that

$$\int_{\Omega} \eta^2 |Dv_h|^2 \, dx \leq C \int_{\Omega} |D\eta|^2 |v_h - b|^2 \, dx$$

for every  $\eta$  as above and every  $b \in \mathbf{R}^2$ , or in another words, if  $v_h$  satisfies the Caccioppoli's inequality, then there exists a  $p > 2$  such that the  $L^p$ -norm of  $Dv_h$  on every set  $\tilde{\Omega}$  compactly contained in  $\Omega$  is bounded by  $C_1 \|v_h\|_{L^2(\Omega)}$ , where  $C_1$  depends only on  $C$ ,  $p$ ,  $\tilde{\Omega}$  and  $\Omega$ . (For a proof of this which is based on the technique of reverse Hölder inequalities see [6].) Using Nirenberg's Lemma again, we see that  $Dv$  is bounded in  $W_{loc}^{1,p}(\Omega)$ . The Hölder continuity of  $Dv$  follows from the Sobolev Imbedding Theorem. The proof is finished.

**COROLLARY.** — *Let  $K$  be a closed connected smooth submanifold of  $M^{2 \times 2}$  such that  $\text{rank}(X - Y) \geq 2$  for any two distinct  $X, Y \in K$ . Suppose moreover*

that  $K$  is "elliptic", or in other words, that for any  $X \in K$  the tangent space to  $K$  passing through  $X$  does not contain rank-one directions. Then every Lipschitz function  $v: \Omega \rightarrow \mathbb{R}^2$  satisfying  $Dv(x) \in K$  for a.e.  $x \in K$  is smooth.

*Proof.* – We notice that the ellipticity condition together with Lemma 1 implies that for each bounded subset  $K_1$  of  $K$  there exists  $\lambda > 0$  such that either

$$\det(X - Y) \geq \lambda |X - Y|^2 \quad \text{for every } X, Y \in K_1$$

or

$$\det(X - Y) \leq -\lambda |X - Y|^2 \quad \text{for every } X, Y \in K_1.$$

We can use Theorem 2 to infer that  $Dv$  is Hölder continuous and that  $v$  belongs to the space  $W_{loc}^{2,2}(\Omega)$ . Since  $Dv(x) \in K$  in  $\Omega$ , the derivatives  $\frac{\partial}{\partial x_i} Dv(x)$  belong to the tangent space of  $K$  at  $Dv(x)$  for a.e.  $x \in \Omega$ . Since

$Dv$  is Hölder continuous and  $K$  is elliptic, we see that  $\frac{\partial}{\partial x_i} v(x)$  can be viewed as solutions of a certain linear first order elliptic system with Hölder continuous coefficients. Therefore  $D^2 v$  is Hölder continuous. (See, for example, [11].) Applying the usual procedure of improving regularity we see that  $v$  must be smooth. The proof is finished.

### 6. EXAMPLES

Classical examples of  $K$ 's which are elliptic in the above sense are

$$K_0 = \{ X \in M^{2 \times 2}, X \text{ is symmetric and Trace } X = 0 \}$$

and

$$K_1 = \{ X \in M^{2 \times 2}, X \text{ is symmetric, positive definite, and } \det X = 1 \}.$$

Clearly  $K_0$  can be viewed as the tangent space to  $K_1$  at the unit matrix.

The following examples arise in connection with problems concerning invariant "wells" which appear in the theory of microstructures. (See, for example, [3], [4], [9], and [10] for motivation). Let  $A_1, \dots, A_m \in M^{2 \times 2}$  with  $\det A_k > 0$  for each  $k = 1, \dots, m$  and let

$$K_w = SO(2) \cdot A_1 \cup \dots \cup SO(2) \cdot A_m.$$

It is easy to check that if  $K_w$  does not contain rank-one connections (*i. e.* rank  $(X - Y) \geq 2$  for any two distinct  $X, Y \in K_w$ ), then there exists  $\nu > 0$  such that  $\det(X - Y) \geq \nu |X - Y|^2$  for each  $X, Y \in K_w$ . We see that in this case Lemma 3 and Theorem 3 can be applied to  $K_w$ . This shows, for example, that if  $K_w$  does not contain rank-one connections, then the deformations  $\phi: \Omega \rightarrow \mathbb{R}^2$  satisfying  $D\phi \in K_w$  a.e. in  $\Omega$  belong to  $C_{loc}^{1,\alpha}(\Omega)$ .

for some  $\alpha > 0$ . Using this it is not difficult to see that if  $K_w$  does not contain rank-one connections, then  $D\phi \in K_w$  a.e. in  $\Omega$  implies that in fact  $D\phi$  is locally constant in  $\Omega$ .

We can also consider continuous families of invariant wells. A simple example is the following: let  $\mu: [0, 1] \rightarrow \mathbf{R}$  and  $\lambda: [0, 1] \rightarrow \mathbf{R}$  be smooth strictly positive functions with  $\mu'(t) > 0$  and  $\lambda'(t) > 0$  for all  $t \in [0, 1]$  and let  $K_c = \bigcup_{t \in [0, 1]} \text{SO}(2) \cdot \begin{pmatrix} \lambda(t) & 0 \\ 0 & \mu(t) \end{pmatrix}$ . It is easy to check that  $K_c$  satisfies the assumptions of Theorem 1 and Theorem 3.

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