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On variational approach to photometric stereo

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ABSTRACT. — In this article we discuss a variational aspect of a system of two equations arising in photometric stereo. Assuming that this system has two distinct solutions, we construct a minimizing sequence for any function lying on a segment joining these two solutions. In general, the system in question may have one, two, four or more than four solutions.

Key words : Minimizing sequences, young measures.

RÉSUMÉ. — Dans cet article, nous étudions l'aspect variationnel d'un système de deux équations intervenant en « vision stéréo ». Sous l'hypothèse que le système a deux solutions distinctes, nous construisons une suite minimisante approchant toutes les fonctions appartenant au segment joignant ces deux solutions. En général, le système étudié peut posséder une, deux, quatre (ou plus) solutions.

1. INTRODUCTION

In this paper we discuss a variational approach to partial differential equations arising in computer vision [9]. The problem of interest is the

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so-caled shape − from − shading problem, in which one seeks a function $u(x_1, x_2)$, representing surface depth in the direction of z-axis, satisfying the irradiance equation

$$\mathrm{R}(u_{x_1}, u_{x_2}) = \mathrm{E}(x_1, x_2)$$

on a domain $\Omega \subset \mathbb{R}_2$. Here R is the so-called reflectance map containing the information on the illumination and surface-reflecting conditions, E is an image formed by orthographic projection of light onto a plane parallel to the (x_1, x_2)-plane and Ω is the image domain.

In this note we are interested in a case where the reflectance map corresponds to the situation in which a distant point − source illuminates a Lambertian surface. For full discussion of this case we refer to papers [4], [5] and [6]. We only briefly mention that if a small portion of a Lambertian surface u with normal direction $\left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, -1\right)$ is illuminated by a distant point-source of unit power in direction (p_1, p_2, p_3), then according to Lambert's law, the emitted radiance and reflectance map are given by cosine of the angle between these two directions. Denoting by $\mathrm{E}(x_1, x_2)$ the corresponding image, then the image irradiance equation in this situation takes the form

$$\frac{p_1 u_{x_1} + p_2 u_{x_2} - p_3}{\sqrt{p_1^2 + p_2^2 + p_3^2}\sqrt{u_{x_1}^2 + u_{x_2}^2 + 1}} = \mathrm{E}(x_1, x_2). \tag{1}$$

By a physical interpretation, E represents the intensity of the reflected light and must be nonnegative. Therefore the domain Ω of this equation consists of the points for which the left − hand side is nonnegative. On the other hand we have by Schwartz inequality $\mathrm{E}(x_1, x_2) \leqq 1$ and consequently $0 \leqq \mathrm{E}(x_1, x_2) \leqq 1$. Recently, some interesting results have been obtained in the case where the light source is situated overhead, that is, $p = (0, 0, -1)$ and the corresponding image irradiance equation takes the form

$$\frac{1}{\sqrt{u_{x_1}^2 + u_{x_2}^2 + 1}} = \mathrm{E}(x_1, x_2).$$

Letting $\mathscr{E}(x_1, x_2) = \mathrm{E}(x_1, x_2)^{-2} - 1$, we arrive at the eikonal equation

$$u_{x_1}^2 + u_{x_2}^2 = \mathscr{E}(x_1, x_2). \tag{2}$$

This equation has attracted the attention of several people [4], [5] and [6], where some results on existence and nonexistence of solutions can be found. These results have been obtained by considering a system of characteristic equations associated with (2). In this recent paper

Kozera [11], discussed the existence of solutions to the system

$$\left.\begin{aligned} \frac{p_1 u_{x_1} + p_2 u_{x_2} - p_3}{\sqrt{p_1^2+p_2^2+p_3^2}\sqrt{u_{x_1}^2+u_{x_2}^2+1}} &= \mathrm{E}_1(x_1, x_2) \\ \frac{q_1 u_{x_1} + q_2 u_{x_2} - q_3}{\sqrt{q_1^2+q_2^2+q_3^2}\sqrt{u_{x_1}^2+u_{x_2}^2+1}} &= \mathrm{E}_2(x_1, x_2). \end{aligned}\right\} \tag{3}$$

This is the case of photometric stereo. Here, the shape of a Lambertian surface is recovered from a pair of image data obtained by illumination from two different light source directions (*see* [9], [11] and [13]). In particular, Kozera [11] has obtained some results guaranteeing the existence of one, two, four or more than four solutions. The method employed in [11] consists of computing u_{x_1} and u_{x_2} and imposing natural conditions on the obtained expressions, which guarantee the existence of solutions. This paper also contains an interesting method showing how one can obtain a solution to (3) by glueing solutions defined on disjoint sets.

In a natural way we associate with (2) the functional $\mathrm{J}(u)$, given by

$$\mathrm{J}(u) = \int_\Omega \big|\, |\mathrm{D}u(x)|^2 - \mathscr{E}(x) \big|\, dx. \tag{4}$$

J is not convex and a minimization technique may not lead to an exact solution of (2). In a recent paper [7], the authors proved that any function u satisfying the inequality $(*)\ |\mathrm{D}u(x)|^2 \leqq \mathscr{E}(x)$ can be regarded as a "minimum" of the functional J, in the sense that there exists a sequence $\{u_n\}$, with $u_n|_{\partial\Omega} = u|_{\partial\Omega}$, such that

$$\lim_{n\to\infty} \int_\Omega \big|\, |\mathrm{D}u_n(x)|^2 - \mathscr{E}(x) \big|\, dx = 0$$

and

$$u_n \rightharpoonup u \text{ weak-}* \text{ in } \mathrm{W}^{1,\infty}(\Omega).$$

The same result continues to hold if the L^1-norm in J is replaced by a L^p-norm with $1 \leqq p < \infty$. This phenomenon occurs regardless of whether or not the equation (2) has an exact solution (for nonexistence result *see* [4], Theorem 1). Therefore, due to this approximating property, one should also consider any function u satisfying $(*)$ as a candidate for a possible shape, which is important in cases when the equation (2) has no exact solution. In practical terms this is due to the fact that the measurement of E is noisy. For this reason a function u satisfying $(*)$ is called a "noisy solution" to (2). In this paper we address the same question for the system (3). We assume that this system has at least two exact solutions u^1 and u^2. To describe our main result, let us introduce the following notation

$$f_1(\mathrm{D}u)=\frac{p_1 u_{x_1}+p_2 u_{x_2}-p_3}{\sqrt{p_1^2+p_2^2+p_3^2}\sqrt{u_{x_1}^2+u_{x_2}^2+1}}$$

and

$$f_2(\mathrm{D}u)=\frac{q_1 u_{x_1}+q_2 u_{x_2}-q_3}{\sqrt{q_1^2+q_2^2+q_3^2}\sqrt{u_{x_1}^2+u_{x_2}^2+1}}$$

and define a functional I by

$$\mathrm{I}(u)=\int_\Omega (|f_1(\mathrm{D}u)-\mathrm{E}_1(x)|+|f_2(\mathrm{D}u)-\mathrm{E}_2(x)|)\,dx. \tag{5}$$

To examine the functional I we introduce a functional E given by

$$\mathrm{E}(u)=\int_\Omega \min(|\mathrm{D}u(x)-\mathrm{D}u^1(x)|,\ |\mathrm{D}u(x)-\mathrm{D}u^2(x)|)\,dx. \tag{6}$$

We show that if $\{u_n\}$ is a sequence in $\mathrm{W}^{1,\infty}(\Omega)$ with $\lim_{n\to\infty}\mathrm{E}(u_n)=0$, then $\lim_{n\to\infty}\mathrm{I}(u_n)=0$. Also, weak-$*$ limit points of $\{\mathrm{D}u_n\}$ in $\mathrm{L}^\infty(\Omega)$, for which $\lim_{n\to\infty}\mathrm{E}(u_n)=0$, lie on the segments joining $\mathrm{D}u^1$ and $\mathrm{D}u^2$. This observation indicates that the "noise phenomenon" should occur for functions of the form $(**)\, u=\lambda u^1+(1-\lambda)u^2$, for some $\lambda\in(0,1)$. In fact, we show that for a given u of the form $(**)$, there exists $\{u_n\}$ in $\mathrm{W}^{1,\infty}(\Omega)$ such that $\lim_{n\to\infty}\mathrm{I}(u_n)=0$, $u_n\rightharpoonup u$ weak-$*$ in $\mathrm{W}^{1,\infty}(\Omega)$ and $u_n|_{\partial\Omega}=u|_{\partial\Omega}$. In both situations, due to the Sobolev compact embedding theorem, up to a subsequence, u_n converges uniformly to u. Consequently, in practice it is difficult to distinguish between u_n and u for large n. Therefore, this also supports the idea that in the case of the eikonal equation (2) any function u satisfying $(*)$ should also be considered as a candidate for a possible shape. In the case of system (3) the same role should be attributed to any function satisfying $(**)$. Finally, we point out that we use Young measures (*see* [2] and [3]) to understand the nature of oscillations of weakly convergent sequences occuring in our approach.

2. PRELIMINARIES AND SOME OBSERVATIONS ON YOUNG MEASURES

Let Ω be bounded domain in $\mathbb{R}_2$ with a Lipschitz boundary $\partial\Omega$. By $\mathrm{W}^{1,p}(\Omega)$, $1\leqq p\leqq\infty$, we denote usual the Sobolev spaces $\mathrm{W}^{1,p}(\Omega)$ (*see* [1]).

Since $\partial\Omega$ is Lipschitz, the elements of $W^{1,p}(\Omega)$ have traces on $\partial\Omega$. For $x \in \mathbb{R}^2$ we write $x=(x_1, x_2)$. We suppose that a Lambertian surface S, represented by the graph of a function $u \in C^1(\bar{\Omega})$, is illuminated from two linearly independent directions, namely $p=(p_1, p_2, p_3)$ and $q=(q_1, q_2, q_3)$. According to the discussion in Section 1, u satisfies the system of equations (3). We assume that $E_i(x)$, $i=1,2$, are continuous on $\bar{\Omega}$. Throughout this paper we assume that the system (3) has at least two distinct solutions. For our purposes, we fix two distinct solutions u^1 and u^2 belonging to $C^1(\bar{\Omega})$. With these two solutions we associate the functional E given by (6).

We describe below a situation, where the system (3) has exactly two distinct solutions. This result is taken from Kozera [11]. Obviously, we identify solutions which differ by a constant.

Let

$$\begin{aligned}\Lambda=\Lambda(x)=\|p\|^2\|q\|^2(1-E_1(x)^2-E_2(x)^2)\\ -\langle p,q\rangle(\langle p,q\rangle-2\|p\|.\|q\|E_1(x)E_2(x))\end{aligned}$$

and

$$\begin{aligned}\sigma^{\pm}(x)=\|p\|^2(p_3\|q\|^2-q_3\langle p,q\rangle)E_1(x)\\ +\|p\|.\|q\|(q_3\|p\|^2-p_3\langle p,q\rangle)E_2(x)\pm(p_1q_2-p_2q_1)\|p\|\sqrt{\Lambda(x)}.\end{aligned}$$

Here $\langle .,. \rangle$ denotes the scalar product in $\mathbb{R}_3$.

Proposition 1. — *Let* $E_i \in C^1(\bar{\Omega})$, $i=1, 2$, *where* Ω *is a simply connected region of* $\mathbb{R}_2$, *and let* $0 \leqq E_i(x) \leqq 1$ *on* Ω, $i=1,2$. *Suppose that* $\Lambda(x)>0$ *on* Ω. *Then a necessary and sufficient condition for the existence of exactly two solutions to* (3) *of class* $C^1(\bar{\Omega})$ *is, for each choice of sign,*

$$\begin{aligned}\frac{\partial}{\partial x_2}\Bigg[\frac{1}{\sigma^{\pm}(x)}(\|p\|^2(q_1\langle p, q\rangle-p_1\|q\|^2)E_1+\|p\|.\|q\|(p_1\langle p, q\rangle\\ -q_1\|p\|^2)E_2\pm(p_3q_2-p_2q_3)\|p\|\sqrt{\Lambda})\Bigg]\\ =\frac{\partial}{\partial x_1}\left(\frac{1}{\sigma^{\pm}(x)}A^{\pm}(x)\right)\\ =\frac{\partial}{\partial x_1}\Bigg[\frac{1}{\sigma^{\pm}(x)}(\|p\|^2(q_2\langle p, q\rangle-p_2\|q\|^2)E_1\\ +\|p\|.\|q\|(p_1\langle p, q\rangle-q_3\|p\|^2)E_2\pm(p_1q_3-p_3q_1)\|p\|\sqrt{\Lambda})\Bigg]\\ =\frac{\partial}{\partial x_1}\left(\frac{1}{\sigma^{\pm}(x)}B^{\pm}(x)\right).\end{aligned}$$

In this case the partial derivatives of u^1 *and* u^2 *are given by*

$$u^1_{x_1}=\frac{A^+(x)}{\sigma^+(x)},\qquad u^1_{x_2}=\frac{B^+(x)(x)}{\sigma^+(x)}$$

and

$$u^2_{x_1}=\frac{A^-(x)}{\sigma^-(x)},\qquad u^2_{x_2}=\frac{B^-(x)}{\sigma^-(x)}.$$

The following example illustrating Proposition 1 is taken from Kozera [11].

Let $p=(0,0,-1)$ and $q=\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right)$ and let

$$E_1(x)=(|x|^2+1)^{-1/2}\quad\text{and}\quad E_2(x)=(x_1+x_2+1)\,(3(|x|^2+1))^{-1/2}.$$

The corresponding irradiance equations have the form

$$\frac{1}{\sqrt{|Du(x)|^2+1}}=\frac{1}{\sqrt{|x|^2+1}},$$
$$\frac{u_{x_1}+u_{x_2}+1}{\sqrt{3}\sqrt{|Du(x)|^2+1}}=\frac{x_1+x_2+1}{\sqrt{3}\sqrt{|x|^2+1}}$$

on $\Sigma=\{x\in\mathbb{R}_2;\ x_1+x_2+1\geqq 0,\ x_1<x_2\}$. In this case $\sigma^\pm=-2$ and $\Lambda=\frac{(x_1-x_2)^2}{3(|x|^2+1)}>0$ on Σ and an easy computation shows that

$$u^1_{x_1}=x_2,\qquad u^1_{x_2}=x_1\qquad\text{and}\qquad u^2_{x_1}=x_1,\qquad u^2_{x_2}=x_2.$$

Consequently, we have $(u^i_{x_1})_{x_2}=(u^i_{x_2})_{x_1}$, $i=1,2$. Therefore there exist exactly two distinct solutions $u^1(x)=x_1x_2$ and $u^2(x)=\frac{|x|^2}{2}$ of the system (3). For conditions guaranteeing the existence of four solutions we refer to the paper [11].

To examine the structure of minimizing sequences of functionals $E(u)$ and $I(u)$, where I is given by (5), we need the following result on Young measures.

THEOREM 1. — *Let* $\{z_j\}$ *be bounded sequence in* $L^1(\Omega,\mathbb{R}_s)$. *Then there exist a subsequence* $\{z_\nu\}$ *of* $\{z_j\}$ *and a family* $\{\nu_x\}$, $x\in\Omega$, *of probability measures on* $\mathbb{R}_2$, *depending measurably on* $x\in\Omega$ *such that for any measurable subset* $A\subset\mathbb{R}^2$

$$f(.\,,z_\nu)\rightharpoonup\langle\nu_x,f(x,.)\rangle\quad\text{in } L^1(A)$$

for every Carathéodory function $f:\Omega\times\mathbb{R}_s\to\mathbb{R}$ *such that* $f(.\,,z_\nu)$ *is sequentially weakly relatively compact in* $L^1(A)$.

Here $\langle \nu_x, f(x,.) \rangle$ denotes the expected value of $f(x,.)$.

We commence with simple results on the functionals I and E, which give the clue to the construction of minimizing sequence of I.

PROPOSITION 2. – *Let* $\{u_n\}$ *be a sequence in* $W^{1,\infty}(\Omega)$. *If* $\lim_{n\to\infty} E(u_n)=0$, *then* $\lim_{n\to\infty} I(u_n)=0$.

Proof. – Let us denote the integrand of the functional E by $F(x,P)$, that is,

$$F(x,P)=\min(|P-Du^1(x)|, |P-Du^2(x)|)$$

for $(x,P)\in\Omega\times\mathbb{R}_2$. Since $\lim_{n\to\infty} E(u_n)=0$, we see that for every $\varepsilon>0$

$$\lim_{n\to\infty} |\{x;\ F(x, Du_n(x))>\varepsilon\}|=0.$$

For a given $\varepsilon>0$ we define sets Ω_1 and Ω_2 by

$$\Omega_1=\{x;\ |Du_n(x)-Du^1(x)|<\varepsilon\}$$

and

$$\Omega_2=\Omega-\{\Omega_1\cup(x;\ F(x, Du_n(x))\geqq\varepsilon)\}.$$

On the set Ω_2 we have $|Du_n(x)-Du^2(x)|<\varepsilon$. We now write

$$\begin{aligned} I(u_n)=&\int_{\Omega_1}(|f_1(Du_n)-E_1|+|f_2(Du_n)-E_2|)\,dx\\ &+\int_{\Omega_2}(|f_1(Du_n)-E_1|+|f_2(Du_n)-E_2|)\,dx\\ &+\int_{F(x,Du_n(x))>\varepsilon}(|f_1(Du_n)-E_1|+|f_2(Du_n)-E_2|)\,dx\\ &=I^1(u_n)+I^2(u_n)+I^3(u_n).\end{aligned}$$

It is easy to see that the functions

$$f_1(P)=\frac{p_1P_1+p_2P_2-p_3}{\|p\|\sqrt{|P|^2+1}}\quad\text{and}\quad f_2(P)=\frac{q_1P_1+q_2P_2-q_3}{\|q\|\sqrt{|P|^2+1}}$$

are Lipschitz on $\mathbb{R}_2$ and let M denote the maximum of the Lipschitz constants of f_1 and f_2. Hence, using the fact that u^i satisfies each equation $f_j(Du)=E_j$, $j=1, 2$, we have

$$I^1(u_m)=\int_{\Omega_1}(|f_1(Du_n)-f_1(Du^1)|+|f_2(Du_n)-f_2(Du^1)|)\,dx\leqq 2M|\Omega|\varepsilon.$$

In a similar way we estimate I^2. Since the functions f_1 and f_2 are bounded, we have

$$I^3(u_n) \leqq C \left| \{ F(Du_n(x)) > \varepsilon \} \right| \to 0 \quad \text{as } \varepsilon \to 0$$

for some constant $C > 0$ depending on $\sup_{\bar{\Omega}} E_i$ and $\sup_{\mathbb{R}_2} f_i$, $i = 1, 2$.

PROPOSITION 3. — *Let* $\{u_n\}$ *be a bounded sequence in* $W^{1,\infty}(\Omega)$ *such that* $u_n \rightharpoonup u$ *weak-*$\ast$ *in* $W^{1,\infty}(\Omega)$ *and* $\lim_{n\to\infty} E(u_n) = 0$. *Then* $\lim_{n\to\infty} I(u_n) = 0$ *and for a.e.* $x \in \Omega\, Du(x)$ *lies on a segment joining the vectors* $Du^1(x)$ *and* $Du^2(x)$.

Proof. — The first part of the assertion is a consequence of Proposition 2. Let $\{\nu_x\}$, $x \in \Omega$, be a family of Young measures corresponding to $\{Du_n\}$. Then for a subsequence of $\{Du_n\}$, denoted again by $\{Du_n\}$, we have

$$0 = \lim_{n\to\infty} E(u_n) = \lim_{n\to\infty} \int_\Omega F(x, Du_n)\, dx = \int_\Omega \langle \nu_x, F(x, .) \rangle\, dx$$

and this implies that $\langle \nu_x, F(x, .) \rangle = 0$ a.e. on Ω and supp $\nu_x \subset [Du^1(x), Du^2(x)]$. Consequently,

$$\nu_x = \lambda(x)\, \delta_{Du^1(x)} + (1 - \lambda(x))\, \delta_{Du^2(x)}$$

and

$$Du(x) = \langle \nu_x, \lambda \rangle = \lambda(x)\, Du^1(x) + (1 - \lambda(x))\, Du^2(x).$$

COROLLARY. — *Let* $\{u_n\}$ *be sequence in* $W^{1,\infty}(\Omega)$ *such that* $\lim_{n\to\infty} (E(u_n) = 0$. *Then all weak-*$\ast$ *limit points of* $\{Du_n\}$ *lie on segments joining* $Du^1(x)$ *and* $Du^2(x)$ *for a.e.* $x \in \Omega$.

In the next section we construct a minimizing sequence with properties described in Section 1 for a function $u = \lambda u^1 + (1 - \lambda) u^2$ for some $\lambda \in (0, 1)$. For simplicity we assume that λ is independent of x.

3. MAIN RESULT

Since a minimizing sequence u_n must agree with a given function $u = \lambda u^1 + (1 - \lambda) u^2$ on $\partial\Omega$, we look for a sequence in the form $u_n = u + \phi_n$, with $\phi_n \in \mathring{W}^{1,\infty}(\Omega)$ and $\phi_n \rightharpoonup 0$ weak-$\ast$ in $W^{1,\infty}(\Omega)$. To obtain some information on the nature of oscillation of the sequence $\{D\phi_n\}$ let us consider its Young measure $\bar{\nu}_x$. Since $\lim_{n \to \infty} \mathrm{E}(u+\phi_n)=0$, we have

$$0=\lim_{n \to \infty} \int_{\Omega} \min(|\mathrm{D}\phi_n+(1-\lambda)(\mathrm{D}u^2-\mathrm{D}u^1)|,$$
$$|\mathrm{D}\phi_n-\lambda(\mathrm{D}u^2-\mathrm{D}u^1)|)\,dx$$
$$=\langle \bar{\nu}_x, \min(\tau+(1-\lambda)(\mathrm{D}u^2-\mathrm{D}u^1)|, |\tau-\lambda(\mathrm{D}u^2-\mathrm{D}u^1)|)\rangle$$

and consequently

$$\bar{\nu}_x=\tau(x)\,\delta_{-(1-\lambda)(\mathrm{D}u^2(x)-\mathrm{D}u^1(x))}+(1-\tau(x))\,\delta_{\lambda(\mathrm{D}u^2(x)-\mathrm{D}u^1(x))}.$$

On the other hand $\mathrm{D}\phi_n \rightharpoonup 0$ weak-$*$ in $\mathrm{L}^\infty(\Omega)$ and this implies that $\langle \bar{\nu}_x, \tau \rangle=0$, that is,

$$-\tau(x)(1-\lambda)(\mathrm{D}u^2(x)-\mathrm{D}u^1(x))+(1-\tau(x))\lambda(\mathrm{D}u^2(x)-\mathrm{D}u^1(x))=0.$$

This identity is equivalent to

$$(\mathrm{D}u^2(x)-\mathrm{D}u^1(x))(\lambda(1-\tau(x))-\tau(x)(1-\lambda))=0.$$

At points x, where $\mathrm{D}u^2(x)-\mathrm{D}u^1(x)\neq 0$, we can take $\lambda=\tau(x)$. Intuitively, this means that we can locally construct ϕ_n as an affine function on two parallel strips with gradient equal to $\lambda(\mathrm{D}u^2-\mathrm{D}u^1)$ and $-(1-\lambda)$ $(\mathrm{D}u^2-\mathrm{D}u^1)$, respectively. Finally, we observe that since $\bar{\nu}_x$ is not a Dirac measure, the sequence $\{\mathrm{D}\phi_n\}$ cannot converge strongly in $\mathrm{L}^p(\Omega)$, $1 \leqq p < \infty$, to 0 (*see* Tartar [12]).

We are now in a position to establish our main result

THEOREM 2. – *Let $u(x)=\lambda u^1(x)+(1-\lambda)u^2(x)$ for some $0<\lambda<1$. Then there exists a sequence $\{u_n\}$ in $\mathrm{W}^{1,\infty}(\Omega)$ with $u_n|_{\partial\Omega}=u|_{\partial\Omega}$ for each n such that $u_n \rightharpoonup u$ weak-$*$ in $\mathrm{W}^{1,\infty}(\Omega)$ and $\lim_{n \to \infty} \mathrm{E}(u_n)=0$ and hence $\lim_{n \to \infty} \mathrm{I}(u_n)=0$.*

Proof. – First, we localize the problem by approximating Ω by a sequence of unions of squares $\mathrm{H}^l=\bigcup_{s=1}^{\mathrm{I}_1} \mathrm{D}_s^l$ with $\mathrm{H}_l \subset \Omega$ and $\lim_{l \to \infty} |\Omega-\mathrm{H}_l|=0$. We assume that the edges of squares are parallel to the coordinate axes with the length $d(\mathrm{D}_s^l)=\dfrac{1}{2^l}$. We denote the centre of the square D_s^l by x_s^l. Given an integer $n>1$ we choose an integer $l_n>1$ such that

$$\int_{\Omega-\mathrm{H}_{l_n}} \mathrm{F}(x, \mathrm{D}u(x))\,dx < \frac{1}{4n} \tag{7}$$

and

$$|Du^i(x) - Du^i(x_s^{l_n})| < \frac{1}{8n|\Omega|}, \qquad i = 1, 2, \tag{8}$$

for all $x \in D_s^{l_n}$. The last inequality follows from the uniform continuity of Du^i on $\bar{\Omega}$. We now define, assuming that $Du^2(x_s^{l_n}) \neq Du^1(x_s^{l_n})$,

$$\psi_s^{l_n}(x) = \begin{cases} \lambda(Du^2(x_s^{l_n}) - Du^1(x_s^{l_n}))x \\ \text{for } \ 0 \leqq (Du^2(x_s^{l_n}) - Du^1(x_s^{l_n}))x \leqq 1-\lambda, \\ -(1-\lambda)((Du^2(x_s^{l_n}) - Du^1(x_s^{l_n}))x - 1) \\ \text{for } \ 1-\lambda \leqq (Du^2(x_s^{l_n}) - Du^1(x_s^{l_n}))x \leqq 1. \end{cases}$$

We see that $\psi_s^{l_n}(x) = 0$ on the lines $(Du^1(x_s^{l_n}) - Du^2(x_s^{l_n}))x = 0$ and $(Du^2(x_s^{l_n}) - Du^1(x_s^{l_n}))x = 1$. We now extend $\psi_s^{l_n}$ into $\mathbb{R}_2$ as a periodic function and the extended function is still denoted by $\psi_s^{l_n}$. Now for each integer $m \geqq 1$ we set

$$\psi_{s,m}^{l_n}(x) = \frac{1}{m}\psi_s^{l_n}(mx) \qquad \text{for} \quad x \in \mathbb{R}_2$$

and denote by $g_{s,m}^{l_n}$ the restriction of $\psi_{s,m}^{l_n}$ to $D_s^{l_n}$. It is clear that

$$\| g_{s,m}^{l_n} \|_{L^\infty(D_s^{l_n})} \leqq \frac{1}{m}$$

and

$$\| Dg_{s,m}^{l_n} \|_{L^\infty(D_s^{l_n})} \leqq |Du^2(x_s^{l_n}) - Du^1(x_s^{l_n})| \leqq \| u^1 \|_{C^1} + \| u^2 \|_{C^1}.$$

We assume that m is sufficiently large to ensure that $\frac{1}{2^{l_n}} - 2\| g_{s,m}^{l_n} \|_{L^\infty(D_s^{l_n})} > 0$ for a fixed n. The integer m will be chosen later. We now denote by $E_{s,m}^{l_n}$ a square contained in $D_s^{l_n}$, with edges parallel to the coordinate axes and of length $\frac{1}{2^{l_n}} - 2\| g_{s,m}^{l_n} \|_{L^\infty(D_s^{l_n})}$ and such that

$$\operatorname{dist}(\partial D_s^{l_n}, E_{s,m}^{l_n}) = \| g_{s,m}^{l_n} \|_{L^\infty(D_s^{l_n})}.$$

We now define a function $h_{s,m}^{l_n}$ on $\partial D_s^{l_n} \cup E_{s,m}^{l_n}$ by

$$h_{s,m}^{l_n}(x) = \begin{cases} 0 & \text{for } \ x \in \partial D_s^{l_n} \\ g_{s,m}^{l_n}(x) & \text{for } \ x \in E_{s,m}^{l_n}. \end{cases}$$

The function $h_{s,m}^{l_n}$ is Lipschitz on its domain of definition and its Lipschitz constant does not exceed $\max(1, |Du^1(x_s^{l_n}) - Du^2(x_s^{l_n})|)$. Let $\phi_{s,m}^{l_n}$ be the Lipschitz extension of $h_{s,m}^{l_n}$ into $\bar{D}_s^{l_n}$ and set $\phi_{s,m}^{l_n}$ to be 0 outside $D_s^{l_n}$. We now choose m_n such that $m_n > n$, $m_n > l_n$ and

$$|D_s^{l_n} - E_{s,\, m_n}^{l_n}| \leqq \frac{1}{8\, M\, n\, |H_{l_n}|}, \tag{9}$$

where $M = \max(\max_{\bar{\Omega}} |Du^2(x) - Du^1(x)|, 1)$. The above construction has been made under the assumption that $Du^2(x_s^{l_n}) \neq Du^1(x_s^{l_n})$. If $Du^1(x_s^{l_n}) = Du^2(x_s^{l_n})$, we define a function $\phi_s^{l_n}$, by $\phi_s^{l_n}(x) = 0$ on $D_s^{l_n}$ and extend it by 0 outside $D_s^{l_n}$. Finally, we set $\phi_n(x) = \sum_{s=1}^{I_{l_n}} \beta_s^n(x)$, where

$$\beta_s^n(x) = \begin{cases} \phi_{s,\, m_n}^{l_n}(x) & \text{if } Du^2(x_s^{l_n}) \neq Du^1(x_s^{l_n}), \\ \phi_s^{l_n}(x) & \text{if } Du^1(x_s^{l_n}) = Du^2(x_s^{l_n}). \end{cases}$$

Let $u_n(x) = u(x) + \phi_n(x)$ and write $H_l = H_l^1 \cup H_l^1$, where H_l^1 is the union of squares D_s^l with $Du^1(x_s^l) = Du^2(x_s^l)$ and H_l^2 is the union of squares with $Du^1(x_s^l) \neq Du^2(x_s^l)$. To show that $\lim_{n \to \infty} E(u_n) = 0$, we write using (7)

$$\begin{aligned} E(u_n) &= \int_\Omega F(x, Du_n)\, dx \\ &= \int_{H_{l_n}} F(x, Du_n)\, dx + \int_{\Omega - H_{l_n}} F(x, Du_n)\, dx \\ &\leqq \frac{1}{4n} + \int_{H_{l_n}^1} F(x, Du_n)\, dx + \int_{H_{l_n}^2} F(x, Du_n)\, dx. \end{aligned} \tag{10}$$

Since $u_n = u$ on $H_{l_n}^1$ and $Du^1(x_s^{l_n}) = Du^2(x_s^{l_n})$ for each $D_s^{l_n} \in H_{l_n}^1$, it follows from (8) that

$$\begin{aligned} &\int_{H_{l_n}^1} F(x, Du_n)\, dx \\ &= \sum_{D_s^{l_n} \in H_{l_n}^1} \int_{D_s^{l_n}} \min((1-\lambda)\, |Du^1(x) - Du^2(x)|,\, \lambda\, |Du^1(x) - Du^2(x)|)\, dx \\ &\leqq \sum_{D_s^{l_n} \in H_{l_n}^1} \int_{D_s^{l_n}} (|Du^1(x) - Du^1(x_s^{l_n})| \\ &\qquad + |Du^2(x_s^{l_n}) - Du^2(x)|)\, dx \leqq \frac{|H_{l_n}^1|}{4n\, |\Omega|} \leqq \frac{1}{4n}. \end{aligned} \tag{11}$$

To estimate $\int_{H^2_{l_n}} F(x, Du_n)\,dx$ we use (9) and (8) to get

$$\int_{H^2_{l_n}} F(x, Du_n)\,dx$$

$$= \sum_{D^{l_n}_s \in H^2_{l_n}} \left[\int_{D^{l_n}_s - E^{l_n}_{s, m_n}} F(x, Du_n)\,ds + \int_{E^{l_n}_{s, m_n}} F(x, Du_n)\,dx\right]$$

$$\leqq \sum_{D^{l_n}_s \in H^2_{l_n}} \int_{E^{l_n}_{s, m_n}} \min (|D\phi_n(x) + (1-\lambda)(Du^2(x) - Du^1(x))|,$$

$$|D\phi_n(x) - \lambda(Du^2(x) - Du^1(x)|)\,dx + \frac{1}{4n}$$

$$\leqq \sum_{D^{l_n}_s \in H^2_{l_n}} \int_{E^{l_n}_{s, m_n}} \min ((1-\lambda)|(Du^2(x^{l_n}_s)$$

$$- Du^1(x^{l_n}_s)) - (Du^2(x) - Du^1(x))|,$$

$$\lambda |(Du^2(x^{l_n}_s) - Du^1(x^{l_n}_s)) - (Du^2(x) - Du^1(x))|)\,dx + \frac{1}{4n}$$

$$\leqq \frac{|H_{l_n}|}{4n|\Omega|} + \frac{1}{4n} \leqq \frac{1}{2n}.$$

Consequently, combining (10), (11) and the last estimate we obtain

$$\int_\Omega F(x, Du_n)\,dx \leqq \frac{1}{n}.$$

It is easy to check that

$$\| D\phi_n \|_{L^\infty(\Omega)} \leqq (\| u^1 \|_{C^1} + \| u^2 \|_{C^1}) \qquad \text{and} \qquad \| \phi_n \|_{L^\infty(\Omega)} \leqq \frac{\lambda(1-\lambda)}{n}.$$

Therefore we may assume that $\phi_n \to 0$ weak-$*$ in $\mathring{W}^{1,\infty}(\Omega)$ and this completes the proof.

To close our paper we point out that elements of our construction are not new and can be traced in variational calculus (*see* [8]).

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