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Exact controllability for semilinear wave equations in one space dimension

by

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ABSTRACT. — The exact controllability of the semilinear wave equation $y''-y_{xx}+f(y)=h$ in one space dimension with Dirichlet boundary conditions is studied. We prove that if $|f(s)|/|s| \log^2|s| \to 0$ as $|s| \to \infty$, then the exact controllability holds in $H_0^1(\Omega) \times L^2(\Omega)$ with controls $h \in L^2(\Omega \times (0, T))$ supported in any open and non empty subset of Ω. The exact controllability time is that of the linear case where $f=0$. Our method of proof is based on HUM (Hilbert Uniqueness Method) and on a fixed point technique. We also show that this result is almost optimal by proving that if f behaves like $-s \log^p(1+|s|)$ with $p>2$ as $|s| \to \infty$, then the system is not exactly controllable. This is due to blow-up phenomena. The method of proof is rather general and applied also to the wave equation with Neumann type boundary conditions.

Key words : Exact controllability, Semilinear wave equation, One space-dimension, Fixed point method.

RÉSUMÉ. — On démontre la contrôlabilité exacte de l'équation des ondes semi-linéaire à une dimension d'espace pour des nonlinéarités f que satisfont $|s|/|s| \log^2|s| \to 0$ lorsque $|s| \to \infty$. La méthode de démonstration combine HUM et une technique de point fixe. En utilisant des arguments

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d'explosion on démontre que la condition de croissance imposée à la nonlinéarité est presque optimale.

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1. INTRODUCTION AND MAIN RESULTS

This paper is concerned with the exact controllability of the semilinear wave equation in one space dimension.

Let be $\Omega=(0, 1)\subset\mathbb{R}$ and $\omega=(l_1, l_2)$ a subinterval with $0\leqq l_1<l_2\leqq 1$. Let be $T>0$ and $f\in C^1(\mathbb{R})$ and let us consider the following semilinear wave equation

$$\left.\begin{array}{ll} y''-y_{xx}+f(y)=h\mathbb{X}_\omega & \text{in } \Omega\times(0, T) \\ y(0, t)=y(1, t)=0 & \text{for } t\in(0, T) \\ y(x, 0)=y^0(x), \quad y'(x, 0)=y^1(x) & \text{for } x\in\Omega \end{array}\right\} \tag{1.1}$$

where $h=h(x, t)\in L^2(\omega\times(0, T))$, $\{y^0, y^1\}\in H_0^1(\Omega)\times L^2(\Omega)$ and $\mathbb{X}_\omega$ denotes the characteristic function of the set ω.

We shall assume the existence of some positive constant $C>0$ such that

$$|f(s)|\leqq C(1+|s|)\log^2(2+|s|), \quad \forall s\in\mathbb{R}.$$

Under this assumption, by the methods of A. Haraux and T. Cazenave [CH] it follows that system (1.1) has a unique global solution

$$y\in C^0([0, T]; H_0^1(\Omega))\cap C^1([0, T]; L^2(\Omega)).$$

The exact controllability problem may be formulated as follows: for any $\{y^0, y^1\}$, $\{z^0, z^1\}\in H_0^1(\Omega)\times L^2(\Omega)$ find a control function $h\in L^2(\omega\times(0, T))$ such that the solution of (1.1) satisfies

$$y(x, T)=z^0(x), \qquad y'(x, T)=z^1(x) \quad \text{for } x\in\Omega. \tag{1.2}$$

Thus, the question is whether we can drive the system from any initial state to any final state in time T by means of the action of a control with support in $\omega\times(0, T)$.

When this property holds true we will say that system (1.1) is exactly controllable in time T.

When the control acts in all of Ω (*i.e.* $\omega=\Omega$) the exact controllability in any time $T>0$ is easy to prove without any restriction on the nonlinearity.

When the support of the control is restricted to some subset $\omega\nsubseteq\Omega$ the situation is much more delicate.

However, in the linear framework ($f=0$), the problem is by now well understood. In one space dimension with $\omega=(l_1, l_2)$, $\Omega=(0, 1)$ the exact

controllability holds in time $T > 2 \max(l_1, 1 - l_2)$ (*see* E. Zuazua [Z3]). Observe that in the one-dimensional case the exact controllability holds for any open and non-empty subset ω of Ω. However, in several space dimensions, the *geometric control* property is needed on the subset ω in order to ensure the exact controllability (*see* C. Bardos, G. Lebeau and J. Rauch [BLR1, BLR2]).

Very recently the first exact controllability results have been proved for the semilinear wave equation.

In [Z4, Z5] we have proved that system (1.1) is exactly controllable when the nonlinearity f is globally Lipschitz. In fact, in [Z4, Z5] we proved this result also in several space dimensions in the case where ω was a neighborhood of the boundary of Ω.

The method of [Z4, Z5] was based on the Hilbert Uniqueness Method (HUM) introduced by J. L. Lions [L1, L2, L3] to study the exact controllability of linear systems and a fixed point technique.

Later on, these results were recovered by I. Lasiecka and R. Triggiani in [LaT1, LaT2] by using global inversion theorems. They also improved in some cases the regularity of the controls given in [Z5] but their results were, as in [Z5], completely restricted to globally Lipschitz nonlinearities.

In a recent paper [Z6], by using in a deeper way the fixed point technique of [Z5] we have proved the exact controllability of system (1.1) (in one space dimension) under the following growth condition on the nonlinearity

$$\lim_{|s| \to \infty} \frac{f(s)}{|s| \log |s|} = 0. \tag{1.3}$$

The goal of this paper is to improve this result by proving the exact controllability under the weaker assumption

$$\lim_{|s| \to \infty} \frac{f(s)}{|s| \log^2 |s|} = 0.$$

In fact, we shall prove that there exists some $\beta_0 > 0$ such that if

$$\limsup_{|s| \to \infty} \frac{|f(s)|}{|s| \log^2 |s|} < \beta_0 \tag{1.4}$$

then the exact controllability of system (1.1) still holds.

Condition (1.4) allows nonlinearities that grow at infinity in a superlinear way. This growth condition might seem to be too restrictive but it is almost optimal since no assumption is done on the sign of the nonlinearity. Indeed, if f behaves at infinity like $-s \log^p(|s|)$ with $p > 2$, then due to blow-up phenomena, the system is not exactly controllable in any time $T > 0$.

To be more precise, let us state our main positive and negative results.

THEOREM 1. — *Let be* $\omega=(l_1, l_2)\neq\varnothing$ *and* $T>T(l_1, l_2)=2\max(l_1; 1-l_2)$. *Then there exists some* $\beta_0>0$ *such that if the nonlinearity* $f\in C^1(\mathbb{R})$ *satisfies* (1.4) *then system* (1.1) *is exactly controllable in time* T.

THEOREM 2. — *Assume that* $\omega\neq\Omega$ *and that* f *satisfies*

$$\liminf_{s\to\infty}\left\{\frac{-f(s)}{s\log^p s}\right\}>0. \tag{1.5}$$

for some $p>2$. *Then, system* (1.1) *is not exactly controllable in any time* $T>0$.

The proof of Theorem 1 is based on the fixed point technique introduced in [Z5] that reduces the exact controllability problem to the obtention of suitable *a priori* estimates for the linearized wave equation with a potential. These estimates will be obtained by techniques that are genuinely one-dimensional. For this reason, we are not able to extend Theorem 1 to several space dimensions.

The proof of Theorem 2 is based on a blow-up argument that applied also in the case of several space variables. It is sufficient to observe that there exist solutions of (1.1) with $h=0$ that blow up in an arbitrarily small time in some point (x_0, t_0) that is out of the influence-region of the control. Then, whatever control h we choose supported in ω, the solution of (1.1) will blow up at time t_0 and therefore it will not be even global in time and in particular, (1.2) will not hold.

All we have said concerns the internal control problem, *i.e.* the case where the control is distributed in the interior of the domain Ω. However, from Theorems 1 and 2 we can also obtain positive and negative results for the following system with boundary control:

$$\left.\begin{array}{lll} y''-y_{xx}+f(y)=0 & & \text{in } \Omega\times(0, T) \\ y(0, t)=v(t), & y(1, t)=0 & \text{for } t\in(0, T) \\ y(0)=y^0, & y'(0)=y^1 & \text{in } \Omega. \end{array}\right\} \tag{1.6}$$

We have the following result.

THEOREM 3. — (*a*) *Let be* $T>2$. *Assume that* $f\in C^1(\mathbb{R})$ *satisfies* (1.4) *with* $\beta_0>0$ *small enough. Then, for every* $\{y^0, y^1\}$, $\{x^0, z^1\}\in H_0^1(\Omega)\times L^2(\Omega)$ *there exists a control* $v\in C([0, T])$ *such that the solution* y *of* (1.6) *satisfies* (1.2).

(*b*) *Assume that the nonlinearity* f *satifies* (1.5) *with* $p>2$, *then, system* (1.6) *is not exactly controllable in any time* $T>0$. *More precisely, there exist initial data* $\{y^0, y^1\}\in H_0^1(\Omega)\times L^2(\Omega)$ *for which the solution of* (1.6) *blows-up in time* $t<T$ *for any control function* $v\in C([0, T])$.

The first statement (*a*) of this theorem is a direct consequence of Theorem 1. It is sufficient to extend the domain Ω as well as the initial and final data and define the control function v as the restriction at $x=0$

of the controlled state $\tilde{y}$ that satisfies an equation of type (1.1) with a control h that is supported in the exterior of Ω.

Since $\tilde{y} \in C([0, T]; H^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$ we deduce that the control function verifies $v \in C^\delta([0, t])$ for any $0 < \delta < \frac{1}{2}$. The solution y of (1.6) is given as the restriction to $\Omega \times (0, T)$ of the solution $\tilde{y}$ defined in the extended domain. On the other hand, the uniqueness of the solution of (1.6) is easy to prove. Therefore, for the control v constructed as above system (1.6) has an unique solution in the class $C([0, T]; V) \cap C^1([0, T]; L^2(\Omega))$ with $V = \{\varphi \in H^1(\Omega) : \varphi(1) = 0\}$.

Theorem 3 applies also in the case where the control functions acts at $x = 1$, *i.e.* with boundary conditions $y(0, t) = 0$, $y(1, t) = w(t)$ for $t \in (0, T)$. When we consider controls at both ends $x = 0$ and $x = 1$, *i.e.* when the boundary conditions are $y(0, t) = v(t)$, $y(1, t) = w(t)$ for $t \in (0, T)$, the exact controllability time reduces to the half: more precisely, the system is then exactly controllable at any time $T > 1$.

This paper is organized as follows. Section 2 is devoted to the development of the fixed point method: Theorem 1 is reduced to the obtention of suitable observability estimates for the wave equation with potential. This estimates are proved in Section 3. In Section 4 we give the proof of Theorem 2 by using blow-up arguments. Section 5 is devoted to the boundary control problem: Theorem 3 is proved. We end with Section 6 where we mention some extensions of the methods and results of this paper as well as some open problems.

Let us complete this introduction with some comments on the bibliographical references that we have not mentioned above.

One of the classic tools that has been used for the study of the controllability of nonlinear system is the Implicit Function Theorem (IFT). L. Markus in [M] and E. B. Lee and L. Markus in [LeM] applied the IFT to deduce local controllability properties for some systems of ODEs (by local controllability we mean that one can drive some initial state to some final state when they are close enough). This technique was after adapted and extended to the wave equation by H. O. Fattorini [F] and W. C. Chewning [CH] (*see* also D. L. Russell's review article [R]), but their results were also of local nature. M. A. Cirina in [Ci] developed a different method for proving local controllability results for some first order hyperbolic systems in one space dimension.

Another classic view-point to study nonlinear control problems is that of using fixed point techniques (*cf.* H. Hermes [H]). D. L. Lukes in [Lu] gives global controllability results for some systems of ODE with nonlinearities that are globally Lipschitz with small Lipschitz constant. In [Lu] the control may enter in the system in a nonlinear way. The work

of N. Carmichael and M. D. Quinn [CaQ] describes some of the applications of fixed point methods in nonlinear control problems.

More recently, K. Naito [N], T. Seidman [Se] and K. Naito and T. Seidman in [NSe] have used Schauder's fixed point Theorem to prove the invariance of the set of reachable states under nonlinear preturbations that grow at infinity in a sublinear way for a number of parabolic and hyperbolic problems. By combining HUM and Schauder's fixed point Theorem we obtained in [Z1, Z2] exact controllability results for wave equations with nonlinearities that were allowed to grow in a sublinear or linear way at infinity.

In the present paper, it is the use of Leray-Schauder's degree theory (instead of Schauder's fixed point Theorem) that allows us to go farther by allowing some nonlinearities that grow at infinity in a superlinear way.

2. DESCRIPTION OF THE FIXED POINT METHOD

In this section we describe the fixed point technique we use in the proof of Theorem 1. The proof of Theorem 1 will be reduced to the obtention of a suitable observability property for the linear wave equation with a bounded potential. This will be done in the next section, concluding the proof of Theorem 1.

We proceed in several steps.

Step 1. — Let us fix the intial and final data $\{y^0, y^1\}$, $\{z^0, z^1\} \in H_0^1(\Omega) \times L^2(\Omega)$ and let us introduce the continuous function

$$g(s) = \begin{cases} \dfrac{f(s)-f(0)}{s} & \text{if } s \neq 0 \\ f'(0) & \text{if } s = 0. \end{cases} \tag{2.1}$$

Given any $\xi \in L^\infty(\Omega \times (0, T))$ we look for a control $h = h(x, t; \xi) \in L^2(\omega \times (0, T))$ such that the solution $y = y(x, t; \xi)$ of

$$\left.\begin{array}{ll} y'' - y_{xx} + g(\xi) y = -f(0) + h \mathbb{X}_\omega & \text{in } \Omega \times (0, T) \\ y(0, t) = y(1, t) = 0 & \text{for } t \in (0, T) \\ y(x, 0) = y^0(x), \quad y'(x, 0) = y^1(x) & \text{in } \Omega \end{array}\right\} \tag{2.2}$$

satisfies

$$y(x, t) = z^0(x), \qquad y'(x, T) = z^1(x). \tag{2.3}$$

To prove this we use HUM.

We first solve the system.

$$\left.\begin{array}{ll} z'' - z_{xx} + g(\xi) z = -f(0) & \text{in } \Omega \times (0, T) \\ z 0, t) = z(1, t) = 0 & \text{for } t \in (0, T) \\ z(x, T) = z^0(x), \quad z'(x, T) = z^1(x) & \text{in } \Omega. \end{array}\right\} \tag{2.4}$$

This system has a unique solution

$$z = z(x, t; \xi) \in C([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$$

and therefore

$$z = (x, 0; \xi) = z_\xi^0 \in H_0^1(\Omega), \qquad z'(x, 0; \xi) = z_\xi^1 \in L^2(\Omega). \tag{2.5}$$

Then, for any $\{\varphi^0, \varphi^1\} \in L^2(\Omega) \times H^{-1}(\Omega)$ we solve

$$\left.\begin{array}{ll} \varphi'' - \varphi_{xx} + g(\xi)\varphi = 0 & \text{in } \Omega \times (0, T) \\ \varphi(0, t) = \varphi(1, t) = 0 & \text{for } t \in t \in (0, T) \\ \varphi(x, 0) = \varphi^0(x), \qquad \varphi'(x, 0) = \varphi^1(x) & \text{in } \Omega \end{array}\right\} \tag{2.6}$$

and

$$\left.\begin{array}{ll} \eta'' - \eta_{xx} + g(\xi)\eta = \varphi \mathbb{X}_\omega & \text{in } \Omega \times (0, T) \\ \eta(0, t) = \eta(1, t) = 0 & \text{for } t \in (0, T) \\ \eta(x, T) = \eta'(x, T) = 0 & \text{in } \Omega. \end{array}\right\} \tag{2.7}$$

We define the linear and continuous operator

$$\Lambda_\xi : L^2(\Omega) \times H^{-1}(\Omega) \to L^2(\Omega) \times H_0^1(\Omega)$$

by

$$\Lambda_\xi \{\varphi^0, \varphi^1\} = \{-\eta'(x, 0), \eta(x, 0)\}. \tag{2.8}$$

The problem reduces to prove the existence of some $\{\varphi^0, \varphi^1\} \in L^2(\Omega) \times H^{-1}(\Omega)$ such that

$$\Lambda_\xi \{\varphi^0, \varphi^1\} = \{-y^1 + z_\xi^1, y^0 - z_\xi^0\}. \tag{2.9}$$

Indeed, if $\{\varphi^0, \varphi^1\}$ is solution of (2.9), then η, the corresponding solution of (2.7), satisfies

$$\eta'(0) = y^1 - z_\xi^1. \qquad \eta(0) = y^0 - z_\xi^0$$

and therefore $y = \eta + z$ satisfies both (2.2) and (2.3).

In order to solve (2.9) we observe that

$$\langle \Lambda_\xi \{\varphi^0, \varphi^1\}, \{\varphi^0, \varphi^1\} \rangle = \int_{\omega \times (0, T)} |\varphi|^2 \, dx \, dt, \qquad \forall \{\varphi^0, \varphi^1\} \in L^2(\Omega) \times H^{-1}(\Omega). \tag{2.10}$$

To see this, it is sufficient to multiply the equation (2.7) by φ and to integrate by parts.

Let us now consider the following wave equation

$$\left.\begin{array}{ll} \varphi'' - \varphi_{xx} + a(x, t)\varphi = 0 & \text{in } \Omega \times (0, T) \\ \varphi(0, t) = \varphi(1, t) = 0 & \text{for } t \in (0, T) \\ \varphi(x, 0) = \varphi^0(x), \qquad \varphi'(x, 0) = \varphi^1(x) & \text{in } \Omega \end{array}\right\} \tag{2.11}$$

with potential $a(x, t) \in L^\infty(\Omega \times (0, T))$.

Let us assume that the following observability property holds for system (2.11): there exists a real positive and continuous function $\alpha \in C([0, \infty])$

such that

$$\|\{\varphi^0, \varphi^1\}\|^2 \leqq \alpha(\|a\|_\infty) \int_{\omega \times (0, T)} |\varphi|^2 \, dx \, dt \tag{2.12}$$

for every $\{\varphi^0, \varphi^1\} \in L^2(\Omega) \times H^{-1}(\Omega)$ and $a \in L^\infty(\Omega \times (0, T))$. In (2.12) $\|.\|$ (resp. $\|.\|_\infty$) denotes the norm in $L^2(\Omega) \times H^{-1}(\Omega)$ [resp. $L^\infty(\Omega \times (0, T)$]. The norm in $H^{-1}(\Omega)$ is given by $\|\varphi\|_{H^{-1}(\Omega)} = \left\| \frac{d}{dx}\left(-\frac{d^2}{dx^2}\right)^{-1} \varphi \right\|_{L^2(\Omega)}$.

Combining (2.10) and (2.12) we obtain

$$\langle \Lambda_\xi \{\varphi^0, \varphi^1\}, \{\varphi^0, \varphi^1\} \rangle \geqq \frac{1}{\alpha(\|g(\xi)\|_\infty)} \|\{\varphi^0, \varphi^1\}\|^2 \tag{2.13}$$

and therefore

$$\Lambda_\xi : \quad L^2(\Omega) \times H^{-1}(\Omega) \to L^2(\Omega) \times H_0^1(\Omega) \text{ is an isomorphism.} \tag{2.14}$$

Then, equation (2.9) has a unique solution

$$\{\varphi^0, \varphi^1\} = \{\varphi^0(x; \xi), \varphi^1(x; \xi)\} \in L^2(\Omega) \times H^{-1}(\Omega)$$

and the function

$$h = \varphi(x, t; \xi)$$

is the desired control such that the solution y of (2.2) satisfies (0.3).

We have defined in a unique way a control $h(x, t; \xi) \in L^2(\omega \times (0, T))$ for system (2.2)-(2.3) and this for every $\xi \in L^\infty(\Omega \times (0, T))$. The solution y of (2.2) belongs to $C([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$ and thanks to the embedding $H_0^1(\Omega) \subset L^\infty(\Omega)$ we deduce that $y \in L^\infty(\Omega \times (0, T))$.

Therefore, we have constructed a nonlinear operator

$$K : \quad L^\infty(\Omega \times (0, T)) \to L^\infty(\Omega \times (0, T))$$

such that $K(\xi) = y$ where y is the solution of (2.2)-(2.3) with the control function $h \in L^2(\Omega \times (0, T))$ defined above.

It is easy to see that the operator K sends bounded sets of $L^\infty(\Omega \times (0, T))$ into bounded sets of

$$C([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega)).$$

This fact, combined with the compactness of embedding

$$C([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega)) \subset L^\infty(\Omega \times (0, T))$$

allows us to prove both the continuity of K from $L^\infty(\Omega \times (0, T))$ to $L^\infty(\Omega \times (0, T))$ and the fact that K maps bounded sets of $L^\infty(\Omega \times (0, T))$ into relatively compact sets of itself.

Therefore the operator

$$K : \quad L^\infty(\Omega \times (0, T)) \to L^\infty(\Omega \times (0, T)) \text{ is compact.} \tag{2.15}$$

Step 2. — We observe that, provided (2.12) holds, it is sufficient to prove the existence of a fixed point of K. Indeed, if $\xi = y \in L^\infty(\Omega \times (0, T))$

is a fixed point of K, then $\xi = y \in C([0, T]; H_0^1(\Omega)) \cap C^1([0, T]); L^2(\Omega))$ and y satisfies both (1.1) and (1.2). Therefore, if $\xi = y$ is a fixed point of K, it is sufficient to choose $h(x, t; \xi)$ as control for the nonlinear system (1.1).

In order to prove the existence of a fixed point for K we use Leray-Schauder's degree Theorem. We define the operator

$$K: \quad [0, 1] \times L^\infty(\Omega \times (0, T)) \to L^\infty(\Omega \times (0, T)) \tag{2.16}$$

such that

$$K(\sigma, \xi) = K_\sigma(\xi)$$

where K_σ is the compact operator defined as in Step 1 but for the nonlinearity σg.

The operator K is compact and $K(0, \xi) = K_0(\xi)$ is independent of ξ. Therefore, in order to conclude the existence of a fixed point for $K = K_1$ it is sufficient to prove that the identity

$$K(\sigma, y) = y \tag{2.16}$$

with $\sigma \in [0, 1]$ and $y \in L^\infty(\Omega \times (0, T))$ implies an uniform bound for y in $L^\infty(\Omega \times (0, T))$.

By construction of K, equation (2.16) is equivalent to the system

$$\left.\begin{array}{ll} y'' - y_{xx} + \sigma f(y) = \varphi \mathbb{X}_\omega & \text{in } \Omega \times (0, T) \\ y(0, t) = y(1, t) = 0 & \text{for } t \in (0, T) \\ y(0) = y^0,\ y'(0) = y^1 & \text{in } \Omega \\ y(T) = z^0,\ y'(T) = z^1 & \text{in } \Omega \\ \varphi'' - \varphi_{xx} + \sigma g(y)\varphi = 0 & \text{in } \Omega \times (0, T) \\ \varphi(0, t) = \varphi(1, t) = 0 & \text{for } t \in (0, T). \end{array}\right\} \tag{2.17}$$

Let us assume for the moment that $f(0) = 0$. Multiplying by φ the equation satisfied by y and integrating by parts in $\Omega \times (0, T)$ we get

$$\int_{\omega \times (0, T)} \varphi^2 \, dx \, dt = \int_\Omega z^1 \varphi(T) \, dx - \langle z^0, \varphi'(T) \rangle - \int_\Omega y^1 \varphi^0 \, dx + \langle y^0, \varphi^1 \rangle. \tag{2.18}$$

In (2.18) $\langle .,. \rangle$ denotes the duality pairing between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$.

From (2.18) we deduce

$$\int_{\omega \times (0, T)} \varphi^2 \, dx \, dt \leqq C_0 \{ \| \varphi^0, \varphi^1 \} \| + \| \{ \varphi(T), \varphi'(T) \} \| \} \tag{2.19}$$

with $C_0 = \max \{ \| \{ y^0, y^1 \} \|_{H_0^1(\Omega) \times L^2(\Omega)}, \| \{ z^0, z^1 \} \|_{H_0^1(\Omega) \times L^2(\Omega)} \}$.

In view of (2.12) and by the time-reversibility of the equation satisfied by φ we deduce

$$\{\|\{\varphi^0, \varphi^1\}\|+\|\{\varphi(T), \varphi'(T)\}\|\}^2 \leqq 4\alpha(\sigma\|g(y)\|_\infty)\int_{\omega\times(0,T)} \varphi^2\,dx\,dt. \quad (2.20)$$

Combining (2.19) and (2.20) we get

$$\|\{\varphi^0, \varphi^1\}\|+\|\{\varphi(T), \varphi'(T)\}\|\leqq 4C_0\alpha(\sigma\|g(y)\|_\infty).$$

We can assume that α is increasing and that $|g|$ is decreasing (resp. increasing) in $(-\infty, 0)$ [resp. $(0, \infty)$]. Then $\alpha(\sigma\|g(y)\|_\infty)\leqq\alpha(|g(\|y\|_\infty)|)$ and therefore

$$\|\{\varphi^0, \varphi^1\}\|+\|\{\varphi(T), \varphi'(T)\}\|\leqq 4C_0\alpha(|g(\|y\|_\infty)|). \quad (2.21)$$

Steep 3. — We need to estimate $\|y\|_\infty$ in terms of $\int_{\omega\times(0,T)} \varphi^2\,dx\,dt$.

Let us consider the following general linear wave equation

$$\left.\begin{array}{ll} u''-u_{xx}+a(x,t)u=b(x,t) & \text{in } \Omega\times(0,T) \\ u(0,t)=u(1,t)=0 & \text{for } t\in(0,T) \\ u(0)=u^0, \quad u'(0)=u^1 & \text{in } \Omega \end{array}\right\} \quad (2.22)$$

and let us define the energy

$$E(t)=\frac{1}{2}\{\|u'(t)\|^2_{L^2(\Omega)}+\|u_x(t)\|^2_{L^2(\Omega)}\}. \quad (2.23)$$

We have the following estimate.

LEMMA 1. — *There exist two positive constants* A, B>0 *that only depend on* Ω *and* T *such that*

$$\left.\begin{array}{c} E(t)\leqq A(E(0)(1+\|a\|_\infty)+\|b\|^2_{L^2(\Omega\times(0,T))})e^{B\sqrt{\|a\|_\infty}\,t}, \\ \forall t\in(0,T). \end{array}\right\} \quad (2.24)$$

for every $a\in L^\infty(\Omega\times(0,T))$ *and every solution of* (2.22) *with* $b\in L^2(\Omega\times(0,T))$ *and* $\{u^0, u^1\}\in H^1_0(\Omega)\times L^2(\Omega)$.

Proof of Lemma 1. — Let be $\delta=\|a\|_\infty$ and let us define the perturbed energy

$$E_\delta(t)=E(t)+\frac{\delta}{2}\|u(t)\|^2_{L^2(\Omega)}.$$

We have

$$\begin{aligned}\frac{dE_\delta(t)}{dt} &= \int_\Omega (\delta - a(x,t))\,u(x,t)\,u'(x,t)\,dx + \int_\Omega b(x,t)\,u'(x,t)\,dx \\ &\leqq 2\delta \|u(t)\|_{L^2(\Omega)} \|u'(t)\|_{L^2(\Omega)} + \|b(t)\|_{L^2(\Omega)} \|u'(t)\|_{L^2(\Omega)} \\ &\leqq (1+2\sqrt{\delta})\,E_\delta(t) + \frac{1}{2}\|b(t)\|^2_{L^2(\Omega)}\end{aligned}$$

and therefore

$$E_\delta(t) \leqq \left(E_\delta(0) + \frac{1}{2}\|b\|^2_{L^2(\Omega\times(0,T))}\right) e^{(1+2\sqrt{\delta})t}, \qquad \forall t \in [0, T]$$

from where (2.24) easily follows. □

Applying (2.24) to the solution y of (2.17) and using the continuity of the embedding $H^1_0(\Omega) \subset L^\infty(\Omega)$ we deduce that, for $C>0$ large enough,

$$\|y\|^2_\infty \leqq C\left(E(0) + \int_{\omega\times(0,T)} \varphi^2\,dx\,dt\right) e^{C\sqrt{|g(\|y\|_\infty)|}\,T} \tag{2.25}$$

From the growth condition (1.4) we deduce that there exists some $d>0$ such that

$$\|g(y)\|_\infty \leqq 2\beta_0 \log^2(\|y\|_\infty) + d^2, \qquad \forall y \in L^\infty(\Omega\times(0,T)). \tag{2.26}$$

Combining (2.25) and (2.26) we get

$$\|y\|^2_\infty \leqq C\left(E(0) + \int_{\omega\times(0,T)} \varphi^2\,dx\,dt\right) e^{C\,dT}\, e^{\sqrt{2\beta_0}\,CT\,|\log(\|y\|_\infty)|}$$

If $\beta_0>0$ is small enough such that $\sqrt{2\beta_0}\,CT \leqq 1$ we deduce that

$$\|y\|_\infty \leqq 1 + C\left(E(0) + \int_{\omega\times(0,T)} \varphi^2\,dx\,dt\right) e^{C\,dT}$$

which combined with (2.19), yields

$$\|y\|_\infty \leqq 1 + C(E(0) + C_0(\|\{\varphi^0, \varphi^1\}\| + \|\{\varphi(T), \varphi'(T)\}\|))\,e^{C\,dT}. \tag{2.27}$$

From (2.21) and (2.27) we get

$$\begin{aligned}&\|\{\varphi^0, \varphi^1\}\| + \|\varphi(T), \varphi'(T)\}\| \\ &\qquad \leqq 4C_0\,\alpha(|g(1 + C(E(0) + C_0(\|\{\varphi^0, \varphi^1\}\| \\ &\qquad\qquad + \|\{\varphi(T), \varphi'(T)\}\|))\,e^{C\,dT})|).\end{aligned} \tag{2.28}$$

Assume that α and g are so that

$$\frac{\alpha(|g(C_1 + C_2 s)|)}{|s|} \to 0 \quad \text{as } |s| \to \infty \tag{2.29}$$

for every C_1, $C_2>0$. Then, from (2.28) we deduce that $\|\{\varphi^0, \varphi^1\}\| + \|\varphi(T), \varphi'(T)\}\|$ is uniformly bounded, which combined with (2.27) yields an uniform bound for $\|y\|_\infty$.

Therefore, the problem reduces to prove (2.29). Assume that we have the estimate

$$\alpha(s) \leqq A' e^{B'\sqrt{s}}, \qquad \forall s > 0 \tag{2.30}$$

for some $A', B' > 0$. Then, clearly, (2.29) follows from the growth condition (1.4) provided $\beta_0 > 0$ is small enough.

Step 4. – Let us return to system (2.17) but now for the case where $f(0) \neq 0$. Instead of (2.18) we have

$$\int_{\omega \times (0, T)} \varphi^2 \, dx \, dt = \int_\Omega z^1 \varphi(T) \, dx - \langle z^0, \varphi'(T) \rangle$$
$$- \int_\Omega y^1 \varphi^0 \, dx + \langle y^0, \varphi^1 \rangle + \sigma f(0) \int_{\omega \times (0, T)} \varphi \, dx \, dt.$$

Assume that we have the following estimate for φ:

$$\int_{\omega \times (0, T)} \varphi^2 \, dx \, dt \leqq \gamma(|g(\|y\|_\infty)|) \|\{\varphi^0, \varphi^1\}\|^2. \tag{2.31}$$

for some increasing function γ such that

$$\gamma(s) \leqq A'' e^{B''\sqrt{s}}, \qquad \forall s \geqq 0 \tag{2.32}$$

for some $A'', B'' > 0$.

Then, proceeding as in steps 3 and 4 we easily obtain the uniform bound for $\|y\|_\infty$.

It remains to prove the observability property (2.12) with α satisfying (2.30) and (2.31) with γ satisfying (2.32). This will be done in the following section (Theorem 4 and Lemma 2) and then the proof of Theorem 1 will be concluded.

3. OBSERVABILITY ESTIMATES FOR THE WAVE EQUATION WITH BOUNDED POTENTIAL

The aim of this section is to prove the following observability result for system (2.11). For proving this estimate we will use the intermediate Lemma 2 which provides (2.31) with γ satisfying (2.32).

THEOREM 4. – *If* $T > 2 \max(l_1, 1 - l_2)$, *there exist two positive constants* $A_1, B_1 > 0$ *such that*

$$\|\{\varphi^0, \varphi^1\}\|^2 \leqq A_1 e^{B_1 \sqrt{\|a\|_\infty}} \int_{\omega \times (0, T)} \varphi^2 \, dx \, dt \tag{3.1}$$

for every $a \in L^\infty(\Omega \times (0, T))$ *and every solution* φ *of* (2.11) *with initial data* $\{\varphi^0, \varphi^1\} \in L^2(\Omega) \times H^{-1}(\Omega)$.

Proof. — We proceed in several steps.

Steps 1. — First we study the behavior of the energy

$$\mathscr{E}(t)=\frac{1}{2}\{\|\varphi(t)\|^2_{L^2(\Omega)}+\|\varphi'(t)\|^2_{H^{-1}(\Omega)}\}. \tag{3.2}$$

We have the following estimate.

LEMMA 2. — *There exist positive constants* A_2, $B_2>0$ *such that*

$$\mathscr{E}(t)\leqq A_2\mathscr{E}(0)(1+\|a\|^2_\infty)e^{B_2\sqrt{\|a\|_\infty}}, \qquad \forall t\in[0, T] \tag{3.3}$$

for every $a\in L^\infty(\Omega\times(0, T))$ *and every solution of* (2.11).

Proof of Lemma 2. — We decompose the solution φ of (2.11) as

$$\varphi=p+q$$

with p and q respectively solutions of

$$\left.\begin{array}{ll} p''-p_{xx}=0 & \text{in } \Omega\times(0, T) \\ p(0, t)=p(1, t) & \text{for } t\in(0, T) \\ p(0)=\varphi^0,\ p'(0)=\varphi^1 & \text{in } \Omega \end{array}\right\} \tag{3.4}$$

and

$$\left.\begin{array}{ll} q''-q_{xx}+aq=-ap & \text{in } \Omega\times(0, T) \\ q(0, t)=q(1, t)=0 & \text{for } t\in(0, T) \\ q(0)=q'(0)=0 & \text{in } \Omega. \end{array}\right\} \tag{3.5}$$

The energy $\mathscr{E}$ is conserved for system (3.4). Therefore

$$\|p(t)\|^2_{L^2(\Omega)}+\|p'(t)\|^2_{H^{-1}(\Omega)}=2\mathscr{E}(0), \qquad \forall t\in[0, T]. \tag{3.6}$$

In order to estimate the energy of q we apply Lemma 1 with $u=q$, $u^0=u^1=0$ and $b=-ap$. We get

$$\begin{aligned}\|q(t)\|^2_{H^1_0(\Omega)}+\|q'(t)\|^2_{L^2(\Omega)}&\leqq 2A\|a\|^2_\infty\|p\|^2_{L^2(\Omega\times(0, T))}e^{B\sqrt{\|a\|_\infty}\,t}\\ &\leqq 4A\|a\|^2_\infty\|T\mathscr{E}(0)e^{B\sqrt{\|a\|_\infty}\,t}.\end{aligned} \tag{3.7}$$

Combining (3.6) and (3.7), (3.3) follows easily. □

Remark. — Observe that Lemma 2 provides (2.31) with γ satisfying (2.32). □

Step 2. — From Lemma 2 we deduce that in fact it is sufficient to prove the existence of t_1, $t_2\in[0, T]$ with $t_1<t_2$ and $C>0$ such that

$$\int_{t_1}^{t_2}[\|\varphi(t)\|^2_{L^2(\Omega)}+\|\varphi'(t)\|^2_{H^{-1}(\Omega)}]\,dt\leqq Ce^{C\sqrt{\|a\|_\infty}}\int_{\omega\times(0, T)}\varphi^2\,dx\,dt. \tag{3.8}$$

Indeed, by (3.3) and the time-revertibility of system (2.11) we have

$$(t_2-t_1)[\|\varphi^0\|^2_{L^2(\Omega)}+\|\varphi^1\|^2_{H^{-1}(\Omega)}]$$
$$\leqq A_2(1+\|a\|^2_\infty)e^{B_2\sqrt{\|a\|_\infty}\,t_2}\int_{t_1}^{t_2}[\|\varphi(t)\|^2_{L^2(\Omega)}+\|\varphi'(t)\|^2_{H^{-1}(\Omega)}]\,dt. \quad (3.9)$$

Combining (3.8)-(3.9), (3.1) follows easily.

We now observe that, in order to get (3.8), it is sufficient to prove

$$\int_{\Omega\times(t_1,\,t_2)}\varphi^2\,dx\,dt\leqq Ce^{C\sqrt{\|a\|_\infty}}\int_{\omega\times(0,\,T)}\varphi^2\,dx\,dt. \quad (3.10)$$

Indeed, multipying by $r(t)\left(\frac{-d^2}{dx^2}\right)^{-1}\varphi$ in equation (2.11) and integrating by parts in $\Omega\times(t_1,\,t_2)$ we get

$$\int_{t_1}^{t_2}r(t)\|\varphi'(t)\|^2_{H^{-1}(\Omega)}=\int_{t_1}^{t_2}r(t)\|\varphi(t)\|^2_{L^2(\Omega)}$$
$$+\int_{\Omega\times(t_1,\,t_2)}r(t)\,a(x,\,t)\,\varphi\left(\frac{-d^2}{dx^2}\right)^{-1}\varphi\,dx\,dt$$
$$-\int_{t_1}^{t_2}r'(t)\left\langle\left(\frac{-d^2}{dx^2}\right)^{-1}\varphi(t),\,\varphi'(t)\right\rangle dt$$
$$-\left[r(t)\left\langle\left(\frac{-d^2}{dx^2}\right)^{-1}\varphi(t),\,\varphi'(t)\right\rangle\right]\Bigg|_{t_1}^{t_2}.$$

Choosing $r\in C^1([t_1,\,t_2])$ such that $r(t_1)=r(t_2)=0$, $r(t)=1$, $\forall\,t\in[t_1',\,t_2']$ with $t_1'=t_1+\frac{(t_2-t_1)}{3}$, $t_2'=t_2-\frac{(t_2-t_1)}{3}$, $\frac{|r'|^2}{r}\in L^\infty(t_1,\,t_2)$ we get

$$\int_{t_1'}^{t_2'}\|\varphi'(t)\|^2_{H^{-1}(\Omega)}\,dt\leqq C(1+\|a\|_\infty)\int_{t_1}^{t_2}\|\varphi(t)\|^2_{L^2(\Omega)}\,dt. \quad (3.11)$$

Combining (3.10) and (3.11) it is easy to get (3.8).

Step 3. – If $\varphi\in L^2(\omega\times(0,\,T))=L^2(l_1,\,l_2;\,L^2(0,\,T))$, by using the equation

$$\varphi_{xx}=\varphi''+a\varphi$$

we deduce that $\varphi_{xx}\in L^2(l_1,\,l_2;\,H^{-2}(0,\,T))$ with

$$\|\varphi_{xx}\|_{L^2(l_1,\,l_2;\,H^2(0,\,T))}\leqq C(1+\|a\|_\infty)\|\varphi\|_{L^2(\omega\times 0,\,T))}.$$

Then, by interpolation we obtain that $\varphi_x\in L^2(l_1,\,l_2;\,H^{-1}(0,\,T))$ with

$$\|\varphi_x\|_{L^2(l_1,\,l_2;\,H^{-1}(0,\,T))}\leqq C(1+\|a\|^{1/2}_\infty)\|\varphi\|_{L^2(\omega\times(0,\,T))}. \quad (3.12)$$

From (3.12) we deduce that, in order to get (3.10) it is sufficient to obtain

$$\int_{\Omega\times(t_1,t_2)} \varphi^2\,dx\,dt \leqq C e^{C\sqrt{\|a\|_\infty}} \left\{ \int_{\omega\times(0,T)} \varphi^2\,dx\,dt + \int_{t_1}^{t_2} \|\varphi_x(x)\|^2_{H^{-1}(\Omega)}\,dx \right\}. \quad (3.13)$$

We now observe that in fact it is sufficient to prove

$$\int_{\tau(x^0)} \varphi^2\,dx\,dt \leqq C e^{C\sqrt{\|a\|_\infty}} \left\{ \int_0^T \varphi^2(x_0,t)\,dt + \|\varphi_x(x_0,t)\|^2_{H^{-1}(0,T)} \right\} \quad (3.14)$$

for any $x_0\in\Omega$ and $T>2\max(x_0, 1-x_0)$ with $\tau(x_0)=\tau_1(x_0)\cup\tau_2(x_0)$ where

$$\tau_1(x_0)=\{(x,t)\in(0,x_0)\times(0,T) : t\in(x_0-x, x+T-x_0)\}$$
$$\tau_2(x_0)=\{(x,t)\in(x_0,l)\times(0,T) : t\in(x-x_0, T-x+x_0)\}.$$

Indeed, integrating (3.14) with respect to those $x_0\in\omega$ for which the time T given in Theorem 4 satisfies $T>2\max(x_0, l-x_0)$ we get (3.13) with $t_1=\max(l_1, 1-l_2)$, $t_2=T-\max(l_1, 1-l_2)$.

Step 4. – Let us finally prove (3.14). We observe that due to finite speed of propagation (=1) in system (2.11) we have

$$\varphi=\psi \quad \text{in } \tau(x_0) \quad (3.15)$$

where $\psi=\psi(x,t)$ is the solution of

$$\left.\begin{array}{ll} \psi_{xx}-\psi''-a\psi=0 & \text{in } (0,T)\times(0,1) \\ \psi(x,0)=\psi(x,T)=0 & \text{for } x\in(0,1) \\ \psi(x_0,t)=\varphi(x_0 t),\ \psi_x(x_0,t)=\varphi_x(x_0,t) & \text{in } (0,T). \end{array}\right\} \quad (3.16)$$

System (3.16) is a wave equation where the roles of the time and space variables has been interchanged. It is an evolution equation with respect to x.

We can apply Lemma 2 to system (3.16). We get

$$\|\psi(x)\|^2_{L^2(0,T)}+\|\psi_x(x)\|^2_{H^{-1}(0,T)} \leqq A_2(1+\|a\|^2_\infty)\,e^{B_2\sqrt{\|a\|_\infty}\max(x_0,1-x_0)} \times\{\|\varphi(x_0)\|^2_{L^2(0,T)}+\|\varphi_x(x_0)\|^2_{H^{-1}(0,T)}\}, \qquad \forall x\in\Omega. \quad (3.17)$$

Combining (3.15) and (3.17) we get (3.14) and this concludes the proof of Theorem 4.

4. PROOF OF THEOREM 2

Due to finite speed of propagation ($=1$) in system (1.1), the points of the set

$$\mathrm{R}=\{(x,t)\in(0,l_1)\times(0,\mathrm{T}) : t<l_1-x\} \cup\{(x,t)\in(l_2,1)\times(0,\mathrm{T}) : t<x-l_2\} \quad (4.1)$$

remain out of the influence-region of the control h that is supported in $\omega\times(0,\mathrm{T})$ with $\omega=(l_1,l_2)$.

Therefore, if $(x_0,t_0)\in\mathrm{R}$ then the solution y of (1.1) restricted to the set

$$\mathrm{D}(x_0,t_0)=\{(x,t) : |x-x_0|<t_0-t\} \quad (4.2)$$

does not depend on the control function h.

In particular, if the restrictions of the initial data to the interval (x_0-t_0, x_0+t_0) are constant functions, *i.e.*

$$y^0|_{(x_0-t_0,\,x_0+t_0)}=\gamma, \qquad y^1|_{(x_0-t_0,\,x_0+t_0)}=\delta \quad (4.3)$$

and $p=p(t)$ is the solution of the differential equation

$$\left.\begin{array}{ll} p''(t)+f(p(t))=0, & t>0 \\ p(0)=\gamma, & p'(0)=\delta \end{array}\right\} \quad (4.4)$$

then the solution y of (1.1) satisfies

$$y(x,t)=p(t), \qquad \forall(x,t)\in\mathrm{D}(x_0,t_0) \quad (4.5)$$

for any $h\in\mathrm{L}^2(\omega\times(0,\mathrm{T}))$.

Assume now that we are able to construct data $\{\gamma,\delta\}$ such that the solution p of (4.3) blows-up in time $t>t_0$. Then, in view of (4.5), we deduce that for initial data $\{y^0,y^1\}\in\mathrm{H}_0^1(\Omega)\times\mathrm{L}^2(\Omega)$ satisfying (4.3) the solution y of (1.1) blows-up in time $t>t_0$ for any control function $h\in\mathrm{L}^2(\omega\times(0,\mathrm{T}))$. In particular, the desired final condition (1.3) will be impossible to achieve for any final data $\{z^0,z^1\}$ and the system (1.1) will not be exactly controllable in any time $\mathrm{T}>0$.

We have reduced the proof of Theorem 2 to show that under the growth condition (1.5) on the nonlinearity f there exist solutions of the differential equation (4.4) that blow up in arbitrarily small time.

Let us choose $\gamma,\delta>0$ such that

$$f(s)\leqq 0, \qquad \forall s\geqq\gamma. \quad (4.6)$$

Then, it is easy to see that the solution p of (4.4) satisfies

$$p(t)\geqq\gamma, \quad p'(t)\geqq\delta, \qquad \forall t\in\mathrm{I}(\gamma,\delta) \quad (4.7)$$

where $\mathrm{I}(\gamma,\delta)$ is the existence interval for equation (4.4).

Multiplying (4.4) by p' we get

$$\frac{1}{2}\frac{d}{dt}(|p'(t)|^2)+\frac{d}{dt}\mathrm{F}(p(t))=0 \tag{4.8}$$

with

$$\mathrm{F}(z)=\int_0^z f(s)\,ds$$

and therefore

$$\frac{1}{2}|p'(t)|^2+\mathrm{F}(p(t))=\frac{1}{2}\delta^2+\mathrm{F}(\gamma). \tag{4.9}$$

Thus

$$p'(t)=[\delta^2+2\mathrm{F}(\gamma)-2\mathrm{F}(p(t))]^{1/2} \tag{4.10}$$

and

$$\mathrm{G}(p(t))-\mathrm{G}(\alpha)=t \tag{4.11}$$

with

$$\mathrm{G}(z)=\int_0^z \frac{ds}{\{\delta^2+2\mathrm{F}(\gamma)-2\mathrm{F}(s)\}^{1/2}}. \tag{4.12}$$

Assume that

$$\lim_{k\to\infty}\int_k^\infty \frac{ds}{(1-2\mathrm{F}(s))^{1/2}}=0. \tag{4.13}$$

Then, given any $\varepsilon>0$ we may choose $\gamma>0$ such that $\mathrm{F}(s)\leqq 0$ for every $s\geqq\gamma$ and

$$\int_\gamma^\infty \frac{ds}{(1-2\mathrm{F}(s))^{1/2}}<\varepsilon. \tag{4.14}$$

For this choice of γ we choose $\delta>0$ such that

$$\delta^2+2\mathrm{F}(\gamma)=1 \tag{4.15}$$

and (4.14) implies that

$$\mathrm{G}(p(t))-\mathrm{G}(\gamma)<\varepsilon$$

and in view of (4.11) we conclude that

$$t<\varepsilon, \qquad \forall\, t\in \mathrm{I}(\gamma,\delta).$$

This means that the solution p of (4.4) with this choice of $\{\gamma,\delta\}$ blows-up in time $t\leqq\varepsilon$.

Therefore, it is sufficient to check that (4.13) holds. In fact, it is sufficient to see that, for $k>0$ large enough,

$$\int_k^\infty \frac{ds}{|\mathrm{F}(s)|^{1/2}}<\infty. \tag{4.16}$$

From the growth condition (1.5) we deduce that

$$\liminf_{s \to \infty} \frac{|F(s)|}{s^2 \log^p s} > 0$$

which implies (4.16) since

$$\int_2^\infty \frac{ds}{s \log^{p/2} s} < \infty$$

for $p > 2$.

This concludes the proof of Theorem 2.

5. BOUNDARY CONTROL

This section is devoted to the proof of Theorem 3. The proof of the second statement (*b*) of this Theorem is analogous to that of Theorem 3. Therefore we concentrate our attention on the proof of the exact controllability result (*a*).

We first observe that, as a consequence of Theorem 1, if $\Omega = (a, b) \subset \mathbb{R}$ and $\omega = (l_1, l_2) \subset \Omega$ with $l_2 > l_1$, then under the growth condition (1.4) with $\beta_0 > 0$ small enough, system (1.1) is exactly controllable in time $T > 2 \max(l_1 - a, b - l_2)$.

Given any $\varepsilon > 0$ let us define the extended domain $\tilde{\Omega} = (-\varepsilon, 1)$ and $\omega = (-\varepsilon, 0)$. Let us extend our initial and final data $\{y^0, y^1\}$, $\{z^0, z^1\} \in H_0^1(\Omega) \times L^2(\Omega)$ by zero outside of Ω to define

$$\tilde{y}^i = \begin{cases} y^i & \text{in } \Omega \\ 0 & \text{in } (-\varepsilon, 0) \end{cases}, \qquad \tilde{z}^i = \begin{cases} z^i & \text{in } \Omega \\ 0 & \text{in } (-\varepsilon, 0), \end{cases} \quad \text{for } i = 0, 1.$$

Then

$$\{\tilde{y}^0, \tilde{y}^1\}, \qquad \{\tilde{z}^0, \tilde{z}^1\} \in H_0^1(\tilde{\Omega}) \times L^2(\tilde{\Omega}).$$

In view of Theorem 1 and since $T > 2$, we deduce that there exists a control $h \in L^2(\omega \times (0, T))$ such that the solution $\tilde{y}$ of

$$\begin{aligned} &\tilde{y}'' - \tilde{y}_{xx} + f(\tilde{y}) = h \mathbb{X}_\omega && \text{in } \tilde{\Omega} \times (0, T) \\ &\tilde{y}(-\varepsilon, t) = \tilde{y}(1, t) = 0 && \text{for } t \in (0, T) \\ &\tilde{y}(0) = \tilde{y}^0(x), \ \tilde{y}'(0) = \tilde{y}^1(x) && \text{in } \tilde{\Omega} \end{aligned}$$

satisfies

$$\tilde{y}(T) = \tilde{z}^0, \qquad \tilde{y}'(T) = \tilde{z}^1 \quad \text{in } \tilde{\Omega}.$$

Let be $y = \tilde{y}|_{\Omega \times (0, T)}$ and $v = \tilde{y}(0, t)$. Then y satisfies clearly (1.6) and (1.2). Therefore the control v answers to the question and since $\tilde{y} \in C([0, T]; H_0^1(\tilde{\Omega})) \cap C^1([0, T]; L^2(\tilde{\Omega}))$ we deduce that, in particular, $v \in C([0, T])$.

This completes the proof of Theorem 3.

6. FURTHER COMMENTS

1. Clearly, Theorems 1 and 2 do not cover the case where the nonlinearity f behaves like $\pm ks \log^2(1+|s|)$ as $|s| \to \infty$ with $k>0$ large. Very probably, in those cases the exact controllability holds since blow-up phenomena do not appear.

2. All the results of this paper extend easily to semilinear wave equations with variable coefficients like

$$y'' - (a(x)y_x)_x + f(y) = h\mathbb{X}_\omega$$

with $a \in W^{1,\infty}(\Omega)$, $a(x) \geqq a_0 > 0$, $\forall x \in \Omega$ for some constant $a_0 > 0$. We refer to [Z3] for the linear case.

3. We have not used in an essential manner the Dirichlet boundary conditions. The same results hold for system (1.1) with Neumann boundary conditions or mixed Dirichlet-Neumann boundary conditions. When considering Neumann boundary conditions the only change to be done in the statement of our theorems is to replace the space $H_0^1(\Omega)$ [resp. $H^{-1}(\Omega)$] by $H^1(\Omega)$ [resp. by $(H^1(\Omega))'$].

4. If the nonlinearity $f \in C^1(\mathbb{R})$ satisfies the sign condition $f(s)s \geqq 0$, $\forall s \in \mathbb{R}$, then all the solutions of (1.1) are global in time. Thus, one could expect the exact controllability of system (1.1). The exact controllability of systems like

$$y'' - y_{xx} + |y|^{p-1} y = h\mathbb{X}_\omega \tag{6.1}$$

with $p>1$ is an open question.

Combining the local controllability results of [Z2] with the stabilization ones of [Z7] one can prove that given any $\{y^0, y^1\}$, $\{z^0, z^1\} \in H_0^1(\Omega) \times L^2(\Omega)$ there exists a time $T>0$ and a control function $h \in L^2(\omega \times (0, T))$ such that the solution of (6.1) with Dirichlet boundary data satisfies both the initial condition $y(0) = y^0$, $y'(0) = y^1$ and the final condition $y(T) = z^0$, $y'(T) = z^1$. However the controllability time T we get in this way is not uniform and tends to infinity when the norm of the initial or final data goes to infinity.

5. The fixed point technique described in Section 2 can be easily adapted to the semilinear wave equation in any space dimension with control on a neighborhood of the boundary of Ω. However, in order to show the existence of a class of nonlinearities that grow at infinity in a superlinear way for which the exact controllability holds, we need an observability result with explicit constants (like Theorem 4) for the wave function plus a potential in several space dimensions. The method we have used in Theorem 4 is genuinely one-dimensional. Therefore the extension of Theorem 1 to several space dimensions is an open problem. However, as we have mentioned in the introduction, the counter-example of Theorem 2 extends easily to several space dimensions.

6. The fixed point method of Section 2 can be also adapted to plate-like semilinear models of the form

$$y'' + \Delta^2 y + f(y) = h \mathbb{X}_\omega$$

with several boundary conditions. However, once again, the obtention of observability results for plate models with potential and explicit constants is an open problem.

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