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<http://www.numdam.org/item?id=AIHPC_1992__9_6_643_0>
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by

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ABSTRACT. — We prove that there exist at least two distinct solutions to the Dirichlet problem for the equation of prescribed mean curvature $\Delta X = 2 H(X) X_u \wedge X_v$, the curvature function $H$ being in a full neighborhood of a suitable constant.

Key words : Equation of prescribed mean curvature, Dirichlet problem, relative minimizer.

1. INTRODUCTION

Let $B = \{ \omega = (u, v) \in \mathbb{R}^2 / |\omega| < 1 \}$ be the unit disc in $\mathbb{R}^2$ with boundary $\partial B$. We consider the Dirichlet problem for the equation of prescribed mean curvature

$$\Delta X = 2 H(X) X_u \wedge X_v, \quad \text{in } B,$$

$$X = X_D, \quad \text{on } \partial B. \quad (1.1)$$

Here, \( X_u = \frac{\partial}{\partial u} X \) and \( X_v = \frac{\partial}{\partial v} X \) denote partial derivative, \( \wedge \) and \( \cdot \) are the exterior and inner product in \( \mathbb{R}^3 \) and \( H: \mathbb{R}^3 \to \mathbb{R} \) is a given function, and \( X_D \) is a given function of class \( C^2 (\mathbb{B} , \mathbb{R}^3) \).

If \( H = H_0 = \text{Const.} \), solutions to (1.1), (1.2) can be characterized as critical points of the functional
\[
E_{H_0} (X) = D(X) + 2H_0 V(X),
\]
in a space of admissible functions satisfying the boundary condition (1.2), where
\[
D(X) = \frac{1}{2} \int_{\mathbb{B}} |\nabla X|^2 \, d\omega
\]
is the Dirichlet integral and
\[
V(X) = \frac{1}{3} \int_{\mathbb{B}} X \cdot X_u \wedge X_v \, d\omega
\]
is the algebraic volume of surface \( X \).

**Theorem 1.1** ([Hi2], [Wet1], [Wet2] and [Stf1]). Suppose \( H = H_0 \in \mathbb{R} \) and let \( X_D \in H^{1,2} (\mathbb{B} , \mathbb{R}^3) \) be given. Assume that either
(i) \( X_D \) is bounded and
\[
|H_0| \cdot \| X_D \|_{L^\infty} < 1,
\]
or
(ii) the condition
\[
H_0^2 D(X_D) < \frac{2}{3} \pi
\]
is satisfied. Then there is a solution \( X \in \{ X_D \} + H_0^{1,2} (\mathbb{B} , \mathbb{R}^3) \) to (1.1), (1.2) which is a strict relative minimizer of \( E_{H_0} \) in this space.

**Remark 1.2.** The observation that the solutions of Hildebrandt, Steffen and Wente are strict relative minima is due to Brezis-Coron [BC].

The existence of a second solution was proved independently by Brezis-Coron [BC] and Struwe [St2] with an important contribution by Steffen [Stf2] as follows

**Theorem 1.3** [Str3]. Let \( X_D \in H^{1,2} \cap L^\infty (\mathbb{B} , \mathbb{R}^3) \) be a non-constant vector, \( H_0 \) any real number different from zero. Suppose \( E_{H_0} \) admits a local minimum \( X \) in the class \( \{ X_D \} + H_0^{1,2} (\mathbb{B} , \mathbb{R}^3) \). Then there exists a solution \( \bar{X} \in \{ X_D \} + H_0^{1,2} (\mathbb{B} , \mathbb{R}^3) \) of (1.1) and (1.2) different from \( X \) and satisfying the condition
\[
E_{H_0} (X) < E_{H_0} (\bar{X}) = \inf_{p \in P} \sup_{X \in \text{im}(p)} E_H (X) < E_{H_0} (X) + \frac{4 \pi}{3 |H_0|^2},
\]

\( \text{im}(p) \) being the image of the line \( p \) in \( \mathbb{R}^3 \).
where
\[ P = \{ p \in C^0([0, 1], \{ X_D \}) + H_0^{1,2}(B, \mathbb{R}^3) \mid p(0) = X, \quad E_{H_0}(p(1)) < E_{H_0}(X) \}. \tag{1.9} \]

For variable curvature functions \( H \) results comparable to Theorem 1.1 have been obtain by Hildebrandt [Hi1] and Steffen [Stf].

**Theorem 1.4 [Hi1].** – Suppose \( H \) is of class \( C^1 \) and let \( X_D \in H^{1,2} \cap L^\infty(B, \mathbb{R}^3) \) be given with \( \| X_D \|_{L^\infty} < 1 \). Then if
\[ h = \text{ess sup}_{|X| \leq 1} H(X) < 1 \]
there exists a solution \( X \in \{ X_D \} + H_0^{1,2}(B, \mathbb{R}^3) \) to (1.1), (1.2) such that \( E_H(X) = \inf \{ E_H(X); X \in M \} \), where \( M \) is given by (2.9) below.

If variable curvature function \( H \) is sufficiently close to a suitable constant, Struwe obtained [Str4].

**Theorem 1.5.** – Suppose \( X_D \in C^2(B, \mathbb{R}^2) \) is non-constant and suppose that for \( H_0 \in \mathbb{R} \setminus \{ 0 \} \) the functional \( E_{H_0} \) admits a relative minimizer in \( \{ X_D \} + H_0^{1,2}(B, \mathbb{R}^3) \). Then there exists a number \( \alpha > 0 \) such that for a dense set \( \mathcal{A} \) of curvature functions \( H \) in the \( \alpha \)-neighborhood of \( H_0 \), the Dirichlet problem (1.1), (1.2) admits at least two distinct regular solutions in \( \{ X_D \} + H_0^{1,2}(B, \mathbb{R}^3) \).

Here the \( \alpha \)-neighborhood of \( H_0 \) is defined as
\[ [H - H_0] = \text{ess sup}_{X \in \mathbb{R}^3} \{ (1 + |X|)(|H(X) - H_0| + |\nabla H(X)|) \}
+ |Q(X) - H_0X| + |\nabla Q(X) - H_0 \text{id}| \leq \alpha, \tag{1.10} \]
where \( Q \) is given by (2.3) below.

In this paper, we improve Theorem 1.5 and obtain that

**Theorem 1.6.** – Suppose \( X_D \in C^2(B, \mathbb{R}^3) \) is non-constant, and suppose that for \( H_0 \in \mathbb{R} \setminus \{ 0 \} \) the functional \( E_{H_0} \) admits a relative minimizer in \( \{ X_D \} + H_0^{1,2} \). Then there exists a number \( \alpha > 0 \) such that if \( [H - H_0] < \alpha \), \( E_H \) admits two solutions in \( \{ X_D \} + H_0^{1,2} \).

From the proof of Theorem 1.5 [Str4], we have a relative minimizer of \( E_H \) for a full \( \alpha \)-neighborhood of \( H_0 \) and another “large” critical point of \( E_H \) for \( H \in \mathcal{A} \). We call the former \( S \)-solution and the latter \( L \)-solution.

First, we show that the \( S \)-solution is also a “strict” relative minimizer – its \( E_H \) energy is less than that of the \( L \)-solution – provided that \( [H - H_0] \) is small enough. Next, we give a priori estimates for solutions of the Dirichlet problem – which are of crucial importance to our result – though they are not given explicitly. Then, we can use the solutions obtained by Struwe in [Str4] – the \( L \)-solutions – for a dense set \( \mathcal{A} \) in

\[ \mathcal{H}_\alpha = \{ H | [H - H_0] < \alpha \} \] to approximate a solution of \( E_H \) for any \( H \in \mathcal{H}_\alpha \) which is different from the \( S \)-solution.

The author would like to thank his supervisor Prof. Wang Guangyin for his constant encouragement and useful suggestions. He also would like to thank the referee for pointing out a few mistakes in the 1st and 2nd versions of this paper.

2. PRELIMINARIES

For variable curvature function \( H \), solutions to (1.1), (1.2) can be characterized as critical points of the functional

\[ E_H(X) = D(X) + 2 V_H(X) \]  (2.1)

in the space \( \{ X_D \} + H_0 \cdot 2(B, \mathbb{R}^2) \). Here, the \( H \)-volume introduced by Hildebrandt is given by

\[ V_H(X) = \frac{1}{3} \int_B Q(X) \cdot X_u \wedge X_v \, dw, \]  (2.2)

where

\[ Q(x_1, x_2, x_3) = \left( \int_0^{x_1} H(s, x_2, x_3) \, ds, \right. \]
\[ \left. \int_0^{x_2} H(x_1, s, x_3) \, ds, \int_0^{x_3} H(x_1, x_2, s) \, ds \right). \]  (2.3)

We list some useful lemmas.

**Lemma 2.1** (Isoperimetric inequality, cf. [Wet1]):

\[ 36 \pi (V(X))^2 \leq D(X)^3, \]  (2.4)

for \( X \in H_0 \cdot 2(B, \mathbb{R}^3) \).

**Lemma 2.2** ([BC], [Str4], Prop. 3.1). - Suppose \( X_D \in C^2(B, \mathbb{R}^3) \) is non-constant, and suppose that for \( H = H_0 \neq 0 \) the functional \( E_{H_0} \) admits a relative minimizer \( X_0 \in \{ X_D \} + C^2 \cap H_0 \cdot 2(B, \mathbb{R}^3) \). Then there exists a radius \( R > 0 \), a function \( X_1 \in \{ X_D \} + C^2 \cap H_0^{1,2}(B, \mathbb{R}^3) \) with \( D(X_1 - X_0) \geq R \), and a continuous path \( p \in C^0([0, 1]; \{ X_D \} + C^2 \cap H_0^{1,2}(B, \mathbb{R}^3)) \) connecting \( X_0 = p(0) \) with \( X_{\pm} = p(1) \) such that the estimates

\[ E_{H_0}(X_1) < \inf \{ E_{H_0}(X); X - X_0 \in H_0^{1,2}, D(X - X_0) \leq R \} \leq E_{H_0}(X_0) \]  (2.5)

\[ < \inf \{ E_{H_0}(X); X - X_0 \in H_0^{1,2}, D(X - X_0) = R \} \]  (2.6)

\[ \leq \sup \{ E_{H_0}(X); X \in p([0, 1]) \} \]  (2.7)

\[ < E_{H_0}(X_0) + 4\pi/3 H_0^3 \]  (2.8)

hold.
DEFINITION 2.3:

\[ M = \{ X \in \{ X_D \} + H_0^{1,2}(B, \mathbb{R}^3); \quad D(X - X_0) \subseteq R \} \]

where \( X_0 \) and \( R \) are as in Lemma 2.2.

LEMMA 2.4 [BC]. Suppose that \( E_{H_0} \) admits a relative minimizer \( X_0 \in \{ X_D \} + C^2 \cap H_0^{1,2}(B, \mathbb{R}^3) \). Then there is \( \delta > 0 \) such that

\[
\int |\nabla \varphi|^2 + 4 H_0 \int X_0 \cdot \varphi_u \land \varphi_v \geq \delta \int |\nabla \varphi|^2, \quad \text{for all } \varphi \in H_0^{1,2}. \quad (2.10)
\]

Let

\[ P = \{ \rho \in C^0([0, 1]; \{ X_D \} + H_0^{1,2}(B, \mathbb{R}^3)), \; p(0) = X_0, \; p(1) = X_1 \} \]

and set

\[
\gamma_{H, \rho} = \inf_{\rho \in P} \sup_{X \in \text{im}(\rho)} E_{H, \rho}(X)
\]

where \( E_{H, \rho}(X) = (1 + \rho) E_{H/(1+\rho)}(X) \). Using Lemma 2.2 we have (see [Str4], (3.5))

\[
E_{H, \rho}(X) < \inf \{ E_H(X); \; X - X_0 \in H_0^{1,2}, \; D(X - X_0) \subseteq R \} \leq E_{H, \rho}(X_0) \quad (2.5)_\rho
\]

\[
< \inf \{ E_H(X); \; X - X_0 \in H_0^{1,2}, \; D(X - X_0) = R \} \leq \inf \{ E_{H, \rho}(X); \ldots \} \quad (2.6)_\rho
\]

\[
\leq \sup \{ E_{H, \rho}(X); \; X \in p([0, 1]) \} \quad (2.7)_\rho
\]

\[
< E_H(X_0) + \beta \leq E_{H, \rho}(X_0) + \beta \quad (2.8)_\rho
\]

and

\[
E_{H, \rho}(X_0) < \gamma_{H, 0} \leq \gamma_{H, \rho} \leq \gamma_{H, \alpha} < E_H(X_0) + \beta, \quad (2.11)
\]

for \( \rho \in [0, \alpha] \). Here \( X_1, X_0 \) and \( P \) are as in Lemma 2.2, \( \alpha \) is small enough and fixed and \( \beta < 4\pi/3 H_0^2 \) is independent of \( H \). Moreover, we have

LEMMA 2.5. There exists a constant number \( \varepsilon_0 \) independent of \( \alpha \) such that

\[
E_{H_0}(X_0) + \varepsilon_0 < \inf \{ E_H(X); \; X - X_0 \in H_0^{1,2}, \; D(X - X_0) = R \}
\]

\[
\leq \inf \{ E_{H, \rho}(X); \; X - X_0 \in H_0^{1,2}, \; D(X - X_0) = R \}
\]

\[
\leq \sup \{ E_{H, \rho}(X); \; X \in p([0, 1]) \}
\]

\[
< E_H(X_0) + \beta - \varepsilon_0 \leq E_{H, \rho}(X_0) + \beta - \varepsilon_0
\]

provided that \( \alpha \) is small enough, where \( X_0, R \) and \( p \) are as in Lemma 2.2.

Proof. Set

\[
\varepsilon_0 = \frac{1}{4} \min \{ (E_{H_0}(X_0) + 4\pi/3 H_0^2 - \sup \{ E_H(X); \; X \in p([0, 1]) \}),
\]

\[
(\inf \{ E_{H_0}(X); \; X - X_0 \in H_0^{1,2}, \; D(X - X_0) = R \} - E_{H_0}(X_0))\}.
\]

It is easy to see that Lemma 2.5 follows from Lemma 2.2 for \( \alpha \) small enough.

Q.E.D.

**Lemma 2.6.** There exists a constant \( c \) independent of \( \alpha \) such that if \( \mathbf{H} \in \mathcal{A} \) (see Theorem 1.5),

\[
D(X - \bar{X}) > c,
\]

where \( X \) (resp. \( \bar{X} \)) is the S-solution (resp. L-solution) to (1.1), (1.2).

**Proof.** It follows the proof of Theorem 1.5 ([Str4], Theorem 3.1) and Lemma 2.5.

Q.E.D.

## 3. THE “STRICT” RELATIVE MINIMA

In this section, we will prove that the S-solution to the Dirichlet problem for the equation of prescribed mean curvature \( \mathbf{H} \) is a “strict” relative minimum in the \( \mathcal{A} \), provided that \( \mathbf{H}_0 \) is small enough. Here \( \mathbf{H}_0 \neq 0 \) is a constant with the property that \( E_{\mathbf{H}_0} \) admits a relative minimizer \( X_0 \in \{ X_0 \} + C^2 \cap H^{1,2}_0 \).

**Lemma 3.1.** There exists a constant number \( \alpha > 0 \) with the property that if \( [\mathbf{H} - \mathbf{H}_0] < \alpha \) there is a constant \( \delta > 0 \) depending only on \( \alpha \) and \( X_0 \) such that

\[
\int |\nabla \varphi|^2 + 4 \int Q(X) \varphi_u \nabla \varphi \geq \delta \int |\nabla \varphi|^2, \quad \text{for any } \varphi \in H^{1,2}_0.
\] (3.1)

Here \( X = X_H \) is the S-solution to (1.1), (1.2).

**Proof.** Let \( X_0 \) be the small solution of \( E_{\mathbf{H}_0} \) in the space \( \{ X_0 \} + H^{1,2}_0 \). By Brezis-Coron [BC]—see Lemma 2.4—there exists a constant \( \delta_1 > 0 \) such that

\[
\int |\nabla \varphi|^2 + 4 \mathbf{H}_0 \int X_0 \varphi_u \nabla \varphi \geq \delta_1 \int |\nabla \varphi|^2, \quad \varphi \in H^{1,2}_0.
\] (3.2)

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Thus for any $\varphi \in H^{1,2}_0$,
\[ \int |\nabla \varphi|^2 + 4 \int Q(X) \varphi_u \wedge \varphi_v = \int |\nabla \varphi|^2 + 4H_0 \int X_0 \varphi_u \wedge \varphi_v + 4 \int (Q(X) - H_0 X) \varphi_u \wedge \varphi_v + 4H_0 \int (X - X_0) \varphi_u \wedge \varphi_v \geq (\delta - 2\alpha) \int |\nabla \varphi|^2 + 4H_0 \int (X - X_0) \varphi_u \wedge \varphi_v. \]

Therefore, Lemma 3.1 follows from the following

**Lemma 3.2.** For any $\varepsilon > 0$, there exists a constant $\alpha > 0$ with the property that for any curvature function $H$ with $[H - H_0] < \varepsilon$, if $X_H$ is the $S$-solution to (1.1), (1.2), then
\[ \|X_H - X_0\|_{L^\infty} < \varepsilon. \] (3.3)

**Proof.** If the Lemma is false, we may assume that there exist $\varepsilon_0 > 0$ and a sequence $\{X_i\}$ of the $S$-solutions of $E_H$ with $H = H_i$ and $[H_i - H_0] \to 0$ as $i \to 0$ such that $\|X_H - X_0\|_{L^\infty} \geq \varepsilon_0$. Noticing that $X_i \in M$, we know that $D(X_i)$ are bounded uniformly in $i$. Thus we may assume that $\{X_i\}$ converges to $X$ weakly in $\{X_D\} + H^{1,2}_0$ for some $X \in \{X_D\} + H^{1,2}_0$. It is easy to see that $X \in M$ (see [Str4]). Recall
\[ M = \{ X \in \{X_D\} + H^{1,2}_0 (B, \mathbb{R}^3); D(X - X_0) \leq R \}. \]

But then
\[ E_{H_0}(X) \geq \inf_{X \in M} E_{H_0}(X) = \inf_{X \in M} \lim_{i \to \infty} E_{H_i}(X) \geq \lim_{i \to \infty} \inf_{X \in M} E_{H_i}(X) = \lim_{i \to \infty} E_{H_i}(X_i). \] (3.4)

Hence, by Theorem 4.5 below, $X_i \to X$ strongly in $\{X_D\} + H^{1,2}_0$ and uniformly in $B$ and $E_{H_0}(X) = \lim_{i \to \infty} E_{H_i}(X_i)$ by (3.4). We have
\[ E_{H_0}(X) = \inf_{X \in M} E_{H_0}(X) = E_{H_0}(X_0). \]

Hence the uniqueness of the small solution of $E_{H_0}$ in $\{X_D\} + H^{1,2}_0$ [BC] shows that $X = X_0$. Therefore, $X_i \to X_0$ uniformly in $B$ which contradicts
the above assumption. This completes the proof of Lemma 3.2.

Q.E.D.

If \( H \) is sufficiently close to \( H_0 \), we have

**Proposition 3.3.** - If \( H \in \mathcal{A} \) and \( X \) is the L-solution to (1.1), (1.2), then \( E_H(X) > E_H(X) \), where \( X \) is the S-solution to (1.1), (1.2).

**Proof.** - Let \( \varphi = X - X \in H_0^{1,2}(B, \mathbb{R}^3) \). Noting that \( X = X + \varphi \) and \( X \) satisfy the equation (1.1), we have

\[
E_H(X) = E_H(X + \varphi) = \frac{1}{2} \int |\nabla (X + \varphi)|^2 + \frac{2}{3} \int Q((X + \varphi)_u \wedge (X + \varphi)_v)
\]

\[
= E_H(X) + \frac{1}{2} \int |\nabla \varphi|^2 + \frac{2}{3} \int Q(\varphi) (X_u \wedge \varphi_e + \varphi_u \wedge X_v)
\]

\[
+ \frac{2}{3} \int (Q(X + \varphi) - Q(\varphi)) \varphi_u \wedge \varphi_e
+ \frac{2}{3} \int Q(\varphi) \varphi_u \wedge \varphi_v + O(\alpha) \left( \int |\nabla \varphi|^2 \right)^{1/2}
\]

(3.5)

by (1.10). Testing (1.1) with \( \varphi \) we get

0 = \int \nabla \varphi \nabla (X + \varphi) + 2 \int H(X + \varphi) \varphi (X + \varphi)_u \wedge (X + \varphi)_v
\]

\[
= \int |\nabla \varphi|^2 + 4 \int Q(X) \varphi_u \wedge \varphi_e + 2 \int Q(\varphi) \varphi_u \wedge \varphi_v
\]

\[
+ O(\alpha) \left( \left( \int |\nabla \varphi|^2 \right)^{1/2} + \int |\nabla \varphi|^2 \right)
\]

(3.6)

by (1.10). From (3.5), (3.6) it is clear

\[
E_H(X) = E_H(X) + \frac{1}{2} \int |\nabla \varphi|^2 + 2 \int Q(X) \varphi_u \wedge \varphi_e
\]

\[
+ \frac{2}{3} \int Q(\varphi) \varphi_u \wedge \varphi_v + \frac{c}{\alpha} \left( \left( \int |\nabla \varphi|^2 \right)^{1/2} + \int |\nabla \varphi|^2 \right)
\]

\[
= E_H(X) + \frac{1}{6} \left( \int |\nabla \varphi|^2 + 4 \int Q(X) \varphi_u \wedge \varphi_v \right)
\]

\[
+ O(\alpha) \left( \left( \int |\nabla \varphi|^2 \right)^{1/2} + \int |\nabla \varphi|^2 \right).
\]

By Lemma 3.1, we get

\[
E_H(X) - E_H(X) \geq \delta \int |\nabla \varphi|^2 - c \alpha \left( \left( \int |\nabla \varphi|^2 \right)^{1/2} + \int |\nabla \varphi|^2 \right).
\]

(3.7)
Therefore, from Lemma 2.6 we have
\[ E_H(X) > E_H(X) \]
promised that \( \alpha \) is small enough.
Q.E.D.

**Proposition 3.4.** — If \( \alpha > 0 \) is small enough, for \( H \in H_\alpha \) there exist a \( \rho_0 > 0 \) and a dense set \( A \) in \( [0, \rho_0] \) such that if \( \rho \in A \), then \( E_{H/(1 + \rho)} \) admits two distinct regular solutions in \( \{X_D\} + H_0^{1,2} \), one is the S-solution \( X_H \) and the other is the L-solution \( X \) with
\[
E_H(X) < \gamma_{H,0} \leq (1 + \rho) E_{H/(1 + \rho)}(X) \leq \gamma_{H,\alpha} < E_H(X_0) + \beta,
\]
where \( \gamma_{H,0}, \gamma_{H,\alpha} \) and \( \beta \) are given in section 2 and \( X_0 \) is the small solution of \( E_{H_0} \).

**Proof.** — Proposition 3.4 follows from the proof of Theorem 1.5 (see [Str4]) and Proposition 3.3.

**4. CONVERGENCE OF SURFACES OF PRESCRIBED MEAN CURVATURE**

As in [Pc], we can also establish a convergence theorem of surfaces of prescribed mean curvature with the Dirichlet boundary condition. Let \( H: \mathbb{R}^3 \to \mathbb{R} \) satisfy
\[
H \in C^1(\mathbb{R}^3, \mathbb{R}) \quad \text{and} \quad \|H\|_{L^\infty(\mathbb{R}^3)} + \|1 + |X|\|_{L^\infty(\mathbb{R}^3)} < +\infty. \tag{4.1}
\]

**Theorem 4.1.** — Let \( H_i \) satisfy (4.1) and \( \|H_i\|_{L^\infty} \leq K \) uniformly, and \( H_i \to H \) a.e. on \( \mathbb{R}^3 \). Suppose \( X_i \in \{X_D\} + H_0^{1,2}(B, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3) \) is a sequence of solutions to (1.1), (1.2) with \( H = H_i \) and \( \int_B |\nabla X_i|^2 \, \omega \leq c \) uniformly. Assume that \( X_i \to X \) weakly in \( H^{1,2}(B, \mathbb{R}^3) \) for some function \( X \in \{X_D\} + H_0^{1,2}(B, \mathbb{R}^3) \). Then \( X_i \to X \) strongly in \( \{X_D\} + H_0^{1,2, \text{loc}}(B \setminus S, \mathbb{R}^3) \) where \( S \) is a finite subset of \( B \). Moreover, \( X \) satisfies
\[
\Delta X = 2H(X)X_u \wedge X_v, \quad \text{in} \ B, \quad \text{X} = X_D, \quad \text{on} \ \partial B. \tag{4.2}
\]

**Proof.** — The proof is similar to that of Proposition 2.6 in [Str4], thus we only sketch it. Set
\[
S = \bigcap_{r > 0} \left\{ w \in B/\lim_{i \to \infty} \inf_{B(w, r) \cap B} |\nabla X_i|^2 \geq \mu_0 \right\}. \tag{4.3}
\]
where $\mu_0$ is a constant like $\mu_0$ in [Str4]. By the same argument of [Str4] or [Pc], we have (by taking subsequence)

$$X_i \to X \text{ strongly in } C^1(\mathbb{B}\setminus S, \mathbb{R}^3),$$

and $S$ is a finite subset of $B$. Moreover, $X$ satisfies (4.2).

**Q.E.D.**

**Lemma 4.2.** — If $X \in C^2(\mathbb{R}^2_+, \mathbb{R}^3)$ satisfies

$$\Delta X = 2H(X)X_u \wedge X_v, \quad \text{in } \mathbb{R}^2_+ \quad \{ X = \text{const.}, \quad \text{on } \partial \mathbb{R}^2_+ \}$$

then $X \equiv \text{const.}$

The Lemma easily follows from [Wet2]. For the convenience of the reader we give a complete proof.

**Proof of Lemma 4.2.** — Note that $X_u \cdot (X_u \wedge X_v) \equiv 0$ in $\mathbb{R}^2_+$, from (4.4) we have

$$X_u \cdot \Delta X = 0, \quad \text{in } \mathbb{R}^2_+$$

It's easy to see that

$$0 = X_u \cdot \Delta X = X_u \cdot \nabla \nabla X$$

$$= \text{div} \left\{ (1, 0) \nabla X \nabla X - \frac{1}{2} (1, 0) |\nabla X|^2 \right\}.$$

By Stokes' formula, we have

$$\int_{\partial \mathbb{R}^2_+} n \cdot (1, 0) \nabla X \nabla X - \frac{1}{2} n \cdot (1, 0) |\nabla X|^2 \, d\omega = 0 \quad (4.5)$$

where $n = (-1, 0)$ is the outer normal to $\mathbb{R}^2_+$ at $\partial \mathbb{R}^2_+$. Since $X \equiv \text{const.}$ on $\partial \mathbb{R}^2_+$, $\nabla X = (\nabla X \cdot n) n$ on $\partial \mathbb{R}^2_+$. Hence, from (4.5) we get

$$\int_{\partial \mathbb{R}^2_+} |\nabla X|^2 \, d\omega = 0.$$

Therefore $X_u \equiv 0$ on $\partial \mathbb{R}^2_+$. By the argument of Wente [Wet3], $X \equiv \text{const.}$ in $\mathbb{R}^2_+$.

**Q.E.D.**

**Lemma 4.3.** — Let $[H - H_0] < \infty$. If $X \in H_0^{1,2}(\mathbb{R}^2, \mathbb{R}^3)$ satisfies

$$\Delta X = 2H(X)X_u \wedge X_v \text{ in } \mathbb{R}^2, \quad \text{and is non-constant then}$$

$$E_H(X) \geq \frac{4\pi}{3H_0} - c_0 [H - H_0], \quad (4.6)$$

where $c_0$ is independent of $H$ and $[H - H_0]$.

**Proof.** — It is easy to prove this lemma, we omit it.
Proposition 4.4. Let \([H - H_0] < \infty\)

\[
\beta_H \geq \frac{4\pi}{3H_0^2} - c_0[H - H_0],
\]

(4.7)

where

\[
\beta_H = \inf \{ \lim_{i \to \infty} \inf (E_{H_i}(X_i) - E_H(X)); X_i \text{ are critical points of } E_{H_i} \}
\]

and \(X_i \to X\) in \(H^{1,2}\) weakly but not strongly.

Proof. For any such sequence \(\{X_i\}\), using Theorem 4.1, we see that \(X_i \to X\) strongly in \(\{X_D\} + H_0^{1,2} (B \setminus S, \mathbb{R}^3)\) where \(S\) is a finite non-empty subset of \(B\) and is defined by (4.3). There are two possibilities: either, (i) \(S \cap \partial B = \emptyset\); or, (ii) \(S \cap \partial B \neq \emptyset\). In case (i) from Section 3 of [Pc], we have a function \(X_0 \in H_0^{1,2} (\mathbb{R}^2, \mathbb{R}^3)\) satisfying \(
\Delta X_0 = 2H(X_0)X_u \wedge X_v \) in \(\mathbb{Z}\) and

\[
\lim_{i \to \infty} \inf (E_{H_i}(X_i) - E_H(X)) \geq E_H(X_0) \geq \frac{4\pi}{3H_0^2} - c_0[H - H_0]
\]

by Lemma 4.3. Therefore, \(\beta_H \geq \frac{4\pi}{3H_0^2} - c_0[H - H_0]\). In case (ii), using the same argument in [BC2] and Lemma 4.2, we also have a “blow up” function \(X\) satisfying \(\Delta X = 2H(X)X_u \wedge X_v\) in \(\mathbb{R}^2\) and

\[
\lim_{i \to \infty} \inf (E_{H_i}(X_i) - E_H(X)) \geq E_H(X_0) \geq \frac{4\pi}{3H_0^2} - c_0[H - H_0]
\]

Q.E.D.

Theorem 4.5. Let \(\alpha\) be fixed as in section 3 and \([H_i - H] < \alpha\) and \(H_i \to H\) a.e. in \(B\). Suppose \(X_i \in \{X_D\} + H_0^{1,2}\) is a sequence of solutions to (1.1), (1.2) with \(H = H_i\) and

\[
|E_{H_i}(X_i)| \leq c < \infty
\]

uniformly in \(i\). Then

\[
D(X_i) \leq c_1
\]

uniformly for another constant number \(c_1\). Moreover, assume \(X_i \to X\) weakly in \(H^{1,2}(B, \mathbb{R}^3)\) for some \(X \in \{X_D\} + H_0^{1,2}\), then \(X\) is a critical point of \(E_H\) in \(\{X_D\} + H_0^{1,2}\) and either

(i) \(X_i \to X\) strongly in \(H^{1,2} \cup L^\infty(B, \mathbb{R}^3)\) with

\[
E_H(X) = \lim_{i \to \infty} \inf E_{H_i}(X_i),
\]

or

(ii) \(E_H(X) \leq \lim_{i \to \infty} \inf E_{H_i}(X_i) - \frac{4\pi}{3H_0^2} + c_0[H - H_0]\).
Proof. Let $X_i$ be the $S$-solution of (1.1)-(1.2) with $H = H_i$ in $\{X_D\} + H_0^{1,2}(B, \mathbb{R}^3)$ (see §3) and $\varphi_i = X_i - X_1 \in H_0^{1,2}$. By (2.7), we have

$$-c + \delta \int |\nabla \varphi_i|^2 \leq E(X_i) - E(X),$$

where $\delta$ depends only on $\sigma$ and $X_D$. Note that $E_{H_i}(X_i)$, $E_{H}(X_i)$ and $D(X_i)$ are bounded uniformly. Hence, $D(X_i) \leq c_1$ uniformly for some constant $c_1$.

Assume that $X_i \to X$ weakly in $H^{1,2}(B, \mathbb{R}^3)$. Now there are two possibilities either, (i) $S = \emptyset$, or, (ii) $S \neq \emptyset$ by Theorem 4.1.

In case (i) $X_i \to X$ strongly in $H^{1,2} \cap L^\infty$ (see [Str4] or [Pc]). In case (ii) by Proposition 4.5. This completes the proof.

Q.E.D.

Remark 4.6. For the Dirichlet problem Theorem 4.5 gives a priori bounds which are of crucial importance to our results.

5. PROOF OF THEOREM 1.6

For any curvature function $H$ with $[H - H_0] < \alpha$, there exists the $S$-solution $X_H$ to (1.1), (1.2). On the other hand, by the results of Struwe [Str4] and proposition 3.3 there exists a sequence of $H_i = H/(1 + \rho_i)$ tending to $H$ such that $E_{H_i}$ admits the $L$-solution $X_i \in \{X_D\} + H_0^{1,2} \cap C^2(B, \mathbb{R}^3)$ with

$$E_H(X_0) < \gamma_{H,0} \leq (1 + \rho_0)E_{H_i}(X_i) \leq \gamma_{H,\alpha} < E_H(X_D) + \beta$$

(see [Str4] or Prop. 3.4), where $\rho_i > 0$ tends to 0 and $\gamma_{H,0}$, $\gamma_{H,\alpha}$, $\beta$ and $X_0$ are as in section 3.

Now from Theorem 4.5, $X_i \to X$ weakly in $H^{1,2}(B, \mathbb{R}^3)$ (by taking subsequence) and $X$ is a critical point of $E_H$ in $\{X_D\} + H_0^{1,2}$ with the property that either,

(i) $X_i \to X$ strongly in $H^{1,2}$, or,

(ii) $X_i \to X$ weakly but not strongly in $H^{1,2}$.

In case (i) $E_H(X) = \lim \inf_{i \to \infty} (1 + \rho_i)E_{H_i}(X_i) \geq \gamma_{H,0}$. In case (ii),

$$E_H(X) \leq \lim \inf_{i \to \infty} (1 + \rho_i)E_{H_i}(X_i) - \beta_H$$

$$\leq \gamma_{H,\alpha} - \beta_H.$$
Therefore, from (2.11), Lemma 3.2 and Proposition 4.4 it is easy to see that in any case
\[ E_H(X) \neq E_H(X_H). \]
This completes the proof of our theorem. Q.E.D.

Remark 5.1. – From (3.7) and Lemma 3.2 case (ii) in the proof of Theorem 1.6 cannot in fact happen for small \( \alpha \).

Remark 5.2. – We expect that for small \( \alpha \) if \( [H - H_0] < \alpha \), \( E_H \) satisfies the Palais-Smale condition in \( (-\infty, E_H(X_0) + \beta_H) \). Here \( X_0 \) is the S-solution of \( E_H \) in \( \{ X_D \} + H_0^{1,2}(B, \mathbb{R}^3) \).

REFERENCES


(Manuscript received December 12, 1990; revised January 6, 1991.)