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# The Dirichlet problem for the equation of prescribed mean curvature

by

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**ABSTRACT.** — We prove that there exist at least two distinct solutions to the Dirichlet problem for the equation of prescribed mean curvature  $\Delta X = 2H(X)X_u \wedge X_v$ , the curvature function  $H$  being in a full neighborhood of a suitable constant.

*Key words :* Equation of prescribed mean curvature, Dirichlet problem, relative minimizer.

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## 1. INTRODUCTION

Let  $B = \{ \omega = (u, v) \in \mathbb{R}^2 \mid |\omega| < 1 \}$  be the unit disc in  $\mathbb{R}^2$  with boundary  $\partial B$ . We consider the Dirichlet problem for the equation of prescribed mean curvature

$$\Delta X = 2H(X)X_u \wedge X_v, \quad \text{in } B, \quad (1.1)$$

$$X = X_D, \quad \text{on } \partial B. \quad (1.2)$$

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*Classification A.M.S. :* 53 A 10, 58 E 99.

Here,  $X_u = \frac{\partial}{\partial u} X$  and  $X_v = \frac{\partial}{\partial v} X$  denote partial derivative,  $\wedge$  and  $\cdot$  are the exterior and inner product in  $\mathbb{R}^3$  and  $H: \mathbb{R}^3 \rightarrow \mathbb{R}$  is a given function, and  $X_D$  is a given function of class  $C^2(\bar{B}, \mathbb{R}^3)$ .

If  $H \equiv H_0 = \text{Const.}$ , solutions to (1.1), (1.2) can be characterized as critical points of the functional

$$E_{H_0}(X) = D(X) + 2H_0 V(X), \tag{1.3}$$

in a space of admissible functions satisfying the boundary condition (1.2), where

$$D(X) = \frac{1}{2} \int_B |\nabla X|^2 d\omega \tag{1.4}$$

is the Dirichlet integral and

$$V(X) = \frac{1}{3} \int_B X \cdot X_u \wedge X_v d\omega \tag{1.5}$$

is the algebraic volume of surface  $X$ .

**THEOREM 1.1** ([Hi2], [Wet1], [Wet2] and [Stf1]). — *Suppose  $H \equiv H_0 \in \mathbb{R}$  and let  $X_D \in H^{1,2}(B, \mathbb{R}^3)$  be given. Assume that either*

(i)  $X_D$  is bounded and

$$|H_0| \cdot \|X_D\|_{L^\infty} < 1, \tag{1.6}$$

or

(ii) the condition

$$H_0^2 D(X_D) < \frac{2}{3} \pi \tag{1.7}$$

is satisfied. Then there is a solution  $\underline{X} \in \{X_D\} + H_0^{1,2}(B, \mathbb{R}^3)$  to (1.1), (1.2) which is a strict relative minimizer of  $E_{H_0}$  in this space.

*Remark 1.2.* — The observation that the solutions of Hildebrandt, Steffen and Wente are strict relative minima is due to Brezis-Coron [BC].

The existence of a second solution was proved independently by Brezis-Coron [BC] and Struwe [St2] with an important contribution by Steffen [Stf2] as follows

**THEOREM 1.3** [Str3]. — *Let  $X_D \in H^{1,2} \cap L^\infty(B, \mathbb{R}^3)$  be a non-constant vector,  $H_0$  any real number different from zero. Suppose  $E_{H_0}$  admits a local minimum  $\underline{X}$  in the class  $\{X_D\} + H_0^{1,2}(B, \mathbb{R}^3)$ . Then there exists a solution  $\bar{X} \in \{X_D\} + H_0^{1,2}(B, \mathbb{R}^3)$  of (1.1) and (1.2) different from  $\underline{X}$  and satisfying the condition*

$$E_{H_0}(\bar{X}) < E_{H_0}(\underline{X}) = \inf_{p \in P} \sup_{X \in \text{im}(P)} E_H(X) < E_{H_0}(\underline{X}) + \frac{4\pi}{3|H_0|^2}, \tag{1.8}$$

where

$$P = \{ p \in C^0([0, 1], \{ X_D \} + H_0^{1,2}(B, \mathbb{R}^3)) \mid p(0) = \underline{X}, E_{H_0}(p(1)) < E_{H_0}(\underline{X}) \}. \quad (1.9)$$

For variable curvature functions H results comparable to Theorem 1.1 have been obtain by Hildebrandt [Hi1] and Steffen [Stf1].

**THEOREM 1.4 [Hi1].** — Suppose H is of class C<sup>1</sup> and let X<sub>D</sub> ∈ H<sup>1,2</sup> ∩ L<sup>∞</sup>(B, ℝ<sup>3</sup>) be given with ||X<sub>D</sub>||<sub>L<sup>∞</sup></sub> < 1. Then if

$$\Delta$$

$$h = \text{ess sup}_{|X| \leq 1} H(X) < 1$$

there exists a solution X ∈ {X<sub>D</sub>} + H<sup>1,2</sup>(B, ℝ<sup>3</sup>) to (1.1), (1.2) such that E<sub>H</sub>(X) = inf { E<sub>H</sub>(X); X ∈ M }, where M is given by (2.9) below.

If variable curvature function H is sufficiently close to a suitable constant, Struwe obtained [Str4].

**THEOREM 1.5.** — Suppose X<sub>D</sub> ∈ C<sup>2</sup>(B, ℝ<sup>2</sup>) is non-constant and suppose that for H<sub>0</sub> ∈ ℝ \ {0} the functional E<sub>H<sub>0</sub></sub> admits a relative minimizer in {X<sub>D</sub>} + H<sup>1,2</sup>(B, ℝ<sup>3</sup>). Then there exists a number α > 0 such that for a dense set A of curvature functions H in the α-neighborhood of H<sub>0</sub>, the Dirichlet problem (1.1), (1.2) admits at least two distinct regular solutions in {X<sub>D</sub>} + H<sup>1,2</sup>(B, ℝ<sup>3</sup>).

Here the α-neighborhood of H<sub>0</sub> is defined as

$$\Delta$$

$$|H - H_0| = \text{ess sup}_{X \in \mathbb{R}^3} \{ (1 + |X|) (|H(X) - H_0| + |\nabla H(X)| + |Q(X) - H_0 X| + |\nabla Q(X) - H_0 \text{id}|) \} \leq \alpha, \quad (1.10)$$

where Q is given by (2.3) below.

In this paper, we improve Theorem 1.5 and obtain that

**THEOREM 1.6.** — Suppose X<sub>D</sub> ∈ C<sup>2</sup>(B, ℝ<sup>3</sup>) is non-constant, and suppose that for H<sub>0</sub> ∈ ℝ \ {0} the functional E<sub>H<sub>0</sub></sub> admits a relative minimizer in {X<sub>D</sub>} + H<sup>1,2</sup>. Then there exists a number α > 0 such that if |H - H<sub>0</sub>| < α, E<sub>H</sub> admits two solutions in {X<sub>D</sub>} + H<sup>1,2</sup>.

From the proof of Theorem 1.5 [Str4], we have a relative minimizer of E<sub>H</sub> for a full α-neighborhood of H<sub>0</sub> and another “large” critical point of E<sub>H</sub> for H ∈ A. We call the former S-solution and the latter L-solution.

First, we show that the S-solution is also a “strict” relative minimizer — its E<sub>H</sub> — energy is less than that of the L-solution — provided that |H - H<sub>0</sub>| is small enough. Next, we give a priori estimates for solutions of the Dirichlet problem — which are of crucial importance to our result — though they are not given explicitly. Then, we can use the solutions obtained by Struwe in [Str4] — the L-solutions — for a dense set A in

$\mathcal{H}_\alpha = \{H \mid |H - H_0| < \alpha\}$  to approximate a solution of  $E_H$  for any  $H \in \mathcal{H}_\alpha$  which is different from the S-solution.

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### 2. PRELIMINARIES

For variable curvature function  $H$ , solutions to (1.1), (1.2) can be characterized as critical points of the functional

$$E_H(X) = D(X) + 2V_H(X) \tag{2.1}$$

in the space  $\{X_D\} + H_0^{1,2}(B, \mathbb{R}^2)$ . Here, the H-volume introduced by Hildebrandt is given by

$$V_H(X) = \frac{1}{3} \int_B Q(X) \cdot X_u \wedge X_v \, dw, \tag{2.2}$$

where

$$Q(x_1, x_2, x_3) = \left( \int_0^{x_1} H(s, x_2, x_3) \, ds, \int_0^{x_2} H(x_1, s, x_3) \, ds, \int_0^{x_3} H(x_1, x_2, s) \, ds \right). \tag{2.3}$$

We list some useful lemmas.

LEMMA 2.1 (Isoperimetric inequality, cf. [Wet1]):

$$36\pi(V(X))^2 \leq D(X)^3, \tag{2.4}$$

for  $X \in H_0^{1,2}(B, \mathbb{R}^3)$ .

LEMMA 2.2 ([BC], [Str4], Prop. 3.1). – Suppose  $X_D \in C^2(\bar{B}, \mathbb{R}^3)$  is non-constant, and suppose that for  $H \equiv H_0 \neq 0$  the functional  $E_{H_0}$  admits a relative minimizer  $\underline{X}_0 \in \{X_D\} + C^2 \cap H_0^{1,2}(B, \mathbb{R}^3)$ . Then there exists a radius  $R > 0$ , a function  $\bar{X}_1 \in \{X_D\} + C^2 \cap H_0^{1,2}(\bar{B}, \mathbb{R}^3)$  with  $D(X_1 - \underline{X}_0) \geq R$ , and a continuous path  $p \in C^0([0, 1]; \{X_D\} + C^2 \cap H_0^{1,2}(\bar{B}, \mathbb{R}^3))$  connecting  $\underline{X}_0 = p(0)$  with  $X_1 = p(1)$  such that the estimates

$$E_{H_0}(X_1) < \inf \{E_{H_0}(X); X - \underline{X}_0 \in H_0^{1,2}, D(X - \underline{X}_0) \leq R\} \leq E_{H_0}(\underline{X}_0) \tag{2.5}$$

$$< \inf \{E_{H_0}(X); X - \underline{X}_0 \in H_0^{1,2}, D(X - \underline{X}_0) = R\} \tag{2.6}$$

$$\leq \sup \{E_{H_0}(X); X \in p([0, 1])\} \tag{2.7}$$

$$< E_{H_0}(\underline{X}_0) + 4\pi/3 H_0^2 \tag{2.8}$$

hold.

DEFINITION 2.3:

$$M = \{ X \in \{ X_D \} + H_0^{1,2}(\mathbb{B}, \mathbb{R}^3); D(X - \underline{X}_0) \leq R \}, \tag{2.9}$$

where  $\underline{X}_0$  and  $R$  are as in Lemma 2.2.

LEMMA 2.4 [BC]. — For  $H_0 \in \mathbb{R} \setminus \{0\}$ . Suppose that  $E_{H_0}$  admits a relative minimizer  $\underline{X}_0 \in \{ X_D \} + C^2 \cap H_0^{1,2}(\mathbb{B}, \mathbb{R}^3)$ . Then there is  $\delta > 0$  such that

$$\int |\nabla \varphi|^2 + 4 H_0 \int \underline{X}_0 \cdot \varphi_u \wedge \varphi_v \geq \delta \int |\nabla \varphi|^2, \text{ for all } \varphi \in H_0^{1,2}. \tag{2.10}$$

Let

$$P = \{ p \in C^0([0, 1]; \{ X_D \} + H_0^{1,2}(\mathbb{B}, \mathbb{R}^3)), p(0) = \underline{X}_0, p(1) = X_1 \}$$

and set

$$\gamma_{H,\rho} = \inf_{p \in P} \sup_{X \in \text{im}(p)} E_{H,\rho}(X)$$

where  $E_{H,\rho}(X) = (1 + \rho) E_{H/(1+\rho)}(X)$ . Using Lemma 2.2 we have (see [Str4], (3.5))

$$E_{H,\rho}(X_1) < \inf \{ E_H(X); X - \underline{X}_0 \in H_0^{1,2}, D(X - \underline{X}_0) \leq R \} \leq E_{H,\rho}(\underline{X}_0) \tag{2.5}_\rho$$

$$< \inf \{ E_H(X); X - \underline{X}_0 \in H_0^{1,2}, D(X - \underline{X}_0) = R \} \leq \inf \{ E_{H,\rho}(X); \dots \} \tag{2.6}_\rho$$

$$\leq \sup \{ E_{H,\rho}(X); X \in p([0, 1]) \} \tag{2.7}_\rho$$

$$< E_H(\underline{X}_0) + \beta \leq E_{H,\rho}(\underline{X}_0) + \beta \tag{2.8}_\rho$$

and

$$E_{H,\rho}(\underline{X}_0) < \gamma_{H,0} \leq \gamma_{H,\rho} \leq \gamma_{H,\alpha} < E_H(\underline{X}_0) + \beta, \tag{2.11}$$

for  $\rho \in [0, \alpha]$ . Here  $X_1$ ,  $\underline{X}_0$  and  $P$  are as in Lemma 2.2,  $\alpha$  is small enough and fixed and  $\beta < 4\pi/3 H_0^2$  is independent of  $H$ . Moreover, we have

LEMMA 2.5. — There exists a constant number  $\varepsilon_0$  independent of  $\alpha$  such that

$$\begin{aligned} E_{H_0}(\underline{X}_0) + \varepsilon_0 &< \inf \{ E_H(X); X - \underline{X}_0 \in H_0^{1,2}, D(X - \underline{X}_0) = R \} \\ &\leq \inf \{ E_{H,\rho}(X); X - \underline{X}_0 \in H_0^{1,2}, D(X - \underline{X}_0) = R \} \\ &\leq \sup \{ E_{H,\rho}(X); X \in p([0, 1]) \} \\ &< E_H(\underline{X}_0) + \beta - \varepsilon_0 \leq E_{H,\rho}(\underline{X}_0) + \beta - \varepsilon_0 \end{aligned}$$

provided that  $\alpha$  is small enough, where  $\underline{X}_0$ ,  $R$  and  $p$  are as in Lemma 2.2.

Proof. — Set

$$\begin{aligned} \varepsilon_0 = \frac{1}{4} \min \{ & (E_{H_0}(\underline{X}_0) + 4\pi/3 H_0^2 - \sup \{ E_H(X); X \in p([0, 1]) \}), \\ & (\inf \{ E_{H_0}(X); X - \underline{X}_0 \in H_0^{1,2}, D(X - \underline{X}_0) = R \} - E_{H_0}(\underline{X}_0)) \}. \end{aligned}$$

It is easy to see that Lemma 2.5 follows from Lemma 2.2 for  $\alpha$  small enough.

Q.E.D.

LEMMA 2.6. — *There exists a constant  $c$  independent of  $\alpha$  such that if  $H \in \mathcal{A}$  (see Theorem 1.5),*

$$D(\underline{X} - \bar{X}) > c,$$

where  $\underline{X}$  (resp.  $\bar{X}$ ) is the S-solution (resp. L-solution) to (1.1), (1.2).

*Proof.* — It follows the proof of Theorem 1.5 ([Str4], Theorem 3.1) and Lemma 2.5.

Q.E.D.

### 3. THE “STRICT” RELATIVE MINIMA

In this section, we will prove that the S-solution to the Dirichlet problem for the equation of prescribed mean curvature  $H$  is a “strict” relative minimum in the space  $\{X_D\} + H_0^{1,2}(B, \mathbb{R}^3)$ , provided that  $[H - H_0]$  is small enough. Here  $H_0 \neq 0$  is a constant with the property that  $E_{H_0}$  admits a relative minimizer  $\underline{X}_0 \in \{X_D\} + C^2 \cap H_0^{1,2}(B, \mathbb{R}^3)$ .

LEMMA 3.1. — *There exists a constant number  $\alpha > 0$  with the property that if  $[H - H_0] < \alpha$  there is a constant  $\delta > 0$  depending only on  $\alpha$  and  $X_D$  such that*

$$\int |\nabla \varphi|^2 + 4 \int Q(\underline{X}) \varphi_u \wedge \varphi_v \geq \delta \int |\nabla \varphi|^2, \quad \text{for any } \varphi \in H_0^{1,2}. \quad (3.1)$$

Here  $\underline{X} = \underline{X}_H$  is the S-solution to (1.1), (1.2).

*Proof.* — Let  $\underline{X}_0$  be the small solution of  $E_{H_0}$  in the space  $\{X_D\} + H_0^{1,2}$ . By Brezis-Coron [BC]—see Lemma 2.4—there exists a constant  $\delta_1 > 0$  such that

$$\int |\nabla \varphi|^2 + 4 H_0 \int \underline{X}_0 \varphi_u \wedge \varphi_v \geq \delta_1 \int |\nabla \varphi|^2, \quad \varphi \in H_0^{1,2}. \quad (3.2)$$

Thus for any  $\varphi \in H_0^{1,2}$

$$\begin{aligned} \int |\nabla \varphi|^2 + 4 \int Q(\underline{X}) \varphi_u \wedge \varphi_v &= \int |\nabla \varphi|^2 + 4 H_0 \int \underline{X}_0 \varphi_u \wedge \varphi_v \\ &+ 4 \int (Q(\underline{X}) - H_0 \underline{X}) \varphi_u \wedge \varphi_v + 4 H_0 \int (\underline{X} - \underline{X}_0) \varphi_u \wedge \varphi_v \\ &\geq (\delta_1 - 2\alpha) \int |\nabla \varphi|^2 + 4 H_0 \int (\underline{X} - \underline{X}_0) \varphi_u \wedge \varphi_v. \end{aligned}$$

Therefore, Lemma 3.1 follows from the following

LEMMA 3.2. — For any  $\varepsilon > 0$ , there exists a constant  $\alpha > 0$  with the property that for any curvature function  $H$  with  $[H - H_0] < \alpha$ , if  $\underline{X}_H$  is the S-solution to (1.1), (1.2), then

$$\|\underline{X}_H - \underline{X}_0\|_{L^\infty} < \varepsilon. \tag{3.3}$$

*Proof.* — If the Lemma is false, we may assume that there exist  $\varepsilon_0 > 0$  and a sequence  $\{\underline{X}_i\}$  of the S-solutions of  $E_H$  with  $H = H_i$  and  $[H_i - H_0] \rightarrow 0$  as  $i \rightarrow \infty$  such that  $\|\underline{X}_{H_i} - \underline{X}_0\|_{L^\infty} \geq \varepsilon_0$ . Noticing that  $\underline{X}_i \in M$ , we know that  $D(\underline{X}_i)$  are bounded uniformly in  $i$ . Thus we may assume that  $\{\underline{X}_i\}$  converges to  $\underline{X}$  weakly in  $\{X_D\} + H_0^{1,2}$  for some  $\underline{X} \in \{X_D\} + H_0^{1,2}$ . It is easy to see that  $\underline{X} \in M$  (see [Str4]). Recall

$$M = \{X \in \{X_D\} + H_0^{1,2}(B, \mathbb{R}^3); D(X - \underline{X}_0) \leq R\}.$$

But then

$$\left. \begin{aligned} E_{H_0}(\underline{X}) &\geq \inf_{X \in M} E_{H_0}(X) \\ &= \inf_{X \in M} \lim_{i \rightarrow \infty} E_{H_i}(X) \\ &\geq \lim_{i \rightarrow \infty} \inf_{X \in M} E_{H_i}(X) \\ &= \lim_{i \rightarrow \infty} E_{H_i}(\underline{X}_i). \end{aligned} \right\} \tag{3.4}$$

Hence, by Theorem 4.5 below,  $\underline{X}_i \rightarrow \underline{X}$  strongly in  $\{X_D\} + H_0^{1,2}$  and uniformly in  $\bar{B}$  and  $E_{H_0}(\underline{X}) = \lim_{i \rightarrow \infty} E_{H_i}(\underline{X}_i)$  by (3.4). We have

$$E_{H_0}(\underline{X}) = \inf_{X \in M} E_{H_0}(X) = E_{H_0}(\underline{X}_0).$$

Hence the uniqueness of the small solution of  $E_{H_0}$  in  $\{X_D\} + H_0^{1,2}$  [BC] shows that  $\underline{X} = \underline{X}_0$ . Therefore,  $\underline{X}_i \rightarrow \underline{X}_0$  uniformly in  $\bar{B}$  which contradicts



the above assumption. This completes the proof of Lemma 3.2.

Q.E.D.

If  $H$  is sufficiently close to  $H_0$ , we have

PROPOSITION 3.3. – *If  $H \in \mathcal{A}$  and  $X$  is the L-solution to (1.1), (1.2), then  $E_H(X) > E_H(\underline{X})$ , where  $\underline{X}$  is the S-solution to (1.1), (1.2).*

Proof. – Let  $\varphi = X - \underline{X} \in H_0^{1,2}(\mathbb{B}, \mathbb{R}^3)$ . Noting that  $X = \underline{X} + \varphi$  and  $\underline{X}$  satisfy the equation (1.1), we have

$$\begin{aligned} E_H(X) &= E_H(\underline{X} + \varphi) \\ &= \frac{1}{2} \int |\nabla(\underline{X} + \varphi)|^2 + \frac{2}{3} \int Q(\underline{X} + \varphi) (\underline{X} + \varphi)_u \wedge (\underline{X} + \varphi)_v \\ &= E_H(\underline{X}) + \frac{1}{2} \int |\nabla \varphi|^2 + \frac{2}{3} \int Q(\varphi) (\underline{X}_u \wedge \varphi_v + \varphi_u \wedge \underline{X}_v) \\ &\quad + \frac{2}{3} \int (Q(\underline{X} + \varphi) - Q(\varphi)) \varphi_u \wedge \varphi_v \\ &\quad + \frac{2}{3} \int Q(\varphi) \varphi_u \wedge \varphi_v + O(\alpha) \left( \int |\nabla \varphi|^2 \right)^{1/2} \end{aligned} \tag{3.5}$$

by (1.10). Testing (1.1) with  $\varphi$  we get

$$\begin{aligned} 0 &= \int \nabla \varphi \nabla(\underline{X} + \varphi) + 2 \int H(\underline{X} + \varphi) \varphi (\underline{X} + \varphi)_u \wedge (\underline{X} + \varphi)_v \\ &= \int |\nabla \varphi|^2 + 4 \int Q(\underline{X}) \varphi_u \wedge \varphi_v + 2 \int Q(\varphi) \varphi_u \wedge \varphi_v \\ &\quad + O(\alpha) \left( \left( \int |\nabla \varphi|^2 \right)^{1/2} + \int |\nabla \varphi|^2 \right) \end{aligned} \tag{3.6}$$

by (1.10). From (3.5), (3.6) it is clear

$$\begin{aligned} E_H(X) &= E_H(\underline{X}) + \frac{1}{2} \int |\nabla \varphi|^2 + 2 \int Q(\underline{X}) \varphi_u \wedge \varphi_v \\ &\quad + \frac{2}{3} \int Q(\varphi) \varphi_u \wedge \varphi_v + c\alpha \left( \left( \int |\nabla \varphi|^2 \right)^{1/2} + \int |\nabla \varphi|^2 \right) \\ &= E_H(\underline{X}) + \frac{1}{6} \left( \int |\nabla \varphi|^2 + 4 \int Q(\underline{X}) \varphi_u \wedge \varphi_v \right) \\ &\quad + O(\alpha) \left( \left( \int |\nabla \varphi|^2 \right)^{1/2} + \int |\nabla \varphi|^2 \right). \end{aligned}$$

By Lemma 3.1, we get

$$E_H(X) - E_H(\underline{X}) \geq \delta \int |\nabla \varphi|^2 - c\alpha \left( \left( \int |\nabla \varphi|^2 \right)^{1/2} + \int |\nabla \varphi|^2 \right). \tag{3.7}$$

Therefore, from Lemma 2.6 we have

$$E_H(X) > E_H(\underline{X})$$

provided that  $\alpha$  is small enough.

Q.E.D.

PROPOSITION 3.4. — *If  $\alpha > 0$  is small enough, for  $H \in \mathcal{H}_\alpha$  there exist a  $\rho_0 > 0$  and a dense set  $A$  in  $[0, \rho_0]$  such that if  $\rho \in A$ , then  $E_{H/(1+\rho)}$  admits two distinct regular solutions in  $\{X_D\} + H_0^{1,2}$ , one is the S-solution  $\underline{X}_H$  and the other is the L-solution  $\bar{X}$  with*

$$E_H(\underline{X}_0) < \gamma_{H,0} \leq (1+\rho) E_{H/(1+\rho)}(\bar{X}) \leq \gamma_{H,\alpha} < E_H(\underline{X}_0) + \beta,$$

where  $\gamma_{H,0}$ ,  $\gamma_{H,\alpha}$  and  $\beta$  are given in section 2 and  $\underline{X}_0$  is the small solution of  $E_{H_0}$ .

*Proof.* — Proposition 3.4 follows from the proof of Theorem 1.5 (see [Str4]) and Proposition 3.3.

#### 4. CONVERGENCE OF SURFACES OF PRESCRIBED MEAN CURVATURE

As in [Pc], we can also establish a convergence theorem of surfaces of prescribed mean curvature with the Dirichlet boundary condition. Let  $H: \mathbb{R}^3 \rightarrow \mathbb{R}$  satisfy

$$\left. \begin{aligned} &H \in C^1(\mathbb{R}^3, \mathbb{R}) \\ &\|H\|_{L^\infty(\mathbb{R}^3)} + \|(1+|X|)|\nabla H(X)\|_{L^\infty(\mathbb{R}^3)} < +\infty. \end{aligned} \right\} \quad (4.1)$$

THEOREM 4.1. — *Let  $H_i$  satisfy (4.1) and  $\|H_i\|_{L^\infty} \leq K$  uniformly, and  $H_i \rightarrow H$  a.e. on  $\mathbb{R}^3$ . Suppose  $X_i \in \{X_D\} + H_0^{1,2}(B, \mathbb{R}^3) \cap C^2(\bar{B}, \mathbb{R}^3)$  is a sequence of solutions to (1.1), (1.2) with  $H = H_i$  and  $\int_B |\nabla X_i|^2 d\omega \leq c$  uniformly. Assume that  $X_i \rightarrow X$  weakly in  $H^{1,2}(B, \mathbb{R}^3)$  for some function  $X \in \{X_D\} + H_0^{1,2}(B, \mathbb{R}^3)$ . Then  $X_i \rightarrow X$  strongly in  $\{X_D\} + H_{0,loc}^{1,2}(B \setminus S, \mathbb{R}^3)$  where  $S$  is a finite subset of  $\bar{B}$ . Moreover,  $X$  satisfies*

$$\left. \begin{aligned} &\Delta X = 2H(X)X_u \wedge X_v, \quad \text{in } B, \\ &X = X_D, \quad \text{on } \partial B. \end{aligned} \right\} \quad (4.2)$$

*Proof.* — The proof is similar to that of Proposition 2.6 in [Str4], thus we only sketch it. Set

$$S = \bigcap_{r>0} \left\{ w \in \bar{B} / \liminf_{i \rightarrow \infty} \int_{B(w,r) \cap B} |\nabla X_i|^2 \geq \mu_0 \right\} \quad (4.3)$$

where  $\mu_0$  is a constant like  $\mu_0$  in [Str4]. By the same argument of [Str4] or [Pc], we have (by taking subsequence)

$$X_i \rightarrow X \quad \text{strongly in } C^1(\bar{B} \setminus S, \mathbb{R}^3),$$

and  $S$  is a finite subset of  $\bar{B}$ . Moreover,  $X$  satisfies (4.2).

Q.E.D.

LEMMA 4.2. — *If  $X \in C^2(\mathbb{R}_+^2, \mathbb{R}^3)$  satisfies*

$$\left. \begin{aligned} \Delta X &= 2H(X)X_u \wedge X_v, & \text{in } \mathbb{R}_+^2 \\ X &= \text{const.}, & \text{on } \partial\mathbb{R}_+^2 \end{aligned} \right\} \quad (4.4)$$

*then  $X \equiv \text{const.}$*

The Lemma easily follows from [Wet2]. For the convenience of the reader we give a complete proof.

*Proof of Lemma 4.2.* — Note that  $X_u \cdot (X_u \wedge X_v) \equiv 0$  in  $\mathbb{R}_+^2$ , from (4.4) we have

$$X_u \cdot \Delta X = 0, \quad \text{in } \mathbb{R}_+^2$$

It's easy to see that

$$\begin{aligned} 0 &= X_u \cdot \Delta X = X_u \cdot \text{div } \nabla X \\ &= \text{div} \left\{ ((1, 0) \nabla X) \nabla X - \frac{1}{2}(1, 0) |\nabla X|^2 \right\}. \end{aligned}$$

By Stokes' formula, we have

$$\int_{\partial\mathbb{R}_+^2} n \cdot ((1, 0) \nabla X) \nabla X - \frac{1}{2} n \cdot (1, 0) |\nabla X|^2 d\omega = 0 \quad (4.5)$$

where  $n = (-1, 0)$  is the outer normal to  $\mathbb{R}_+^2$  at  $\partial\mathbb{R}_+^2$ . Since  $X \equiv \text{const.}$  on  $\partial\mathbb{R}_+^2$ ,  $\nabla X = (\nabla X \cdot n)n$  on  $\partial\mathbb{R}_+^2$ . Hence, from (4.5) we get

$$\int_{\partial\mathbb{R}_+^2} |\nabla X|^2 d\omega = 0.$$

Therefore  $X_n \equiv 0$  on  $\partial\mathbb{R}_+^2$ . By the argument of Wente [Wet3],  $X \equiv \text{const.}$  in  $\mathbb{R}_+^2$ .

Q.E.D.

LEMMA 4.3. — *Let  $[H - H_0] < \infty$ . If  $X \in H_0^{1,2}(\mathbb{R}^2, \mathbb{R}^3)$  satisfies  $\Delta X = 2H(X)X_u \wedge X_v$  in  $\mathbb{R}^2$ , and is non-constant then*

$$E_H(X) \geq \frac{4\pi}{3H_0^2} - c_0[H - H_0], \quad (4.6)$$

*where  $c_0$  is independent of  $H$  and  $[H - H_0]$ .*

*Proof.* — It is easy to prove this lemma, we omit it.

PROPOSITION 4.4. — *Let  $[H - H_0] < \infty$*

$$\beta_H \geq \frac{4\pi}{3H_0^2} - c_0[H - H_0], \tag{4.7}$$

where

$$\beta_H = \inf \left\{ \liminf_{i \rightarrow \infty} (E_{H_i}(X_i) - E_H(X)); X_i \text{ are critical points of } E_{H_i} \text{ and } X_i \rightarrow X \text{ in } H^{1,2} \text{ weakly but not strongly} \right\}.$$

*Proof.* — For any such sequence  $\{X_i\}$ , using Theorem 4.1, we see that  $X_i \rightarrow X$  strongly in  $\{X_D\} + H_{0,loc}^{1,2}(B \setminus S, \mathbb{R}^3)$  where  $S$  is a finite non-empty subset of  $\bar{B}$  and is defined by (4.3). There are two possibilities: either, (i)  $S \cap \partial B = \emptyset$ ; or, (ii)  $S \cap \partial B \neq \emptyset$ . In case (i) from Section 3 of [Pc], we have a function  $X_0 \in H_0^{1,2}(\mathbb{R}^2, \mathbb{R}^3)$  satisfying  $\Delta X_0 = 2H(X_0)X_{0u} \wedge X_{0v}$  in  $\mathbb{R}^2$  and

$$\liminf_{i \rightarrow \infty} (E_{H_i}(X_i) - E_H(X)) \geq E_H(X_0) \geq \frac{4\pi}{3H_0^2} - c_0[H - H_0]$$

by Lemma 4.3. Therefore,  $\beta_H \geq \frac{4\pi}{3H_0^2} - c_0[H - H_0]$ . In case (ii), using the same argument in [BC2] and Lemma 4.2, we also have a “blow up” function  $X$  satisfying  $\Delta X = 2H(X)X_u \wedge X_v$  in  $\mathbb{R}^2$  and

$$\liminf_{i \rightarrow \infty} (E_{H_i}(X_i) - E_H(X)) \geq E_H(X_0) \geq \frac{4\pi}{3H_0^2} - c_0[H - H_0]$$

Q.E.D.

THEOREM 4.5. — *Let  $\alpha$  be fixed as in section 3 and  $[H_i - H] < \alpha$  and  $H_i \rightarrow H$  a.e. in  $B$ . Suppose  $X_i \in \{X_D\} + H_0^{1,2}$  is a sequence of solutions to (1.1), (1.2) with  $H = H_i$  and*

$$|E_{H_i}(X_i)| \leq c < \infty$$

uniformly in  $i$ . Then

$$D(X_i) \leq c_1$$

uniformly for another constant number  $c_1$ . Moreover, assume  $X_i \rightarrow X$  weakly in  $H^{1,2}(B, \mathbb{R}^3)$  for some  $X \in \{X_D\} + H_0^{1,2}$ , then  $X$  is a critical point of  $E_H$  in  $\{X_D\} + H_0^{1,2}$  and either

(i)  $X_i \rightarrow X$  strongly in  $H^{1,2} \cup L^\infty(B, \mathbb{R}^3)$  with

$$E_H(X) = \liminf_{i \rightarrow \infty} E_{H_i}(X_i),$$

or

$$(ii) E_H(X) \leq \liminf_{i \rightarrow \infty} E_{H_i}(X_i) - \frac{4\pi}{3H_0^2} + c_0[H - H_0].$$

*Proof.* — Let  $X_i$  be the S-solution of (1.1)-(1.2) with  $H=H_i$  in  $\{X_D\} + H_0^{1,2}(B, \mathbb{R}^3)$  (see §3) and  $\varphi_i = X_i - \underline{X}_i \in H_0^{1,2}$ . By (2.7), we have

$$-c + \delta \int |\nabla \varphi_i|^2 \leq E(X_i) - E(\underline{X}_i),$$

where  $\delta$  depends only on  $\alpha$  and  $X_D$ . Note that  $E_{H_i}(X_i)$ ,  $E_{H_i}(X_i)$  and  $D(X_i)$  are bounded uniformly. Hence,  $D(X_i) \leq c_1$  uniformly for some constant  $c_1$ .

Assume that  $X_i \rightarrow X$  weakly in  $H^{1,2}(B, \mathbb{R}^3)$ . Now there are two possibilities either, (i)  $S = \emptyset$ , or, (ii)  $S \neq \emptyset$  by Theorem 4.1.

In case (i)  $X_i \rightarrow X$  strongly in  $H^{1,2} \cap L^\infty$  (see [Str4] or [Pc]). In case (ii)

$$\liminf_{i \rightarrow \infty} E_{H_i}(X_i) - E_H(X) \geq \beta_H \geq \frac{4\pi}{3H_0} - c_0[H - H_0],$$

by Proposition 4.5. This completes the proof.

Q.E.D.

*Remark 4.6.* — For the Dirichlet problem Theorem 4.5 gives *a priori* bounds which are of crucial importance to our results.

### 5. PROOF OF THEOREM 1.6

For any curvature function  $H$  with  $[H - H_0] < \alpha$ , there exists the S-solution  $\underline{X}_H$  to (1.1), (1.2). On the other hand, by the results of Struwe [Str4] and proposition 3.3 there exists a sequence of  $H_i = H/(1 + \rho_i)$  tending to  $H$  such that  $E_{H_i}$  admits the L-solution  $X_i \in \{X_H\} + H_0^{1,2} \cap C^2(\bar{B}, \mathbb{R}^3)$  with

$$E_H(X_0) < \gamma_{H,0} \leq (1 + \rho_i) E_{H_i}(X_i) \leq \gamma_{H,\alpha} < E_H(\underline{X}_D) + \beta$$

(see [Str4] or Prop. 3.4), where  $\rho_i > 0$  tends to 0 and  $\gamma_{H,0}$ ,  $\gamma_{H,\alpha}$ ,  $\beta$  and  $X_0$  are as in section 3.

Now from Theorem 4.5,  $X_i \rightarrow X$  weakly in  $H^{1,2}(B, \mathbb{R}^3)$  (by taking subsequence) and  $X$  is a critical point of  $E_H$  in  $\{X_D\} + H_0^{1,2}$  with the property that either,

- (i)  $X_i \rightarrow X$  strongly in  $H^{1,2}$ , or,
- (ii)  $X_i \rightarrow X$  weakly but not strongly in  $H^{1,2}$ .

In case (i)  $E_H(X) = \liminf_{i \rightarrow \infty} (1 + \rho_i) E_{H_i}(X_i) \geq \gamma_{H,0}$ . In case (ii),

$$\begin{aligned} E_H(X) &\leq \liminf_{i \rightarrow \infty} (1 + \rho_i) E_{H_i}(X_i) - \beta_H \\ &\leq \gamma_{H,\alpha} - \beta_H. \end{aligned}$$

Therefore, from (2.11), Lemma 3.2 and Proposition 4.4 it is easy to see that in any case

$$E_H(X) \neq E_H(\underline{X}_H).$$

This completes the proof of our theorem.

Q.E.D.

*Remark 5.1.* – From (3.7) and Lemma 3.2 case (ii) in the proof of Theorem 1.6 cannot in fact happen for small  $\alpha$ .

*Remark 5.2.* – We expect that for small  $\alpha$  if  $[H - H_0] < \alpha$ ,  $E_H$  satisfies the Palais-Smale condition in  $(-\infty, E_H(\underline{X}_0) + \beta_H)$ . Here  $\underline{X}_0$  is the S-solution of  $E_H$  in  $\{X_D\} + H_0^{1,2}(B, \mathbb{R}^3)$ .

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