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by

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ABSTRACT. — We prove that there exist at least two distinct solutions to the Dirichlet problem for the equation of prescribed mean curvature $\Delta X = 2H(X)X_u \wedge X_v$, the curvature function $H$ being in a full neighborhood of a suitable constant.

Key words : Equation of prescribed mean curvature, Dirichlet problem, relative minimizer.

1. INTRODUCTION

Let $B = \{ \omega = (u, v) \in \mathbb{R}^2| |\omega| < 1 \}$ be the unit disc in $\mathbb{R}^2$ with boundary $\partial B$. We consider the Dirichlet problem for the equation of prescribed mean curvature

$$\Delta X = 2H(X)X_u \wedge X_v, \quad \text{in } B,$$

$$X = X_D, \quad \text{on } \partial B. \tag{1.1} \tag{1.2}$$

Here, $X_u = \frac{\partial}{\partial u} X$ and $X_v = \frac{\partial}{\partial v} X$ denote partial derivative, $\wedge$ and $\cdot$ are the exterior and inner product in $\mathbb{R}^3$ and $H: \mathbb{R}^3 \to \mathbb{R}$ is a given function, and $X_D$ is a given function of class $C^2 (B, \mathbb{R}^3)$.

If $H = H_0 = \text{Const.}$, solutions to (1.1), (1.2) can be characterized as critical points of the functional

$$E_{H_0} (X) = D(X) + 2H_0 V(X),$$

in a space of admissible functions satisfying the boundary condition (1.2), where

$$D(X) = \frac{1}{2} \int_B |\nabla X|^2 \, d\omega$$

is the Dirichlet integral and

$$V(X) = \frac{1}{3} \int_B X \cdot X_u \wedge X_v \, d\omega$$

is the algebraic volume of surface $X$.

**Theorem 1.1** ([Hi2], [Wet2] and [Stf1]). Suppose $H = H_0 \in \mathbb{R}$ and let $X_D \in H^{1,2} (B, \mathbb{R}^3)$ be given. Assume that either

(i) $X_D$ is bounded and

$$|H_0| \cdot \|X_D\|_{L^\infty} < 1,$$

or

(ii) the condition

$$H_0^2 D(X_D) < \frac{2}{3} \pi$$

is satisfied. Then there is a solution $X \in \{X_D\} + H_0^{1,2} (B, \mathbb{R}^3)$ to (1.1), (1.2) which is a strict relative minimizer of $E_{H_0}$ in this space.

**Remark 1.2.** The observation that the solutions of Hildebrandt, Steffen and Wente are strict relative minima is due to Brezis-Coron [BC].

The existence of a second solution was proved independently by Brezis-Coron [BC] and Struwe [St2] with an important contribution by Steffen [Stf2] as follows

**Theorem 1.3** [Str3]. Let $X_D \in H^{1,2} \cap L^{\infty} (B, \mathbb{R}^3)$ be a non-constant vector, $H_0$ any real number different from zero. Suppose $E_{H_0}$ admits a local minimum $X$ in the class $\{X_D\} + H_0^{1,2} (B, \mathbb{R}^3)$. Then there exists a solution $X \in \{X_D\} + H_0^{1,2} (B, \mathbb{R}^3)$ of (1.1) and (1.2) different from $X$ and satisfying the condition

$$E_{H_0} (X) < E_{H_0} (X) = \inf_{p \in P} \sup_{X \in \text{im}(p)} E_H (X) < E_{H_0} (X) + \frac{4 \pi}{3 |H_0|^2},$$

$$E_{H_0} (X) < E_{H_0} (X) = \inf_{p \in P} \sup_{X \in \text{im}(p)} E_H (X) < E_{H_0} (X) + \frac{4 \pi}{3 |H_0|^2},$$

where

$$E_{H_0} (X) = D(X) + 2H_0 V(X).$$
where
\[ P = \{ p \in C^0([0, 1], \{X_D\} + H_0^{1,2}(B, \mathbb{R}^3)) \mid p(0) = X, \quad E_{H_0}(p(1)) < E_{H_0}(X) \}. \] (1.9)

For variable curvature functions \( H \) results comparable to Theorem 1.1 have been obtained by Hildebrandt [Hi1] and Steffen [Stf1].

**Theorem 1.4 [Hi1].** Suppose \( H \) is of class \( C^1 \) and let \( X_D \in H^{1,2} \cap L^\infty(B, \mathbb{R}^3) \) be given with \( \|X_D\|_{L^\infty} < 1 \). Then if
\[ h = \text{ess sup} \frac{H(X)}{|X| \leq 1} \]
there exists a solution \( X \in \{X_D\} + H_0^{1,2}(B, \mathbb{R}^3) \) to (1.1), (1.2) such that
\[ E_H(X) = \inf \{ E_H(X); X \in M \}, \]
where \( M \) is given by (2.9) below.

If variable curvature function \( H \) is sufficiently close to a suitable constant, Struwe obtained [Str4].

**Theorem 1.5.** Suppose \( X_D \in C^2(B, \mathbb{R}^2) \) is non-constant and suppose that for \( H_0 \in \mathbb{R} \setminus \{0\} \) the functional \( E_{H_0} \) admits a relative minimizer in \( \{X_D\} + H_0^{1,2}(B, \mathbb{R}^3) \). Then there exists a number \( \alpha > 0 \) such that for a dense set \( \mathcal{A} \) of curvature functions \( H \) in the \( \alpha \)-neighborhood of \( H_0 \), the Dirichlet problem (1.1), (1.2) admits at least two distinct regular solutions in \( \{X_D\} + H_0^{1,2}(B, \mathbb{R}^3) \).

Here the \( \alpha \)-neighborhood of \( H_0 \) is defined as
\[ \alpha \]
\[ [H - H_0] = \text{ess sup} \{ (1 + |X|) (|H(X) - H_0| + |\nabla H(X)|) \]
\[ + |Q(X) - H_0 X| + |\nabla Q(X) - H_0 \text{id}| \} \leq \alpha, \] \( (1.10) \)
where \( Q \) is given by (2.3) below.

In this paper, we improve Theorem 1.5 and obtain that

**Theorem 1.6.** Suppose \( X_D \in C^2(B, \mathbb{R}^3) \) is non-constant, and suppose that for \( H_0 \in \mathbb{R} \setminus \{0\} \) the functional \( E_{H_0} \) admits a relative minimizer in \( \{X_D\} + H_0^{1,2} \). Then there exists a number \( \alpha > 0 \) such that if \( [H - H_0] < \alpha \), \( E_H \) admits two solutions in \( \{X_D\} + H_0^{1,2} \).

From the proof of Theorem 1.5 [Str4], we have a relative minimizer of \( E_H \) for a full \( \alpha \)-neighborhood of \( H_0 \) and another “large” critical point of \( E_H \) for \( H \in \mathcal{A} \). We call the former \( S \)-solution and the latter \( L \)-solution.

First, we show that the \( S \)-solution is also a “strict” relative minimizer—its \( E_H \)-energy is less than that of the \( L \)-solution—provided that \( [H - H_0] \) is small enough. Next, we give a priori estimates for solutions of the Dirichlet problem—which are of crucial importance to our result—though they are not given explicitly. Then, we can use the solutions obtained by Struwe in [Str4]—the \( L \)-solutions—for a dense set \( \mathcal{A} \) in

\[ \mathcal{H}_x = \{ H | [H - H_0] < \alpha \} \] to approximate a solution of \( E_H \) for any \( H \in \mathcal{H}_x \) which is different from the S-solution.

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2. PRELIMINARIES

For variable curvature function \( H \), solutions to (1.1), (1.2) can be characterized as critical points of the functional

\[ E_H(X) = D(X) + 2 V_H(X) \]

in the space \( \{ X_D \} + H_0^{1,2}(B, \mathbb{R}^2) \). Here, the H-volume introduced by Hildebrandt is given by

\[ V_H(X) = \frac{1}{3} \int_B Q(X) \cdot X_u \land X_v \, d\nu, \]

where

\[ Q(x_1, x_2, x_3) = \left( \int_0^{x_1} H(s, x_2, x_3) \, ds, \int_0^{x_2} H(x_1, s, x_3) \, ds, \int_0^{x_3} H(x_1, x_2, s) \, ds \right). \]

We list some useful lemmas.

**Lemma 2.1** (Isoperimetric inequality, cf. [Wet1]):

\[ 36 \pi (V(X))^2 \leq D(X)^3, \]

for \( X \in H_0^{1,2}(B, \mathbb{R}^3) \).

**Lemma 2.2** ([BC], [Str4], Prop. 3.1). - Suppose \( X_D \in C^2(\bar{B}, \mathbb{R}^3) \) is non-constant, and suppose that for \( H \equiv H_0 \neq 0 \) the functional \( E_{H_0} \) admits a relative minimizer \( X_0 \in \{ X_D \} + C^2 \cap H_0^{1,2}(B, \mathbb{R}^3) \). Then there exists a radius \( R > 0 \), a function \( X_1 \in \{ X_D \} + C^2 \cap H_0^{1,2}(\bar{B}, \mathbb{R}^3) \) with \( D(X_1 - X_0) \geq R \), and a continuous path \( \rho \in C^0([0, 1]; \{ X_D \} + C^2 \cap H_0^{1,2}(\bar{B}, \mathbb{R}^3)) \) connecting \( X_0 = \rho(0) \) with \( X_1 = \rho(1) \) such that the estimates

\[ E_{H_0}(X_1) < \inf \{ E_{H_0}(X); X - X_0 \in H_0^{1,2}, D(X - X_0) \leq R \} \leq E_{H_0}(X_0) \]

\[ < \inf \{ E_{H_0}(X); X - X_0 \in H_0^{1,2}, D(X - X_0) = R \} \]

\[ \leq \sup \{ E_{H_0}(X); X \in \rho([0, 1]) \} \]

\[ < E_{H_0}(X_0) + 4 \pi/3 H_0^3 \]

hold.
DEFINITION 2.3:

\[ M = \{ X \in \{ X_D \} + H_0^{1,2}(B, \mathbb{R}^3); \ D(X - X_0) \leq R \}, \]  

(2.9)

where \( X_0 \) and \( R \) are as in Lemma 2.2.

LEMMA 2.4 [BC]. – For \( \Omega \in \mathbb{R} \setminus \{0\} \), Suppose that \( E_{H_0} \) admits a relative minimizer \( X_0 + C2 \in H^{1,2}(B, \mathbb{R}^3) \). Then there is \( \delta > 0 \) such that

\[ \int |\nabla \varphi|^2 + 4 \int X_0 \cdot \varphi_n + \varphi_r \geq \delta \int |\nabla \varphi|^2, \text{ for all } \varphi \in H_0^{1,2}. \]  

(2.10)

Let

\[ P = \{ p \in C^0 ([0, 1]); \{ X_D \} + H_0^{1,2}(B, \mathbb{R}^3), p(0) = X_0, p(1) = X_1 \} \]

and set

\[ \gamma_{H, \rho} = \inf_{P} \sup_{X \in \text{im}(p)} E_{H, \rho}(X) \]

where \( E_{H, \rho}(X) = (1 + \rho) E_{H/(1+\rho)}(X) \). Using Lemma 2.2 we have (see [Str4], (3.5))

\[ E_{H, \rho}(X) < \inf \{ E_H(X); X - X_0 \in H_0^{1,2}, D(X - X_0) \leq R \} \leq E_{H, \rho}(X_0) \]

(2.5)_\rho

\[ < \inf \{ E_H(X); X - X_0 \in H_0^{1,2}, D(X - X_0) = R \} \]

\[ \leq \inf \{ E_{H, \rho}(X); \ldots \} \]  

(2.6)_\rho

\[ \leq \sup \{ E_{H, \rho}(X); X \in p ([0, 1]) \} \]

(2.7)_\rho

\[ < E_H(X_0) + \beta \leq E_{H, \rho}(X_0) + \beta \]  

(2.8)_\rho

and

\[ E_{H, \rho}(X_0) < \gamma_{H, 0} \leq \gamma_{H, \rho} \leq \gamma_{H, \alpha} < E_H(X_0) + \beta, \]  

(2.11)

for \( \rho \in [0, \alpha] \). Here \( X_1, X_0 \) and \( P \) are as in Lemma 2.2, \( \alpha \) is small enough and fixed and \( \beta < 4 \pi/3 \Omega^2 \) is independent of \( H \). Moreover, we have

LEMMA 2.5. – **There exists a constant number \( \varepsilon_0 \) independent of \( \alpha \) such that**

\[ E_{H_0}(X_0) + \varepsilon_0 < \inf \{ E_H(X); X - X_0 \in H_0^{1,2}, D(X - X_0) = R \} \]

\[ \leq \inf \{ E_{H, \rho}(X); X - X_0 \in H_0^{1,2}, D(X - X_0) = R \} \]

\[ \leq \sup \{ E_{H, \rho}(X); X \in p ([0, 1]) \} \]

\[ < E_H(X_0) + \beta - \varepsilon_0 \leq E_{H, \rho}(X_0) + \beta - \varepsilon_0 \]

provided that \( \alpha \) is small enough, where \( X_0, R \) and \( p \) are as in Lemma 2.2.

**Proof.** – Set

\[ \varepsilon_0 = \frac{1}{4} \min \{ (E_{H_0}(X_0) + 4 \pi/3 \Omega^2 - \sup \{ E_H(X); X \in p ([0, 1]) \}), \]

\[ \inf \{ E_{H_0}(X); X - X_0 \in H_0^{1,2}, D(X - X_0) = R \} - E_{H_0}(X_0) \} \].

It is easy to see that Lemma 2.5 follows from Lemma 2.2 for \( \alpha \) small enough.

**Q.E.D.**

**Lemma 2.6.** There exists a constant \( c \) independent of \( \alpha \) such that if \( H \in \mathcal{A} \) (see Theorem 1.5),

\[
D(X - \overline{X}) > \epsilon,
\]

where \( X \) (resp. \( \overline{X} \)) is the S-solution (resp. L-solution) to (1.1), (1.2).

**Proof.** It follows the proof of Theorem 1.5 ([Str4], Theorem 3.1) and Lemma 2.5.

**Q.E.D.**

### 3. THE "STRICT" RELATIVE MINIMA

In this section, we will prove that the S-solution to the Dirichlet problem for the equation of prescribed mean curvature \( H \) is a "strict" relative minimum in the \( \mathcal{B} \), provided that \( \epsilon \) is small enough. Here \( \epsilon \) is a constant with the property that \( E_\epsilon \) admits a relative minimizer \( X_0 \in \{ X_D \} + C^2 \cap H^{1,2}_0 (B, \mathbb{R}^3) \).

**Lemma 3.1.** There exists a constant number \( \alpha > 0 \) with the property that if \( |H - H_0| < \alpha \) there is a constant \( \delta > 0 \) depending only on \( \alpha \) and \( X_D \) such that

\[
\int |\nabla \varphi|^2 + 4 \int Q(X) \varphi_u \wedge \varphi_v \geq \delta \int |\nabla \varphi|^2, \quad \text{for any } \varphi \in H^{1,2}_0. \tag{3.1}
\]

Here \( X = X_H \) is the S-solution to (1.1), (1.2).

**Proof.** Let \( X_0 \) be the small solution of \( E_{H_0} \) in the space \( \{ X_D \} + H^{1,2}_0 \). By Brezis-Coron [BC]—see Lemma 2.4—there exists a constant \( \delta_1 > 0 \) such that

\[
\int |\nabla \varphi|^2 + 4 H_0 \int X_0 \varphi_u \wedge \varphi_v \geq \delta_1 \int |\nabla \varphi|^2, \quad \varphi \in H^{1,2}_0. \tag{3.2}
\]
Thus for any $\varphi \in H^1_0$,
\[
\int |\nabla \varphi|^2 + 4 \int Q(X) \varphi_u \wedge \varphi_v = \int |\nabla \varphi|^2 + 4 H_0 \int X_0 \varphi_u \wedge \varphi_v + 4 \int (Q(X) - H_0 X) \varphi_u \wedge \varphi_v + 4 H_0 \int (X - X_0) \varphi_u \wedge \varphi_v
\] 
\[
\geq (\delta_1 - 2 \alpha) \int |\nabla \varphi|^2 + 4 H_0 \int (X - X_0) \varphi_u \wedge \varphi_v.
\]

Therefore, Lemma 3.1 follows from the following

**Lemma 3.2.** For any $\varepsilon > 0$, there exists a constant $\alpha > 0$ with the property that for any curvature function $H$ with $|H - H_0| < \alpha$, if $X_H$ is the $S$-solution to (1.1), (1.2), then

\[
\|X_H - X_0\|_{L^\infty} < \varepsilon.
\] (3.3)

**Proof.** If the Lemma is false, we may assume that there exist $\varepsilon_0 > 0$ and a sequence $\{X_i\}$ of the $S$-solutions of $E_H$ with $H = H_i$ and $|H_i - H_0| \to 0$ as $i \to 0$ such that $\|X_{H_i} - X_0\|_{L^\infty} \geq \varepsilon_0$. Noticing that $X_i \in M$, we know that $D(X_i)$ are bounded uniformly in $i$. Thus we may assume that $\{X_i\}$ converges to $X$ weakly in $\{X_D\} + H^1_0$ for some $X \in \{X_D\} + H^1_0$. It is easy to see that $X \in M$ (see [Str4]). Recall

\[
M = \{X \in \{X_D\} + H^1_0 \in \mathbb{B}, \mathbb{R}^3; D(X - X_0) \leq R\}.
\]

But then

\[
E_{H_0}(X) \geq \inf_{X \in M} E_{H_0}(X) = \inf_{X \in M} \lim_{i \to \infty} E_{H_i}(X) \geq \lim_{i \to \infty} \inf_{X \in M} E_{H_i}(X) = \lim_{i \to \infty} E_{H_i}(X_i).
\] (3.4)

Hence, by Theorem 4.5 below, $X_i \to X$ strongly in $\{X_D\} + H^1_0$ and uniformly in $\mathbb{B}$ and $E_{H_0}(X) = \lim_{i \to \infty} E_{H_i}(X_i)$ by (3.4). We have

\[
E_{H_0}(X) = \inf_{X \in M} E_{H_0}(X) = E_{H_0}(X_0).
\]

Hence the uniqueness of the small solution of $E_{H_0}$ in $\{X_D\} + H^1_0$ [BC] shows that $X = X_0$. Therefore, $X_i \to X_0$ uniformly in $\mathbb{B}$ which contradicts...
If \( H \) is sufficiently close to \( H_0 \), we have

**PROPOSITION 3.3.** If \( H \in \mathcal{A} \) and \( X \) is the L-solution to (1.1), (1.2), then \( E_H(X) > E_H(X) \), where \( X \) is the S-solution to (1.1), (1.2).

**Proof.** Let \( \varphi = X - X \in \mathcal{H}_{A,2}^1(B, \mathbb{R}^3) \). Noting that \( X = X + \varphi \) and \( X \) satisfy the equation (1.1), we have

\[
E_H(X) = E_H(X + \varphi)
\]

by (1.10). Testing (1.1) with \( \varphi \) we get

\[
0 = \int \nabla \varphi \cdot \nabla (X + \varphi) + 2 \int H(X + \varphi) \varphi (X + \varphi)_u \wedge (X + \varphi)_v
\]

\[
= \int |\nabla \varphi|^2 + 4 \int \mathcal{Q}(X) \varphi_u \wedge \varphi_v + 2 \int \mathcal{Q}(\varphi) \varphi_u \wedge \varphi_v
\]

\[
+ O(\alpha) \left( \left( \int |\nabla \varphi|^2 \right)^{1/2} + \int |\nabla \varphi|^2 \right) \quad (3.5)
\]

by (1.10). From (3.5), (3.6) it is clear

\[
E_H(X) = E_H(X) + \frac{1}{2} \int |\nabla \varphi|^2 + 2 \int \mathcal{Q}(X) \varphi_u \wedge \varphi_v
\]

\[
+ \frac{2}{3} \int \mathcal{Q}(\varphi) \varphi_u \wedge \varphi_v + c \alpha \left( \left( \int |\nabla \varphi|^2 \right)^{1/2} + \int |\nabla \varphi|^2 \right)
\]

\[
= E_H(X) + \frac{1}{6} \left( \int |\nabla \varphi|^2 + 4 \int \mathcal{Q}(X) \varphi_u \wedge \varphi_v \right)
\]

\[
+ O(\alpha) \left( \left( \int |\nabla \varphi|^2 \right)^{1/2} + \int |\nabla \varphi|^2 \right).
\]

By Lemma 3.1, we get

\[
E_H(X) - E_H(X) \geq \delta \int |\nabla \varphi|^2 - c \alpha \left( \left( \int |\nabla \varphi|^2 \right)^{1/2} + \int |\nabla \varphi|^2 \right). \quad (3.7)
\]
Therefore, from Lemma 2.6 we have
\[ E_H(X) > E_H(X) \]
provided that \( \alpha \) is small enough.

Q.E.D.

**Proposition 3.4.** If \( \alpha > 0 \) is small enough, for \( H \in \mathcal{H}_2 \) there exist a \( \rho_0 > 0 \) and a dense set \( A \) in \([0, \rho_0]\) such that if \( \rho \in A \), then \( E_{H/(1+\rho)} \) admits two distinct regular solutions in \( \{ X_D \} + H_0^{1,2} \), one is the S-solution \( X_H \) and the other is the L-solution \( X \) with
\[ E_H(X_0) < \gamma_{H,0} \leq (1 + \rho) E_{H/(1+\rho)}(X) \leq \gamma_{H,\alpha} < E_H(X_0) + \beta, \]
where \( \gamma_{H,0} \), \( \gamma_{H,\alpha} \) and \( \beta \) are given in section 2 and \( X_0 \) is the small solution of \( E_{H_0} \).

**Proof.** Proposition 3.4 follows from the proof of Theorem 1.5 (see [Str4]) and Proposition 3.3.

## 4. CONVERGENCE OF SURFACES OF PRESCRIBED MEAN CURVATURE

As in [Pc], we can also establish a convergence theorem of surfaces of prescribed mean curvature with the Dirichlet boundary condition. Let \( H: \mathbb{R}^3 \rightarrow \mathbb{R} \) satisfy
\[ H \in C^1(\mathbb{R}^3, \mathbb{R}) \]
\[ \| H \|_{L^{\infty}(\mathbb{R}^3)} + \| (1 + |X|) \|_{L^\infty(\mathbb{R}^3)} < + \infty. \]  

**Theorem 4.1.** Let \( H_i \) satisfy (4.1) and \( \| H_i \|_{L^{\infty}} \leq K \) uniformly, and \( H_i \rightarrow H \) a.e. on \( \mathbb{R}^3 \). Suppose \( X_i \in \{ X_D \} + H_0^{1,2}(B, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3) \) is a sequence of solutions to (1.1), (1.2) with \( H = H_i \) and \( \int_B |\nabla X_i|^2 \, d\omega \leq c \) uniformly. Assume that \( X_i \rightarrow X \) weakly in \( H^{1,2}(B, \mathbb{R}^3) \) for some function \( X \in \{ X_D \} + H_0^{1,2}(B, \mathbb{R}^3) \). Then \( X_i \rightarrow X \) strongly in \( \{ X_D \} + H_0^{1,2}(B \setminus S, \mathbb{R}^3) \) where \( S \) is a finite subset of \( B \). Moreover, \( X \) satisfies
\[ \Delta X = 2 H(X) X_u \wedge X_v, \quad \text{in } B, \]
\[ X = X_D, \quad \text{on } \partial B. \]  

**Proof.** The proof is similar to that of Proposition 2.6 in [Str4], thus we only sketch it. Set
\[ S = \bigcap_{r > 0} \left\{ w \in \bar{B}/\lim_{i \rightarrow \infty} \inf_{B(w, r) \cap B} |\nabla X_i|^2 \geq \mu_0 \right\} \]  

where $\mu_0$ is a constant like $\mu_0$ in [Str4]. By the same argument of [Str4] or [Pc], we have (by taking subsequence)

$$X_i \to X \text{ strongly in } C^1(\mathbb{B} \setminus S, \mathbb{R}^3),$$

and $S$ is a finite subset of $B$. Moreover, $X$ satisfies (4.2).

Q.E.D.

**Lemma 4.2.** If $X \in C^2(\mathbb{R}^2_+, \mathbb{R}^3)$ satisfies

$$\Delta X = 2H(X)X_u \wedge X_v, \quad \text{in } \mathbb{R}^2_+$$

$$X = \text{const.}, \quad \text{on } \partial \mathbb{R}^2_+$$

then $X \equiv \text{const.}$

The Lemma easily follows from [Wet2]. For the convenience of the reader we give a complete proof.

**Proof of Lemma 4.2.** Note that $X_u, (X_u \wedge X_v) \equiv 0$ in $\mathbb{R}^2_+$, from (4.4) we have

$$X_u \cdot \Delta X = 0, \quad \text{in } \mathbb{R}^2_+$$

It's easy to see that

$$0 = X_u \cdot \Delta X = X_u \cdot \text{div} \nabla X$$

$$= \text{div} \left\{ (1, 0) \nabla X \right\} = \frac{1}{2} (1, 0) |\nabla X|^2$$

By Stokes' formula, we have

$$\int_{\partial \mathbb{R}^2_+} n \cdot ((1, 0) \nabla X) \nabla X - \frac{1}{2} n \cdot (1, 0) |\nabla X|^2 \, d\omega = 0 \quad (4.5)$$

where $n = (-1, 0)$ is the outer normal to $\mathbb{R}^2_+$ at $\partial \mathbb{R}^2_+$. Since $X \equiv \text{const.}$ on $\partial \mathbb{R}^2_+$, $\nabla X = (\nabla X \cdot n)n$ on $\partial \mathbb{R}^2_+$. Hence, from (4.5) we get

$$\int_{\partial \mathbb{R}^2_+} |\nabla X|^2 \, d\omega = 0.$$

Therefore $X_u \equiv 0$ on $\partial \mathbb{R}^2_+$. By the argument of Wente [Wet3], $X \equiv \text{const.}$ in $\mathbb{R}^2_+$.

Q.E.D.

**Lemma 4.3.** Let $[H - H_0] < \infty$. If $X \in H^1_0(\mathbb{R}^2_+, \mathbb{R}^3)$ satisfies

$$\Delta X = 2H(X)X_u \wedge X_v \text{ in } \mathbb{R}^2_+, \text{ and is non-constant}$$

then

$$E_H(X) \geq \frac{4 \pi}{3 H_0} - c_0 [H - H_0], \quad (4.6)$$

where $c_0$ is independent of $H$ and $[H - H_0]$.

**Proof.** It is easy to prove this lemma, we omit it.
**Proposition 4.4.** — Let \([H - H_0] < \infty\)

\[
\beta_H \geq \frac{4\pi}{3H_0^2} - c_0[H - H_0],
\]

where

\[
\beta_H = \inf \{ \lim \inf_{i \to \infty} (E_{H_i}(X_i) - E_H(X)) ; \text{X}_i \text{ are critical points of } E_{H_i} \text{ and } X_i \to X \text{ in } H^{1,2} \text{ weakly but not strongly} \}
\]

**Proof.** — For any such sequence \(\{X_i\}\), using Theorem 4.1, we see that \(X_i \to X\) strongly in \(\{X_D\} + H^{1,2}_{0,loc}(B \setminus S, \mathbb{R}^3)\) where \(S\) is a finite non-empty subset of \(B\) and is defined by (4.3). There are two possibilities: either, (i) \(S \cap \partial B = \emptyset\); or, (ii) \(S \cap \partial B \neq \emptyset\). In case (i) from Section 3 of [Pc], we have a function \(X_0 \in H^{1,2}_{0,loc}(\mathbb{R}^2, \mathbb{R}^3)\) satisfying \(\Delta X_0 = 2H(X_0)X_{Ou} \wedge X_{Ov} \text{ in } \mathbb{R}^2\) and

\[
\lim_{i \to \infty} \inf_{\text{X}} (E_{H_i}(X_i) - E_H(X)) \geq E_H(X_0) \geq \frac{4\pi}{3H_0^2} - c_0[H - H_0]
\]

by Lemma 4.3. Therefore, \(\beta_H \geq \frac{4\pi}{3H_0^2} - c_0[H - H_0]\). In case (ii), using the same argument in [BC2] and Lemma 4.2, we also have a “blow up” function \(X\) satisfying \(\Delta X = 2H(X)X_u \wedge X_v\) in \(\mathbb{R}^2\) and

\[
\lim_{i \to \infty} \inf_{\text{X}} (E_{H_i}(X_i) - E_H(X)) \geq E_H(X_0) \geq \frac{4\pi}{3H_0^2} - c_0[H - H_0]
\]

Q.E.D.

**Theorem 4.5.** — Let \(\alpha\) be fixed as in section 3 and \([H_1 - H] < \alpha\) and \(H_i \to H\) a.e. in \(B\). Suppose \(X_i \in \{X_D\} + H^{1,2}_{0,loc}(B, \mathbb{R}^3)\) is a sequence of solutions to (1.1), (1.2) with \(H = H_i\) and

\[
|E_{H_i}(X_i)| \leq c < \infty
\]

uniformly in \(i\). Then

\[
D(X_i) \leq c_1
\]

uniformly for another constant number \(c_1\). Moreover, assume \(X_i \to X\) weakly in \(H^{1,2}(B, \mathbb{R}^3)\) for some \(X \in \{X_D\} + H^{1,2}_{0,loc}\), then \(X\) is a critical point of \(E_H\) in \(\{X_D\} + H^{1,2}_{0,loc}\) and either

(i) \(X_i \to X\) strongly in \(H^{1,2} \cup L^\infty(B, \mathbb{R}^3)\) with

\[
E_H(X) = \lim_{i \to \infty} \inf_{\text{X}} E_{H_i}(X_i),
\]

or

(ii) \(E_H(X) \leq \lim_{i \to \infty} \inf_{\text{X}} E_{H_i}(X_i) - \frac{4\pi}{3H_0^2} + c_0[H - H_0]\).
Proof. Let $X_i$ be the $S$-solution of (1.1)-(1.2) with $H = H_i$ in $\{X_D\} + H_0^{1,2}(\mathbb{B}, \mathbb{R}^3)$ (see §3) and $E_{H_i}(X_i) = \int_0^1 \int_{\mathbb{B}} |\nabla X_i|^2 = \int_0^1 \int_{\mathbb{B}} |\nabla X_i|^2 - E(X_i) - E(X)\),

where $\delta$ depends only on $\alpha$ and $X_D$. Note that $E_{H_i}(X_i)$, $E_{H}(X_i)$ and $D(X_i)$ are bounded uniformly. Hence, uniformly for some constant $c_1$.

Assume that $X_i \to X$ weakly in $H^{1,2}(\mathbb{B}, \mathbb{R}^3)$. Now there are two possibilities either, (i) $S = \emptyset$, or, (ii) $S \neq \emptyset$ by Theorem 4.1.

In case (i) $X_i \to X$ strongly in $H^{1,2} \cap L^\infty$ (see [Str4] or [Pc]). In case (ii)

\[ \lim \inf \ E_{H_i}(X_i) - E_H(X) \geq \beta_H \geq \frac{4\pi}{3H_0} - c_0[H - H_0], \]

by Proposition 4.5. This completes the proof.

Q.E.D.

Remark 4.6. For the Dirichlet problem Theorem 4.5 gives a priori bounds which are of crucial importance to our results.

5. PROOF OF THEOREM 1.6

For any curvature function $H$ with $[H - H_0] < \alpha$, there exists the $S$-solution $X_H$ to (1.1), (1.2). On the other hand, by the results of Struwe [Str4] and proposition 3.3 there exists a sequence of $H_i = H/(1 + \rho_i)$ tending to $H$ such that $E_{H_i}$ admits the L-solution $X_i \in \{X_D\} + H_0^{1,2} \cap C^2(\mathbb{B}, \mathbb{R}^3)$ with

\[ E_H(X_0) < \gamma_{H,0} \leq (1 + \rho_i)E_{H_i}(X_i) \leq \gamma_{H,\cdot} < E_H(X_D) + \beta \]

(see [Str4] or Prop. 3.4), where $\rho_i > 0$ tends to 0 and $\gamma_{H,0}$, $\gamma_{H,\cdot}$, $\beta$ and $X_0$ are as in section 3.

Now from Theorem 4.5, $X_i \to X$ weakly in $H^{1,2}(\mathbb{B}, \mathbb{R}^3)$ (by taking subsequence) and $X$ is a critical point of $E_H$ in $\{X_D\} + H_0^{1,2}$ with the property that either,

(i) $X_i \to X$ strongly in $H^{1,2}$, or,

(ii) $X_i \to X$ weakly but not strongly in $H^{1,2}$.

In case (i) $E_H(X) = \lim \inf \ (1 + \rho_i) E_{H_i}(X_i) \geq \gamma_{H,0}$. In case (ii),

\[ E_H(X) \leq \lim \inf \ (1 + \rho_i) E_{H_i}(X_i) - \beta_H \]

\[ \leq \gamma_{H,\cdot} - \beta_H. \]

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Therefore, from (2.11), Lemma 3.2 and Proposition 4.4 it is easy to see that in any case
\[ E_H(X) \neq E_H(X_H). \]
This completes the proof of our theorem.

Q.E.D.

Remark 5.1. From (3.7) and Lemma 3.2 case (ii) in the proof of Theorem 1.6 cannot in fact happen for small \( \alpha \).

Remark 5.2. We expect that for small \( \alpha \) if \( [H-H_0] < \alpha \), \( E_H \) satisfies the Palais-Smale condition in \( (-\infty,E_H(X_0)+\beta_H) \). Here \( X_0 \) is the \( S \)-solution of \( E_H \) in \( \{ X_D \} + H^{1/2}(B,\mathbb{R}^3) \).

REFERENCES


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