ANNALES DE L'I. H. P., SECTION C

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Annales de l'I. H. P., section C, tome 9, nº 5 (1992), p. 573-595
<http://www.numdam.org/item?id=AIHPC_1992__9_5_573_0>

Ann. Inst. Henri Poincaré,

Vol. 9, n° 5, 1992, p. 573-595. *Analyse non linéaire*

Bounded approximants to monotone operators on Banach spaces

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ABSTRACT. — It is shown that for any maximal monotone set-valued operator T on a real Banach space E, there is a sequence $\{T_n\}$ of bounded maximal monotone operators which have nonempty values at each point and which converge to T in a reasonable sense. Better convergence properties are shown to hold when T is in a new proper subclass of maximal monotone operators (the "locally" maximal monotone operators), a subclass which coincides with the entire class in reflexive spaces. The approximation method is patterned on the one which results when the (maximal monotone) subdifferential ∂f of a proper lower semicontinuous convex function f is approximated by a sequence of bounded subdifferentials $\{\partial f_n\}$, where each f_n is the (continuous and convex) inf-convolution of f with the function $n\|\,.\,\|$. The main advantage of this approximation scheme over the classical Moreau-Yosida approximation method is that it exists in non-reflexive Banach spaces.

Key words : Maximal monotone operators, Banach spaces, convex functions, subdifferentials, Mosco convergence.

Classification A.M.S. : 47 H 04, 47 H 05, 58 C 20, 46 G 05.

RÉSUMÉ. — On montre que, pour chaque opérateur maximal monotone T sur un espace de Banach E, il existe une suite $\{T_n\}$ d'opérateurs maximaux monotones bornés et non vides en chaque point, tel que $T_n \to T$ dans un sens raisonnable. De meilleures propriétés de convergence sont obtenues quand T est « localement maximal monotone », une sous-classe nouvelle de celle des opérateurs maximaux monotones et qui coïncide avec celle-ci dans les espaces réflexifs. La méthode d'approximation procède selon l'exemple de l'approximation d'une sous-différentielle ∂f (où f est propre, convexe, et semi-continue inférieur) au moyen d'une suite $\{\partial f_n\}$ de sous-différentielles, où chaque f_n est la inf-convolution (continue et convexe) de f avec la fonction $n\|\cdot\|$. L'avantage principal de cette méthode sur la méthode classique de Moreau-Yosida est de rester applicable dans les espaces de Banach non réflexifs.

1. INTRODUCTION

The primary goal of this paper is to present a method for approximating an arbitrary maximal monotone operator T on a real Banach space E by a sequence of "nicer" maximal monotone operators. A distant secondary goal (which arose while working on the primary one) is to highlight the fact (*see* Section 5) that some basic open questions about monotone operators on general Banach spaces are still open.

There already exists a well known and useful approximation method, called the Yosida (or Moreau-Yosida) approximation, but its definition requires that the space E be reflexive (and that it be renormed to be strictly convex with strictly convex dual); we will briefly compare the two methods at the end of Section 5. Our method was motivated by the special case when $T=\partial f$, the (maximal monotone) subdifferential of a lower semicontinuous convex proper function f on E. We start by examining this special case in some detail, since it not only motivates our method, but provides a guide as to what may be possible in the general case.

As will be shown below, the subdifferential ∂f can be approximated in a reasonable sense by the subdifferentials ∂f_n of the inf-convolutions (or "epi-sums")

$$f_n(x)=\inf\{f(y)+n\|x-y\|: y\in E\}, \qquad x\in E.$$

(The approximating sequence $\{f_n\}$ was originally introduced by Hausdorff [Hau] for any lower-bounded lower semicontinuous function f of a real variable.) Specifically, recall that the extended real-valued lower semicontinuous convex function f on the real Banach space E is said to be *proper* provided $f(x) > -\infty$ for all $x \in E$ and its (convex) essential domain

$$\operatorname{dom} f = \{ x \in E : f(x) < +\infty \}$$

is nonempty. Given such a function, one can define, for all sufficiently large n, the sequence of convex, everywhere finite functions f_n as above. [It is obvious that $f_n(x) < \infty$ for all n. Moreover, since the subdifferential $\partial f(x)$ is nonempty for at least one $x \in E$ – say it contains the element x^* – it is easily seen that $f_n(x) > -\infty$ for all x provided $n > \| x^* \|$. We assume below that this is always the case, that is, that n is sufficiently large that $f_n(x) > -\infty$ for all x.]

Some well-known elementary properties of the inf-convolution of f with $n \| . \|$ (see, for instance, [La] and [H-U$_1$]) are listed below.

NOTATION. – We denote by B^* and S^* the closed unit ball and the unit sphere of E^*, respectively. The *essential* dom*ain* of ∂f is denoted by $D(\partial f) = \{ x \in E : \partial f(x) \neq 0 \}$, while its *range* is $R(\partial f) = \{ x^* \in E^* : x^* \in \partial f(x)$ for some $x \in E \}$.

1.1. PROPOSITION. – *With f as above, the sequence $\{ f_n \}$ has the following properties:*

(i) *Each f_n is convex and Lipschitzian, with Lipschitz constant n.*

(ii) $f_n(x) \leqq f_{n+1}(x) \leqq f(x)$ *for each $x \in E$ and each n.*

(iii) $f_n(x) \to f(x)$ *for each $x \in E$.*

(iv) $D(\partial f_n) = E$ *for all n.*

(v) $\partial f_n(x) = \partial f(x) \cap n B^*$ *if and only if $f_n(x) = f(x)$; equivalently, if and only if $\partial f(x)$ intersects $n B^*$.*

As noted above, our aim is to develop an approximation scheme for arbitrary maximal monotone operators which will generalize the relationship between ∂f and the sequence of subdifferentials $\{ \partial f_n \}$; this latter relationship is described in more detail in the next proposition.

1.2. PROPOSITION. – *The subdifferentials ∂f_n converge to ∂f in the following sense: for any $x \in E$,*

$$\partial f(x) \cap n B^* \subset \partial f_n(x) \subset \overline{R(\partial f)} \cap n B^* \subset n B^*. \tag{1}$$

$$\textit{If } x \in D(\partial f), \quad \textit{then } \partial f_n(x) = \partial f(x) \cap n B^* \tag{2}$$

for all sufficiently large n and

$$\partial f_n(x) \setminus \partial f(x) \subset n S^* \quad \textit{for all } x \in E. \tag{3}$$

Proof. – Everything in assertion (1) except the second inclusion follows from Proposition 1.1. Since f_n has Lipschitz constant n, it follows that $\partial f_n(x) \subset n B^*$. To show that any $x^* \in \partial f_n(x)$ is in the norm closure of $R(\partial f)$, note first that by definition of $f_n(x)$, for each $k = 1, 2, 3, \ldots$ there exists $z_k \in \mathrm{E}$ such that

$$f(z_k)+n\|x-z_k\|<f_n(x)+1/k.$$

For all $y\in\mathrm{E}$ we have $f_n(y)\leqq f(y)$, so if $x^*\in\partial f_n(x)$, then

$$\begin{aligned}\langle x^*, y-z_k\rangle+\langle x^*, z_k-x\rangle&=\langle x^*, y-x\rangle\\ &\leqq f_n(y)-f_n(x)<f(y)-f(z_k)-n\|x-z_k\|+1/k.\end{aligned}$$

Thus, for each k, there exists $\varepsilon_k>0$ such that for all $y\in\mathrm{E}$,

$$\langle x^*, y-z_k\rangle<f(y)-f(z_k)+\varepsilon_k, \tag{$*$}$$

where $\varepsilon_k=\langle x^*, x-z_k\rangle-n\|x-z_k\|+1/k$. [Take $y=z_k$ in $(*)$ to see that $\varepsilon_k>0$ and use the fact that $\|x^*\|\leqq n$ to see that $\varepsilon_k\leqq 1/k$.] Now, $(*)$ implies that $x^*\in\partial_{\varepsilon_k} f(z_k)$ (the ε_k-subdifferential of f at z_k, *see*, for instance, [Ph], p. 48), so by the Brøndsted-Rockafellar theorem, we can conclude that for each k there exists $y_k\in\mathrm{E}$ and $y_k^*\in\partial f(y_k)$ such that $\|y_k-z_k\|\leqq\sqrt{\varepsilon_k}$ and $\|x^*-y_k^*\|\leqq\sqrt{\varepsilon_k}$. In particular, there exists $\{y_k^*\}\subset\mathrm{R}(\partial f)$ such that $\|x^*-y_k^*\|\to 0$.

Assertion (2) follows from Proposition 1.1 (v).

To prove (3), we may obviously assume that $\partial f_n(x)\diagdown\partial f(x)\neq 0$, which implies that $\partial f(x)\cap n\mathrm{B}^*=\varnothing$ [otherwise, by Proposition 1.1 (v), we would have $\partial f_n(x)=\partial f(x)\cap n\mathrm{B}^*\subset\partial f(x)$]; by Proposition 1.1 (ii) and (v), we must have $f_n(x)<f(x)$. Fix α such that $f_n(x)<\alpha<f(x)$; then there exists $d>0$ such that

$$f(u)+n\|x-u\|<\alpha \quad\text{implies}\quad \|x-u\|>d.$$

Indeed, if this were not true we could choose a sequence $u_k\to x$ such that

$$f(u_k)+n\|x-u_k\|<\alpha$$

and hence, by lower semicontinuity of f, we would have

$$f(x)\leqq\liminf\{f(u_k)+n\|x-u_k\|\}\leqq\alpha<f(x).$$

Suppose, now, that $x^*\in\partial f_n(x)$ and that $\varepsilon>0$ is sufficiently small that $f_n(x)+d\varepsilon<\alpha$. By definition of $f_n(x)$, there exists $u\in\mathrm{E}$ such that

$$f(u)+n\|x-u\|<f_n(x)+d\varepsilon,$$

hence $\|x-u\|\geqq d$. Since $x^*\in\partial f_n(x)$ we know that

$$\langle x^*, u-x\rangle\leqq f_n(u)-f_n(x)\leqq f(u)-f_n(x)$$

and therefore

$$\langle x^*, x-u\rangle\geqq f_n(x)-f(u)>n\|x-u\|-d\varepsilon\geqq(n-\varepsilon)\|x-u\|,$$

so that $\|x^*\|\geqq n-\varepsilon$ for all sufficiently small $\varepsilon>0$; we conclude that $\|x^*\|=n$.

In the following corollary to Proposition 1.1, the relationship between the subdifferentials ∂f_n and ∂f is used to obtain new information about the latter. Recall that a subdifferential ∂f is said to be *locally bounded* at the point $x \in D(\partial f)$ provided there exists a neighborhood U of x such that $\partial f(U)$ is a bounded set.

1.3. COROLLARY. – *Suppose that f is a proper lower semicontinuous convex function on the Banach space* E *and that $x_0 \in \operatorname{bdry} D(\partial f)$; then ∂f is not locally bounded at x_0.*

Proof. – Suppose, to the contrary, that for some $n \geqq 1$ there exists an open convex neighborhood U of x_0 such that $\partial f(x) \subset n B^*$ for each $x \in D(\partial f) \cap U$. By Proposition 1.1 (v), this implies that $f(x) = f_n(x)$ for each such x. Since $0 \leqq f - f_n$ is lower semicontinuous, and since (by the Brøndsted-Rockafellar theorem), $D(\partial f)$ is dense in $\operatorname{dom}(f)$, we conclude that $f = f_n$ in $U \cap \operatorname{dom}(f)$. From Proposition 1.1 (v) again, it follows that $\partial f(x) \cap n B^* = \partial f_n(x) \neq 0$ for all $x \in U \cap \operatorname{dom}(f)$, so that $U \cap \operatorname{dom}(f) = U \cap D(\partial f)$. Since U is a neighborhood of a boundary point of the convex set $\operatorname{dom}(f)$, the Bishop-Phelps theorem implies that there exists $y \in \overline{\operatorname{dom}(f)} \cap U$ and $y^* \neq 0$ in E^* such that $\langle y^*, y \rangle = \sup\{\langle y^*, x \rangle : x \in \operatorname{dom}(f)\}$. Since y is in the closure of $\operatorname{dom}(f)$ and in U, there exists a sequence $\{x_k\} \subset \operatorname{dom}(f) \cap U$ such that $x_k \to y$. By the lower semicontinuity of f, we have

$$f(y) \leqq \liminf_{k \to \infty} f(x_k) = \liminf_{k \to \infty} f_n(x_k) = f_n(y) < \infty,$$

hence

$$y \in U \cap \operatorname{dom}(f) = U \cap D(\partial f).$$

But (as is easily verified) $\partial f(y) + \lambda y^* \subset \partial f(y) \subset n B^*$ for all $\lambda \geqq 0$, a contradiction which completes the proof.

2. APPROXIMATIONS TO MONOTONE OPERATORS

In this section we will present our scheme for approximating an arbitrary maximal monotone operator on a Banach space. First, we recall some basic definitions.

DEFINITIONS. – A subset M of $E \times E^*$ is said to be *monotone* provided $\langle x^* - y^*, x - y \rangle \geqq 0$ whenever (x, x^*), $(y, y^*) \in M$. A set-valued mapping $T : E \to 2^{E^*}$ is *monotone* provided its graph

$$G(T) = \{(x, x^*) \in E \times E^* : x^* \in T(x)\}$$

is a monotone set. The *effective domain* $D(T)$ of T is the set of all points $x \in E$ for which $T(x)$ is nonempty and T is said to be *locally bounded* if each point of E has a neighborhood U such that $T(U) \equiv \bigcup\{T(x) : x \in U\}$

is contained in some ball in E^*. A monotone operator T is said to be *maximal monotone* if its graph is maximal (under inclusion) in the family of all monotone subsets of $E \times E^*$. The *inverse* T^{-1} of T is defined by

$$T^{-1}(x^*) = \{x \in E : x^* \in T(x)\}, \qquad x^* \in E^*.$$

Since $G(T^{-1}) \subset E^* \times E$ and $G(T) \subset E \times E^*$ are (within a permutation) the same set, T^{-1} is maximal if and only if T is maximal.

As is well known, the subdifferential of a proper lower semicontinuous convex function on a Banach space is a special case of a maximal monotone operator. [This fundamental fact, first proved by Rockafellar [Ro_5], has recently been given an remarkably short proof by S. Simons [Si].] Since our scheme is modeled on this special case, we first see what Proposition 1.2 would look like if it were about general monotone operators instead of subdifferentials.

2.1. GOALS. — *Let* T *be a maximal monotone operator on the Banach space* E. *Determine when there exists a sequence* $\{T_n\}$ *of maximal monotone operators on* E *such that*

(i) $D(T_n) = E$ *for all* n.
(ii) $T(x) \cap n B^* \subset T_n(x)$ *for each* $x \in E$ *and all* n.
(iii) $R(T_n) \subset \overline{R(T)} \cap n B^*$ *for all* n.
(iv) *If* $x \in D(T)$, *then* $T_n(x) = T(x) \cap n B^*$ *for all sufficiently large* n.
(v) $T_n(x) \setminus T(x) \subset n S^*$ *for all* $x \in E$.

Our construction will meet some (but not all) of these goals. We first need a lemma.

2.2. LEMMA. — *Suppose that* $T : E \to 2^{E^*}$ *is monotone, locally bounded and has closed graph* (*in the norm* $\times$ *weak* topology in* $E \times E^*$). *If the set* $T(x)$ *is nonempty and convex for each* $x \in E$, *then* T *is maximal monotone.*

Proof. — Let $\tilde{T}$ be a maximal monotone operator on E which contains T; we will show that $\tilde{T} = T$. To that end, take the closure in the product space $E \times (E^*, weak^*)$ of the graph of T; this will be the graph of a set-valued mapping $\bar{T}$ which, using the local boundedness, is readily seen to be monotone. Since T has closed graph, $\bar{T} = T$. By a known theorem (*see*, for instance, [Ph], Th. 7.13), for each $x \in E$, the set $\tilde{T}(x)$ is the weak* closed convex hull of $\bar{T}(x) = T(x)$. Since $T(x)$ is weak* closed and convex for each $x \in E$, we have $T(x) = \tilde{T}(x)$.

DEFINITION. — A sequence of sets $\{X_n\}$ in the Banach space E is said to be *Mosco-convergent* to the subset $X \subset E$ provided the following conditions hold:

(*a*) For all $x \in X$ there exists $x_n \in X_n$ such that $\|x_n - x\| \to 0$ and
(*b*) If $x \in E$ is such that $x_k \to x$ in the weak topology, where $x_k \in X_{n_k}$ for each k, then $x \in X$.

If the space E is a dual space and if in condition (*b*) the weak topology is replaced by the weak* topology, we say that $\{X_n\}$ is *weak* Mosco-convergent* to X.

See [At] for an extensive discussion of Mosco convergence.

NOTATION. — If A is a subset of E*, the weak* closed convex hull of A is denoted by $\overline{co}$ A.

2.3. THEOREM. — *Suppose that* T *is a maximal monotone operator on* E *and for* $n=1, 2, 3, \ldots$ *let* $C_n=\overline{co}[R(T)\cap n B^*]$. *There exists a sequence* $\{T_n\}$ *of maximal monotone operators on* E *such that*

(i) $D(T_n)=E$ *for all n.*

(ii) $T(x)\cap n B^*\subset T_n(x)$ *for each* $x\in E$ *and all n.*

(iii) $R(T_n)\subset C_n\subset n B^*$ *for all n.*

(iv) *For any* $x_0\in \operatorname{int} D(T)$, *there exists an open neighborhood* U *of* x_0 *in* D(T) *and* $n_0\geqq 1$ *such that* $T_n=T$ *in* U *for all* $n\geqq n_0$.

(v) *For all* $x\in E$, *the sequence of sets* $\{T_n(x)\}$ *is weak* Mosco-convergent to* $T(x)$.

Proof. — For each $n\geqq 1$ define S_n by

$$S_n(x)=T(x)\cap n B^*, \qquad x\in E,$$

where B* is the weak* compact unit ball in E*. These set-valued maps are obviously monotone. Next, consider the family, ordered by inclusion, of all monotone subsets of $E\times C_n$. By applying Zorn's lemma, we can obtain a maximal monotone subset of $E\times C_n$ containing the graph $G(S_n)$ of S_n; this defines a monotone operator T_n. By a special case of an extension theorem due to Debrunner and Flor [D-F] (*see* below) for each $x\in E$, there exists $x^*\in C_n$ such that $G(T_n)\cup\{(x, x^*)\}$ is a monotone subset of $E\times C_n$. The maximality of T_n implies that $G(T_n)$ contains the point (x, x^*), so we conclude that $D(T_n)$ is all of E. It follows easily from the maximality of T_n that it has closed graph (in $E\times(C_n$, *weak**)) and that for all $x\in E$, $T_n(x)$ is nonempty, weak* closed and convex. From Lemma 2.2 we conclude that T_n is maximal monotone in $E\times E^*$, not merely in $E\times C_n$. Properties (i), (ii) and (iii) are obvious consequences of the foregoing construction.

To prove (iv), consider a point $x\in \operatorname{int} D(T)$. By local boundedness of T (*see* [Ro_1], [B-F] or [Ph], p. 29), there exists an open neighborhood U of x_0 in D(T) and $n_0\geqq 1$ such that $T(x)\subset n B^*$ for all $x\in U$. Suppose, now, that $n\geqq n_0$, hence that $T(U)\subset n B^*$. In view of this inclusion, $T|_U=S_n|_U\subset T_n|_U$. [The latter inclusion means that $G(S_n|_U)\subset G(T_n|_U)$.] Now, it is known (*see*, for instance, [Ph], Cor. 7.8), that $T|_U$ is maximal in U. (That is, its graph in $U\times E^*$ is maximal among all the monotone subsets of $U\times E^*$.) It follows that $T|_U=T_n|_U$ for each $n\geqq n_0$, which was to be shown.

Finally, to prove (v), we check (*a*) and (*b*) in the definition of weak* Mosco-convergence. For (*a*), suppose that $x^* \in T(x)$; since $x^* \in S_n(x) \subset T_n(x)$ provided $n > \| x^* \|$, we need only take $x_n^* = x^*$. For (*b*), suppose that $x^* \in E^*$ and that $x_k^* \in T_{n_k}(x)$ converges weak* to x^*. By maximality of T, to see that $x^* \in T(x)$, it suffices to show that $0 \leqq \langle x^* - y^*, x - y \rangle$ whenever $y \in D(T)$ and $y^* \in T(y)$. But for $n_k \geqq \| y^* \|$ we have $y \in S_{n_k}(y) \subset T_{n_k}(y)$ and hence $0 \leqq \langle x_k^* - y^*, x - y \rangle$, so weak* convergence yields the desired inequality.

The proof above used the following very special case of an extension theorem of Debrunner and Flor [D-F]. Since the proof of the latter is simpler in this special case, we have included it for the sake of completeness. As in the original proof, the key tool is a variant of the Farkas lemma (on the existence of solutions for finite systems of linear inequalities). A different proof, using an appropriate form of the Hahn-Banach theorem, has been given by Kønig and Neumann [K-N].

2.4. Lemma (Debrunner-Flor). — *Suppose that* C *is a weak* compact convex subset of* E* *and that* $M \subset E \times C$ *is a monotone set. For any* $x_0 \in E$ *there exists* $x_0^* \in C$ *such that* $\{(x_0, x_0^*)\} \cup M$ *is a monotone set.*

Proof. — For each element $(y, y^*) \in M$ let

$$C(y, y^*) = \{ x^* \in C : \langle x^* - y^*, x_0 - y \rangle \geqq 0 \}.$$

Each of these sets is the intersection with C of a weak* closed half-space, hence is weak* compact, and the theorem will be proved if we can produce a point x_0^* in the intersection $\bigcap \{ C(y, y^*) : (y, y^*) \in M \}$. By compactness, it suffices to prove that this family of sets has the finite intersection property. Suppose, then, that

$$(y_1, y_1^*), (y_2, y_2^*), \ldots, (y_n, y_n^*) \in M.$$

We will show that there exist $\lambda_j \geqq 0$, $j = 1, 2, \ldots, n$, such that $\Sigma \lambda_j = 1$ and $\Sigma \lambda_j y_j^* \in \bigcap \{ C(y_i, y_i^*) : i = 1, 2, \ldots, n \}$. Let $x_0^* = \Sigma \lambda_j y_j^*$; since it is a convex combination, x_0^* will be in C, so it suffices to find constants $\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_n$ which solve the system of linear inequalities

$$\sum_{j=1}^{n} \lambda_j \langle y_j^* - y_i^*, x_0 - y_i \rangle \equiv \langle \Sigma \lambda_j y_j^* - y_i^*, x_0 - y_i \rangle \geqq 0, \; i = 1, \ldots, n,$$
$$-\lambda_0 + \Sigma \lambda_j \geqq 0,$$
$$\lambda_0 - \Sigma \lambda_j \geqq 0,$$
$$\lambda_0 > 0,$$
$$\lambda_j \geqq 0, \; j = 1, 2, \ldots, n.$$

(We then divide by λ_0 to get $\Sigma \lambda_j = 1$.) By a variant of the Farkas lemma [Tu], if this system fails to have a solution, then there is a solution

$\mu_1, \mu_2, \ldots, \mu_{n+2}$ of the system

$$\sum_{i=1}^{n} \langle y_j^* - y_i^*, x_0 - y_i \rangle \mu_j + \mu_{n+1} - \mu_{n+2} \leqq 0, j = 1, 2, \ldots, n,$$
$$-\mu_{n+1} + \mu_{n+2} < 0,$$
$$\mu_i \geqq 0, \; i = 1, 2, \ldots, n+2.$$

This implies that there exist nonnegative μ_i's which satisfy

$$\sum_{i=1}^{n} \mu_i \langle y_j^* - y_i^*, x_0 - y_i \rangle < 0, \qquad j = 1, 2, \ldots, n.$$

Evidently, the μ_i's are not all zero, so if we multiply the *j-th* inequality by μ_j and sum over j, we get

$$\sum_{j=1}^{n} \sum_{i=1}^{n} \mu_i \mu_j \langle y_j^* - y_i^*, x_0 - y_i \rangle < 0.$$

Now, the terms in this double sum can be paired so that it becomes

$$\sum_{i<j} \mu_i \mu_j [\langle y_j^* - y_i^*, x_0 - y_i \rangle + \langle y_i^* - y_j^*, x_0 - y_j \rangle]$$
$$= \sum_{i<j} \mu_i \mu_j \langle y_j^* - y_i^*, y_j - y_i \rangle,$$

which, by monotonicity, is nonnegative, a contradiction which completes the proof.

We will see in the next two sections that, under additional hypotheses, Goal 2.1 (v) is valid. Assuming that it is (in fact, that a slightly weaker version is valid), we can show that the range of a maximal monotone operator has convex closure. This fact, along with some closely related results, has been proved in reflexive spaces by Rockafellar [Ro$_3$]. We first require a somewhat technical geometric lemma.

2.5. LEMMA. — *Let* E *be a Banach space and* $x^* \in E^*$ *with* $\|x^*\| = 2$. *If* $0 < r < 1/2$, *define*

$$C = co(\{x^*, -x^*\} \cup r B^*)$$

and $A = C \diagdown 2 r B^*$. *Then, letting* $\alpha = 2 - \dfrac{2 r^2}{2 - r}$, *we have* $1 < \alpha < 2$ *and*

$$A \cup [(\overline{co}\, A) \cap \operatorname{bdry} C] \subset \{-x^*, x^*\} + \alpha B^*.$$

Proof. — First, suppose that $y^* \in A$, so that it can be represented in the form $y^* = \lambda x^* + (1 - |\lambda|) u^*$, where $|\lambda| \leqq 1$, $u^* \in r B^*$ and $\|y^*\| > 2r$. The triangle inequality gives $2|\lambda| + (1 - |\lambda|) r > 2r$ so that $|\lambda| > r/(2-r)$. Now we calculate

$$\begin{aligned}\|\operatorname{sgn}(\lambda)x^*-y^*\| &= \|\operatorname{sgn}(\lambda)x^*-\lambda x^*-(1-|\lambda|)u^*\|\\ &\leqq (1-|\lambda|)2+(1-|\lambda|)r\\ &=(1-|\lambda|)(2+r)\\ &<[1-r/(2-r)](2+r)\\ &=2-2r^2/(2-r)=\alpha\end{aligned}$$

as claimed. The same representation of y^* shows that its distance from the segment $[-x^*, x^*]$ is less than or equal to

$$\|y^*-\lambda x^*\|=\|(1-|\lambda|)u^*\|\leqq r(1-|\lambda|)\leqq r\left(1-\frac{r}{2-r}\right)\equiv\beta.$$

This shows that $A\subset[-x^*, x^*]+\beta B^*$, so the same is true of $\overline{co}\, A$. Thus, if $y^*\in[\overline{co}\, A]\cap \operatorname{bdry} C$, then we have $y^*=\mu x^*+z^*$ where $\|z^*\|\leqq\beta$, $|\mu|\leqq 1$ and $y^*\notin \operatorname{int} C$. If $|\mu|=1$, then $\|y^*-\operatorname{sgn}(\mu)x^*\|=\|z^*\|\leqq\beta<\alpha$. If $|\mu|<1$, we can write $y^*=\mu x^*+(1-|\mu|)(1-|\mu|)^{-1}z^*$; since $y^*\notin\operatorname{int} C$, we must have $\|(1-|\mu|)^{-1}z^*\|\geqq r$, which shows that $\beta\geqq r(1-|\mu|)$. Now

$$\begin{aligned}\|\operatorname{sgn}(\mu)x^*-y^*\| &= \|\operatorname{sgn}(\mu)x^*-\mu x^*-z^*\|\\ &\leqq(1-|\mu|)2+\beta\\ &\leqq(2/r)\beta+\beta=\alpha\end{aligned}$$

as we wanted.

DEFINITION. — *By a* homothet *of a monotone operator* T *we mean any operator of the form* $\lambda T+u^*$, *where* $\lambda>0$ *and* $u^*\in E^*$.

It is easily seen that $\overline{R(\lambda T+u^*)}=\lambda\overline{R(T)}+u^*$ and that homothets of T are (maximal) monotone if and only if T is (maximal) monotone.

2.6. PROPOSITION. — *Suppose that* T *is a maximal monotone operator on* E *and that for each equivalent dual ball* B^* *on* E^* (*with unit sphere* S^*) *and every homothet* $\tilde{T}$ *of* T, *there is a maximal monotone extension* T_1 *of the operator* $x\to\tilde{T}(x)\cap B^*$ *such that*

(i) $D(T_1)=E$,

(ii) $R(T_1)\subset\overline{co}[R(\tilde{T})\cap B^*]$ *and*

(iii) $R(T_1)\diagdown R(\tilde{T})\subset S^*$.

Then the norm closure of R(T) *is convex.*

Proof. — Suppose that $\overline{R(T)}$ is not convex. Thus, it is not midpoint convex and it follows by a simple argument that there exist two points in R(T) whose midpoint is not in $\overline{R(T)}$. By changing to an appropriate homothet of T, we may assume that x^* and $-x^*$ are in R(T) with $\|x^*\|=1$ but for some $r\in(0, 1/2)$ the range of T does not intersect $2rB^*$. Define an equivalent dual ball by

$$C=co(\{2x^*, -2x^*\}\cup rB^*).$$

By our hypothesis on E we can find a maximal monotone extension T_1 of $x \to T(x) \cap C$ such that $D(T_1)=E$ and $R(T_1) \subset \overline{co}[R(T) \cap C]$ while $R(T_1) \setminus R(T) \subset \text{bdry}\, C$.

We now apply Lemma 2.5, with $2x^*$ in place of x^* and with

$$\alpha = 2 - \frac{2r^2}{2-r} \quad \text{and} \quad A = C \setminus 2r B^*$$

as before. Since $R(T_1) \cap R(T) \subset C \cap R(T) \subset A$ and

$$R(T_1) \setminus R(T) \subset \overline{co}[R(T) \cap C] \cap \text{bdry}\, C \subset \overline{co}\, A \cap \text{bdry}\, C,$$

we conclude that

$$R(T_1) \subset \{-2x^*, 2x^*\} + \alpha B^*.$$

However, for each $x \in E$ the set $T_1(x)$ is convex and therefore cannot intersect both $-2x^* + \alpha B^*$ and $2x^* + \alpha B^*$. Since the operator T_1 is norm-to-weak* upper semicontinuous, the sets $T_1^{-1}(-2x^* + \alpha B^*)$ and $T_1^{-1}(2x^* + \alpha B^*)$ are closed; moreover, they are disjoint and their union is all of E. Connectedness of E shows that one of them must be empty, contradicting the fact that the sets $\pm 2x^* + \alpha B^*$ contain $\pm x^*$, respectively.

3. LOCALLY MAXIMAL MONOTONE OPERATORS

In this section we will show that it is possible to obtain the kind of conclusions we want for the approximating operators T_n, provided we assume that the operator T belongs to the following special subclass of maximal monotone operators.

DEFINITION. — *A monotone operator* T *on a Banach space* E *is said to be* locally maximal monotone *if, for each* $x \in E$ *and* $x^* \notin T(x)$, *and each open convex subset* U *of* E* *which contains* x^* *and intersects* R(T), *there is* $z \in E$ *and* $z^* \in T(z) \cap U$ *such that* $\langle z^* - x^*, z - x \rangle < 0$.

Since E* is a permissible choice for U, it is clear that a locally maximal monotone operator is maximal monotone. We will see later that an example of Gossez [Go_2] can be used to show that not every maximal monotone operator is locally maximal monotone. Note that the term "locally" really refers to the *range* of T; in fact, it is straightforward to see that T is locally maximal monotone if and only if, for each open convex $U \subset E^*$ which intersects R(T), the graph of the monotone operator $x \to T(x) \cap U$ is maximal among all monotone subsets of $E \times U$.

The following theorem shows that locally maximal monotone operators satisfy most of our goals.

3.1. Theorem. — *Suppose that* E *is a Banach space and that* T *is a locally maximal monotone operator on* E. *Then there exists a sequence* $\{T_n\}$ *of maximal monotone operators such that*

(i) $D(T_n)=E$ *for all* n.

(ii) $T(T_n)\subset \overline{co}[R(T)\cap nB^*]\subset nB^*$ *for all* n.

(iii) $T(x)\cap nB^*\subset T_n(x)$ *for each* $x\in E$ *and all* n.

(iv) $T_n(x)\setminus T(x)\subset nS^*$ *for each* $x\in E$ *and all* n.

(v) $T_n(x)=T(x)\cap nB^*$, *provided* $T(x)\cap \operatorname{int} nB^*\neq 0$.

(vi) *If* $x\in D(T)$, *then* $T_n(x)=T(x)\cap nB^*$ *for all sufficiently large* n.

(vii) *If* $x\notin D(T)$, *then* $T_n(x)\subset nS^*$ *for all* n.

Proof. — Assertions (i), (ii) and (iii) are valid for any maximal monotone operator, as shown in the first three assertions in Theorem 2.3. To prove (iv), suppose there exists $x^*\in T_n(x)\setminus T(x)$ with $\|x^*\|<n$ and let $U=\operatorname{int} nB^*$. Then U is an open convex set containing x^* and intersecting $R(T)$, so local maximality of T provides $z\in E$ and $z^*\in T(z)\cap U\subset T_n(z)$ such that $\langle z^*-x^*, z-x\rangle<0$, contradicting the monotonicity of T_n. To prove (v), suppose that $x^*\in T_n(x)$. By hypothesis, there exists $y^*\in T(x)\cap U\subset T_n(x)$, hence the open line segment between y^* and x^* is contained in $T_n(x)\cap U$, thus in $T(x)\cap U$, by part (iv). Since $T(x)$ is closed, it follows that $x^*\in T(x)$, so $T_n(x)=T(x)\cap B^*$. Finally parts (vi) and (vii) are immediate from (v) and (iv), respectively.

This result will allow us to attain our goals in *reflexive* Banach spaces for arbitrary maximal monotone operators (*see* Section 4), once we have shown that in such spaces they are locally maximal monotone. To this end, we reformulate the definition.

3.2. Proposition. — *A monotone operator* T *on* E *is locally maximal monotone if and only if it satisfies the following condition: for any weak* closed convex and bounded subset* C *of* E^* *such that* $R(T)\cap \operatorname{int} C\neq 0$ *and for each* $x\in E$ *and* $x^*\in \operatorname{int} C$ *with* $x^*\notin T(x)$, *there exists* $z\in E$ *and* $z^*\in T(z)\cap C$ *such that* $\langle x^*-z^*, x-z\rangle<0$.

Proof. — In one direction, if T is locally maximal monotone and C is given, let $U=\operatorname{int} C$. In the other direction, if U is open and convex in E^*, if $u\in E$ and $x\in E$ with $u^*\in T(u)\cap U$ and $x^*\in U$ but $x^*\notin T(x)$, then there exists $\varepsilon>0$ such that $u^*+\varepsilon B^*\subset U$ and $x^*+\varepsilon B^*\subset U$. By convexity, $C\equiv[u^*, x^*]+\varepsilon B^*$ is a weak* closed, convex and bounded subset of U which can be used to verify that U has the required property.

We recall the following definition.

Definition. — If S and T are monotone operators, their *sum* $S+T$ is the monotone operator defined, using vector addition of sets, by $(S+T)(x)=S(x)+T(x)$ provided $x\in D(S)\cap D(T)\equiv D(S+T)$, while $(S+T)(x)=0$ otherwise.

The only use of reflexivity in the next result is in the application of Rockafellar's theorem [Ro_2], which asserts that the sum of two maximal monotone operators is maximal monotone, provided the domain of one of them intersects the interior of the domain of the other (and the space is reflexive).

3.3. PROPOSITION. – *If* E *is reflexive and* T *is maximal monotone on* E, *then it is locally maximal monotone.*

Proof. – Suppose that C is weak* closed and convex, that $\operatorname{int} C \cap R(T) \equiv \operatorname{int} C \cap D(T^{-1}) \neq 0$, that $x \in E$ and that $x^* \in \operatorname{int} C$ but $x^* \notin T(x)$. Let $S^{-1} = \partial I_C$ be the (maximal monotone) subdifferential of the indicator function of C. Since $\operatorname{int} D(S^{-1}) = \operatorname{int} C$ and since the latter intersects $D(T^{-1})$, Rockafellar's theorem implies that $T^{-1} + S^{-1}$ is maximal monotone. Now $x^* \notin T(x)$ implies that $x \notin T^{-1}(x^*)$, and since $S^{-1}(x^*) = \{0\}$, we see that $x \notin T^{-1}(x^*) + S^{-1}(x^*)$. By maximality of $T^{-1} + S^{-1}$, there exists $z^* \in D(T^{-1}) \cap D(S^{-1}) \equiv R(T) \cap C$ and $z \in T^{-1}(z^*)$ such that $\langle x^* - z^*, x - z \rangle < 0$. Thus, T satisfies the condition in Proposition 3.2 and is therefore locally maximal monotone.

The following proposition shows that if T is locally maximal monotone, then the ranges of our approximating operators are contained in the norm closure of the range of T, as suggested by Goal 2.1 (iii).

3.4. PROPOSITION. – *Let* T *be a locally maximal monotone operator on a Banach space* E *with* $R(T) \cap \operatorname{int} n B^* \neq 0$. *Then* $R(T_n) \subset \overline{R(T)}$ *for any monotone operator* T_n *such that* $T(x) \cap n B^* \subset T_n(x)$ *for all* $x \in E$.

Proof. – Suppose, to the contrary, that $x^* \in T_n(x) \setminus \overline{R(T)}$. Let $\varepsilon > 0$ be sufficiently small such that $\operatorname{dist}(x^*, R(T)) > 2n\varepsilon$ and $R(T) \cap (n-\varepsilon) B^* \neq 0$. Let $y^* \in R(T) \cap (n-\varepsilon) B^*$ and consider the open set

$$U = [x^*, y^*] + \varepsilon^2 \operatorname{int} B^*.$$

Local maximality shows that there is $z^* \in T(z) \cap U$ such that $\langle x^* - z^*, x - z \rangle < 0$. Since $x^* \in T_n(x)$ and $T_n(z) \supset T(z) \cap n B^*$, we must have $\| z^* \| > n$. Now for some v^* with $\| v^* \| < \varepsilon^2$ and some $0 \leqq \lambda \leqq 1$ we have $z^* = \lambda x^* + (1-\lambda) y^* + v^*$ so that

$$\begin{aligned} n < \| z^* \| &\leqq \lambda \| x^* \| + (1-\lambda) \| y^* \| + \varepsilon^2 \\ &\leqq n\lambda + (1-\lambda)(n-\varepsilon) + \varepsilon^2 = n - (1-\lambda)\varepsilon + \varepsilon^2 \end{aligned}$$

and hence $1 - \lambda < \varepsilon$. Now

$$\begin{aligned} \| z^* - x^* \| &= \| (1-\lambda)(y^* - x^*) + v^* \| \\ &< (1-\lambda)(2n-\varepsilon) + \varepsilon^2 < 2n\varepsilon < \operatorname{dist}(x^*, R(T)), \end{aligned}$$

which contradicts the fact that $z^* \in R(T)$.

3.5. THEOREM. – *If* T *is a locally maximal monotone operator on the Banach space* E, *then* $\overline{R(T)}$ *is convex.*

Proof. — Since the property of being locally maximal monotone does not depend on which (equivalent) norm we have on E and since (as is readily verified) T is locally maximal monotone if and only if the same is true of each homothet of T, Theorem 3.1 guarantees that T satisfies the hypotheses of Proposition 2.6, hence the conclusion of the latter that $\overline{R(T)}$ is convex.

It follows from this that *not every maximal monotone operator is locally maximal monotone*: Gossez [Go$_2$] has exhibited a maximal monotone operator T on l_1 for which $\overline{R(T)}$ is not convex. [That Gossez's example is actually maximal can readily be deduced from our Lemma 2.2.]

The foregoing observation also implies that property (iv) of Theorem 3.1 (the fact that $T_n(x)\setminus T(x)\subset n S^*$) cannot hold in general. This is a consequence of the following proposition, which gives a sufficient condition on E that every maximal monotone operator be locally maximal monotone.

3.6. Proposition. — *Suppose that* E *is a Banach space such that for each equivalent renorming of* E, *whenever a maximal monotone operator* T *on* E *admits a maximal monotone operator* T_1 *satisfying*

(i) $R(T_1)\subset B^*$ *and*

(ii) $T(x)\cap B^*\subset T_1(x)$ *for each* $x\in E$, *then it also satisfies*

(iii) $T_1(x)\setminus T(x)\subset S^*$ *for each* $x\in E$.

Then every maximal monotone operator on E *is locally maximal.*

Proof. — Suppose that T is maximal monotone and that for some $x\in E$ and $x^*\notin T(x)$, there is an open convex subset U of E^* containing x^* such that $u^*\in U\cap T(u)$ for some $u\in E$. The element $y^*=\frac{1}{2}(x^*+u^*)$ is in U. Let $\tilde{x}^*=x^*-y^*$, $\tilde{u}^*=u^*-y^*$, $\tilde{U}=U-y^*$ and $\tilde{T}=T-y^*$. Then $\tilde{T}$ is maximal monotone, $\tilde{x}^*\notin\tilde{T}(x)$, $\tilde{u}^*\in\tilde{U}\cap\tilde{T}(u)$ and both $\tilde{x}^*$ and $\tilde{u}^*(=-\tilde{x}^*)$ are in $\tilde{U}$. Renorm E so that the new equivalent dual ball is

$$B_1^*\equiv[-\tilde{x}^*,\tilde{x}^*]+\varepsilon B^*,$$

where $\varepsilon>0$ is sufficiently small that $B_1^*\subset\tilde{U}$. If T were not locally maximal monotone, then we would have $\langle z^*-x^*, z-x\rangle\geqq 0$ for all $z^*\in T(z)\cap U$ and hence $\langle\tilde{z}^*-\tilde{x}^*, z-x\rangle\geqq 0$ for all $\tilde{z}^*\in\tilde{T}(z)\cap\tilde{U}$. Define S_1 to be the monotone operator whose graph is the union of $\{(x,\tilde{x}^*)\}$ and the graph of $x\to\tilde{T}(x)\cap B_1^*$. By Zorn's lemma, it is contained in a maximal monotone subset of $E\times B_1^*$ which is therefore the graph of a monotone operator T_1. By applying the Debrunner-Flor Lemma 2.4 and Lemma 2.2 as in the proof of Theorem 2.3, we conclude that T_1 is a maximal monotone operator on E satisfying (i) $R(T_1)\subset B_1^*$ and (ii) $\tilde{T}(z)\cap B_1^*\subset T_1(z)$ for all $z\in E$. By hypothesis, it satisfies (iii): every element of $T_1(x)\setminus\tilde{T}(x)$ must have B_1^*-norm equal to 1. Since $\tilde{x}^*\in\operatorname{int} B_1^*$, we conclude that $\tilde{x}^*\in\tilde{T}(x)$, a contradiction which completes the proof.

4. THE REFLEXIVE CASE

The following theorem is an immediate corollary of Theorem 3.1 and Proposition 3.3.

4.1. THEOREM. — *Suppose that* E *is reflexive, and that* T *is maximal monotone. Then there exists a sequence* $\{T_n\}$ *of maximal monotone operators such that*

(i) $D(T_n)=E$ *for all n.*
(ii) $R(T_n)\subset\overline{co}[R(T)\cap nB^*]\subset nB^*$ *for all n.*
(iii) $T(x)\cap nB^*\subset T_n(x)$ *for each* $x\in E$ *and all n.*
(iv) $T_n(x)\setminus T(x)\subset nS^*$ *for each* $x\in E$.
(v) $T_n(x)=T(x)\cap nB^*$, *provided* $T(x)\cap\operatorname{int} nB^*\neq 0$.
(vi) *If* $x\in D(T)$, *then* $T_n(x)=T(x)\cap nB^*$ *for all sufficiently large n.*
(vii) *If* $x\notin D(T)$, *then* $T_n(x)\subset nS^*$ *for all n.*

The next theorem was arrived at by reformulating Theorem 4.1 in terms of the *inverses* of all the monotone operators involved, then interchanging the roles of E and E* and renaming the operators.

NOTATION. — If T is a monotone operator on E, then for any subset $A\subset E$, the operator $T|_A$ is defined by $T|_A(x)=0$ if $x\notin A$ and $T|_A(x)=T(x)$ if $x\in A$. The closed unit ball in E is denoted by B.

4.2. THEOREM. — *Suppose that* E *is reflexive and that* T *is a maximal monotone operator on* E. *Then there exists a sequence of maximal monotone operators* $\{T_n\}$ *with the following properties:*

(i) $R(T_n)=E^*$ *for each n.*
(ii) $D(T_n)\subset\overline{co}[D(T)\cap nB]$ *for each n.*
(iii) $T|_{nB}(x)\subset T_n(x)$ *for each n and each* $x\in E$.
(iv) $T_n(x)=T(x)$ *for each* $x\in\operatorname{int} nB$.

Proof. — Assertions (i), (ii) and (iii) are easily seen to be equivalent to the corresponding assertions in Theorem 4.1, applied to the inverses of the operators T and T_n. To prove (iv), note that (iii) implies that $T(x)\subset T_n(x)$ whenever $x\in nB$. The reverse inclusion follows by contradiction from part (iv) of Theorem 4.1 (again applied to inverses).

It would be very satisfying if we knew that the approximating operators constructed in Theorem 2.3 were, when applied to the case $T=\partial f$, necessarily the same as the subdifferentials ∂f_n of the inf-convolutions f_n of Proposition 1.2. That this is indeed the case in reflexive spaces, at least, will be a consequence of a uniqueness theorem (below); we first require a definition and a lemma.

DEFINITION. — For simplicity of notation, we let A_n denote the maximal monotone subdifferential operator $\partial(n\|\cdot\|)$.

It is clear that $A_n(x)=\{x^*\in E^*:\|x^*\|\leqq n$ and $\langle x^*, x\rangle=n\|x\|\}$; from this it follows readily that for any $x^*\in E^*$,

$$A_n^{-1}(x^*)=\begin{cases}\varnothing, & \text{if } \|x^*\|>n;\\ \{0\}, & \text{if } \|x^*\|<n;\\ \{x\in E:\langle x^*, x\rangle=n\|x\|\} & \text{if } \|x^*\|=n.\end{cases}$$

4.3. LEMMA. — *Suppose that* T *is maximal monotone and that* T_n *is a maximal monotone extension of* $S_n(x)=T(x)\cap nB^*$, $(x\in E)$ *which satisfies* $D(T_n)=E$ *and* $R(T_n)\subset nB^*$; *then* $T_n^{-1}=T_n^{-1}+A_n^{-1}$. *Furthermore, if* E *is reflexive and* $R(T)\cap \operatorname{int} nB^*\neq 0$, *then* $T_n^{-1}=T^{-1}+A_n^{-1}$.

Proof. — Note that $T_n^{-1}+A_n^{-1}$ is a monotone operator; also $D(T_n^{-1})\subset nB^*$ and $0\in A_n^{-1}(x^*)$ for each $x^*\in D(T_n^{-1})$. Thus, $T_n^{-1}+A_n^{-1}$ is a monotone extension of T_n^{-1} so maximality of T_n yields the first equality. To prove the second one, note that

$$D(T^{-1})\cap \operatorname{int} D(A_n^{-1})=R(T)\cap \operatorname{int} nB^*\neq 0.$$

This, together with reflexivity, implies that $T^{-1}+A_n^{-1}$ is maximal monotone [Ro2]. By the first equality, we have $T_n^{-1}=T_n^{-1}+A_n^{-1}\supset T^{-1}+A_n^{-1}$, since A_n^{-1} is empty outside of nB^*. The maximality of $T^{-1}+A_n^{-1}$ shows that equality holds.

The following theorem shows that, in reflexive spaces the approximating operators T_n of Theorem 2.3 are (for all sufficiently large n) unique; in fact, we need only assume that $R(T_n)\subset nB^*$ (rather than $R(T_n)\subset C_n$).

4.4. THEOREM. — *Suppose that* E *is reflexive, that* T *is maximal monotone on E and that for each n,* T_n *is a maximal monotone operator satisfying*

(i) $D(T_n)=E$ *and* $R(T_n)\subset nB^*$

(ii) $T(x)\cap nB^*\subset T_n(x)$ *for all* $x\in E$.

If n is sufficiently large that $R(T)\cap \operatorname{int} nB^*\neq 0$, *then* $\tilde{T}_n=T_n$ *whenever* $\tilde{T}_n$ *is a maximal monotone operator on* E *which satisfies* (i) *and* (ii).

Proof. — It suffices to show that $\tilde{T}_n(x)\subset T_n(x)$ for all $x\in E$; equality of $\tilde{T}_n$ and T_n follows from the maximality of $\tilde{T}_n$. Suppose, then, that $x^*\in\tilde{T}_n(x)$ and define

$$\tilde{S}_n(y)=\begin{cases}[T(x)\cap nB^*]\cup\{x^*\} & \text{if } y=x\\ T(y)\cap nB^*, & \text{if } y\neq x.\end{cases}$$

By the same use of the Debrunner-Flor lemma as in the proof of Theorem 2.3, there exists a maximal monotone operator V_n which extends

$\tilde{S}_n$ and satisfies $D(V_n)=E$ and $R(V_n)\subset n B^*$. By the second assertion in Lemma 4.3 (applied to V_n, then to T_n), we have

$$V_n^{-1}=T^{-1}+A_n^{-1}=T_n^{-1}$$

and hence $T_n(x)=V_n(x)\ni x^*$.

4.5. COROLLARY. – *Suppose that* E *is reflexive and that* f *is a proper semicontinuous convex function on* E. *Suppose, further, that* $R(\partial f)\cap \operatorname{int} n B^*\neq 0$ *and that* T_n *is a maximal monotone operator satisfying* (i) *and* (ii) *of Proposition* 4.4. *Then* $T_n=\partial f_n$.

Conclusion (v) of Theorem 4.1 can be improved slightly if the norm in the dual of the reflexive space E is *strictly convex*; that is, if $\|tx^*+(1-t)y^*\|<1$ whenever x^* and y^* are distinct elements in E^* of norm 1 and $0<t<1$.

4.6. PROPOSITION. – *Suppose that* E *is reflexive and that the norm in* E^* *is strictly convex. If* $x\in E$ *is such that* $T(x)\cap n B^*\neq 0$, *then* $T_n(x)=T(x)\cap n B^*$.

Proof. – Let $x^*\in T_n(x)$ and choose $y^*\in T(x)\cap n B^*\subset T_n(x)$; without loss of generality, we assume that $y^*\neq x^*$. Now, for $0<t<1$, define $x_t^*=(1-t)x^*+ty^*$. Then $x_t^*\in T_n(x)$ and $\|x^*\|<n$, since E^* is strictly convex. From Theorem 4.1 (iv), we conclude that $x_t^*\in T(x)$ and, since the latter is closed, $x^*\in T(x)$.

The following simple example shows that the foregoing proposition may fail if E^* is not assumed to be strictly convex.

4.7. *Example.* – *Let* T *be the maximal monotone operator from the two-dimensional space* $E=l_1^{(2)}$ *to* $E^*=l_\infty^{(2)}$ *defined by* $T(x_1, x_2)=(x_2, -x_1)$. *Then* $T(0,1)\cap B^*\neq 0$, *but* $T_1(0,1)\neq T(0,1)\cap B^*$.

Proof. – It can be verified directly that T is monotone and maximal. It is obvious that $T(0,0)\in \operatorname{int} B^*$, so by the second part of Lemma 4.3,

$$\begin{aligned} T_1^{-1}(1,1)&=T^{-1}(1,1)+A_1^{-1}(1,1)\\ &=\{(-1,1)\}+\{(x_1,x_2):|x_1|+|x_2|=x_1+x_2\}\\ &=\{(-1,1)\}+\{(x_1,x_2):x_1\geqq 0,\ x_2\geqq 0\}.\end{aligned}$$

In particular, $(0,1)\in T_1^{-1}(1,1)$, hence $(1,1)\in T_1(0,1)$ but

$$(1,1)\notin T(0,1)\cap B^*=\{(1,0)\}.$$

It is worth recalling at this point that any reflexive Banach space can be renormed so that its dual is strictly convex. (*See*, for instance [Da], p. 160.)

5. REMARKS AND QUESTIONS

While the primary aim of this paper has been to develop a potentially useful approximation scheme for maximal monotone operators on arbitrary Banach spaces, the attempt to do so has highlighted the fact that there is much more to learn about the structure and behavior of such operators. There has not been much progress in this endeavor since Rockafellar's papers some twenty years ago (particularly notable examples being [Ro_1]-[Ro_5]; *see*, also, [Go_1]). For instance, the fundamental fact that the sum $S+T$ of two maximal monotone operators S and T is again maximal (and monotone), provided $D(S) \cap \operatorname{int} D(T)$ is nonempty, has only been proved in reflexive spaces; we know of no counter-example for nonreflexive spaces.

5.1. QUESTION. — *Suppose that* S *and* T *are maximal monotone operators on the Banach space* E *such that* $D(S) \cap \operatorname{int} D(T) \neq 0$. *Must* $S+T$ *be maximal?*

Using additional hypotheses on S and T, Brezis, Crandall and Pazy [B-C-P] have proved maximality of $S+T$ (in reflexive spaces, unfortunately) without assuming that either $D(S)$ or $D(T)$ has nonempty interior. The following example shows that, even in a two-dimensional space, some additional restriction is needed (beyond $D(S) \cap D(T)$ merely being nonempty).

5.2. EXAMPLE. — *In the plane* R^2 *let* $C = \{(x, y) : y \geqq x^2\}$ *and let* L *be the x-axis. Further, define* $S = \partial I_C$ *and* $T = \partial I_L$; *then these subdifferentials are maximal monotone, but their monotone sum* $S+T$, *with domain* $D(S+T) = C \cap L \equiv \{(0, 0)\}$, *is not maximal.*

Proof. — As in [Ph], p. 54, $(S+T)(0,0) = \{(0, y); y \in R\}$, so its graph $G(S+T)$ is a proper subset of the monotone set $\{(0,0)\} \times R^2$.

The only use of reflexivity in proving uniqueness of our approximating operators was in Lemma 4.3, where it was used to show that a sum (namely, $T^{-1} + A_n^{-1}$) is maximal monotone. It is conceivable that the special nature of $A_n \equiv \partial (n \| \cdot \|)$ might allow this conclusion without reflexivity, but the following example shows that, even when T is the subdifferential of a very simple convex function, this need not be true.

5.3. EXAMPLE. — *Let* E *be a Banach space and suppose that* E *is not reflexive, so there exists* $x^* \in E^*$ *which does not attain its norm. Define* $f(x) = \langle x^*, x \rangle^2$ $(x \in E)$ *and let* $T = \partial f$. *Then* $T^{-1} + A_n^{-1}$ *is not maximal.*

Proof. — It is straightforward to compute that

$$R(T) (= D(T^{-1}) = \{ rx^* : r \in R \}.$$

On the other hand, from the description of A_n^{-1} given at the beginning of Section 4, and the fact that x^* does not attain its supremum on the unit ball, it follows that $A_n^{-1}(rx^*)=\{0\}$ for $\|rx^*\|\leqq n$. Thus, $D(T^{-1}+A_n^{-1})=\{rx^*:|r|\leqq n\}$ and $T^{-1}+A_n^{-1}=T^{-1}|_{nB^*}$, which is not maximal, it graph being a proper subset of the graph of T^{-1}.

The foregoing example sheds some light on the maximality of "parallel sums", as defined below. (The terminology arises from analogy with the formula giving the joint resistance of two resistors in parallel.)

DEFINITION. — Suppose that S and T are monotone operators on E. Their *parallel sum* $S:T$ is defined to be the operator $(S^{-1}+T^{-1})^{-1}$. The domain of this operator is defined (in an indirect way) as the range of $S^{-1}+T^{-1}$, where this latter sum has domain equal to $R(S)\cap R(T)$.

This notion has been studied by Passty [Pa] and [To], as well as by [Lu] (who gave a different—but equivalent—definition and used the terminology "inf-convolution").

Other fundamental questions also remain open in nonreflexive spaces; for instance, rather little is known about the structure of either the range or domain of an arbitrary maximal monotone operator. Rockafellar [Ro_3] has shown that in reflexive spaces, both D(T) and R(T) have convex closures and, as mentioned earlier, Gossez [Go_2] has shown that R(T) need not have convex closure when $E=l_1$, but the following question appears to be open.

5.4. QUESTION. — *Suppose that* T *is maximal monotone on the Banach space* E. *Must* $\overline{D(T)}$ *be convex?*

Note that this is true when $T=\partial f$ (where f is a proper lower semicontinuous convex function on E). Another result about ∂f (*see* Corollary 1.3) which remains open for general maximal monotone operators leads to the next question. (Relevant results are to be found in [B-F] and [Ro_1].)

5.5. QUESTION. — *Suppose that* T *is maximal monotone. Is it true that* T *fails to be locally bounded at each point of the boundary of* D(T)? *What if* D(T) *is assumed to be convex, or to have convex closure?*

As shown in Section 3, not every maximal monotone operator is locally maximal monotone, but the following question is open.

5.6. QUESTION. — *Suppose that f is a proper lower semicontinuous convex function on the Banach space* E. *Must the maximal monotone operator* $T=\partial f$ *be locally maximal monotone?* (*)

(*) This question has been answered in the affirmative by S. Simons, "Subdifferentials are locally maximal monotone", *Bull. Australian Math. Soc.* (to appear).

An affirmative answer to this last question would be a generalization of the fundamental and nontrivial fact that such subdifferentials are maximal monotone.

Another question arising in this paper is whether Corollary 4.5 is valid in nonreflexive spaces; that is, whether the approximating operators in Theorem 2.3 coincide with the subdifferentials ∂f_n of Section 1 when $T = \partial f$. More generally, *is it true that the uniqueness result in Theorem* 4.4 *is valid in nonreflexive spaces?*

The referee has suggested an interesting question, concerning the possibility of a form of convergence of $\{T_n\}$ to T different from the weak* Mosco convergence of Theorem 2.3. We first require a definition.

5.7. Definition. — A sequence of convex subsets $\{C_n\}$ of a normed linear space is said to converge to another convex set C in the *bounded Hausdorff topology* if, for every bounded subset A, the Hausdorff distance from $C_n \cap A$ to $C \cap A$ tends to zero. We say that a sequence $\{T_n\}$ of monotone operators on E is *graph-convergent* to the monotone operator T in the *bounded Hausdorff sense* provided the sequence $G(T_n)$ of graphs in $E \times E^*$ converges to the graph $G(T)$ of T in the bounded Hausdorff topology.

Note that in using this notion we may assume, without loss of generality, that the bounded subset $A \subset E \times E^*$ is of the form $r(B \times B^*)$, for some $r > 0$. For more about graph-convergence, see [A-N-T].

5.8. Proposition. — *Suppose that* T *is a maximal monotone operator on* E. *The approximating sequence* T_n *is graph-convergent to* T *in the bounded Hausdorff sense if* T *is either locally maximal monotone or is the subdifferential of a proper lower semicontinuous convex function on* E.

Proof. — Suppose that $A = r(B \times B^*) \subset E \times E^*$. The key to the proof is the fact that, in either of the two cases of interest, if $n > r$, then $G(T_n) \cap A = G(T) \cap A$, so that the Hausdorff distance between these sets is zero. Indeed, one inclusion is valid for any maximal monotone T: if $(x, x^*) \in G(T) \cap A$, then $x^* \in r B^* \subset n B^*$, so by Theorem 2.3 (ii), $x^* \in T_n(x)$, hence $(x, x^*) \in G(T_n) \cap A$. For the reverse inclusion, suppose that $(x, x^*) \in G(T_n) \cap A$. This implies that $\|x^*\| \leqq r$ and $x^* \in T_n(x)$, so if $x^* \notin T(x)$, then by Proposition 1.2 (if T is a subdifferential) or Theorem 3.1 (ii) (if T is locally maximal monotone), we must have $\|x^*\| = n$, a contradiction which completes the proof.

It remains an open question whether the foregoing proposition is valid for arbitrary maximal monotone operators.

We next compare our approximation scheme with the Moreau-Yosida approximations. Because the latter are only defined in strictly convex reflexive spaces having strictly convex duals, we necessarily restrict our attention to such spaces. Rather than repeat their definition, we simply describe their convergence properties. Now, the Moreau-Yosida approximations use a *continuous* parameter $\lambda>0$, with $\lambda\to 0$; for purposes of comparison, we replace λ by $1/n$. (Aternatively, one could reverse this; *see* below.) One can now formulate the Moreau-Yosida convergence properties as follows:

5.9. THEOREM. — *Suppose that* T *is a maximal monotone operator on* E, *normalized so that* $0\in T(0)$. *Then there exists a sequence of* single-valued *maximal monotone operators* $\{T_n\}$ *with the following properties:*

(i) $0\in T_n(0)$ *and* $D(T_n)=E$ *for each* n.

(ii) *For each* $x\in D(T)$, $\{T_n(x)\}$ *converges in the weak topology to the unique point of least norm in* $T(x)$.

(iii) *For each* $x\notin \overline{D(T)}$, $\|T_n(x)\|\to\infty$.

This result should be compared with Theorem 4.1 and Proposition 4.6.

It is perhaps more illuminating to see what happens when the two methods are applied to a subdifferential. As we have shown in Section 4, when $T=\partial f$, then (ar least for all sufficiently large n) we have $T_n=\partial f_n$, where f_n is the inf-convolution of f and $n\|\,.\,\|$. In this case, the Moreau-Yosida approximations become the subdifferentials of the inf-convolutions of f with the functions $(n/2)\|\,.\,\|^2$.

If, in the original definition of the approximating sequence $\{T_n\}$ one were to replace n throughout by $1/\lambda$ and consider $\lambda\to 0^+$, one would obtain an approximating family $\{T_\lambda\}$ which would have precisely the same convergence properties as $\lambda\to 0^+$ as $\{T_n\}$ does for $n\to\infty$. The only changes would be in Theorem 2.3 (v) and Proposition 5.8, where Mosco convergence (resp. graph-convergence) would have to be reformulated so as to apply to the continuous family of sets $\{T_\lambda(x)\}$, a straightforward task.

We conclude with some historical remarks about the inf-convolution approximations f_n to a function f. It was first used by F. Hausdorff [Hau] in 1919 to give a much simpler proof of R. Baire's [Bai] theorem that any lower semicontinuous function of a real variable which is bounded below is in the first Baire class. Hausdorff attributes the definition (with no bibliographical reference) to M. Pasch. The Hausdorff-Pasch construction was apparently independently rediscovered and published in 1934 by both H. Whitney [Wh] (footnote p. 63) and E. J. McShane [McS] who used it to extend a continuous (resp. Lipschitzian) function defined on a subset of Euclidean space. [In fact, Whitney (who used an equivalent "supremum" form of the definition), described the inf-convolution of f with $h(\|\,.\,\|)$, where h is any continuous nondecreasing real-valued function on the right half line for which $h(0)=0$. With $h(t)=nt$, for instance, one obtains the f_n's, while with $h(t)=\frac{1}{2\lambda}t^2$ one gets the Moreau-Yosida approximations.]

In 1980, J.-B. Hiriart-Urruty [H-U_1] carried out a systematic investigation of the Hausdorff-Pasch construction for convex functions; the same year, he, too, used it to extend Lipschitzian functions [H-U_2]. More recently, it has been applied by [No] and [Ph], among others.

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(*Manuscript received December* 5, 1990;
revised May 21, 1991.)