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Front propagation for reaction-diffusion equations of bistable type

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ABSTRACT. — We present a direct PDE approach to study the behavior as $\varepsilon \to 0$ of the solution u^ε of the reaction-diffusion equation: $u_t^\varepsilon - \varepsilon \Delta u^\varepsilon = (1/\varepsilon) f(u^\varepsilon)$ in $\mathbb{R}^N \times (0, \infty)$ in the case when f is the derivative of a bistable potential. Such singular perturbation problems arise in the

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study of large time wavefront propagations generated by such equations following a method introduced by M. Freidlin.

RÉSUMÉ. — Nous présentons une méthode EDP directe pour étudier le comportement, quand $\varepsilon \to 0$, de la solution u^ε de l'équation de réaction-diffusion : $u_t^\varepsilon - \varepsilon \Delta u^\varepsilon = (1/\varepsilon) f(u^\varepsilon)$ dans $\mathbb{R}^N \times (0, \infty)$ dans le cas où f est la dérivée d'un potentiel bistable. De tels problèmes de perturbation singulière interviennent dans l'étude asymptotique des fronts de propagations générés par de telles équations, en suivant une méthode introduite par M. Freidlin.

1. INTRODUCTION

In this paper we present a direct PDE approach to study the behavior as $\varepsilon \to 0$ of the parabolic initial value problem

$$\left.\begin{aligned} u_t^\varepsilon &= \varepsilon \Delta u^\varepsilon + \frac{1}{\varepsilon} f(u^\varepsilon) \quad \text{in } \mathbb{R}^N \times (0, \infty), \\ u^\varepsilon(x, 0) &= u_0(x) \quad \text{in } \mathbb{R}^N, \end{aligned}\right\} \tag{0.1}$$

where the function $u \mapsto f(u)$ is of "bistable type", *i.e.* has two zeros h_- and h_+ at which f' is negative and (for simplicity) has only one other zero between them. In the sequel, we normalize $h_\pm = \pm 1$ and we denote the third zero by $\mu \in (-1, 1)$. Moreover, again for simplicity, we will present the main arguments only for the special cubic nonlinearity

$$f(u) = 2(u - \mu)(1 - u^2); \tag{0.2}$$

the general case follows along the same lines.

A general motivation for considering such types of singular perturbation problems is the study of front propagations. In general, reaction-diffusion equations (or systems) model physical, chemical or biological phenomena in which fronts develop naturally for large times; e. g. flame propagation in combustion, phase transitions or evolutions of populations. To study this question, Freidlin ([Fr1], [Fr2]) introduced a scaling of order ε^{-1} in both space and time. Such a scaling transforms equations like

$$u_t = \Delta u + f(u) \quad \text{in } \mathbb{R}^N \times (0, \infty) \tag{0.3}$$

into

$$u_t^\varepsilon = \varepsilon \Delta u^\varepsilon + \frac{1}{\varepsilon} f(u^\varepsilon) \quad \text{in } \mathbb{R}^N \times (0, \infty), \tag{0.4}$$

where $u^\varepsilon(x,t)=u((x/\varepsilon),(t/\varepsilon))$. The basic idea is that (0.4) should reproduce in finite time the properties of (0.3) as $t\to\infty$, while the scaling in the x-variable suggests that the front should remain in a bounded region in space for bounded times. Then the initial data for (0.4) corresponds to initializing the position of the front. Also, since the reaction rate in (0.4) is large compared to the diffusion rate, the fronts develop quickly, given that the solution is close to the pure reaction equation $u_t^\varepsilon=(1/\varepsilon)f(u^\varepsilon)$.

Equation (0.4) is a non conservative of "type A" Ginzburg-Landau equation in the terminology of [GSS]. The model is thought to describe the dynamics of phase boundaries as internal surfaces in various alloys (*see* e. g. [AC]). In the case $\mu=0$, (0.4) was proposed in [AC] as a model for the motion of antiphase boundaries in crystalline solids. There are also a number of other situations where Ginzburg-Landau type dynamics lead through singular limits to geometric models for phase boundary motion (*see* e. g. [Ca], [CaF], [P]).

Our approach is a natural continuation of a general program initiated by Evans and Souganidis [ES1] (*see* also Evans and Souganidis [ES2], Barles, Evans and Souganidis [BES]) in order to use viscosity solution related methods to study singular perturbation problems. To give a formal justification of our approach as well as to explain our results, we consider the one-dimensional problem

$$u_t^\varepsilon=\varepsilon\frac{\partial^2 u^\varepsilon}{\partial x^2}+\frac{1}{\varepsilon}f(u^\varepsilon)\quad\text{in }\mathbb{R}\times(0,\infty),\tag{0.5}$$

which is associated to the reaction-diffusion

$$u_t=\frac{\partial^2 u}{\partial x^2}+f(u)\quad\text{in }\mathbb{R}\times(0,\infty).\tag{0.6}$$

In this case, it is well known that, as $t\to\infty$, $u(x,t)$ tends to either 1 or -1, which are the two stable equilibria of the dynamical system

$$\dot{v}=f(v),\tag{0.7}$$

or, equivalently, the two local minima of the bistable potential F defined by $F'=f$. The front may be seen as the set separating the regions where $u\approx 1$ and where $u\approx -1$. Moreover, (0.6) has a traveling wave solution [AW], *i.e.* a solution of the form $q(x-\alpha t)$, where q is the unique (up to translations) strictly increasing solution of

$$\alpha q'+q''+f(q)=0\quad\text{in }\mathbb{R},\tag{0.8}$$

which connects the stable equilibria of (0.7), *i.e.* $q(\pm\infty)=\pm 1$. The velocity α is also determined uniquely, since the potential is bistable. In the special case of the cubic nonlinearity (0.2), an easy computation yields

$$\left.\begin{aligned}q(\xi)&=th(\xi)\quad(\xi\in\mathbb{R})\\ \alpha&=2\mu.\end{aligned}\right\}\tag{0.9}$$

We now consider (0.5) with f as in (0.2) together with the initial datum

$$u^\varepsilon(x, 0) = 1_{(0, +\infty)} - 1_{(-\infty, 0)} \quad \text{in } \mathbb{R},$$

which corresponds to initializing the front at $x=0$. (Here 1_A denotes the characteristic function of the set A.) In this setting, we expect that the solution to (0.6) converges to q as $t \to \infty$ and therefore, as $\varepsilon \to 0$,

$$u^\varepsilon(x, t) \approx q\left(\frac{x - \alpha t}{\varepsilon}\right).$$

Hence,

$$\lim_{\varepsilon \to 0} u^\varepsilon(x, t) = 1_{(0, +\infty)}(x - \alpha t) - 1_{(-\infty, 0)}(x - \alpha t).$$

In other words, the front propagates with constant speed α.

At this point we should remark that Fife [Fi] and Rubinstein, Sternberg and Keller [RSK] suggested via formal expansions, that asymptotically the speed of the front is in fact $\alpha + \varepsilon K + o(\varepsilon)$, where K is the mean curvature of the front. Moreover, relation (0.9) shows that the case $\mu = 0$, studied by Allen and Cahn [AC], corresponds to $\alpha = 0$; in this case the speed of the front is given by εK, as expected. This was proved by Bronsard and Kohn [BK1] in the radial case, using energy type estimates. The general case (still with $\alpha = 0$), but only for the case of classical motion, was proved by DeMottoni and Schatzman [DS1], [DS2] by finding appropriate error estimates on the asymptotic expansion of u^ε and, more recently, by Chen [Ch] using sub- and super-solutions. (For results in one space dimension, *see* [CP], [FH], [BK2].) In the case where the motion is not classical, the result was recently obtained by Evans, Soner and Souganidis [ESS] using viscosity solution techniques and the notion of generalized motion by mean curvature.

In this paper we will prove that, at the $\varepsilon = 0$ level, the front moves with speed α in the sense that we will now make precise. To this end we introduce the set

$$G_0 = \{ x \in \mathbb{R}^N \mid u_0(x) > \mu \}, \tag{0.10}$$

and make the assumption

$$\left.\begin{array}{l} \mathbb{R}^N = G_0 \cup \{ x \in \mathbb{R}^N \mid u_0(x) < \mu \} \cup \partial G_0 \\ \text{and} \\ \qquad \partial G_0 \text{ is smooth.} \end{array}\right\} \tag{0.11}$$

With this assumption, $\partial G_0 = \{ x \in \mathbb{R}^N \mid u_0(x) = \mu \}$ is the position of the front at time $t = 0$. A very special case of initial datum is then $u_0 = 1_{G_0} - 1_{G_0^c}$. We also define the signed distance to ∂G_0 by

$$d(x) = \left\{ \begin{array}{ll} d(x, \partial G_0) & \text{if } x \in G_0, \\ -d(x, \partial G_0) & \text{if } x \in G_0^c. \end{array} \right\} \tag{0.12}$$

Finally, throughout the paper we assume

$$-1 \leqq u_0 \leqq 1 \quad \text{in } \mathbb{R}^N. \tag{0.13}$$

Our result is:

THEOREM. — (i) $\lim_{\varepsilon \to 0} u^\varepsilon(x, t) = 1$ *uniformly on compact subsets of*

$$\{(x, t) \in \mathbb{R}^N \times (0, \infty) \mid d(x) > \alpha t\}$$

(ii) $\lim_{\varepsilon \to 0} u^\varepsilon(x, t) = -1$ *uniformly on compact subsets of*

$$\{(x, t) \in \mathbb{R}^N \times (0, \infty) \mid d(x) < \alpha t\}$$

(iii) $u^\varepsilon(x, t) = q((v(x, t) + o(1))/\varepsilon)$ *as* $\varepsilon \to 0$, *where* v *is the unique viscosity solution of*

$$\left.\begin{array}{l} v_t + \beta_+ (|Dv|^2 - 1) + \alpha = 0 \quad \textit{in } \{d(x) > \alpha t, t > 0\}, \\ v_t - \beta_- (|Dv|^2 - 1) + \alpha = 0 \quad \textit{in } \{d(x) < \alpha t, t > 0\}, \\ v(x, t) = 0 \quad \textit{on } \{d(x) = \alpha t\}, \\ v(x, 0) = \begin{cases} +\infty & \textit{if } u_0(x) = 1, \\ -\infty & \textit{if } u_0(x) = -1, \\ 0 & \textit{if } -1 < u_0(x) < 1, \end{cases} \end{array}\right\} \tag{0.14}$$

where $\beta_- = (\alpha - \sqrt{\alpha^2 - 4f'(-1)/2})$ *and* $\beta_+ = (\alpha + \sqrt{\alpha^2 - 4f'(1)/2})$. *Moreover, the limit above is locally uniform on compact subsets of* $\mathbb{R}^N \times (0, \infty)$.

The main consequence of this result is that at time t the position of the front separating the regions where $u^\varepsilon \to 1$ and $u^\varepsilon \to -1$ is given by the equation $d(x) = \alpha t$. The geometrical interpretation of this result is that the front propagates with a constant normal velocity α ([B]).

A slightly less precise result has been already obtained by Gärtner [G] using a combination of probabilistic and PDE techniques. In fact, Gärtner worked on more general parabolic equations

$$u_t^\varepsilon = \varepsilon (a_{ij}(x) u_{x_j}^\varepsilon)_{x_i} + \frac{1}{\varepsilon} f(x, u^\varepsilon) \quad \text{in } \mathbb{R}^N \times (0, \infty),$$

where f again comes from a bistable potential. Here we provide a simpler, purely PDE proof which is less local in character. In addition, we give the precise rate with which u^ε approaches ± 1 via the WKB-type approximation stated in the theorem. The general case has been recently obtained by Barles, Soner and Souganidis [BSS].

Our methods rely on the theory of viscosity solution, which was introduced by Crandall and Lions [CrL]. One of the basic steps in this approach is to make a change of the unknown of the form

$$u^\varepsilon(x, t) = q\left(\frac{v^\varepsilon(x, t)}{\varepsilon}\right). \tag{0.15}$$

Such a transformation is a natural extension of the logarithmic change of variables introduced in the context of stochastic control by Fleming [Fl] and later applied in several contexts involving large deviations for diffusion processes (*cf.* [EI], [FlS], etc.) or problems related to front propagation (*cf.* [ES1], [ES2], [BES]). The main technical tool for our approach was developed in a paper by Barles and Perthame ([BP]) (*see* also Ishii [I]). This tool allows passage to the limit in singular perturbation problems with almost no estimates at all; the compensating factor being the strong uniqueness of the limiting equations. In the case at hand, however, the WKB type change (0.15) gives rise, at the limit, to

$$v_t + (\beta_+ \operatorname{sgn}^+(v) + \beta_- \operatorname{sgn}^-(v))(|Dv|^2 - 1) + \alpha = 0 \quad \text{in } \mathbb{R}^N \times (0, \infty), \qquad (0.16)$$

which possesses strong non-uniqueness properties.

The paper is organized as follows: Section 1 describes a reduction process to the "radial case", *i.e.* when

$$u^\varepsilon(x, 0) = \theta\, 1_B - 1_{B^c}, \qquad (0.17)$$

where $\theta \in (0, 1)$ and B is some ball in $\mathbb{R}^N$. In section 2 we obtain all the information needed to perform the limiting process described in Section 3. Finally, in Section 4 we identify the limit and, thus, conclude the proof of the theorem.

SECTION 1

In this section we reduce the question of the asymptotic behavior, as $\varepsilon \to 0$, of (0.1) to the case where the initial data are given by (0.17).

LEMMA 1.1. — *Assume that the theorem holds for initial data given by* (0.17). *Then the theorem holds for any initial data satisfying the general assumptions.*

Proof. — The proof is based on using solutions associated with initial data like (0.17) as subsolutions of (0.1).

To this end, let $(\bar{x}, \bar{t}) \in \mathbb{R}^N \times (0, \infty)$ be such that $d(\bar{x}, \partial G_0) < \alpha \bar{t}$; there exists $x_0 \in \operatorname{Int}(G_0)$ such that $|\bar{x} - x_0| < \alpha \bar{t}$ and a ball B centered at x_0 such that $u_0 \geqq \theta > 0$ on B. Since $u_0 \geqq -1$ in $\mathbb{R}^N$, we have

$$u_0 \geqq \theta\, 1_B - 1_{B^c} \quad \text{in } \mathbb{R}^N.$$

If $\tilde{u}^\varepsilon$ denotes the solution of (0.1) with initial datum given by the right hand side of the above inequality, the maximum principle yields

$$\tilde{u}^\varepsilon \leqq u^\varepsilon \leqq 1 \quad \text{in } \mathbb{R}^N \times (0, \infty).$$

Moreover, since $B \subset G_0$,

$$d(x, \partial B) \leqq d(x, \partial G_0) < \alpha \bar{t}.$$

The assumptions of the lemma together with the two observations above imply that $u^\varepsilon \to 1$ in a neighborhood of $(\bar{x}, \bar{t})$, since the same holds for $\tilde{u}^\varepsilon$. To prove the second part of the theorem, we argue in the same way but we change u^ε to $-u^\varepsilon$; α is then changed to $-\alpha$ and d to $-d$.

Finally, the last part of the theorem follows by an argument similar to the above, comparing this time the functions v^ε defined by (0.15) with $\tilde{v}^\varepsilon$ defined by $\tilde{u}^\varepsilon = q((\tilde{v}^\varepsilon/\varepsilon))$. Since there is an explicit formula for $\lim_{\varepsilon \to 0} \tilde{v}_\varepsilon$ (which is computed below), the conclusion follows easily. □

LEMMA 1.2. — *Assume that the third part of the theorem holds for initial data given by* (0.17). *Then the other two parts hold too.*

Proof. — This is an immediate consequence of the transformation (0.15), the properties of the traveling wave q and the fact that the solution v of (0.14) is strictly positive when $d(x) > \alpha t$ and strictly negative when $d(x) < \alpha t$. The third fact follows by computing the solution of (0.14) using the Lax-Oleinik formula ([L], [LSV]). This is done by first observing that

$$v(x, t) = v_0(x, t) - \alpha t$$

for small t, where v_0 is the solution of (0.14) with $\alpha = 0$. In the case $\alpha < 0$, this equality is true for all $t > 0$. If $\alpha > 0$, the equality is valid until the region $\{d(x) > \alpha t, t > 0\}$ disappears. Then one computes v_0 using the classical Lax-Oleinik formula.

Another argument consists in applying the maximum principle for viscosity solutions of (0.14). For example, if we consider the region $\{d(x) > \alpha t, t > 0\}$, then the maximum principle yields

$$v(x, t) \geqq 0 \quad \text{in } \{d(x) \geqq \alpha t\},$$

since 0 is a viscosity solution in this region. On the other hand, if there exists a point $(x_0, t_0) \in \{d(x) > \alpha t, t > 0\}$ such that $v(x_0, t_0) = 0$, then (x_0, t_0) is a minimum point of v and from the definition of viscosity solutions, one has $-\beta_+ + \alpha \geqq 0$, which is impossible. □

SECTION 2

The remainder of the paper is devoted to the proof of the theorem in the radial case. For the sake of clarity, we will make few more reductions. Henceforth, we will assume that the nonlinearity f is given by (0.2). This allows us to use the explicit formula of q given by (0.9). The general case follows along the same lines but one needs to use the detailed behavior of q and its derivatives near $\pm\infty$ and to appropriately modify all the arguments. Finally, we will present the whole proof only in the case where

$\alpha=\mu=0$ and we will indicate the minor changes needed when $\alpha\neq 0$ at the end of Section 4.

In this section we obtain, in a somewhat non-standard way, various estimates needed for passing to the limit. Moreover, we will deal with resolving the non-uniqueness feature of (0.16).

PROPOSITION 2.1. — *Let u^ε be the solution of* (0.1) *with f given by* (0.2) *with $\mu=0$ and initial data as in* (0.17). *Then*

$$-t\leqq \liminf_{\substack{\varepsilon\to 0\\ (y,s)\to(x,t)}} \varepsilon\log(1-u^\varepsilon(y,s))\leqq \limsup_{\substack{\varepsilon\to 0\\ (y,s)\to(x,t)}} \varepsilon\log(1-u^\varepsilon(y,s))\leqq 0 \tag{2.1}$$

and

$$-\frac{1}{2}t-\frac{[d(x)^-]^2}{4t}\leqq \liminf_{\substack{\varepsilon\to 0\\ (y,s)\to(x,t)}} \varepsilon\log(1+u^\varepsilon(y,s))$$
$$\leqq \limsup_{\substack{\varepsilon\to 0\\ (y,s)\to(x,t)}} \varepsilon\log(1+u^\varepsilon(y,s))\leqq \frac{1}{2}t-\frac{[d(x)^-]^2}{4t}, \tag{2.2}$$

where d is the signed distance to ∂B and $d(x)^-=-\sup(-d(x),0)$.

One may recognize in (2.1) and (2.2) large deviations type estimates. We may in fact loosely interpret these two estimates as the analogue to the probabilistic part in Gärtner's proof [G]. The proof of Proposition 2.1 is based on ideas introduced in [EI], [FlS], etc. for the treatment of large deviation problems using PDE techniques.

Proof of Proposition 2.1. — We only prove (2.2), since (2.1) follows along the same lines. To this end, we introduce the transformation

$$u^\varepsilon=-1+\exp\left(\frac{w_A^\varepsilon}{\varepsilon}\right)-\exp\left(-\frac{A}{\varepsilon}\right) \tag{2.3}$$

where $A>0$. This exponential change comes up naturally in our context as the tail of the traveling wave near $+\infty$. The term $\exp(-(A/\varepsilon))$ is a correction introduced to assure the *a priori* boundedness of w_A^ε. The initial value problem by w_A^ε is

$$\left.\begin{aligned} &w_{A,t}^\varepsilon-\varepsilon\Delta w_A^\varepsilon-|Dw_A^\varepsilon|^2=\frac{2u^\varepsilon(1-(u^\varepsilon)^2)}{u^\varepsilon+1+\exp(-(A/\varepsilon))}\quad \text{in } \mathbb{R}^N\times(0,\infty),\\ &w^\varepsilon(x,0)=\begin{cases}\varepsilon\log\left(1+\theta+\exp\left(-\dfrac{A}{\varepsilon}\right)\right), & \text{if } x\in B,\\ -A, & \text{if } x\in B^c.\end{cases}\end{aligned}\right\} \tag{2.4}$$

Estimating the right-hand side of the equation in (2.4), we obtain

$$-\frac{1}{2} \leqq w^{\varepsilon}_{A,t} - \varepsilon \Delta w^{\varepsilon}_{A} - |Dw^{\varepsilon}_{A}|^2 \leqq \frac{1}{2} \quad \text{in } \mathbb{R}^N \times (0, \infty).$$

We now introduce the functions

$$\bar{w}_A(x,t) = \limsup_{\substack{\varepsilon \to 0 \\ (y,s) \to (x,t)}} w^{\varepsilon}_A(y,s) \qquad \text{and} \qquad \underline{w}_A(x,t) = \liminf_{\substack{\varepsilon \to 0 \\ (y,s) \to (x,t)}} w^{\varepsilon}_A(y,s).$$

In view of the results of [BP], $\bar{w}_A$ and $\underline{w}_A$ satisfy in the viscosity sense the inequalities

$$\bar{w}_{A,t} - |D\bar{w}_A|^2 \leqq \frac{1}{2} \qquad \text{and} \qquad \underline{w}_{A,t} - |D\underline{w}_A|^2 \geqq -\frac{1}{2}, \quad \text{respectively.}$$

Let z_A be the unique viscosity solution of

$$\left.\begin{aligned} & z_{A,t} - |Dz_A|^2 = 0 \quad \text{in } \mathbb{R}^N \times (0,\infty), \\ & z_A(x,0) = \begin{cases} 0 & \text{if } x \in B, \\ -A & \text{if } x \in B^c, \end{cases} \end{aligned}\right\} \tag{2.5}$$

Using the comparison results of Crandall, Lions and Souganidis [CrLS], which are based on the regularizing properties of (2.5), we obtain

$$z_A(x,t) - \frac{t}{2} \leqq \underline{w}_A(x,t) \leqq z_A(x,t) + \frac{t}{2} \quad \text{in } \mathbb{R}^N \times (0,\infty). \tag{2.6}$$

But z_A is given explicitely by the Lax-Oleinik formula, *i.e.*

$$z_A(x,t) = \sup_{y \in \mathbb{R}^N} \left\{ z_A(y,0) - \frac{|x-y|^2}{4t} \right\}.$$

On the other hand, it is easy to check that

$$\underline{w}_A(x,t) = \sup\{-A, \liminf_{\substack{\varepsilon \to 0 \\ (y,s) \to (x,t)}} \varepsilon \log(1 + u^{\varepsilon}(y,s))\}$$

and

$$\bar{w}_A(x,t) = \sup\{-A, \limsup_{\substack{\varepsilon \to 0 \\ (y,s) \to (x,t)}} \varepsilon \log(1 + u^{\varepsilon}(y,s))\}.$$

We conclude by letting $A \to +\infty$ in the formulae for z_A, $\underline{w}_A$ and $\bar{w}_A$ and by using (2.6). $\square$

Remark. — Proposition 2.1 and, especially, its proof, are a non-standard replacement for L^∞ estimates for the functions v^ε defined by (0.15). Indeed, since

$$\arg th(t) = \frac{1}{2}(\log(t+1) - \log(1-t)),$$

inequalities (2.1) and (2.2) will provide the desired estimates for $\limsup v^\varepsilon$ and $\liminf v^\varepsilon$.

Next we use Proposition 2.1 to obtain some information about the limiting behavior of the u^ε's as $\varepsilon \to 0$.

PROPOSITION 2.2. — *Under the same hypotheses as Proposition* 2.1, $\lim_{\varepsilon \to 0} u^\varepsilon = 1$ *uniformly on compact subsets of the cone*

$$C = \{ (x, t) \in \mathbb{R}^N \times (0, \infty) \mid d(x) > 2\sqrt{2t} \}.$$

Proof. — Consider the function w^ε defined by $w^\varepsilon = (\theta - u^\varepsilon)^+$. Since $t \mapsto t^+$ is a convex function, classical arguments yield that w^ε satisfies

$$w_t^\varepsilon - \varepsilon \Delta w^\varepsilon \leqq -2 u^\varepsilon (1 - (u^\varepsilon)^2) \quad \text{in } \mathbb{R}^N \times (0, \infty). \tag{2.7}$$

Moreover, since $-1 \leqq u^\varepsilon \leqq 1$ by the maximum principle, (2.7) yields

$$w_t^\varepsilon - \varepsilon \Delta w^\varepsilon \leqq 2 w^\varepsilon \quad \text{in } \mathbb{R}^N \times (0, \infty),$$

with

$$w^\varepsilon (x, 0) = \begin{cases} 0 & \text{if } x \in B, \\ 1 + \theta & \text{if } x \in B^c. \end{cases}$$

We now introduce the transformation

$$w^\varepsilon (x, t) = \exp\left(\frac{z^\varepsilon (x, t)}{\varepsilon} \right) - \exp\left(-\frac{A}{\varepsilon} \right).$$

Working along the lines of the proof of Proposition 2.1 we obtain

$$w^\varepsilon (x, t) \leqq \exp\left(\frac{z (x, t) + 0(1)}{\varepsilon} \right),$$

where

$$z (x, t) = -\frac{[d(x)^+]^2}{4t} + 2t.$$

Therefore, $w^\varepsilon \to 0$ as $\varepsilon \to 0$ on compact subsets of C, or, equivalently,

$$\underline{u}(x, t) = \liminf_{\substack{\varepsilon \to 0 \\ (y, s) \to (x, t)}} u^\varepsilon (y, s) \geqq \theta > 0 \quad \text{on } C.$$

On the other hand, taking lim sup on both sides of (0.1), we get

$$-2 \underline{u} (1 - \underline{u}^2) \geqq 0 \quad \text{in } \mathbb{R}^N \times (0, \infty).$$

Combining the last two inequalities yields

$$\underline{u} = 1 \quad \text{on } C.$$

Finally, since $-1 \leqq u^\varepsilon \leqq 1$ in $\mathbb{R}^N \times (0, \infty)$, we have

$$\bar{u}(x, t) = \limsup_{\substack{\varepsilon \to 0 \\ (y, s) \to (x, t)}} u^\varepsilon(y, s) \leqq 1 \quad \text{in } \mathbb{R}^N \times (0, \infty),$$

and, therefore,

$$\underline{u} = \bar{u} = 1 \quad \text{on C}.$$

In view of the definition of $\underline{u}$ and $\bar{u}$, the last equality concludes the proof. □

SECTION 3

In this section we find the equation satisfied by the limit of the functions v^ε (defined by (0.15)) as $\varepsilon \to 0$. Since we do not know *a priori* that this limit exists (recall that we only have L^∞ bounds on v^ε), we need to employ again the lim sup and lim inf passage to the limit. To this end, we define the functions

$$\bar{v}(x, t) = \limsup_{\substack{\varepsilon \to 0 \\ (y, s) \to (x, t)}} v^\varepsilon(y, s) \quad \text{in } \mathbb{R}^N \times (0, \infty) \tag{3.1}$$

and

$$\underline{v}(x, t) = \liminf_{\substack{\varepsilon \to 0 \\ (y, s) \to (x, t)}} v_\varepsilon(y, s) \quad \text{in } \mathbb{R}^N \times (o, \infty), \tag{3.2}$$

and we identify the equations satisfied by them. Before stating the results, we want to remind the reader that we are only treating the case of f as in (0.2) with $\alpha = \mu = 0$.

PROPOSITION 3.1. – *The functions $\bar{v}$ and $\underline{v}$ defined by* (3.1) *and* (3.2) *are locally bounded in* $\mathbb{R}^N \times (0, \infty)$ *and respectively viscosity sub- and super-solutions of* (0.16) *in* $\mathbb{R}^N \times (0, \infty)$. *Moreover, they are respectively viscosity sub- and super-solution of*

$$v_t + 2(|Dv|^2 - 1) = 0 \quad \text{in C}. \tag{3.3}$$

Before we proceed with the proof, we need to explain in what sense is (0.16) satisfied when $v = 0$, *i.e.* when there is a discontinuity in the problem. The notion of viscosity solution for equations involving discontinuities was first given by H. Ishii [I]. In the case of (0.16), this notion is reduced to saying that on the set where $\bar{v}$ and $\underline{v}$ are equal to zero, they are respectively sub- and super-solution of the relaxed problems

$$v_t + \min(|Dv|^2 - 1, 1 - |Dv|^2) \leqq 0$$

and

$$v_t + \max(|Dv|^2 - 1, 1 - |Dv|^2) \geqq 0.$$

A final comment is that the second part of Proposition 3.1 gives better information on the limiting equation in C. This is a very important fact, for it will allow us to deal with the non-uniqueness of (0.16).

Proof of Proposition 3.1. — Since

$$v^\varepsilon = \frac{1}{2}\{\varepsilon \log(1+u^\varepsilon) - \varepsilon \log(1-u^\varepsilon)\},$$

we have

$$\bar{v}(x,t) \leqq \frac{1}{2}\{\limsup_{\substack{\varepsilon \to 0\\(y,s)\to(x,t)}} \varepsilon \log(1+u^\varepsilon(y,s)) - \liminf_{\substack{\varepsilon \to 0\\(y,s)\to(x,t)}} \varepsilon \log(1-u^\varepsilon(y,s))\},$$

which, in view of (2.1) and (2.2), yields

$$\bar{v}(x,t) \leqq \frac{3t}{4} - \frac{[d(x)^-]^2}{8t} \quad \text{in } \mathbb{R}^N \times (0,\infty).$$

Similarly using (2.1)-(2.2) and the properties of lim inf and lim sup, we obtain the estimates

$$\bar{v}(x,t) \geqq -\frac{t}{4} - \frac{[d(x)^-]^2}{8t} \quad \text{in } \mathbb{R}^N \times (0,\infty),$$

$$\underline{v}(x,t) \leqq \frac{3t}{4} - \frac{[d(x)^-]^2}{8t} \quad \text{in } \mathbb{R}^N \times (0,\infty),$$

and

$$\underline{v}(x,t) \geqq -\frac{t}{4} - \frac{[d(x)^-]^2}{8t} \quad \text{in } \mathbb{R}^N \times (0,\infty).$$

We now turn to the passage to the limit. To this end, we recall that v^ε solves the equation

$$v_t^\varepsilon - \varepsilon \Delta v^\varepsilon + 2\,\mathrm{th}\left(\frac{v^\varepsilon}{\varepsilon}\right)(|Dv^\varepsilon|^2 - 1) = 0 \quad \text{in } \mathbb{R}^N \times (0,\infty), \tag{3.4}$$

and we introduce the Hamiltonian H_ε defined by

$$H_\varepsilon(r,p,M) = -\varepsilon\,\mathrm{tr}(M) + 2\,\mathrm{th}\left(\frac{r}{\varepsilon}\right)(|p|^2 - 1),$$

for $r \in \mathbb{R}$, $p \in \mathbb{R}^N$ and $M \in S^N$, the space of $N \times N$ symmetric matrices. Here $\mathrm{tr}(M)$ denotes the trace of M. Employing the results of [BP] (*see* also Barles and Souganidis [BS]), using that v^ε solves the equation

$$v_t^\varepsilon + H_\varepsilon(v^\varepsilon, Dv^\varepsilon, D^2 v^\varepsilon) = 0 \quad \text{in } \mathbb{R}^N \times (0,\infty)$$

and that $\bar{v}$ and $\underline{v}$ are locally bounded in $\mathbb{R}^N \times (0, \infty)$, we obtain that $\bar{v}$ and $\underline{v}$ are respectively viscosity sub- and super-solutions of

$$\bar{v}_t + \underline{H}(\bar{v}, D\bar{v}, D^2 \bar{v}) \leqq 0 \quad \text{in } \mathbb{R}^N \times (0, \infty)$$

and

$$\underline{v}_t + \bar{H}(\underline{v}, D\underline{v}, D^2 \underline{v}) \geqq 0 \quad \text{in } \mathbb{R}^N \times (0, \infty),$$

with $\underline{H}$ and $\bar{H}$ given by

$$\underline{H}(r, p, M) = \liminf_{\substack{\varepsilon \to 0 \\ (s, q, N) \to (r, p, M)}} H_\varepsilon(s, q, N)$$

and

$$\bar{H}(r, p, M) = \limsup_{\substack{\varepsilon \to 0 \\ (s, q, N) \to (r, p, M)}} H_\varepsilon(s, q, N).$$

Now, to conclude the proof of the first part of Proposition 3.1, we only need to compute $\underline{H}$ and $\bar{H}$. It is straightforward that

$$\underline{H}(r, p, M) = \bar{H}(r, p, M) = 2 \operatorname{sgn}(u)(|p|^2 - 1) \qquad \text{if} \quad u \neq 0,$$

while, if $u = 0$,

$$\underline{H}(0, p, M) = \min(|p|^2 - 1, 1 - |p|^2)$$

and

$$\bar{H}(0, p, M) = \max(|p|^2 - 1, 1 - |p|^2).$$

Finally, it remains to prove the last point. To this end, observe that (3.4) may be rewritten as

$$v_t^\varepsilon - \varepsilon \Delta v^\varepsilon + 2 u^\varepsilon (|Dv^\varepsilon|^2 - 1) = 0 \quad \text{in } \mathbb{R}^N \times (0, \infty). \tag{3.5}$$

We want to pass to the limit as $\varepsilon \to 0$ in (3.5) for (x, t) in the cone C. This time, we consider the Hamiltonian

$$\tilde{H}_\varepsilon(x, t, p, M) = -\varepsilon \operatorname{tr}(M) + 2 u^\varepsilon(x, t)(|p|^2 - 1)$$

for $(x, t) \in C$, $p \in \mathbb{R}^N$ and $M \in S^N$. Arguing as before and using Proposition 2.2, we find that $\tilde{H}_\varepsilon$ converges uniformly on compact subsets of $C \times \mathbb{R}^N \times S^N$ to the Hamiltonian

$$\tilde{H}(x, t, p, M) = 2(|p|^2 - 1) \quad \text{in } C \times \mathbb{R}^N \times S^N.$$

Hence, $\bar{v}$ and $\underline{v}$ are respectively viscosity sub- and super-solutions of (3.3). $\square$

SECTION 4

In this section we identif the limit of the v^ε's as $\varepsilon \to 0$.

PROPOSITION 4.1. — *Let* $\underline{v}$ *and* $\bar{v}$ *be defined by* (3.1) *and* (3.2) *respectively. Then*

$$\underline{v}=\bar{v}=v \quad in \ \mathbb{R}^N \times (0, \infty), \tag{4.1}$$

where

$$v(x,t)=\begin{cases} 2t & if \ \ d(x) \geqq 2t, \\ d(x) & if \ \ -4t \leqq d(x) \leqq 2t, \\ -\dfrac{[d(x)]^2}{8t} - 2t & if \ \ d(x) \leqq -4t. \end{cases} \tag{4.2}$$

Before we give the proof of Proposition 4.1, we conclude the proof of the theorem in the radial case, when $\alpha = 0$.

Proof of Theorem. — It is immediate that (4.1) yields the uniform convergence of v^ε to v on compact subsets of $\mathbb{R}^N \times (0, \infty)$. Therefore,

$$u^\varepsilon(x,t) = \operatorname{th}\left(\frac{v(x,t)+o(1)}{\varepsilon}\right) \quad \text{in } \mathbb{R}^N \times (0, \infty), \tag{4.3}$$

where $o(1)$ is uniform on compact subsets of $\mathbb{R}^N \times (0, \infty)$. Now since $v>0$ in $\{d>0\}$ and $v<0$ in $\{d<0\}$, (4.3) implies the local uniform convergence of u^ε to -1 in $\{d<0\}$ and to 1 in $\{d>0\}$. Finally, the uniqueness of v follows as in [CrLS]. $\square$

Proof of Proposition 4.1. — We begin by noting that we cannot use any standard comparison result (based on viscosity solutions) for (0.16). To see this, let

$$H(u,p) = 2 \operatorname{sgn}(u)(|p|^2 - 1)$$

and observe there is no $\gamma \in \mathbb{R}$ such that the map $u \mapsto H(u,p) + \gamma u$ is nondecreasing for all $p \in \mathbb{R}^N$, the latter being a necessary condition for uniqueness. Indeed, for $|p|<1$, we have formally $(\partial H/\partial u)\ (0,p) = -\infty$. However, when $|p| \geqq 1$, the map $u \mapsto H(u,p)$ is nondecreasing. The main idea of the proof is to build ad hoc sub- and super-solutions of (0.16), with gradients having norms bigger than 1 in an as large as possible subset of $\mathbb{R}^N \times (0, \infty)$. In fact, we really want this outside C, since in the cone C we have additional information regarding the equation satisfied by $\underline{v}$ and $\bar{v}$.

We will organize the construction of the above mentioned auxiliary functions in a series of lemmas. We present their proofs at the end of the section.

LEMMA 4.2. — *The function* $w(x, t) = d(x) - \alpha t$ *is a viscosity supersolution of* (0.16) *in* $\mathbb{R}^N \times (0, \infty)$. *Moreover, it is a solution of*

$$|D_x w|^2 = 1 \quad \text{in } \mathbb{R}^N \times (0, \infty). \tag{4.3}$$

We stated this lemma also for the non-zero speed cases, because this will be the only additional argument needed to treat, in particular, the $\alpha > 0$ case.

LEMMA 4.3. — *For all* $\theta \geqq 0$ *and* $\theta < \beta < 2 + \theta$, *the functions defined by*

$$v_{\theta, \beta}(x, t) = \begin{cases} (\beta - \theta)\, t & \text{if } d(x) \geqq \beta t, \\ d(x) - \theta t & \text{if } -4t \leqq d(x) \leqq \beta t, \\ -\dfrac{[d(x)]^2}{8t} - (2+\theta)\, t & \text{if } d(x) < 0, \end{cases}$$

are viscosity subsolutions of (0.16) *in* $\mathbb{R}^N \times (0, t_\theta)$, *where* $t_\theta = (1/\theta) \max_{x \in \mathbb{R}^N} d(x)$.

The main property of the subsolution in Lemma 4.3 is that

$$|Dv_{\theta, \beta}| = 1 \qquad \text{if } d(x) > \beta t.$$

Now we continue with the proof of the proposition. First we compare the subsolution $\bar{v}$ with the supersolution w (with $\alpha = 0$). Using $\bar{v}(.,0) \leqq w(.,0)$ in $\mathbb{R}^N$, the fact that w satisfies (4.3) together with the upper bound for $\bar{v}$ obtained in Proposition 3.2 (needed here to take care of difficulties when $|x| \to \infty$), we can apply standard comparison arguments (*cf.* [CrIL]) to get

$$\bar{v}(x, t) \leqq w(x, t) = d(x) \quad \text{in } \mathbb{R}^N \times (0, \infty). \tag{4.4}$$

In particular $\bar{v}(x, t) < 0$ in $\{(x, t) \in \mathbb{R}^N \times (0, \infty) \mid d(x) < 0\}$.
Next, we turn to the comparison of $\underline{v}$ and $v_{\theta, \beta}$ for some θ, β. We choose $\theta = 1$ and $\beta = 3$. Using the lower bound for $\underline{v}$ obtained in Proposition 3.2, we see that

$$v_{1,3}(x, t) - \underline{v}(x, t) \leqq C t \quad \text{in } \mathbb{R}^N \times (0, t_1), \tag{4.5}$$

for some constant $C > 0$. Moreover, $|Dv_{1,3}(x, t)| \geqq 1$ if $d(x) < 3t$, and, hence,

$$|Dv_{1,3}| \geqq 1 \quad \text{in a neighborhood of } C^c.$$

Employing again standard comparison results for viscosity solutions we obtain

$$v_{1,3}(x, t) \leqq \underline{v}(x, t) \quad \text{in } \mathbb{R}^N \times (0, t_1). \tag{4.6}$$

In particular, $\underline{v}(x, t) > 0$ if $d(x) > t$, which yields

$$\underline{v}_t + |D\underline{v}|^2 - 1 \geqq 0 \quad \text{in } C_1 = \{(x, t) \in \mathbb{R}^N \times (0, \infty) \mid d(x) > t\}.$$

We repeat now the above argument using C_1 in place of C and $v_{0,\,3/2}$ in place of $v_{1,\,3}$. We obtain

$$v_{0,\,3/2}(x,t) \leqq \underline{v}(x,t) \quad \text{in } \mathbb{R}^N \times (0,\infty), \tag{4.7}$$

which implies

$$\underline{v}(x,t) > 0 \quad \text{in } \{(x,t) \in \mathbb{R}^N \times (0,\infty) \,|\, d(x) > 0\}.$$

Combining (4.4), (4.6) and the inequality $\underline{v} \leqq \bar{v}$, which holds by the very definition of $\underline{v}$ and $\bar{v}$, we deduce that $\bar{v}$ and $\underline{v}$ are respectively viscosity sub- and super-solutions of (0.14) with $\alpha = 0$. Applying again standard comparison results we obtain

$$\bar{v} \leqq v \leqq \underline{v} \quad \text{in } \mathbb{R}^N \times (0,\infty),$$

since v is the solution of (0.14) computed by the Lax-Oleinik formula.

The last inequality clearly implies $\underline{v} = \bar{v} = v$ in $\mathbb{R}^N \times (0,\infty)$. □

Proof of Lemma 4.2. – The function w is concave in (x, t). Therefore, for $\varphi \in C^1(\mathbb{R}^N \times (0,\infty))$, if $(x_0, t_0) \in \mathbb{R}^N \times (0,\infty)$ is a minimum point of $w - \varphi$, then w is differentiable at (x_0, t_0). With this observation the rest of the proof is standard and we will omit it. □

Proof of Lemma 4.3. – Let z_β be the unique solution of

$$\begin{gathered} z_t + \beta(|Dz|^2 - 1) = 0 \quad \text{in } B \times (0,\infty), \\ z(x,t) = 0 \quad \text{on } \partial B \times (0,\infty), \\ z(x,0) = 0 \quad \text{in } B. \end{gathered}$$

It is easy to check that

$$z_\beta(x,t) = \inf(d(x), \beta t) \quad \text{in } \mathbb{R}^N \times (0,\infty),$$

hence,

$$v_{\theta,\,\beta}(x,t) = z_\beta(x,t) - \theta t \quad \text{in } B \times (0,\infty). \tag{4.9}$$

The claim of the lemma can be easily checked at every point of differentiability of $v_{\theta,\,\beta}$ by simply computing the derivatives. On the other hand $v_{\theta,\,\beta}$ is positive in a neighborhood of $\{d(x) = \beta t\}$ and solves [by (4.9)]

$$v_t + \beta(|Dv|^2 - 1) + \theta = 0.$$

Hence

$$v_t + 2(|Dv|^2 - 1) = (2 - \beta)|Dv|^2 + (\beta - \theta) - 2.$$

The right hand side of the equality above is non negative. Indeed, if $\beta > 2$ this follows from the assumption $\beta \leqq \theta + 2$, while if $\beta < 2$,

$$(2 - \beta)|Dv|^2 + (\beta - \theta) - 2 \leqq (2 - \beta) + (\beta - \theta) - 2 = -\theta < 0,$$

since $|Dv| \leqq 1$. The same argument holds at the point (x_0, t_0), where x_0 is the center of the ball B, as long as $v_{\theta,\,\beta}(x_0, t) > 0$ *i.e.* for $t \leqq t_\theta = (1/\theta) \max_{x \in \mathbb{R}^N} d(x)$. □

We conclude this section with a short discussion of the case of non-zero speed. The proof follows along the same lines, the only difference being that, in the case where $\alpha>0$, the region where $u^\varepsilon(.,t)\to 1$ disappears in finite time. Indeed, if $\alpha<0$, it is easy to check that

$$v_\varepsilon(x,t)\to v(x,t)-\alpha t \quad \text{locally uniformly in } \mathbb{R}^N\times(0,\infty),$$

where v is given in Proposition 4.1. Roughly speaking, this is due to the fact that the region where the sign of $v(x,t)-\alpha t$ is different from the sign of $v(x,t)$, is included in the region where $|Dv|=1$.

If $\alpha>0$, this result remains true only up to the time $t_\alpha=1/\alpha \max\limits_{x\in\mathbb{R}^N} d(x)$.

Past this time, we know by Lemma 4.2, that

$$\limsup_{\varepsilon\to 0} v^\varepsilon(x,t)<0 \quad \text{in } \mathbb{R}^N\times(t_\alpha,+\infty).$$

It is then easy to check that

$$v^\varepsilon\to\tilde{v} \quad \text{locally uniformly in } \mathbb{R}^N\times(t_\alpha,+\infty),$$

where $\tilde{v}$ is the unique solution of

$$\tilde{v}_t-2(|D\tilde{v}|^2-1)+\alpha=0 \quad \text{in } \mathbb{R}^N\times(t_\alpha,+\infty)$$
$$\tilde{v}(x,t_\alpha)=v(x,t_\alpha)-\alpha t_\alpha \quad \text{in } \mathbb{R}^N. \quad \square$$

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