ANNALES DE L'I. H. P., SECTION C

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Annales de l'I. H. P., section C, tome 9, nº 4 (1992), p. 465-477
<http://www.numdam.org/item?id=AIHPC_1992__9_4_465_0>

Ann. Inst. Henri Poincaré,

Vol. 9, n° 4, 1992, p. 465-478. *Analyse non linéaire*

Radially symmetric solutions of a class of problems of the calculus of variations without convexity assumptions

by

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ABSTRACT. — We show that the functional

$$I(u)=\int_B a(|x|)\,u(x)\,dx+\int_B h(|x|,\Delta u(x)-\lambda u(x))\,dx$$

attains a minimum on the space $W_0^{2,p}(B)$.

Key words : Calculus of Variations, convex functionals, rotation group, radially symmetric solutions.

RÉSUMÉ. — Nous montrons que le fonctionnel

$$I(u)=\int_B a(|x|)\,u(x)\,dx+\int_B h(|x|,\Delta u(x)-\lambda\,u(x))\,dx$$

atteint le minimum sur l'espace $W_0^{2,p}(B)$.

Classification A.M.S. : 49 A 22.

1. INTRODUCTION

In this paper, we deal with a class of integrals of the Calculus of Variations without convexity assumptions on their integrands. More precisely, we consider problems of the form:

$$\left.\begin{array}{c} \textit{Minimize} \displaystyle\int_{B} g(|x|, u(x))\, dx + \int_{B} h(|x|, \Delta u(x) - \lambda u(x))\, dx \\ u \in W^{2,p}(B) \cap W_0^{1,p}(B) \\ \dfrac{\partial u}{\partial n} = 0 \quad \text{on } \partial B, \end{array}\right\} \qquad (P)$$

where B is the unit ball and λ is a non negative number, and we seek radially symmetric solutions. Our existence result concerns the case when $g(|x|, u) = a(|x|)\, u$.

In order to gain insight into the mathematical difficulties raised by this type of problems, it is worthwhile to consider first the case $g \equiv 0$. For this case, let us remark that the problem of seeking the minimum on the larger space $W^{2,p} \cap W_0^{1,p}$ offers no difficulty. In fact, it is enough, in this case, to take any (symmetric) selection σ, in L^p, from the map $x \mapsto \operatorname{argmin}\{h(|x|, .)\}$ and consider the (symmetric) solution u_1 to the Dirichlet problem

$$\left.\begin{array}{c} \Delta u - \lambda u = \sigma(|x|) \\ u = 0 \quad \text{on } \partial B. \end{array}\right\} \qquad (DP)$$

Then u_1 is a solution to the given minimization problem. Hence, in general, this problem admits several solutions, obtained simply as solutions to Dirichlet problems. However this procedure cannot be used for the same minimization problem under the additional condition $\dfrac{\partial u}{\partial n} = 0$ on ∂B, since the corresponding Dirichlet problem would be overdetermined, so that, even for this case, a more complex approach is needed.

There are many papers devoted to the existence of solutions to problem (P) that avoid the convexity assumption on h: however all of them prove existence of solutions by imposing conditions implying that every solution to problem (P**), *i.e.* problem (P) where h is replaced by h^{**}, is in fact a solution to problem (P). The method of proof goes by showing that along any solution to (P**), the functions h and h^{**} have to coincide almost everywhere, otherwise the Euler-Lagrange equation would be violated. This method cannot possibly be applied to cases where there are solutions to (P**) that are not solutions to (P): in particular it cannot be applied to the simple case when $g \equiv 0$ and $\lambda = 0$, because, in general, (P**) has solutions that are not solutions to (P). As an example, take $n = 2$,

$h(s)=i(s)$, i the indicator function of the set $\{-1,+1\}$. A computation shows that the function u_2 defined by

$$u_2(x)=\begin{cases}\dfrac{1}{2}\left(-\dfrac{|x|^2}{2}+\log\sqrt{2}\right), & \text{if } |x|\leqq\dfrac{1}{\sqrt{2}}\\ \dfrac{1}{2}\left(\dfrac{|x|^2}{2}-\log|x|\right)-\dfrac{1}{4}, & \text{if } \dfrac{1}{\sqrt{2}}<|x|\leqq 1,\end{cases}$$

satisfies the boundary conditions: $u_2=0$ and $\dfrac{\partial u_2}{\partial n}=0$, on ∂B and has a Laplacian taking values either $+1$ or -1, *i.e.* u_2 is a solution to the original problem. However, the convexified problem, where h^{**} is the indicator function of the interval $[-1,+1]$ has, among others, the solution u_3 identically zero, *i.e.*, in this simple case, there are solutions to the relaxed problem that are not solutions to the original problem. Moreover, since the method mentioned above uses the Euler-Lagrange equation pointwise, some regularity conditions on the functions g and h have to be imposed. In that spirit Aubert-Tahraoui [A-T2], in case h independent of x and $g(x,.)$ convex, proposed a method based on Duality Theory as presented in [E-T], by generalizing their earlier idea in dimension one (*see* [A-T1]), where the required hypothesis was $g_u(x,u)\neq 0$. Raymond ([R], Annexe 1) gives a direct proof by the Euler-Lagrange equation, by imposing the more general condition, already considered by Aubert-Tahraoui [A-T2],

$$\sum_{i=1}^{n} h^{**}_{sx_i x_i}(x,s)+g_u(x,u)\neq 0,$$

for the case h depending on x. These papers seek the minimum on the space $W_0^{2,p}$. An existence result in the space $W^{2,p}\cap W_0^{1,p}$ has been given in [T]. Our result, that contains the case $g\equiv 0$ and allows h to be lower semicontinuous in s and measurable in (x,s), neither contains nor is contained in any of the papers mentioned above.

Liapunov's theorem has been used as a tool to prove existence of a solution for a different minimum problem in [C-C]. In this paper we had, in particular, to extend the applicability of this theorem to a more complex operator and boundary conditions.

2. NOTATIONS AND PRELIMINARIES

Throughout this paper, n is an integer, p is a real number such that $2\leqq n<p$. B is the unit ball of $\mathbf{R}^n$ with boundary ∂B. For fixed $\lambda\geqq 0$ we equip the space $W^{2,p}(B)\cap W_0^{1,p}(B)$ with the norm $\|\Delta u-\lambda u\|_{L^p}$.

$\mathbf{SO(n)}$ denotes the Rotation Group in $\mathbf{R^n}$ which has as elements the orthogonal matrices $A \in M(n)$ such that $\det(A)=1$: it is a compact and connected topological group (*see* [DNF]). Therefore, given $u \in W^{2,p}(B)$, the integral

$$\int_{SO(n)} u(Ax)\, d\mu(A)$$

is well defined, where μ is a left (or right) Haar measure on $\mathbf{SO(n)}$ with $\mu(\mathbf{SO(n)})=1$ (*see* [C]). By the definition of $\mathbf{SO(n)}$, we have that its elements preserve the inner product, *i.e.*

$$\langle Ax, Ay \rangle = \langle x, y \rangle, \qquad \forall A \in \mathbf{SO(n)}.$$

Hence $|Ax|=|x|$. Furthermore, fixed $x \in \mathbf{R^n}$, it is not difficult to show that:

$$\{Ax \in \mathbf{R^n} : A \in \mathbf{SO(n)}\} = \{y \in \mathbf{R^n} : |y|=|x|\}.$$

We have the following

PROPOSITION 1. – *Let* $u \in W^{2,p}(B) \cap W_0^{1,p}(B)$ *such that* $\dfrac{\partial u}{\partial n}=0$ *on* ∂B. *Define* $\bar{u}: B \to \mathbf{R}$ *by*

$$\bar{u}(x) = \int_{SO(n)} u(Ax)\, d\mu(A).$$

Then

(*a*) $\bar{u} \in W^{2,p}(B) \cap W_0^{1,p}(B)$;

(*b*) $\dfrac{\partial \bar{u}}{\partial n} = 0$ *on* ∂B;

(*c*) $\bar{u}(.)$ *is radially symmetric*;

(*d*) $\Delta \bar{u}(x) = \displaystyle\int_{SO(n)} \Delta u(Ax)\, d\mu(A)$.

Proof. – Since $|Ax|=|x|$ we have that $\bar{u}$ vanishes on ∂B since u does so. We use Tonelli-Fubini's theorem (*see* [C]) to prove that $\bar{u} \in W^{2,p}(B)$. Let $A \in M(n)$, $A=(a_{ij})$, $|x|=1$

$$\begin{aligned}\frac{\partial \bar{u}}{\partial n}(x) &= \langle \nabla \bar{u}(x), x \rangle \\ &= \sum_{j=1}^{n} \frac{\partial \bar{u}}{\partial x_j}(x)\, x_j = \sum_{j=1}^{n} \int_{SO(n)} \sum_{i=1}^{n} \frac{\partial u}{\partial \xi_i}(Ax)\, a_{ij} x_j\, d\mu(A) \\ &= \int_{SO(n)} \sum_{i=1}^{n} \frac{\partial u}{\partial \xi_i}(\xi)\, \xi_i\, d\mu(A) = \int_{SO(n)} \langle \nabla u(\xi), \xi \rangle\, d\mu(A)\end{aligned}$$

where $\xi = A x$ and $\xi_i = \sum_{j=1}^{n} a_{ij} x_j$. Since $|\xi| = |x|$: claim (*b*) is proved. (*c*) follows from the previous remark and (*d*) is a consequence of the definition of $\mathbf{SO}(\mathbf{n})$. ■

Let I be any interval in $\mathbf{R}$ and denote by $\mathscr{L}$ the σ-algebra of (Lebesgue) measurable subsets of I and, by $\mathscr{B}(\mathbf{R})$ the σ-algebra of $\mathbf{R}$. We denote by $\mathscr{L} \otimes \mathscr{B}(\mathbf{R})$ the product σ-algebra on $I \times \mathbf{R}$ generated by all the sets of the form $A \times B$ with $A \in \mathscr{L}$ and $B \in \mathscr{B}(\mathbf{R})$. We recall that a function $f: I \times \mathbf{R} \to \bar{\mathbf{R}}$ is called $\mathscr{L} \otimes \mathscr{B}(\mathbf{R})$-measurable or simply measurable if the inverse image under f of every closed subset of $\bar{\mathbf{R}}$ is measurable.

Let $\xi \mapsto h^{**}(r, \xi)$ be the bipolar of the function $\xi \mapsto h(r, \xi)$. We have the following

PROPOSITION 2 ([E-T] Prop. I.4.1; Lemma IX.3.3; Prop. IX.3.1). – (*a*) *$h^{**}(r, \xi)$ is the largest convex (in ξ) function not larger than $h(r, \xi)$.*
(*b*) *Let $h: I \times \mathbf{R} \to \bar{\mathbf{R}}$ be such that:*

(h_1) *h is $\mathscr{L} \otimes \mathscr{B}(\mathbf{R})$-measurable;*

(h_2) *$\xi \mapsto h(r, \xi)$ is lower semicontinuous for almost all r in I;*

(h_3) *there exists a positive constant α_1 such that $h(r, \xi) \geqq \alpha |\xi|^p - \beta(r)$, where the function $r \mapsto r^{n-1} \beta(r)$ is in $L^1(I)$. Then*

$$h^{**}(r, \xi) = \min \left\{ \sum_{i=1}^{2} \lambda_i h(r, \xi_i) : \xi = \sum_{i=1}^{2} \lambda_i \xi_i;\ \lambda_i \geqq 0;\ \sum_{i=1}^{2} \lambda_i = 1 \right\}.$$

(*c*) *Let $z(.)$ be measurable. Then there exist measurable $p_i: I \to [0, 1]$ and measurable $v_i: I \to \mathbf{R}$, $i = 1, 2$, such that:*

$$\sum_{i=1}^{2} p_i(r) = 1;\ z(r) = \sum_{i=1}^{2} p_i(r) v_i(r);\ h^{**}(r, z(r)) = \sum_{i=1}^{2} p_i(r) h(r, v_i(r)).$$

Finally, we state a version of the famous Liapunov's theorem.

PROPOSITION 3 ([Ce] 16.1.V). – *Let A be a measurable subset of $\mathbf{R}^n$ with finite (Lebesgue) measure, let $f_1, \ldots, f_k$ be a integrable functions from A to $\mathbf{R}^m$ and let $p_1, \ldots, p_k$ be a measurable functions from A to [0, 1] such that:*

$$\sum_{i=1}^{k} p_i(x) = 1 \quad \text{a.e.} \quad on\ A.$$

Then, there exists a measurable partition of A, $(A_i)_i$, $i = 1, \ldots, k$ such that

$$\sum_{i=1}^{k} \int_A p_i(x) f_i(x)\, dx = \sum_{i=1}^{k} \int_{A_i} f_i(x)\, dx.$$

3. MAIN RESULT

We shall assume the following hypothesis.

HYPOTHESES (H). — Set I to be [0, 1]. The map $a: \mathrm{I} \to \mathbf{R}$ is such that $r \mapsto r^{n-1} a(r)$ is in $\mathrm{L}^{p'}(\mathrm{I})$ with p' the exponent conjugate to p. The map $h: \mathrm{I} \times \mathbf{R} \to \bar{\mathbf{R}}$ is such that

(h_1) h is $\mathscr{L} \otimes \mathscr{B}(\mathbf{R})$-measurable;

(h_2) $\xi \mapsto h(r, \xi)$ is lower semicontinuous for almost all r in I.

Moreover:

(h_3) there exists a positive constant α, such that $h(r, \xi) \geqq \alpha |\xi|^p - \beta(r)$ where the function $r \mapsto r^{n-1} \beta(r)$ is in $\mathrm{L}^1(\mathrm{I})$.

THEOREM 1. — *Let h and a satisfy hypothesis* (H) *and λ be non-negative. Assume that the functional $\int_{\mathrm{B}} h(|x|, \Delta u(x) - \lambda u(x))\, dx$ has a finite value for some u in $\mathrm{W}^{2,p}(\mathrm{B}) \cap \mathrm{W}_0^{1,p}(\mathrm{B})$ such that $\frac{\partial u}{\partial n} = 0$ on $\partial \mathrm{B}$. Then the problem*

$$\left.\begin{aligned} &\textit{Minimize} \int_{\mathrm{B}} a(|x|) u(x)\, dx + \int_{\mathrm{B}} h(|x|, \Delta u(x) - \lambda u(x))\, dx \\ &u \in \mathrm{W}^{2,p}(\mathrm{B}) \cap \mathrm{W}_0^{1,p}(\mathrm{B}), \\ &\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial \mathrm{B}, \end{aligned}\right\} \qquad (\mathrm{P}_0)$$

admits at least one radially symmetric solution.

Proof. — In (*a*) below we show that the relaxed problem admits at least one radially symmetric solution; in (*b*) we write several functions as convex combinations and apply Liapunov's theorem to begin defining a candidate for a solution to the original problem; (*c*) is a technical integrability result and in (*d*) we complete the construction of the solution.

(*a*) We consider the relaxed problem

$$\left.\begin{aligned} &\textit{Minimize} \int_{\mathrm{B}} a(|x|) u(x)\, dx + \int_{\mathrm{B}} h^{**}(|x|, \Delta u(x) - \lambda u(x))\, dx \\ &u \in \mathrm{W}^{2,p}(\mathrm{B}) \cap \mathrm{W}_0^{1,p}(\mathrm{B}), \\ &\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial \mathrm{B}, \end{aligned}\right\} \qquad (\mathrm{P}_0^{**})$$

Clearly, h^{**} satisfies the growth condition (h_3), therefore a well known result (*see* [E-T]) assures that problem (P_0^{**}) has a solution $\hat{u}$. We claim that we can assume the function $\hat{u}$ to be radially symmetric. If it is not so, we can consider the function $\bar{u}: \mathrm{B} \to \mathbf{R}$ defined by

$$\bar{u}(x) = \int_{\mathrm{SO}(n)} \hat{u}(\mathrm{A}x)\, d\mu(\mathrm{A}) \qquad (1)$$

instead of $\hat{u}$, which by Proposition 1 belongs to $W^{2,p}(B) \cap W_0^{1,p}(B)$, is such that $\dfrac{\partial \bar{u}}{\partial n}=0$ on ∂B and $\bar{u}(x)=\bar{u}(|x|)$. Let us show that $\bar{u}$ is another solution to problem (P_0^{**}). Jensen inequality and (*d*) of Proposition 1 imply

$$h^{**}(|x|, \Delta\bar{u}(x)-\lambda\bar{u}(x)) \leqq \int_{SO(n)} h^{**}(|x|, \Delta\hat{u}(Ax)-\lambda\hat{u}(Ax))\, d\mu(A) < +\infty. \quad (2)$$

Moreover, using Tonelli-Fubini's theorem (*see* [C]) we have

$$\int_B \int_{SO(n)} h^{**}(|x|, \Delta\hat{u}(Ax)-\lambda\hat{u}(Ax))\, d\mu(A)\, dx = \int_{SO(n)} \int_B h^{**}(|x|, \Delta\hat{u}(Ax)-\lambda\hat{u}(Ax))\, dx\, d\mu(A),$$

but

$$\int_B h^{**}(|x|, \Delta\hat{u}(Ax)-\lambda\hat{u}(Ax))\, dx = \int_B h^{**}(|y|, \Delta\hat{u}(y)-\lambda\hat{u}(y))\, dy, \qquad \forall A \in \mathbf{SO(n)},$$

so that

$$\int_B \int_{SO(n)} h^{**}(|x|, \Delta\hat{u}(Ax)-\lambda\hat{u}(Ax))\, d\mu(A)\, dx = \int_{SO(n)} \int_B h^{**}(|x|, \Delta\hat{u}(Ax)-\lambda\hat{u}(Ax))\, dx\, d\mu(A) = \int_B h^{**}(|y|, \Delta\hat{u}(y)-\lambda\hat{u}(y))\, dy.$$

Hence, from (2) it follows that

$$\int_B h^{**}(|x|, \Delta\bar{u}(x)-\lambda\bar{u}(x))\, dx \leqq \int_B h^{**}(|y|, \Delta\hat{u}(y)-\lambda\hat{u}(y))\, dy$$

Similarly we can show that

$$\int_B a(|x|)\,\bar{u}(x)\, dx = \int_B a(|x|)\,\hat{u}(x)\, dx,$$

i.e. $\bar{u}$ is a radially symmetric solution to problem (P_0^{**}).

(*b*) Using spherical coordinates we obtain

$$\int_B h^{**}(|x|, \Delta\hat{u}(x) - \lambda\,\hat{u}(x))\,dx = n\,\omega_n \int_0^1 r^{n-1} h^{**}\left(r, \hat{u}''(r) + \frac{n-1}{r}\hat{u}'(r) - \lambda\,\hat{u}(r)\right) dr \quad (3)$$

where ω_n denotes the volume of the unit ball B. By (*c*) of Proposition 2 there exist measurable functions p_i and v_i, $i=1,2$; such that

$$\sum_{i=1}^{2} p_i(r) = 1;\, p_i(r) \geqq 0, \qquad i=1,2; \quad (4)$$

$$\sum_{i=1}^{2} p_i(r)\,v_i(r) = \hat{u}''(r) + \frac{n-1}{r}\hat{u}'(r) - \lambda\,\hat{u}(r); \quad (5.a)$$

$$\sum_{i=1}^{2} p_i(r)\,h(r, v_i(r)) = h^{**}\left(r, \hat{u}''(r) + \frac{n-1}{r}\hat{u}'(r) - \lambda\,\hat{u}(r)\right). \quad (5.b)$$

On the other hand by Lusin's theorem there exist a sequence $(K_j)_j$ of disjoint compact subsets of I and a null set N, such that $I = N \cup (\bigcup_j K_j)$ and the restriction of each of the maps $r \mapsto h(r, v_i(r))$ to each K_j is continuous.

Consider the two functions φ and ψ, both belonging to

$$W^{2,p}(B) \cap W_0^{1,p}(B),$$

where φ is the (radially symmetric, *see* for instance [G-T]) solution to the Dirichlet problem

$$\left.\begin{array}{l} \Delta\varphi - \lambda\varphi = 1 \\ \varphi = 0 \quad \text{on } \partial B. \end{array}\right\} \quad (6)$$

and ψ is the (radially symmetric) solution to the problem

$$\left.\begin{array}{l} \Delta\psi - \lambda\psi = a(|x|) \\ \psi = 0 \quad \text{on } \partial B. \end{array}\right\} \quad (7)$$

At this point we apply Liapunov's theorem to construct from $\hat{u}$ a new function u, that will be a solution to the original problem.

By Proposition 3 there exists a measurable partition of each K_j, $(E_{i,j})_i$, $i=1,2$, such that: for every j,

$$\sum_{i=1}^{2} \int_{K_j} p_i(r)\,r^{n-1} h(r, v_i(r))\,dr = \sum_{i=1}^{2} \int_{K_j} \chi_{E_{i,j}}(r)\,r^{n-1} h(r, v_i(r))\,dr; \quad (8)$$

$$\sum_{i=1}^{2} \int_{K_j} p_i(r)\,r^{n-1} v_i(r)\,dr = \sum_{i=1}^{2} \int_{K_j} \chi_{E_{i,j}}(r)\,r^{n-1} v_i(r)\,dr; \quad (9)$$

$$\sum_{i=1}^{2}\int_{K_j} p_i(r)\,r^{n-1}\,v_i(r)\,\varphi(r)\,dr=\sum_{i=1}^{2}\int_{K_j}\chi_{E_{i,j}}(r)\,r^{n-1}\,v_i(r)\,\varphi(r)\,dr; \tag{10}$$

$$\sum_{i=1}^{2}\int_{K_j} p_i(r)\,r^{n-1}\,v_i(r)\,\psi(r)\,dr=\sum_{i=1}^{2}\int_{K_j}\chi_{E_{i,j}}(r)\,r^{n-1}\,v_i(r)\,\psi(r)\,dr. \tag{11}$$

(*c*) We claim first that the map

$$r\mapsto\sum_{j=1}^{\infty}\sum_{i=1}^{2}\chi_{E_{i,j}}(r)\,r^{n-1}\,h(r,v_i(r))$$

belongs to $L^1(I)$. If it is so,

$$\Big|\sum_{i,j}\chi_{E_{i,j}}(r)\,r^{(n-1)/p}\,v_i(r)\Big|^p=\sum_{i,j}\chi_{E_{i,j}}(r)\,r^{n-1}\,|v_i(r)|^p$$

$$\leqq\frac{1}{\alpha}\sum_{i,j}\chi_{E_{i,j}}(r)\,r^{n-1}\,(h(r,v_i(r))+\beta(r)),$$

i.e. the map

$$r\mapsto\sum_{i,j}\chi_{E_{i,j}}(r)\,r^{(n-1)/p}\,v_i(r)$$

belongs to $L^p(I)$ or, equivalently, the map

$$x\mapsto\sum_{i,j}\chi_{E_{i,j}}(|x|)\,v_i(|x|) \tag{12}$$

belongs to $L^p(B)$. To prove the previous claim, first, notice that, from (5.*b*), the map

$$r\mapsto\sum_{i}p_i(r)\,r^{n-1}\,h(r,v_i(r))$$

is integrable. On the other hand the sequence of maps

$$z_m(r)=\sum_{j\leqq m}\Big\{\sum_{i}\chi_{E_{i,j}}(r)\,r^{n-1}\,(h(r,v_i(r))+\beta(r))\Big\}$$

is monotone non decreasing and

$$\int_0^1 z_m(r)\,dr=\sum_{j\leqq m}\int_{K_j}\sum_{i}\chi_{E_{i,j}}(r)\,r^{n-1}\,(h(r,v_i(r))+\beta(r))\,dr.$$

Set $S_m=\bigcup_{j\leqq m}K_j$. By (8), the right hand side equals

$$\sum_{j\leqq m}\int_{K_j}\sum_{i}p_i(r)\,r^{n-1}\,(h(r,v_i(r))+\beta(r))\,dr$$

$$=\int_0^1\chi_{S_m}(r)\,r^{n-1}\left(h^{**}\left(r,\hat{u}''(r)+\frac{n-1}{r}\hat{u}'(r)-\lambda\,\hat{u}(r)\right)+\beta(r)\right)dr$$

$$\leqq\int_0^1 r^{n-1}\left(h^{**}\left(r,\hat{u}''(r)+\frac{n-1}{r}\hat{u}'(r)-\lambda\,\hat{u}(r)\right)+\beta(r)\right)dr<\infty.$$

Hence

$$\int_0^1 \sum_{i,j} \chi_{E_{i,j}}(r)\, r^{n-1}(h(r, v_i(r)) + \beta(r))\, dr = \int_0^1 (\lim_m z_m(r))\, dr$$
$$= \lim_m \int_0^1 z_m(r)\, dr = \int_0^1 r^{n-1}\left(h^{**}\left(r, \hat{u}''(r) + \frac{n-1}{r}\hat{u}'(r) - \lambda \hat{u}(r)\right) + \beta(r)\right) dr.$$

The latter implies

$$\int_0^1 \sum_{i,j} \chi_{E_{i,j}}(r)\, r^{n-1}\, h(r, v_i(r))\, dr$$
$$= \int_0^1 r^{n-1}\, h^{**}\left(r, \hat{u}''(r) + \frac{n-1}{r}\hat{u}'(r) - \lambda \hat{u}(r)\right) dr, \quad (13)$$

so that the claim is proved.

(*d*) Since $E_{i,j}$, $i=1, 2$, is a partition of K_j, we have

$$h(r, \sum_{i,j} \chi_{E_{i,j}}(r)\, v_i(r)) = \sum_{i,j} \chi_{E_{i,j}}(r)\, h(r, v_i(r)).$$

Therefore from (3) and (13) it follows that

$$\int_B h^{**}(|x|, \Delta\hat{u}(x) - \lambda \hat{u}(x))\, dx = \int_B h(|x|, \sum_{i,j} \chi_{E_{i,j}}(|x|)\, v_i(|x|))\, dx. \quad (14)$$

Now, let u be the (unique) radially symmetric solution to the Dirichlet problem

$$\Delta u - \lambda u = \sum_{i,j} \chi_{E_{i,j}}(|x|)\, v_i(|x|) \quad (15)$$
$$u = 0 \quad \text{on } \partial B. \quad (16)$$

We actually know that $u \in W^{2,p}(B)$ ([G-T]), *i.e.* that

$$u \in W^{2,p}(B) \cap W_0^{1,p}(B).$$

Notice that, from (12), the right hand side of (15) is in $L^p(B)$. We claim that the function u is a solution to problem (P_0). To infer it, we shall prove that:

$\frac{\partial u}{\partial n} = 0$ on ∂B or, equivalently,

in spherical coordinates, that $u'(1)=0$; (17)

$$\int_B h^{**}(|x|, \Delta\hat{u}(x) - \lambda \hat{u}(x))\, dx = \int_B h(|x|, \Delta u(x) - \lambda u(x))\, dx; \quad (18)$$

$$\int_B a(|x|)\, \hat{u}(x)\, dx = \int_B a(|x|)\, u(x)\, dx. \quad (19)$$

Ad (17). First, remark that (15) in spherical coordinates implies

$$r^{n-1} u'(r) = \lambda \int_0^r s^{n-1} u(s)\, ds + \int_0^r s^{n-1} \sum_{i,j} \chi_{E_{i,j}}(s)\, v_i(s)\, ds. \tag{20}$$

Then

$$\begin{aligned}
u'(1) &= \lambda \int_0^1 s^{n-1} u(s)\, ds + \int_0^1 s^{n-1} \sum_{i,j} \chi_{E_{i,j}}(s)\, v_i(s)\, ds \\
&= \lambda \int_0^1 s^{n-1} u(s)\, ds + \sum_j \int_{K_j} s^{n-1} \sum_i \chi_{E_{i,j}}(s)\, v_i(s)\, ds \\
&= \lambda \int_0^1 s^{n-1} u(s)\, ds + \sum_j \int_{K_j} s^{n-1} \sum_i p_i(s)\, v_i(s)\, ds \\
&= \lambda \int_0^1 s^{n-1} u(s) + \int_0^1 s^{n-1} \sum_i p_i(s)\, v_i(s)\, ds \\
&= \lambda \int_0^1 s^{n-1} u(s) + \int_0^1 (s^{n-1} \hat{u}'(s))'\, ds - \lambda \int_0^1 s^{n-1} \hat{u}(s)\, ds \\
&= \lambda \int_0^1 s^{n-1} (u(s) - \hat{u}(s))\, ds,
\end{aligned}$$

where we have used (9) and (5.a) and $\hat{u}'(1)=0$. On the other hand, by taking into account (6), we have

$$\begin{aligned}
n\,\omega_n \int_0^1 s^{n-1} (u(s) - \hat{u}(s))\, ds &= \int_B (u(x) - \hat{u}(x))\, dx \\
&= \int_B (\Delta\varphi(x) - \lambda\varphi(x))(u(x) - \hat{u}(x))\, dx.
\end{aligned}$$

From Green's Formula we have

$$\int_B \Delta\varphi(x)(u(x) - \hat{u}(x))\, dx = \int_B \varphi(x)(\Delta u(x) - \Delta\hat{u}(x))\, dx$$

so that

$$\begin{aligned}
n\,\omega_n \int_0^1 s^{n-1} (u(s) - \hat{u}(s))\, ds
&= \int_B \varphi(x)(\Delta u(x) - \lambda u(x) - \Delta\hat{u}(x) + \lambda \hat{u}(x))\, dx \\
&= n\,\omega_n \int_0^1 s^{n-1} \Big\{ \sum_{i,j} \chi_{E_{i,j}}(s)\, v_i(s) - \sum_i p_i(s)\, v_i(s) \Big\}\, \varphi(s)\, ds.
\end{aligned}$$

The last integral equals

$$n\,\omega_n \sum_j \left\{ \int_{K_j} s^{n-1} \sum_i \chi_{E_{i,j}}(s)\, v_i(s)\, \varphi(s)\, ds - \int_{K_j} s^{n-1} \sum_i p_i(s)\, v_i(s)\, \varphi(s)\, ds \right\}$$

and, from (10) we conclude that

$$u'(1) = \lambda \int_0^1 s^{n-1} (u(s) - \hat{u}(s))\, ds = 0.$$

Ad (18). This is a straightforward consequence from (14) and (15).

Ad (19). From the definition of ψ,

$$\int_B a(|x|)\, u(x)\, dx = \int_B (\Delta\psi(x) - \lambda\psi(x))\, u(x)\, dx.$$

By means of Green's Formula the right hand side can be written as

$$\int_B (\Delta u(x) - \lambda\, u(x))\, \psi(x)\, dx.$$

Taking in account (15) in spherical coordinates, the last integral equals

$$\begin{aligned} n\,\omega_n \int_0^1 r^{n-1} \sum_{i,j} \chi_{E_{i,j}}(r)\, v_i(r)\, \psi(r)\, dr &= n\,\omega_n \sum_j \int_{K_j} r^{n-1} \sum_i \chi_{E_{i,j}}(r)\, v_i(r)\, \psi(r)\, dr \\ &= n\,\omega_n \sum_j \int_{K_j} r^{n-1} \sum_i p_i(r)\, v_i(r)\, \psi(r)\, dr \\ &= n\,\omega_n \int_0^1 r^{n-1} \sum_i p_i(r)\, v_i(r)\, \psi(r)\, dr \\ &= \int_B (\Delta\hat{u}(x) - \lambda\,\hat{u}(x))\, \psi(x)\, dx \\ &= \int_B (\Delta\psi(x) - \lambda\psi(x))\, \hat{u}(x)\, dx = \int_B a(|x|)\, \hat{u}(x)\, dx, \end{aligned}$$

where we have used (11), (5.a) and (7), and the proof is complete. This proves that u is a radially symmetric solution to problem (P_0). ■

Remark. – In the scalar case; $n=1$, $p>1$, B is the interval $]-1, +1[$. In this situation the proof remains as before (setting $n=1$), except only that we take

$$\bar{u}(x) = \frac{1}{2}\hat{u}(x) + \frac{1}{2}\hat{u}(-x)$$

instead of that defined in (1).

REFERENCES

[A-T1] G. Aubert and R. Tahraoui, Théorèmes d'existence pour des problèmes du calcul des variations..., *J. Diff. Eq.*, Vol. **33**, 1979, pp. 1-15.

[A-T2] G. Aubert and R. Tahraoui, Théorèmes d'existence en Optimisation non Convexe, *Appl. Anal.*, Vol. **18**, 1984, pp. 75-100.

[Ce] L. Cesari, *Optimization-Theory and Applications*, Springer-Verlag, New York, 1983.

[C] D. L. Cohn, *Measure Theory*, Birkhäuser Boston, 1980.

[C-C] A. Cellina and G. Colombo, On a Classical Problem of the Calculus of Variations Without Convexity Assumptions, *Ann. Inst. Henri Poincaré*, Vol. 7, No. 2, 1990, pp. 97-106.

[DNF] B. Doubrovine, S. Novikov and A. Fomenko, *Géométrie Contemporaine Méthodes et Applications*, Première Partie, MIR, Moscou, 1982.

[E-T] I. Ekeland and R. Teman, *Convex Analysis and Variational Problems*, North-Holland, Amsterdam, 1976.

[G-T] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin, 1983.

[R] J. P. Raymond, *Problèmes de Calcul des Variations et de Contrôle Optimal: Existence et Régularité des Solutions*, Thèse d'habilitation.

[T] R. Tahraoui, Sur une classe de fonctionnelles non convexes et applications, SIAM *J. Math. Anal.*, Vol. **31**, 1990, pp. 37-52.

(*Manuscript received October* 8, 1990;
revised May 13, 1991.)