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Infinite cup length in free loop spaces with an application to a problem of the N-body type

by

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ABSTRACT. — Cup lengths in the cohomology of the space of free loops (over fields of finite characteristic) are computed and results applied to prove the existence of infinitely many solutions of a Hamiltonian system of N-body type.

RÉSUMÉ. — « Cup » longueurs dans la cohomologie de l'espace des lacets libres (sur corps de caractéristique finie) sont obtenues et sont appliquées à démontrer, l'existence d'une infinité de solutions d'un système Hamiltonien à N-corps.

1. INTRODUCTION

Let E denote the Sobolev space $W_T^{1,2}(\mathbb{R}^n)$ of T -periodic functions from the reals \mathbb{R} to Euclidean n -space \mathbb{R}^n . Then, if U is an open set in \mathbb{R}^n , set

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$E(U)$ equal to those $q \in E$ whose range is in U . The topology of $E(U)$ plays an important role in many problems in Analysis. For example, if U is connected, simply connected and not contractible, it is known that the (Lusternik-Schnirelmann) category of $E(U)$ is infinite [FH1] and this fact is instrumental in establishing the existence of infinitely many solutions to singular Hamiltonian systems, where U is the domain of the corresponding potential function (see e. g., [AC], [BR], [C], [R1]). A special case of this situation is the so-called N -body problem in \mathbb{R}^k , where $E = W_T^{1,2}(\mathbb{R}^{kN})$ and for non-collision solutions the subset $E(U)$, where $U = F_N(\mathbb{R}^k)$ is the N -th configuration space of \mathbb{R}^k , plays an essential role (see e. g. [BR]). The proof of the above result [FH1] that $\text{cat } E(U)$ is infinite does not require knowledge of the cup length in the cohomology of $E(U)$. In fact, the cup length of $E(U)$ may very well be finite over certain coefficient fields. For example, when $n \geq 3$ is odd, and $U = \mathbb{R} - \{0\}$, the cup length of $E(U)$ over the field of rationals \mathbb{Q} is 1 [VS].

The topological analogue of $E(U)$ is the space of free loops ΛU , where for any topological space M , $\Lambda M = \mathcal{C}^0(S^1, M)$, the space of continuous maps from the circle S^1 to the space M . $E(U)$ has the same homotopy type as ΛM [K] so that category and cup length of $E(U)$ may be studied through ΛU . For example, the general result in [FH1] states that for simply connected space M with non-trivial finitely generated cohomology (over some field), ΛM has infinite category and this implies the corresponding result for $E(U)$ mentioned above. (The case where M is finite dimensional but not necessarily simply connected or of finite type is contained in [FH2].)

Even though the cup length in the cohomology of ΛM does not play a role in the above general results in [FH1], nevertheless, it is still useful to know that over some fields, depending on M , the cup length is infinite. We will show in this note, that for spheres S^m , ΛS^m has infinite cup length over \mathbb{Z}_2 and for complex projective space $\mathbb{C}P^n$, $\Lambda \mathbb{C}P^n$ has infinite cup length over \mathbb{Z}_r , where r divides $n+1$. Incidentally, $\Lambda \mathbb{C}P^n$ has only finite cup length over the rational field [VS]. The first result will allow us to compute the relative category [F1] of the pair $(\Lambda M, \Lambda N)$, where M is a wedge of spheres and N a "subwedge". In this case both ΛM and ΛN have infinite category and the relative category $\text{cat}(\Lambda M, \Lambda N)$ is shown to be infinite using a cup length type argument. This result provides an alternative tool for the topological part of a theorem of Bahri-Rabinowitz [BR] of 3-body type and should be useful in the general case $N > 3$ (see § 2).

For our N -body problem application, we will require the category of a certain subspace of ΛS^m described as follows. Let $\mathbb{Z}_2 = \{1, \zeta\}$ act on ΛS^m by the action $(\zeta q)(t) = -q\left(t + \frac{1}{2}\right)$, $0 \leq t \leq 1$, where q is considered

1-periodic. Let $\Lambda_0 S^m$ denote the fixed point set under this action. $\Lambda_0 S^m$ fibers over S^m but for m even, this fibration does not admit a section, which is a requirement for the main tool in [FH1]. Nevertheless, we show that the cup length of $\Lambda_0 S^m$ over \mathbb{Z}_2 is infinite and hence the category of $\Lambda_0 S^m$ is infinite. As an application of these results, it follows that the subspace $E(F_N(\mathbb{R}^k))$ of the Sobolev space $W_T^{1,2}(\mathbb{R}^{kN})$ corresponding to the free loops $\Lambda F_N(\mathbb{R}^k)$ on the N-th configuration space of \mathbb{R}^k , also has infinite cup length over \mathbb{Z}_2 . Furthermore, if we let $\Lambda_0(F_N(\mathbb{R}^k))$ denote the subspace of $\Lambda(F_N(\mathbb{R}^k))$ which is the fixed point set of the \mathbb{Z}_2 -action defined above, then $\Lambda_0(F_N(\mathbb{R}^k))$, has infinite cup length over \mathbb{Z}_2 and hence $\text{cat } \Lambda_0(F_N(\mathbb{R}^k)) = \infty$. This result is the key to proving a critical point theorem for the functional associated with a problem of N-body type, which improves a result of Coti-Zelati [C] who minimizes the appropriate functional to obtain a critical point. In addition to yielding an unbounded sequence of critical values, the theorem (see Section 3) allows a non-autonomous potential $V(q, t)$ which is C^1 and $T/2$ -periodic in t .

2. CUP LENGTH IN SOME FREE LOOP SPACES

We employ the following notation. I is the unit interval $[0, 1]$; M^I is the space of maps from I to a space M ; ΛM is the free loop space on M given by $\Lambda M = \{ \alpha \in M^I : \alpha(0) = \alpha(1) \}$; and $\Omega(M) = (\Omega M, *)$, the space of based loops, *i. e.* loops $\alpha \in \Lambda M$ such that $\alpha(0) = \alpha(1) = * \in M$. If we consider the (Hurewicz) fibration.

$$\Omega(M) \rightarrow M^I \xrightarrow{q} M \times M \tag{1}$$

where $q(\alpha) = (\alpha(0), \alpha(1))$, then the diagonal map $\Delta : M \rightarrow M \times M$ induces the fibration.

$$\Omega M \rightarrow \Lambda M \xrightarrow{p} M \tag{2}$$

where $p(\alpha) = \alpha(0) = \alpha(1)$. We will also make use of the induced fibrations:

$$\begin{array}{ccc} P^1 M & \xrightarrow{\hat{i}_1} & M^1 \\ p_1 \downarrow & & \downarrow q \\ M & \xrightarrow{i_1} & M \times M \end{array} \quad \begin{array}{ccc} P^2 M & \xrightarrow{\hat{i}_2} & M^1 \\ p_2 \downarrow & & \downarrow q \\ M & \xrightarrow{i_2} & M \times M \end{array} \tag{3}$$

where $i_1(x) = (*, x)$, $i_2(x) = (x, *)$ and $P^1 M, P^2 M$ are contractible.

The Leray-Serre spectral sequences of (1), (2), p_1 and p_2 will be denoted by $(E^{p,q}, d)$, $(\bar{E}^{p,q}, \bar{d})$, $(E^{p,q}, d')$, $(E^{p,q}, d'')$.

We consider first the case $M = S^{m+1}$, $m+1$ even, $m \geq 1$ and prove several lemmas under this assumption. If u is a generator of $E_{m+1}^{m+1,0} = H^{m+1}(M)$, we define $x \in H^m(\Omega M)$ by $d'_{m+1}x = u$. Furthermore, we define $y \in H^{2m}(\Omega M)$ by $d'_{m+1}y = ux$. With these definitions, the \mathbb{Z} -cohomology of ΩS^{m+1} has the form $H^*(S^m) \otimes H^*(\Omega S^{2m+1})$ where the first factor has generator x in dimension m and the second factor is a divided polynomial algebra with generators $y_1, y_2, \dots, y_k, \dots$ in dimensions $2km$ and $y = y_1$ (see [1]).

2.1. LEMMA. - $d''_{m+1}x = -u$.

Proof. - This is a simple calculation using the reverse map.

$$\begin{array}{ccc}
 P^2 M & \xrightarrow{v} & P^1 M \\
 p_2 \searrow & & \swarrow p_1 \\
 & & M
 \end{array} \tag{4}$$

where $(v\alpha)(t) = \alpha(1-t)$. Then, $v_0 = v|_{\Omega M}$, has the property that $v_0^*(x) = -x$ and hence $d''_{m+1}x = d''_{m+1}v_0^*(-x) = d'_{m+1}(-x) = u$.

Comparing, the spectral sequences (SS) of p_1 and p_2 with that of q we have:

2.2. LEMMA. - $d_{m+1}(x) = u \times 1 - 1 \times u$ in $H^{m+1}(S^{m+1}) = E_{m+1}^{m+1,0}$.

We consider next the differential operator $d_{m+1}: E_{m+1}^{m+1,m} \rightarrow E_{m+1}^{2m+2,0}$.

2.3. LEMMA. - (a) $d_{m+1}((1 \times u)x) = -u \times u$

(b) $d_{m+1}((u \times 1)x) = u \times u$

and the kernel of

$$d_{m+1}: E_{m+1}^{m+1,m} \rightarrow E_{m+1}^{2m+2,0}$$

is generated by $((u \times 1)x + (1 \times u)x)$.

Proof. - To prove (a) consider

$$d_{m+1}((1 \times u)x) = (-1)^{m+1}(1 \times u)(1 \times u - u \times 1) = 1 \times u^2 - u \times u = -u \times u$$

(b) follows from a similar argument. Thus

$$d_{m+1}((1 \times u)x + (u \times 1)x) = 0$$

and an easy argument shows that the kernel of $d_{m+1}^{m+1,m}$ has $(u \times 1)x + (1 \times u)x$ as generator.

We now consider the diagram:

$$\begin{array}{ccc}
 \Lambda M & \xrightarrow{\hat{\Delta}} & M^{\dagger} \\
 p \downarrow & & \downarrow q \\
 M & \xrightarrow{\Delta} & M \times M
 \end{array} \tag{5}$$

Recall that \bar{d} is the differential in the SS for p , and $x \in \bar{E}_{m+1}^{0,m}$, $y \in \bar{E}_{m+1}^{0,2m}$.

2.4. LEMMA. — $\bar{d}_{m+1}(x) = 0$, $\bar{d}_{m+1}(y) = 2ux$

Proof. — First observe that from Lemma 2.2.

$$\bar{d}_{m+1}(x) = \bar{d}_{m+1} \hat{\Delta}^*(x) = \Delta^* d_{m+1}(x) = \Delta^*(u \times 1 - 1 \times u) = u - u = 0.$$

Since, $M^1 \sim M$, in the SS for q , it follows that

$$d_{m+1}(y) = (1 \times u)x + (u \times 1)x \in E_{m+1}^{m,m+1}$$

and comparison with the SS for p_1 forces the plus sign. Hence,

$$\bar{d}_{m+1}(y) = \bar{d}_{m+1} \hat{\Delta}^*(y) = \Delta^* d_{m+1}(y) = 2ux.$$

2.5. LEMMA. — Let $i: \Omega M \rightarrow \Lambda M$, denote the inclusion map, where $M = S^{m+1}$ as above. Then,

$$i^*: H^q(\Lambda M; \mathbb{Z}_2) \rightarrow H^q(\Omega M; \mathbb{Z}_2), \quad q \geq 0$$

is surjective.

Proof. — Consider the terms, $E_{m+1}^{0,*} = H^*(\Omega M)$ in the spectral sequence for p over \mathbb{Z} . Let y_k denote one of the generators of the divided polynomial algebra $H^*(\Omega S^{2m+1})$. Then, using induction,

$$\bar{d}_{m+1}(y_1 y_{k-1}) = k \bar{d}_{m+1}(y_k) = (2ux)y_{k-1} + y, \quad (2uxy_{k-2}) = k 2uxy_{k-1}$$

Therefore, $\bar{d}_{m+1}(y_k) = 2uxy_{k-1}$. Hence, over \mathbb{Z}_2 , $\bar{d}_{m+1}(y_k) = 0$. This is sufficient to verify the lemma.

2.6. THEOREM. — If $M = S^{m+1}$, $m+1$ even, then as algebras,

$$H^*(\Lambda S^{m+1}; \mathbb{Z}_2) \simeq H^*(S^{m+1}, \mathbb{Z}_2) \otimes_{\mathbb{Z}_2} H^*(\Omega S^{m+1}; \mathbb{Z}_2).$$

Proof. — This is immediate from the Leray-Hirsch theorem and the fact that i^* is an isomorphism in dimensions which are multiplies of m .

The case when $M = S^{m+1}$ with $m+1$ odd is considerably easier. $H^*(\Omega S^{m+1})$ is the divided polynomial algebra on generators y_1, \dots, y_k, \dots and in the SS for p , $\bar{d}_{m+1}y_k = 0$, so that i^* is surjective over \mathbb{Z} as well as over \mathbb{Z}_2 .

2.7. THEOREM. — If $M = S^{m+1}$, $m+1$ odd, then as algebras over \mathbb{Z} ,

$$H^*(\Lambda S^m) \simeq H^*(S^{m+1}) \otimes H^*(\Omega S^{m+1}).$$

2.8. COROLLARY. — If $M = S^{m+1}$, $m \geq 1$, the cup length of $H^*(\Lambda S^{m+1}; \mathbb{Z}_2)$ is infinite.

Proof. — $H^*(\Lambda S^{m+1}; \mathbb{Z}_2)$ contains the divided polynomial algebra over \mathbb{Z}_2 on generators y_1, y_2, \dots, y_k and calculating binomial coefficients mod 2 we find that the cup product

$$y_2 \cdot y_4 \cdot y_8 \cdot \dots \cdot y_{2^k} = y_r, \quad r = 2^{k+1} - 2$$

is non-zero for all $k \geq 1$.

2.9. *Remark.* — Corollary 2.8 implies that the category of ΛS^{m+1} , $m \geq 1$, is infinite. However, the direct argument in [FH1] is simpler. Nevertheless, we will need Corollary 2.8 later on to compute a relative cup product. Our next example cannot be handled using [FH1].

Let f denote any map $f: M \rightarrow M$ and consider its graph $1 \times f: M \rightarrow M \times M$. Let $\Lambda_f M$ denote the total space of the induced fibration as in the following diagram:

$$\begin{array}{ccc}
 \Omega M & & \Omega M \\
 \downarrow & & \downarrow \\
 \Lambda_f M & \xrightarrow{\hat{f}} & M^1 \\
 p_f \downarrow & & \downarrow^q \\
 M & \xrightarrow{1 \times f} & M \times M
 \end{array}$$

An important case for us in the application to be given in Section 3, is $M = S^{m+1}$, $m+1$ even, and f the antipodal map. (If $m+1$ is odd, f is homotopic to the identity and $\Lambda_f M \sim \Lambda M$). Notice that in this case p_f does not admit a section which is why [FH1] does not apply.

Let \tilde{d} denote the differential in the SS over \mathbb{Z} for the fiber map p_f . The analogue of Lemma 2.4 is:

2.10. LEMMA. — $\tilde{d}_{m+1}(x) = -2u$, $\tilde{d}_{m+1}(y) = 0$.

Proof:

$$\tilde{d}_{m+1}(x) = \tilde{d}_{m+1} \hat{f}^*(x) = (1 \times f)^* d_{m+1}(x) = (1 \times f)^*(u \times 1 - 1 \times u) = 2u.$$

Furthermore,

$$\begin{aligned}
 \tilde{d}_{m+1}(y) &= \tilde{d}_{m+1} \hat{f}^*(y) \\
 &= (1 \times f)^* d_{m+1}(y) = (1 \times f)^*((u \times 1)x + (1 \times u)x) = ux - ux = 0.
 \end{aligned}$$

2.11. LEMMA. — Let $i: \Omega M \rightarrow \Lambda_f M$, where $M = S^{m+1}$, $m+1$ even, and f the antipodal map. Then,

$$i^*: H^q(\Lambda_f M; \mathbb{Z}_2) \rightarrow H^q(\Omega M; \mathbb{Z}_2), \quad q \geq 0$$

is surjective. Furthermore, over \mathbb{Z} , the image of

$$i^*: H^*(\Lambda_f M) \rightarrow H^*(\Omega M)$$

contains the divided polynomial algebra in $H^*(\Omega M)$.

Proof. — It follows by induction that $\tilde{d}_{m+1} y_k = 0$ over \mathbb{Z} where, as above, the y_k are generators of the polynomial algebra $H^*(\Omega S^{2m+1})$. This observation suffices to prove the lemma.

2.12. COROLLARY. — $\Lambda_f M$, where $M = S^{m+1}$, $m \geq 1$, has infinite cup length over \mathbb{Z} and \mathbb{Z}_2 and hence $\Lambda_f M$ has infinite category.

2.13. COROLLARY:

$$H^*(\Lambda_f M; \mathbb{Z}_2) \simeq H^*(S^{m+1}, \mathbb{Z}_2) \otimes_{\mathbb{Z}_2} H^*(\Omega S^{m+1}; \mathbb{Z}_2),$$

as algebras.

Our next example is the computation on the cup length of $\Lambda \mathbb{C}P^n$. We will make use of the fact that $\Omega \mathbb{C}P^n$ has the same homotopy type as $S^1 \times \Omega S^{2n+1}$. Working over \mathbb{Z} and employing the diagrams (3), (4) and (5), let x denote a generator of $H^1(\Omega \mathbb{C}P^n)$, u a generator of $H^2(\mathbb{C}P^n)$, and $y_1, y_2, \dots, y_k, \dots$ the generators of the divided polynomial algebra in $H^*(\Omega \mathbb{C}P^n)$ corresponding to $H^*(\Omega S^{2n+1})$. Also let $y_1 = y$, to conform to some previous notation. We may assume that $d'_2(x) = u$ in the SS for p_1 . Then, Lemmas 2.1-2.2 obtain with only notational adjustments and $d''_2(x) = -u$, $d_2(x) = 1 \times u - u \times 1$ and $\bar{d}_2(x) = 0$. $\bar{d}_2(u^k x) = 0$, $k = 1, \dots, n$ and the differential operators \bar{d}_j are all trivial for $2 \leq j \leq 2n-1$. At the level \bar{E}_{2n}^{**} , we have

$$\bar{E}_{2n}^{i,0} = \langle u^i \rangle, \bar{E}_{2n}^{i,1} = \langle u^i x \rangle, \bar{E}_{2n}^{0,2n} = \langle y \rangle,$$

$i = 0, \dots, n$, where $\langle \ \rangle$ indicates “generated by”.

We will need the following in the SS for q in (5).

2.14. LEMMA. — Let $u_1 = 1 \times u$ and $u_2 = u \times 1$ in $H^*(\mathbb{C}P^n \times \mathbb{C}P^n)$. Then, in the SS for q the element

$$w = (u_1^n + u_1^{n-1} u_2 + \dots + u_1 u_2^{n-1} + u_2^n) x \in \bar{E}_{2n}^{2n,1}$$

is a d_2 cocycle, i.e., $d_2(w) = 0$. w is a generator of kernel d_2 chosen so that $d_{2n}(y) = w$. Therefore, in the SS for p in (5) we have $\bar{d}_{2n}(y) = \bar{d}_{2n}(y_1) = (n+1)u^n x$.

Proof:

$$\begin{aligned} d_2 w &= (u_1^n + u_1^{n-1} u_2 + \dots + u_1 u_2^{n-1} + u_2^n) d_2(x) \\ &= (u_1^n + u_1^{n-1} u_2 + \dots + u_1 u_2^{n-1} + u_2^n) (u_1 - u_2) = u_1^{n+1} - u_2^{n+1} = 0. \end{aligned}$$

On the other hand, if

$$(a_1 u_1^n + a_2 u_1^{n-1} u_2 + \dots + a_n u_1 u_2^{n-1} + a_{n+1} u_2^n) (u_1 - u_2) = 0$$

we have, by equating coefficients, $a_1 = a_2 = \dots = a_{n+1}$ and w generates $\ker d_2$. Thus, $d_2 y = \pm w$ since $H^{2n+1}(M^1) = 0$ and w cannot survive. There is no loss of generality if we stipulate that $d_2(y) = w$. Finally, in the SS for p , we have

$$\bar{d}_{2n}(y) = \bar{d}_{2n} \hat{\Delta}^*(y) = \hat{\Delta}^* \bar{d}_{2n} y = (n+1)u^n x.$$

2.15. LEMMA. — On the SS for p we have

$$d_{2n}(y_k) = (n+1)u^n x y_{k-1}.$$

Proof. — We use induction on k .

$$\begin{aligned} d_{2n}(y_1 \cdot y_{k-1}) &= (n+1)u^n xy_{k-1} + y_1 d_{2n}y_{k-1} \\ &= (n+1)u^n xy_{k-1} + y_1(n+1)u^n xy_{k-2} \\ &= (n+1)u^n x[y_{k-1} + (k-1)y_{k-1}] = k(n+1)u^n xy_{k-1} = kd_{2n}(y_k). \end{aligned}$$

Hence, $d_{2n}(y_k) = (n+1)u^n xy_{k-1}$

2.16. COROLLARY. — *If r is a prime which divides $n+1$, then in the SS for p over \mathbb{Z}_r we have $d_{2n}(x) = 0$ and $d_{2n}(y_k) = 0$ for all $k \geq 1$. Hence, the inclusion map $i: \Omega \mathbb{C} P^n \rightarrow \Lambda \mathbb{C} P^n$ induces surjections*

$$i^*: H^q(\Lambda \mathbb{C} P^n; \mathbb{Z}_r) \rightarrow H^q(\Omega \mathbb{C} P^n; \mathbb{Z}_r)$$

and hence, as algebras,

$$H^*(\Lambda \mathbb{C} P^n; \mathbb{Z}_r) \simeq H^*(\mathbb{C} P^n; \mathbb{Z}_r) \otimes_{\mathbb{Z}_r} H^*(\Omega \mathbb{C} P^n; \mathbb{Z}_r).$$

We need to extend some of these results to configuration spaces. First let $M = F_N(\mathbb{R}^k)$, the N -th configuration space of Euclidean k -space \mathbb{R}^k . Recall, that

$$F_N(\mathbb{R}^k) = \{(x_1, \dots, x_N) \in (\mathbb{R}^k)^N, x_i \neq x_j \text{ for } i \neq j\}.$$

Also, the projection $p_N: F_N(\mathbb{R}^k) \rightarrow F_{N-1}(\mathbb{R}^k)$ given by

$$p_N(x_1, \dots, x_N) = (x_1, \dots, x_{N-1})$$

is locally trivial with fiber $\mathbb{R}^k - Q_N$, where Q_N is a set of $(N-1)$ points. In particular, $p_2: F_2(\mathbb{R}^k) \rightarrow \mathbb{R}^k$, with fiber $\mathbb{R}^k - 0$. Hence $F_2(\mathbb{R}^k) \sim \mathbb{R}^k - 0$ and we have for $k \geq 3$

$$H^*(\Lambda F_2(\mathbb{R}^k); \mathbb{Z}_2) \simeq H^*(S^{k-1}; \mathbb{Z}_2) \otimes_{\mathbb{Z}_2} H^*(\Omega S^{k-1}; \mathbb{Z}_2)$$

so that $\Lambda F_2 \mathbb{R}^k$ has infinite cup length over \mathbb{Z}_2 . It is easy to see that p_N admits a section for $N \geq 3$. In fact we will produce an equivariant section which will be needed by the next example.

2.17. LEMMA. — *For $N \geq 3$, $p_N: F_N(\mathbb{R}^k) \rightarrow F_{N-1}(\mathbb{R}^k)$ admits a section σ with the property that $\sigma(-x) = -\sigma(x)$.*

Proof. — Let

$$\alpha = \alpha(x_1, \dots, x_{N-1}) = \min_{i \neq j} |x_i - x_j|.$$

Define

$$x_N = x_N(x_1, \dots, x_{N-1}) = \left(1 - \frac{\alpha}{2|x_2 - x_1|}\right)x_1 + \left(\frac{\alpha}{2|x_2 - x_1|}\right)x_2$$

and set

$$\sigma(x_1, \dots, x_{N-1}) = (x_1, \dots, x_{N-1}, x_N).$$

2.18. THEOREM. — $\Lambda F_N(\mathbb{R}^k)$ has infinite cup length over \mathbb{Z}_2 for $k \geq 3$, $N \geq 2$.

Proof. — By the previous lemma $\Lambda p_N: \Lambda F_N(\mathbb{R}^k) \rightarrow \Lambda F_{N-1}(\mathbb{R}^k)$ admits a section for $N \geq 3$ and hence by induction the result follows.

We now consider the configuration space analogue of Corollary 2.12. We define $\Lambda_A F_N(\mathbb{R}^k)$ as a pull-back by the diagram

$$\begin{array}{ccc} \Lambda_A F_N(\mathbb{R}^k) & \longrightarrow & [F_N(\mathbb{R}^k)]^A \\ p_A \downarrow & & \downarrow q \\ F_N(\mathbb{R}^k) & \xrightarrow{1 \times A} & F_N(\mathbb{R}^k) \times F_N(\mathbb{R}^k) \end{array}$$

where $A(x_1, \dots, x_N) = (-x_1, \dots, -x_N)$.

Thus, $\Lambda_A F_N(\mathbb{R}^k)$ is the space of paths $q = (q_1, \dots, q_N)$ in $F_N(\mathbb{R}^k)$ such that $q_i(1) = -q_i(0)$. Then, the fibration

$$p_N: F_N(\mathbb{R}^k) \rightarrow F_{N-1}(\mathbb{R}^k), \quad N \geq 3$$

induces

$$\bar{p}_N: \Lambda_A F_N(\mathbb{R}^k) \rightarrow \Lambda_A F_{N-1}(\mathbb{R}^{k-1}).$$

\bar{p}_N admits a section using the section σ of Lemma 2.17. It is easy to identify $\Lambda_A F_2(\mathbb{R}^k)$, up to homotopy type, with $\Lambda_f S^{k-1}$ of Corollary 2.12. Combining the remarks we obtain:

2.19. THEOREM. — For $N \geq 2$, $k \geq 3$, the cup length of $\Lambda_A F_N(\mathbb{R}^k)$ is infinite.

Our final computation concerns the relative category of a certain pair which can be estimated by considering $H^*(X; A)$ as a module over $H^*(X)$. In [BR], Bahri and Rabinowitz exploited a purely topological result that the free loop spaces $\Lambda F_3(\mathbb{R}^k)$ and $\Lambda F_2(\mathbb{R}^k)$ were not of the same homotopy type to prove a theorem of the 3-body type concerning the existence of an unbounded sequence of critical values without a symmetry condition on the potential (see Section 3). This topological result is derived from a result of Vigué-Poirier and Sullivan [VS] to the effect that the rational Betti numbers of $\Lambda F_3(\mathbb{R}^k)$, $k \geq 3$, were unbounded, while those of $\Lambda F_2(\mathbb{R}^k)$ were bounded. The following result (theorem) provides an alternative tool for the Bahri-Rabinowitz theorem and will, hopefully, play a role in the case $N > 3$.

First we recall one of the definitions of relative category introduced in [F1], [F2].

2.20. DEFINITION. — Let (X, A) be a topological pair. A categorical cover for (X, A) of length n is an $(n+1)$ -tuple of open sets (V_0, V_1, \dots, V_n) such that $\cup V_j \supset X$, V_0 deforms in X to A relative to A , and V_i , $i \geq 1$, deforms in X to a point. $\text{cat}(X, A)$ is the minimum length

of such categorical covers if such categorical covers exist. Otherwise, set $\text{cat}(X, A) = \infty$.

The next result is the analogue for cup length in this setting, using the fact that over any commutative ring of coefficients, $H^*(X, A; R)$ is a module over $H^*(X; R)$. Although, the result depends on coefficients we will not display it in the notation.

2.21. PROPOSITION [F1]. — *If there exist n elements u_1, \dots, u_n in $H^*(X)$ of positive dimension such that the product $u_1 u_2 \dots u_n$ is not in the annihilator of $H^*(X, A)$, then $\text{cat}(X, A) > n$.*

Now, let $M = S_1 \vee \dots \vee S_m$ denote a wedge of spheres of dimension ≥ 2 and M' a “subwedge” which we take to be $S_1 \vee \dots \vee S_k, k < m$. We employ Proposition 2.21 and Corollary 2.8 to prove the following.

2.22. THEOREM. — $\text{cat}(\Lambda M, \Lambda M') = \infty$.

Proof. — We work with \mathbb{Z}_2 coefficients.

Consider the diagram

$$\begin{array}{ccccccc}
 0 & \rightarrow & H^q(\Lambda M, \Lambda M') & \xrightarrow{j^*} & H^q(\Lambda M) & \xrightarrow{i^*} & H^q(\Lambda M') \rightarrow 0 \\
 & & & & r^* \downarrow & & \nearrow 0 \\
 & & & & H^q(\Lambda S_m) & &
 \end{array}$$

where i^* surjects because $\Lambda M'$ is a retract of ΛM and r^* injects, where $r: \Lambda M \rightarrow \Lambda S_m$ is a retraction which takes $\Lambda M'$ to a point. Since $H^*(\Lambda S_m)$ has infinite cup length the result follows.

2.23. Remark. — In the Lusternik-Schnirelmann method it is useful to know that when the category of a space X is infinite, there are compact subsets of arbitrarily high category in X . When the cup length of X , using singular cohomology, is infinite over some coefficient field, this is automatic [FH1]. We are indebted to Luis Montejano for suggesting the use of “infinite dimensional topology” to verify that when X is an ANR (sep.met.) and has infinite category, then X has compact subsets of arbitrarily high category. For example, if X is a Hilbert manifold modelled on a Hilbert space H , then by a result of D. Henderson [H], $X = P \times H$, where P is a locally finite polyhedron. If X has infinite category, then so does P . Since P is σ -compact, it is now an exercise to show that P has subpolyhedra of arbitrarily high category in P .

In the next section it will be necessary to apply some of the computations of this section to the corresponding Sobolev spaces. Let $W_T^{1,2}(\mathbb{R}^{kN})$ denote the Sobolev space of T periodic functions $q: \mathbb{R} \rightarrow \mathbb{R}^{kN}$ which are absolutely continuous and have square summable first derivatives. q can be represented by $q = ((q_1, \dots, q_N))$ where each $q_i: \mathbb{R} \rightarrow \mathbb{R}^k$. The inner product for

$W_T^{1,2}(\mathbb{R}^{kN})$ is given by

$$\langle f, g \rangle = \int_0^T \langle \dot{f}(t), \dot{g}(t) \rangle dt + \int_0^T \langle f(t), g(t) \rangle dt.$$

Let $C_T^0(\mathbb{R}^{kN})$ denote the Banach space of continuous T-periodic functions with the uniform norm. We may readily identify $C_T^0(\mathbb{R}^{kN})$ with $\Lambda \mathbb{R}^{kN}$. It is a well-known fact [K] that the inclusion $i: W_T^{1,2}(\mathbb{R}^{kN}) \rightarrow C_T^0(\mathbb{R}^{kN})$ is a continuous injection, whose image is dense in $C_T^0(\mathbb{R}^{kN})$. Define an open set $\Lambda(N) \subset W_T^{1,2}(\mathbb{R}^{kN})$ as follows:

$$\Lambda(N) = \{ (q_1, \dots, q_N) \in W_T^{1,2}(\mathbb{R}^{kN}) : q_i(t) \neq q_j(t) \text{ for } 1 \leq t \leq T \}$$

Then,

$$\bar{i} = i|_{\Lambda(N)} : \Lambda(N) \rightarrow \Lambda F_N(\mathbb{R}^k) \subset \Lambda \mathbb{R}^{kN}$$

where $\Lambda F_N(\mathbb{R}^k)$ is an open subset of $\Lambda \mathbb{R}^{kN}$. Then, by a theorem of Palais [P], \bar{i} is a homotopy equivalence. Thus $\Lambda(N)$ has both infinite cup length over \mathbb{Z}_2 and infinite category. Now, introduce an action of

$\mathbb{Z}_2 = \{1, \zeta\}$ on $W_T^{1,2}(\mathbb{R}^{kN})$ by $(\zeta q)(t) = -q\left(t + \frac{T}{2}\right)$, $0 \leq t \leq 1$. Let E_0

denote the fixed point set of this action, namely those q such that $\zeta q = q$. Then E_0 is a closed Hilbert subspace of $W_T^{1,2}(\mathbb{R}^{kN})$. Let C_0 denote the corresponding subspace of $C_T^0(\mathbb{R}^{kN})$. Then, again E_0 continuously injects into C_0 , with image dense in C_0 . Let $\Lambda_0(N) = \Lambda(N) \cap E_0$ and $\Lambda_0 F_N(\mathbb{R}^{kN}) = C_0 \cap \Lambda F_N(\mathbb{R}^k)$. Then, by the same argument as above, $\Lambda_0(N)$ and $\Lambda_0 F_N(\mathbb{R}^k)$ have the same homotopy type. But $\Lambda_0 F_N(\mathbb{R}^k)$ may be identified with $\Lambda_A F_N(\mathbb{R}^k)$ of the theorem 2.19. Thus, $\Lambda_0(N)$ has infinite cup length over \mathbb{Z}_2 and infinite category. Summarizing:

2.25. THEOREM. — *Let*

$$\Lambda(N) = \{ (q_1, \dots, q_N) \in W_T^{1,2}(\mathbb{R}^{kN}) : q_i(t) \neq q_j(t), 0 \leq t \leq T \}$$

$$\Lambda_0(N) = \left\{ (q_1, \dots, q_N) \in \Lambda(N) : q_i(t) = -q_i\left(t + \frac{T}{2}\right), 0 \leq t \leq T, 1 \leq i \leq N \right\}.$$

Then, both $\Lambda(N)$ and $\Lambda_0(N)$ have infinite cup length over \mathbb{Z}_2 and hence infinite category (with compact subsets of arbitrarily high category).

3. A HAMILTONIAN SYSTEM OF THE N-BODY TYPE

Consider a potential function $V: F_N(\mathbb{R}^k) \times \mathbb{R} \rightarrow \mathbb{R}$ of the following form:

$$V(q_1, \dots, q_N) = \frac{1}{2} \sum_{1 \leq i \neq j \leq N} V_{ij}((q_i - q_j), t)$$

and the following properties for $1 \leq i \neq j \leq N$, $q \in \mathbb{R}^k - 0$, $0 \leq t \leq T$.

(V₁) $V_{ij} \in C^1(\mathbb{R}^k - \{0\} \times \mathbb{R}, \mathbb{R})$, $V_{ij}(q, t) \leq 0$.

(V₂) $V_{ij}(q, t) = V_{ji}(q, t)$ and $V_{ij}(q, t) = V_{ij}\left(q, t + \frac{T}{2}\right)$.

(V₃) $V_{ij}(q, t) \rightarrow -\infty$ as $q \rightarrow 0$ uniformly in t .

(V₄) There exists $U_{ij} \in C^1(W - 0; \mathbb{R})$ on a neighborhood W of 0 in \mathbb{R}^k such that:

(a) $U_{ij}(q) \rightarrow +\infty$ as $q \rightarrow 0$,

(b) $-V_{ij}(q, t) \geq |U'_{ij}(q)|^2$, $q \in W - 0$, $t \in [0, T]$.

(V₄) was introduced by Gordon [G] and called the Strong Force Condition. Consider the following Hamiltonian system

$$m\ddot{q} + V_q(q, t) = 0, \quad q = (q_1, \dots, q_N) \tag{HS}$$

where $m = (m_1, \dots, m_N)$ is the mass vector with $m_i > 0$. The functional corresponding to (HS) has the form

$$I(q) = \sum_{i=1}^N \frac{m_i}{2} \int_0^T |\dot{q}_i(t)|^2 dt - \int_0^T V(q_1(t), \dots, q_N(t), t) dt \tag{*}$$

The arguments employed do not depend on the values m_i so we assume the masses $m_i = 1$ and write

$$I(q) = \frac{1}{2} \int_0^T |\dot{q}|^2 dt - \int_0^T V(q, t) dt \tag{*}$$

If we let E denote the Sobolev space $W_1^{1,2}(\mathbb{R}^{kN})$ of T -periodic, absolutely continuous functions with L^2 derivatives, then if (V₁) holds, $I(q)$ is C^1 and bounded from below by 0 on the open subset

$$\Lambda(N) = \{q \in W_1^{1,2}(\mathbb{R}^{kN}) : q_i(t) \neq q_j(t), 1 \leq i \neq j \leq N, 0 \leq t \leq T\}$$

which corresponds to $W_1^{1,2}(F_N(\mathbb{R}^k))$. We also define a closed subspace E_0 of E as follows: Let $\mathbb{Z}_2 = \{1, \zeta\}$ denote the group of order 2 with non-trivial element ζ . Define the action of \mathbb{Z}_2 on E by $(\zeta q)(t) = -q\left(t + \frac{1}{2}\right)$.

Then,

$$E_0 = \{q \in E, \zeta q = q\}.$$

We also set $\Lambda_0(N) = E_0 \cap \Lambda(N)$.

3.1. THEOREM. — *If the potential V satisfies (V₁)–(V₄), then (*) possesses an unbounded sequence of critical values.*

3.2. Remark. — The Coti-Zelati result [C] proves that when V is autonomous and T -periodic, that the minimum of (*) is a critical value. The Bahri-Rabinowitz result [BR] for $N = 3$, assumes no symmetry such as $V_{ij} = V_{ji}$, but imposes conditions on behaviour of V and V' at infinity.

Before proceeding with the proof of Theorem 3.1 we observe that

$$(1) \quad V(-q, t) = V(q, t), \quad q \in \Lambda(N), \quad t \in [0, T].$$

and

$$(2) \quad I(\zeta q) = I(q), \quad q \in \Lambda(N).$$

3.3. PROPOSITION. — *Let I_0 denote $I|_{\Lambda_0(N)}$. Then critical points of I_0 are critical points of I .*

Proof. — This is a general phenomenon. Namely if a functional I is invariant under the action of a finite group G , then critical points of the restriction I_0 to the fixed point set of the action are always critical points of I . In our case, if $u \in E$, then $u + \zeta u \in E_0$ and

$$I'(q)(u + \zeta u) = 2I'(q)(u), \quad q \in \Lambda_0(N)$$

and hence if q is a critical point for I_0 , $I'(q)$ vanishes on E .

Theorem 3.1 Now follows from the following theorem.

3.4. THEOREM. — *If V satisfies $(V_1) - (V_4)$, then $I_0 = I|_{\Lambda_0(N)}$ possesses an unbounded sequence of critical values.*

The proof will be broken down into a series of lemmas.

3.5. LEMMA (Gordon's Lemma [G]). — *If V satisfies (V_1) , (V_3) , (V_4) and if a sequence q^n in $\Lambda_0(N)$ converges weakly to $q \in E_0$, then if $q \in \partial\Lambda_0(N)$, then $I_0(q^n) \rightarrow +\infty$, where $\partial\Lambda_0(N)$, is the boundary of $\Lambda_0(N)$ in E_0 .*

3.6. LEMMA. — *If V satisfies $(V_1) - (V_4)$, I_0 satisfies the Palais-Smale condition (PS) on $\Lambda_0(N)$.*

Proof. — Let q^n denote a sequence in $\Lambda_0(N)$ such that $I_0(q^n) \rightarrow s \geq 0$ and $I'_0(q^n) \rightarrow 0$. Then, we may assume $I_0(q^n) \leq s + 1$ and hence

$$\int_0^T |\dot{q}_n(s)|^2 ds \leq 2(s + 1).$$

Then, since $q_n \in \Lambda_0(N)$

$$\int_t^{t+(1/2)} \dot{q}_n(s) ds = q_n\left(t + \frac{1}{2}\right) - q_n(t) = -2q_n(t)$$

it follows easily that the sequence q_n is bounded in the $W_T^{1,2}$ norm. Again, by standard arguments [R1], there is a subsequence, also denoted by q_n , such that q_n converges weakly to $q \in E_0$, and Gordon's lemma implies that $q_0 \in \Lambda_0(N)$. Furthermore, I' has the form $I'(q) = q - \mathcal{P}q$, where $\mathcal{P}q_n$ has a (strongly) convergent subsequence in E_0 . Since $I'(q_n) \rightarrow 0$, it follows that a subsequence of q_n converges strongly to q and I_0 is (PS) on $\Lambda_0(N)$.

The next lemma merely isolates the deformation theorem we employ (see [R2]).

3.7. LEMMA. — Let Ω denote an open set in a Hilbert space E and $I: \Omega \rightarrow \mathbb{R}$ a C^1 functional which is bounded from below. Suppose I satisfies (PS) on Ω and we have the condition that when $q_n \rightarrow q \in \partial\Omega$, $q_n \in \Omega$, then $I(q_n) \rightarrow +\infty$, i. e. I is unbounded at the boundary $\partial\Omega$. Then, for $c \in \mathbb{R}$, $\bar{\varepsilon} > 0$ and U a neighborhood of $K_c = \{q \in \Omega: I(q) = c \text{ and } I'(q) = 0\}$, there is an $\varepsilon > 0$, $\varepsilon < \bar{\varepsilon}$ and a deformation $\varphi: \Omega \times I \rightarrow \Omega$ such that

- (1) $\varphi_0 = \text{identity}$, $\varphi_t: \Omega \rightarrow \Omega$ is a homeomorphism, $t \in I$.
- (2) $\varphi(q, t) = q$ if $|I(q) - c| \geq \bar{\varepsilon}$, $t \in I$.
- (3) $\varphi(q, 1) \in I^{c-\varepsilon}$ if $q \in I^{c+\varepsilon} - U$, where

$$I^a = \{q \in \Omega: I(q) < a\}.$$

- (4) If $K_c = \emptyset$ we may take $U = \emptyset$.

3.8. Remark. — The proof of Lemma 3.7 requires only a minor modification of the proof of theorem A.4 in [R2]. Following the notation in [R2], let

$$A \equiv \{u \in \Omega \mid I(u) \leq c - \hat{\varepsilon}\} \cup \{u \in E \mid I(u) \geq c + \hat{\varepsilon}\}$$

and

$$B \equiv \{u \in \Omega \mid c - \varepsilon \leq I(u) \leq c + \varepsilon\}$$

Then, A is closed in Ω but $A \cup (E - \Omega) = A_1$ is closed in E . B is closed in Ω but also closed in E because I is unbounded at $\partial\Omega$. Furthermore, $A_1 \cap B = \emptyset$ because $\varepsilon < \hat{\varepsilon}$. After requiring the cutoff function to be 0 on A_1 and I on B , the proof proceeds verbatim and the Ω is forced to be invariant under the flow.

Next, the corresponding abstract critical point theorem.

3.9. LEMMA. — Let Ω denote an open set in a Hilbert space E and $f: \Omega \rightarrow \mathbb{R}$ a C^1 functional which is bounded from below. Suppose f satisfies (PS) on Ω and f is unbounded at the boundary $\partial\Omega$. If $\text{cat } \Omega = +\infty$ then f possesses an unbounded sequence of critical values.

Proof. — The proof is quite standard after making a few remarks. First, since Ω is a Hilbert manifold, Ω possesses compact subsets of arbitrarily high category (see Section 2.) Thus, if we set $\Sigma_j = \{X \subset \Omega: \text{cat } X \geq j\}$ and

$$c_j = \inf_{\Sigma_j} \sup_X f(x), \quad j \geq 1$$

we see that the c_j are finite and $c_j \leq c_{j+1}$ with $c_1 = \inf_{\Omega} f$. The usual arguments apply to show that the c_j are all critical values, and $\lim c_j = +\infty$ (see [R2]).

Proof of Theorem 3.3. — The proof is an immediate application of Lemma 3.2 to Lemma 3.9.

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