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A nonexistence result for a nonlinear equation involving critical Sobolev exponent

by

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ABSTRACT. — Given any constant $C>0$, we show that there exists smooth bounded nonstarshaped domains $\mathbf{U}$ in $\mathbb{R}^N$ ($N \geqq 5$), such that the problem

$$\mathscr{P}_N(\mathbf{U}) \begin{cases} -\Delta u = u^{(N+2)/(N-2)} & \text{in } \mathbf{U}, \\ u=0 & \text{on } \partial\mathbf{U}, \\ u>0 & \text{in } \mathbf{U}, \end{cases}$$

has no solution u, whose energy, $\int_{\mathbf{U}} |\nabla u|^2$, is less than C.

Key words : Elliptic equations, concentration compactness principle, critical Sobolev exponents, moving plane principle.

RÉSUMÉ. — Étant donnée une constante $C>0$ arbitraire, nous montrons qu'il existe des ouverts bornés réguliers non étoilés $\mathbf{U}$ de $\mathbb{R}^N$ ($N \geqq 5$), tels que le problème

$$\mathscr{P}_N(\mathbf{U}) \begin{cases} -\Delta u = u^{(N+2)/(N-2)} & \text{dans } \mathbf{U}, \\ u=0 & \text{sur } \partial\mathbf{U}, \\ u>0 & \text{dans } \mathbf{U}, \end{cases}$$

ne possède pas de solution u, dont l'énergie, $\int_{\mathbf{U}} |\nabla u|^2$, est plus petite que C.

Classification A.M.S. : 35 A 99, 35 J 60, 35 B 40, 35 B 45.

INTRODUCTION

Let $\mathbf{U}$ be any smooth open bounded domain in $\mathbb{R}^N$. For $N \geqq 5$, consider the problem:

$$\mathscr{P}_N(\mathbf{U}) \begin{cases} -\Delta u = u^p & \text{in } \mathbf{U}, \\ u = 0 & \text{on } \partial\mathbf{U}, \\ u > 0 & \text{in } \mathbf{U}, \end{cases}$$

where $p = \dfrac{N+2}{N+2}$ is the critical Sobolev exponent.

It is well-known that if $\mathbf{U}$ is starshaped, $\mathscr{P}_N(\mathbf{U})$ has no solution [P] and if $\mathbf{U}$ has a nontrivial topology, Bahri and Coron [B.C] have shown that $\mathscr{P}_N(\mathbf{U})$ has a solution. On the other hand, Dancer [D_1] and independently Ding [D_2], were able to construct a contractible domain $\mathbf{D}$, such that $\mathscr{P}_N(\mathbf{D})$ has a solution.

Then, the question arises whether there exists an open domain $\mathbf{U}$, smooth, bounded and not starshaped, with a trivial topology, on which $\mathscr{P}_N(\mathbf{U})$ has no solution.

We define the energy $\mathbf{E}_\mathbf{U}(v)$, where $v \in H_0^1(\mathbf{U})$ as follows:

$$\mathbf{E}_\mathbf{U}(v) = \int_\mathbf{U} |\nabla v|^2.$$

We shall denote by S the Sobolev constant,

$$S = \inf_{u \in H_0^1(\mathbf{U}),\ \|u\|_{p+1} = 1} \int_\mathbf{U} |\nabla u|^2,$$

which does not depend on the choice of the domain $\mathbf{U}$.

The main results of our paper are the following:

THEOREM 1. — *Let* η *be any real number strictly less than* $S^{N/2}$. *Then there exists a bounded domain* $\mathscr{CL}$ *which is not starshaped such that* $\mathscr{P}_N(\mathscr{CL})$ *has no solution whose energy is less than* $2S^{N/2} - \eta$.

THEOREM 2. — *Assume* $5 \leqq N \leqq 8$. *Then for any constant* $C > S^{N/2}$, *there exists a bounded domain* $\mathscr{CL}$ *which is not starshaped such that* $\mathscr{P}_N(\mathscr{CL})$ *has no solution whose energy is less than* C.

These theorems call for a remark. We construct a nonstarshaped domain such that our problem has no solution with a prescribed bound for the energy. We believe the result to be true without the energy constraint. Also, the statement of Theorem 2 contains a technical condition on the dimension. This condition is used in estimates concerning the interaction terms (*see* Appendix **B** and [B]). We believe the result to be true for all dimensions, even in dimensions four and three.

This paper is divided in two parts. In the first part, we construct an explicit sequence of open sets $\mathbf{\Omega}_\varepsilon$ which are not starshaped and converge to the unit ball of $\mathbb{R}^N$. Using the method of "moving planes" of Alexandroff, in the same way as in [S], in [G.N.N] and in [HB.N], we give some geometrical properties of any solution of $\mathscr{P}_N(\boldsymbol{\Omega}_\varepsilon)$. In the second part, we suppose that $\mathscr{P}_N(\boldsymbol{\Omega}_\varepsilon)$ has a solution u_ε which satisfies $\mathbf{E}_{\Omega_\varepsilon}(u_\varepsilon) \leqq C$, C being a given constant. We use the concentration compactness principle introduced in [P.L.L] to study the behavior of u_ε. By the generalization of the method developed in [R.L], we analyze the location of the concentration points of u_ε, when ε goes to zero. Finally, a connection between the geometrical part and the concentration points is displaid. A contradiction comes out from those facts. Our $\mathscr{CL}$ is chosen to be $\boldsymbol{\Omega}_\varepsilon$, for ε small enough.

I. GEOMETRICAL PROPERTIES OF THE SOLUTIONS

A. Construction of Ω_ε

We set:

$$\mathbb{R}^N = \{ x = (x', x_N), x_N \in \mathbb{R}, x' \in \mathbb{R}^{N-1} \},$$

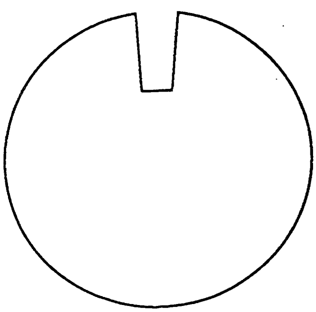
B will denote the open unit ball in $\mathbb{R}^N$ and we consider the points $P = (0, 1)$ and $M_\rho = (0, \rho)$, where $\rho < -1$ is a fixed constant. For $\varepsilon > 0$, $B(P, \varepsilon)$ is the ball centered at P with radius ε (which is going to be small), C_ε is the closed cone with vertex M_ρ consisting of all those rays which intersect the sphere $\partial B(P, \varepsilon)$ in other words:

$$C_\varepsilon = \left\{ (x', x_N), |x'| \leqq \frac{\varepsilon (x_N - \rho)}{1 - \rho} \right\}.$$

Then, l being a fixed constant in $]0, 1[$, we define the required Ω_ε as follows:

$$\Omega_\varepsilon = B \setminus (C_\varepsilon \cap \{ x \in \mathbb{R}^N, x_N \geqq l \}).$$

For each ε small enough, Ω_ε has a trivial topology, is not starshaped and not conformal to a starshaped domain. By smoothing the corners, we may work as if Ω_ε were a smooth domain without changing the nature of our arguments.



The picture of a projection of Ω_ε.

B. The moving planes principle

In what follows, we suppose that $\mathscr{P}_N(\Omega_\varepsilon)$ has a solution, denoted by u. The classical results of regularity [B.K] say that $u \in C^{1,\alpha}(\Omega_\varepsilon)$. Next we have:

LEMMA 2. — *Let $x_0 \in \Omega_\varepsilon$ be such that:*

$$u(x_0) = \|u\|_{L^\infty},$$

then:

$$x_0 \in \{x_N \leqq 0\} \cap \Omega_\varepsilon.$$

We postpone the proof of this lemma until the end of this section. We start by introducing some notations. Let λ be any nonnegative real number. Then we denote:

$$T_\lambda = \{x_N = \lambda\},$$
$$\Sigma^\lambda = \Omega_\varepsilon \cap \{x_N > \lambda\},$$
$$x_\lambda = (x', 2\lambda - x_N), \qquad \text{where} \quad x = (x', x_N),$$

x^λ is the reflection of x across T_λ,

$$\Lambda = \left\{ \lambda \in [0, (1-\varepsilon^2)^{1/2}[, \forall x \in \Sigma^\lambda,\ u(x) \leqq u(x^\lambda) \text{ and } \frac{\partial u}{\partial x_N}(x) < 0 \right\}.$$

LEMMA 3. — *Let Λ be defined as above. Then we have:*

$$\Lambda \neq \varnothing.$$

Proof. — By the Hopf Lemma [G.N.N], it follows that:

$$\frac{\partial u}{\partial x_N} < 0 \quad \text{on } \partial\Omega_\varepsilon \diagdown \partial B,$$

and by the Serrin Lemma:

$$\forall A \in \partial B \cap C_\varepsilon \cap \{x_N \geqq 0\},$$

either

$$\frac{\partial u}{\partial x_N}(A) < 0 \qquad \text{or} \qquad \frac{\partial u}{\partial x_N}(A) = 0, \qquad \frac{\partial^2 u}{\partial x_N^2}(A) < 0.$$

Then, for all points A of $\partial B \cap C_\varepsilon \cap \{x_N \geqq 0\}$, there is some $\varepsilon(A) > 0$ such that:

$$\frac{\partial u}{\partial x_N} < 0 \quad \text{in } B(A, \varepsilon(A)) \cap \Omega_\varepsilon.$$

On the other hand, by compactness there is a finite number of points $A_1, \ldots, A_q$ in $\partial B \cap C_\varepsilon \cap \{x_N \geqq 0\}$ such that:

$$\partial B \cap C_\varepsilon \cap \{x_N \geqq 0\} \subseteqq \bigcup_{k=1}^{q} B(A_k, \varepsilon(A_k)).$$

We set:

$$B_k = B(A_k, \varepsilon(A_k)) \cap \Omega_\varepsilon.$$

Consider k and j such that $B_k \cap B_j \neq \varnothing$. We define:

$$r_{k,j} = \operatorname{Max}\{d(x, \partial B \cap C_\varepsilon \cap \{x_N \geqq 0\}), x \in B_k \cap B_j\},$$
$$\delta = \operatorname{Min}\{r_{k,j}, k, j \text{ such that } B_k \cap B_j \neq \varnothing\}.$$

Then, it is easy to verify that:

$$1 - \frac{\delta}{2} \in \Lambda,$$

which proves the lemma.

Now, let $\lambda \in \Lambda$ and $x \in \Sigma^\lambda$. We set:

$$v(x) = u(x^\lambda),$$
$$w_\lambda(x) = v(x) - u(x).$$

PROPOSITION 4. – *If* $w_\lambda \not\equiv 0$ *in* Σ^λ, *then:*

$$w_\lambda > 0 \quad \text{in } \Sigma^\lambda,$$

and

$$\frac{\partial u}{\partial x_N} < 0 \quad \text{on } T_\lambda \cap \Omega_\varepsilon.$$

Proof. – Let $c(x)$ be defined by:

$$c(x) = -\frac{v^p(x) - u^p(x)}{v(x) - u(x)}.$$

Since v still satisfies: $-\Delta v = v^p$ in Σ^λ, and we have chosen λ in Λ, w_λ satisfies:

$$-\Delta w_\lambda + c(x) w_\lambda = 0 \quad \text{in } \Sigma^\lambda,$$
$$w_\lambda \geqq 0 \quad \text{in } \Sigma^\lambda,$$
$$w_\lambda = 0 \quad \text{on } T_\lambda \cap \Omega_\varepsilon.$$

The function $c(x)$ is clearly a continuous function. Consequently by the strong maximum principle, we obtain the fact that: $w_\lambda > 0$ in Σ^λ. On the other hand, again by the Hopf Lemma, we see that: $\dfrac{\partial w_\lambda}{\partial x_N} > 0$ in $T_\lambda \cap \bar{\Omega}_\varepsilon$.

Since the following equality holds:

$$\frac{\partial w_\lambda}{\partial x_N} = -2\frac{\partial u}{\partial x_N},$$

the result follows.

COROLLARY 5. — *Let* $\lambda \in \Lambda$, *such that* $\lambda > 0$. *Then with the notations introduced above*, $w_\lambda > 0$ *in* Σ^λ.

Proof. — According to Proposition 4, it suffices to show that there exists a point $y_0 \in \Sigma^\lambda$ such that: $w_\lambda(y_0) \neq 0$. Let $x_n \in \Sigma^\lambda$ be a sequence which converges to some point $x \in \partial\Omega_\varepsilon$. Because $\lambda > 0$, $x^\lambda \notin \partial\Omega_\varepsilon$. Then, it is obvious that: $u(x^\lambda) > 0$. On the other hand:

$$u(x_n) \xrightarrow[n \to \infty]{} u(x) = 0,$$
$$u(x_n^\lambda) \xrightarrow[n \to \infty]{} u(x^\lambda) > 0.$$

This shows that for n large enough, $w_\lambda(x_n) > 0$.

We consider now:

$$\mu = \inf \Lambda$$

In order to prove Lemma 2, we have to establish that $\mu = 0$. We start with:

LEMMA 6. — *Let* μ *be defined as above. Then* $\mu \in \Lambda$.

Proof. — By definition, $\mu > 0$ and there exists a sequence λ_k such that:

$$\lambda_k \xrightarrow[k \to \infty]{} \mu, \qquad \lambda_k > 0, \qquad \lambda_k \in \Lambda.$$

Let x be any point in Σ^μ. Then clearly, there is k_0 such that, $\forall k \geqq k_0$, $x \in \Sigma^{\lambda_k}$. It follows that:

$$\frac{\partial u}{\partial x_N}(x) < 0 \qquad \text{and} \qquad u(x) < u(x^{\lambda_k}).$$

Clearly, passing to the limit: $u(x) \leqq u(x^\mu)$, and the lemma is proved.

We are now ready to prove Lemma 2. Arguing by contradiction, we suppose that $\mu \neq 0$. Then there is a non decreasing sequence μ_k of strictly positive reals, a sequence of points $x_k \in \Sigma^{\mu_k}$ such that

$$u(x_k) > u(x_k^{\mu_k}),$$
$$u_k \xrightarrow[k \to \infty]{} \mu.$$

Let x be a limit point (passing to subsequence) of x_k. Then, $x \in \bar{\Sigma}^\mu$, and consequently by Lemma 6,

$$u(x) \leqq u(x^\mu).$$

It follows that:

$$u(x)=u(x^{\mu}).$$

Then, by Lemma 6 and Corollary 5 of Proposition 4, we have necessarily:

$$x\in T_{\mu}.$$

On the other hand, for every integer k, there exists ξ_k on the line segment $[x_k, x_k^{\mu_k}]$ such that:

$$\frac{u(x_k)-u(x_k^{\mu_k})}{2((x_k)_N-\mu_k)}=\frac{\partial u}{\partial x_N}(\xi_k).$$

According to Proposition 4 and Lemma 6,

$$\exists\, k_0,\quad \forall\, k\geqq k_0,\qquad \frac{\partial u}{\partial x_N}(\xi_k)<0,$$

and consequently:

$$\forall\, k\geqq k_0,\quad u(x_k)<u(x_k^{\mu_k}).$$

This is a contradiction, showing that $\mu=0$, and Lemma 2 is proved.

Applying the technique of the moving plane in all directions, one is led to:

THEOREM 7. — *There exists a compact set* $\mathbf{K}\subset B\cap\{x_N\leqq 0\}$, *which does not depend on* ε, *such that for all* $\varepsilon>0$, *for all solutions* u *of* $\mathscr{P}_N(\boldsymbol{\Omega}_\varepsilon)$:

$$\forall\, x\in\Omega_\varepsilon \textit{ such that } \nabla u(x)=0, \quad \textit{then } x\in\mathbf{K}.$$

In order words, all critical points of the solution u of the problem $\mathscr{P}_N(\boldsymbol{\Omega}_\varepsilon)$ are contained in a compact set **K**, which does not depend on ε, and which lies in the lower half ball. For the proof, apply the same procedure as in Lemma 2, but in all possible directions.

II. AN APPLICATION OF THE CONCENTRATION COMPACTNESS PRINCIPLE

We are now in position to prove Theorems 1 and 2. We shall suppose that $\mathscr{P}_N(\boldsymbol{\Omega}_\varepsilon)$ has a solution u_ε, whose energy is bounded by a constant C which does not depend on ε. From the facts that

$$B\diagdown\{x=(0,x_N),\ l\leqq x_N\leqq 1\}$$

has the same capacity as B and that $\mathscr{P}_N(B)$ has no solution by Pohozaev's identity, it follows that the sequence u_ε, extended to B by zero outside Ω_ε, converges weakly to zero in $H_0^1(B)$. Arguing as in the first part of [R.L], which relies on the work of P. L. Lions, it is easily seen that:

$$\exists\, a_1, a_2, \ldots, a_k \in \bar{\mathrm{B}}, \quad \text{such that} \,|\nabla u_\varepsilon|^2 \underset{\varepsilon \to 0}{\longrightarrow} \mathrm{S}^{N/2} \sum_{j=1}^{j=k} \delta_{a_j},$$

where δ_{a_j} is the Dirac mass at a_j, and the convergence is the weak convergence of measures. We shall say that a_j is a concentration point of u_ε. We start with a uniform convergence result:

THEOREM 8. — *Let* K *be any compact subset of* B, *which does not contain the points* a_j, $j=1, \ldots, k$. *Then, the sequence* u_ε *converges uniformly to zero on the compact* K.

Proof. — This is a direct application of Theorem A.2, proved in Appendix A, which is in the same spirit as the regularity result of Brezis-Kato [B.K].

From the equation satisfied by u_ε, it follows that:

$$u_\varepsilon^{2^*} \underset{\varepsilon \to 0}{\longrightarrow} \mathrm{S}^{N/2} \sum_{j=1}^{j=k} \delta_{x_j},$$

where $2^* = p+1 = \dfrac{2\mathrm{N}}{\mathrm{N}-2}$.

Let K′ be an other compact subset which strictly contains K and does not contain the points a_j. From (1) it follows that:

$$\int_{\mathrm{K}'} u_\varepsilon^{2^*} \underset{\varepsilon \to 0}{\longrightarrow} 0.$$

Then apply Theorem A.2 with $a_\varepsilon = u_\varepsilon^{p-1}$.

A direct and basic consequence of Theorem 8 combined with Theorem 7 can be stated as follows:

PROPOSITION 9. — *Let* **K** *be the compact set introduced in Theorem* 8. *Then all the concentration points of the sequence* u_ε *are contained in* **K**.

We now set;

$$\delta(a, \lambda) = \frac{c_0 \lambda^{(\mathrm{N}-2)/2}}{(\lambda^2 + |x-a|^2)^{(\mathrm{N}-2)/2}},$$

where c_0 is such that:

$$-\Delta\delta(a, \lambda) = \delta(a, \lambda)^p.$$

We denote by $\mathrm{P}_\varepsilon \delta(a, \lambda)$ the orthogonal projection onto $\mathrm{H}_0^1(\Omega_\varepsilon)$ of the functions $\delta(a, \lambda)$; that is the unique solution of the problem:

$$\begin{cases} -\Delta \mathrm{P}_\varepsilon \delta(a, \lambda) = -\Delta\delta(a, \lambda) & \text{in } \Omega_\varepsilon, \\ \mathrm{P}_\varepsilon \delta(a, \lambda) = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases}$$

The following statement describes the decomposition of the function u_ε in terms of the functions $P_\varepsilon \delta(a, \lambda)$. This result is now classical in the context of the critical Sobolev exponent, and the reader can consult [B] and [B.C]. Here, the assumption on the bound of the energy of u_ε is essential.

THEOREM 10. — *There exists an integer k', a sequence $(a_{1,\varepsilon}, \ldots, a_{k',\varepsilon})$ included in $(\Omega_\varepsilon)^{k'}$, a sequence $(\lambda_{1,\varepsilon}, \ldots, \lambda_{k',\varepsilon})$ in $\mathbb{R}_+^{k'}$, a sequence v_ε in $H_0^1(\Omega_\varepsilon)$, whose norm in $H_0^1(\Omega_\varepsilon)$ goes to zero as $\varepsilon \to 0$ such that:*

(i) $k' \geqq k$ (see *Theorem* 8)*;*

(ii) $\forall i=1, \ldots, k'$, $\exists j_i$ *such that* $a_{i,\varepsilon} \xrightarrow[\varepsilon \to 0]{} a_{j_i}$*;*

(iii) $u_\varepsilon = \sum_{i=1}^{i=k'} P_\varepsilon \delta(a_{i,\varepsilon}, \lambda_{i,\varepsilon}) + v_\varepsilon$*;*

(iv) $\forall\ i=1, \ldots, k'$:

$$(P_\varepsilon \delta(a_{i,\varepsilon}, \lambda_{i,\varepsilon}), v_\varepsilon) = \left(\frac{\partial P_\varepsilon \delta(a_{i,\varepsilon}, \lambda_{i,\varepsilon})}{\partial \lambda}, v_\varepsilon\right) = \left(\frac{\partial P_\varepsilon \delta(a_{i,\varepsilon}, \lambda_{i,\varepsilon})}{\partial a}, v_\varepsilon\right) = 0;$$

(v) $\forall i=1, \ldots, k'$:

$$\lambda_{i,\varepsilon}\, d(a_{i,\varepsilon}, \partial\Omega_\varepsilon) \xrightarrow[\varepsilon \to 0]{} \infty;$$

(vi) $\forall i \neq j$:

$$\varepsilon_{i,j} = \left(\frac{\lambda_{i,\varepsilon}}{\lambda_{j,\varepsilon}} + \frac{\lambda_{j,\varepsilon}}{\lambda_{i,\varepsilon}} + \lambda_{i,\varepsilon}\lambda_{j,\varepsilon}\, d^2(a_{i,\varepsilon}, a_{j,\varepsilon})\right)^{-1} \xrightarrow[\varepsilon \to 0]{} 0.$$

In (iv), the expressions in the parenthesis denote the scalar product in the space $H_0^1(\Omega_\varepsilon)$.

Assume first that $E_{\Omega_\varepsilon}(u_\varepsilon) \leqq 2\, S^{N/2} - \eta$, η being as in Theorem 1. Then, arguing as in [R.L], using Theorems 7 and Proposition 9, we find that $k = k' = 1$. Then, we obtain a contradiction between Proposition 9 and Lemma 10 in [R.L], which states that $d(a_{1,\varepsilon}, \Omega_\varepsilon) \to 0$ as $\varepsilon \to 0$, and Theorem 1 is proved.

To conclude, the generalization of the techniques used in [R.L] leads to the following:

PROPOSITION 11. — *Assume $5 \leqq N \leqq 8$. Then there exists an index i_0, a sequence ε_n which goes to zero as $n \to +\infty$, such that the sequence $d(a_{i_0,\varepsilon_n}, \Omega_{\varepsilon_n}) \to 0$ as $n \to +\infty$. In other words the sequence a_{i_0,ε_n} cannot converge to a point of the compact set* **K**.

The proof of this proposition, which is rather technical, is given in Appendix **B**.

The contradiction between Proposition 11 and Proposition 9 is now clear, and shows that for ε small enough, the problem $\mathscr{P}_N(\Omega_\varepsilon)$ cannot have a solution, which is the claim of Theorem 2.

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APPENDIX A

Let Ω be a smooth bounded domain in $\mathbb{R}^N$. We consider a function $a(x)$ in the space $L^{N/2}(\Omega)$ and we denote by P_a the problem:

$$P_a \begin{cases} -\Delta u = a(x)\, u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We recall the following result due to Brezis and Kato [B.K]:

THEOREM A.1. – *Let $u \in H_0^1(\Omega)$ be any solution of P_a. Then for all $t \geqq 1$, u is in $L^t(\Omega)$.*

Let $a_\varepsilon \geqq 0$ be a sequence in the space $L^{N/2}(\Omega)$, a compact subset K of Ω, such that:

$$\int_K a_\varepsilon^{N/2}(x)\, dx \to 0, \quad \text{as } \varepsilon \to 0. \tag{A.1}$$

We have:

THEOREM A.2. – *Let $u_\varepsilon \in H_0^1(\Omega)$ solution of P_{a_ε} such that the sequence (u_ε) converges weakly to 0 in $H_0^1(\Omega)$, and $u_\varepsilon > 0$. Then for every compact subset K' of* int K *and for all real number $t \geqq 1$ we have:*

$$\int_{K'} u_\varepsilon^t \to 0, \quad \textit{as } \varepsilon \to 0. \tag{A.2}$$

Proof. – Let K' be a compact subset of int K. We consider a C^∞ cut-off function φ, such that:

$$0 \leqq \varphi \leqq 1,$$
$$\varphi = 1 \quad \text{on } K',$$
$$\varphi = 0 \quad \text{on } \Omega \setminus K.$$

Let α be any real number greater than 1. By theorem A.1 and the fact that $u_\varepsilon > 0$, the function $\varphi u_\varepsilon^\alpha$ belongs to $H_0^1(\Omega)$. Hence, it makes sens to multiply P_{a_ε} by $\varphi u_\varepsilon^\alpha$, which leads to:

$$\int_K -\Delta(u_\varepsilon)\varphi u_\varepsilon^\alpha = \int_K a_\varepsilon \varphi u_\varepsilon^{\alpha+1}. \qquad (A.3)$$

By Hölder inequality:

$$\int_K a_\varepsilon \varphi u_\varepsilon^{\alpha+1} \leqq \| a_\varepsilon \|_{L^{N/2}(K)} \| \varphi^{1/(\alpha+1)} u_\varepsilon \|^{\alpha+1}_{(2^*/2)(\alpha+1)}, \qquad (A.4)$$

where $2^* = \dfrac{2N}{N-2}$.

On the other hand, by integrating by parts, we obtain:

$$\int_K -\Delta(u_\varepsilon)\varphi u_\varepsilon^\alpha = \frac{4\alpha}{(\alpha+1)^2}\int_K |\nabla(\varphi^{1/2} u_\varepsilon^{(\alpha+1)/2})|^2 - \frac{\alpha-1}{\alpha+1}\int_K \nabla u_\varepsilon . \nabla\varphi . u_\varepsilon^\alpha - \frac{4\alpha}{(\alpha+1)^2}\int_K u_\varepsilon^{\alpha+1}|\nabla\varphi^{1/2}|^2. \qquad (A.5)$$

In the same way:

$$\int_K \nabla u_\varepsilon . \nabla\varphi . u_\varepsilon^\alpha = \frac{1}{\alpha+1}\int_K u_\varepsilon^{\alpha+1}(-\Delta\varphi). \qquad (A.6)$$

On the other hand, by Sobolev inequality,

$$\int_K |\nabla(\varphi^{1/2} u_\varepsilon^{(\alpha+1)/2})|^2 \geqq S \| \varphi^{1/(\alpha+1)} u_\varepsilon \|^{\alpha+1}_{(2^*/2)(\alpha+1)},$$

where S is the Sobolev constant.

Now, using the fact that φ is a C^∞ function, combining (A.3) to (A.6), we see that there exists $C(K, K', \alpha) > 0$ such that:

$$\| \varphi^{1/(\alpha+1)} u_\varepsilon \|^{\alpha+1}_{(2^*/2)(\alpha+1)} \leqq \| a_\varepsilon \|_{L^{N/2}(K)} \| \varphi^{1/(\alpha+1)} u_\varepsilon \|^{\alpha+1}_{(2^*/2)(\alpha+1)} + C(K, K', \alpha)\int_K u_\varepsilon^{\alpha+1}.$$

By (A.1), we can choose ε small enough such that

$$\| a_\varepsilon \|_{L^{N/2}(K)} \leqq \frac{1}{2},$$

which leads to:

$$\| u_\varepsilon \|^{\alpha+1}_{(2^*/2)(\alpha+1)}(K') \leqq 2C(K, K', \alpha)\int_K u_\varepsilon^{\alpha+1}.$$

But we assume that u_ε converges weakly to 0 in $H_0^1(\Omega)$. Hence, if we choose any α such that:

$$\alpha < 2^* - 1,$$

we see that:

$$\int_K u_\varepsilon^{\alpha+1} \to 0, \quad \text{as } \varepsilon \to 0.$$

This proves that:

$$\| u_\varepsilon \|_{(2^*/2)(\alpha+1)}^{\alpha+1} (K') \to 0, \quad \text{as } \varepsilon \to 0.$$

Now, taking into account that $\frac{2^*}{2} > 1$, we can reiterate the process, and the proof is complete.

APPENDIX B

We prove in this appendix Proposition 11. We use the notations of part II, and in particular Theorem 10. These computations, which take into account the interaction between the singularities, were originally introduced by Bahri and Coron ([B] and [B.C]), who were the pionneers of those kinds of arguments.

As in [R.L], we start with an estimate of the $H_0^1(\Omega_\varepsilon)$ norm of v_ε. The following holds:

LEMMA B.1. — *We have the estimate*:

$$\| v_\varepsilon \| H_0^1(\Omega_\varepsilon) = \begin{cases} O(\sum_{i \neq j} \varepsilon_{i,j}^{3/2} |\operatorname{Log} \varepsilon_{i,j}|^{3/2} + \sum_i \xi_{i,\varepsilon}^3) \ \textit{if}\ N=5, \\ O(\sum_{i \neq j} \varepsilon_{i,j}^2 |\operatorname{Log} \varepsilon_{i,j}|^2 + \sum_i |\operatorname{Log} \xi_{i,\varepsilon}|^{2/3} \xi_{i,\varepsilon}^4) \ \textit{if}\ N=6, \\ O(\sum_{i \neq j} \varepsilon_{i,j}^2 |\operatorname{Log} \varepsilon_{i,j}|^2 + \sum_i \xi_{i,\varepsilon}^{(N+2)/2}) \ \textit{if}\ N>6, \end{cases}$$

where $\xi_{i,\varepsilon} = \dfrac{1}{d(a_{i,\varepsilon}, \partial\Omega_\varepsilon) \lambda_{i,\varepsilon}}$.

Proof. — In what follows, we do not write systematically the index ε, and we shall write $P\delta_i$ for $P_\varepsilon \delta(a_{i,\varepsilon}, \lambda_{i,\varepsilon})$. We multiply the equation satisfied by u, whose expression is given by (iii) in Theorem 10, by v and obtain, taking into account (iv):

$$\int |\nabla v|^2 = \int (\sum_i P\delta_i + v)^p v. \tag{B.1}$$

On the other hand, we have:

$$(\sum_i P\delta_i + v)^p = (\sum_i P\delta_i)^p + p(\sum_i P\delta_i)^{p-1} v + O(|v|^p) + (\sum_i P\delta_i)^{p-2} \operatorname{Inf}\{(\sum_i P\delta_i)^2, v^2\},$$

which leads to:

$$\int (\sum_i \mathrm{P}\,\delta_i + v)^p v = \int (\sum_i \mathrm{P}\,\delta_i)^p v + p \int (\sum_i \mathrm{P}\,\delta_i)^{p-1} v^2$$
$$+ O\left(\int |v|^{p+1} + \int (\sum_i \mathrm{P}\,\delta_i)^{p-2} \operatorname{Inf}\{(\sum_i \mathrm{P}\,\delta_i)^2, v^2\} v\right). \quad (\mathrm{B}.2)$$

On the other hand, we know by [B] that there exists a constant ρ, which just depends on the dimension, such that:

$$\rho \int |\nabla v|^2 \leqq \int |\nabla v|^2 - p \sum_i \int \delta_i^{p-1} v^2. \quad (\mathrm{B}.3)$$

We now combine (B.1), (B.2) and (B.3) and obtain:

$$\rho \int |\nabla v|^2 \leqq \int (\sum_i \mathrm{P}\,\delta_i)^p v - p \sum_i \int \delta_i^{p-1} v^2 + p \int (\sum_i \mathrm{P}\,\delta_i)^{p-1} v^2$$
$$+ O\left(\int |v|^{p+1} + \int (\sum_i \mathrm{P}\,\delta_i)^{p-2} \operatorname{Inf}\{(\sum_i \mathrm{P}\,\delta_i)^2, v^2\} v\right). \quad (\mathrm{B}.4)$$

We now study each of the terms appearing in (B.4). Using the usual expansion, we have:

$$\int (\sum_i \mathrm{P}\,\delta_i)^p v = \int (\sum_i \mathrm{P}\,\delta_i^p) v + O\left(\sum_{i \neq j} \int \mathrm{P}\,\delta_i^{p-1} \operatorname{Inf}(\mathrm{P}\,\delta_i, \mathrm{P}\,\delta_j) v\right). \quad (\mathrm{B}.5)$$

We set:

$$\varphi_i = \delta_i - \mathrm{P}\,\delta_i.$$

By orthogonality,

$$\int \delta_i^p v = \int \nabla \mathrm{P}\,\delta_i \nabla v = 0,$$

then:

$$\int \mathrm{P}\,\delta_i^p v = O\left(\int \varphi_i \delta_i^{p-1} v\right). \quad (\mathrm{B}.6)$$

We shall denote by B_i the ball centered at a_i, whose radius is $d_i = d(a_i, \partial\Omega)$. We have:

$$\mathrm{J}_i = \int_\Omega \varphi_i \delta_i^{p-1} |v| = \int_{\mathrm{B}_i} \varphi_i \delta_i^{p-1} |v| + \int_{\Omega \setminus \mathrm{B}_i} \varphi_i \delta_i^{p-1} |v|.$$

By the Maximum Principle:

$$0 \leqq \varphi_i \leqq \delta_i \qquad \text{and} \qquad \|\varphi_i\|_{\mathrm{L}^\infty} = O(\xi_i^{(\mathrm{N}-2)/2} d_i^{(2-\mathrm{N})/2}),$$

so that, as in [R.L], we obtain:

$$\int_\Omega \varphi_i \delta_i^{p-1} |v| = \begin{cases} O(\xi_i^3) & \text{if } N=5, \\ O(|\operatorname{Log} \xi_i|^{2/3} \xi_i^4) & \text{if } N=6, \\ O(\xi_i^{(N+2)/2}) & \text{if } N>6. \end{cases} \tag{B.7}$$

We look at the interaction term:

$$\left| \sum_{i \neq j} \int \mathrm{P}\,\delta_i^{p-1} \operatorname{Inf}(\mathrm{P}\,\delta_i, \mathrm{P}\,\delta_j)\, v \right|,$$

which is less than

$$\sum_{i \neq j} \int \mathrm{P}\,\delta_i^{p-1} \mathrm{P}\,\delta_j |v| = O\left(\| v \| \sum_{i \neq j} \left\{ \int \mathrm{P}\,\delta_i^{((p-1)(p+1))/P}\, \mathrm{P}\,\delta_j^{(p+1)/p} \right\}^{p/(p+1)} \right).$$

But for any strictly positive real number α, we clearly have:

$$\mathrm{P}\,\delta_i^\alpha \leqq 2^\alpha \delta_i^\alpha,$$

(use the inequality satisfied by φ_i) and by [B], we know that:

$$\left\{ \int \delta_i^{((p-1)(p+1))/p}\, \delta_j^{(p+1)/p} \right\}^{p/(p+1)} = O(\varepsilon_{i,j}^{((N+2)(N-2)\theta)/4N} |\operatorname{Log} \varepsilon_{i,j}|^{((N+2)(N-2)\theta)/4N}),$$

where $\theta = \min\left(\dfrac{(p-1)(p+1)}{p}, \dfrac{p+1}{p} \right)$, which leads to:

$$\sum_{i \neq j} \int \mathrm{P}\,\delta_i^{p-1} \operatorname{Inf}(\mathrm{P}\,\delta_i, \mathrm{P}\,\delta_j)\, v = O(\| v \| \varepsilon_{i,j}^{((N+2)(N-2)\theta)/4N} |\operatorname{Log} \varepsilon_{i,j}|^{((N+2)(N-2)\theta)/4N}). \tag{B.8}$$

On the other hand:

$$p \int (\sum_i \mathrm{P}\,\delta_i)^{p-1} v^2 = p \int (\sum_i \mathrm{P}\,\delta_i^{p-1}) v^2 + O\left\{ \int (\sum_i \mathrm{P}\,\delta_i)^{p-2} \operatorname{Inf}(\mathrm{P}\,\delta_i, \mathrm{P}\,\delta_j)\, v^2 \right\}. \tag{B.9}$$

Expanding $\mathrm{P}\,\delta_i^{p-1}$, we have:

$$\int \mathrm{P}\,\delta_i^{p-1} v^2 = \int \delta_i^{p-1} v^2 + O\left(\int \delta_i^{p-2} \varphi_i v^2 \right).$$

Looking at (B.4), we know that the term $\int \delta_i^{p-1} v^2$ vanishes. On the other hand, by Sobolev and Hölder inequalities, we have:

$$\int \delta_i^{p-2} \varphi_i v^2 = O\left\{ \|v\|^2 \left(\int \varphi_i^{(p+1)/(p-1)} \delta_i^{((p-2)(p+1))/(p-1)} \right)^{(p-1)/(p+1)} \right\}, \quad (\text{B}.10)$$

and it is easy to check that

$$f_i = \left(\int \varphi_i^{(p+1)/(p-1)} \delta_i^{((p-2)(p+1))/(p-1)} \right)^{(p-1)/(p+1)}$$

is a quantity which goes to zero as ε goes to zero. It remains the term:

$$\int (\sum_i \mathrm{P}\,\delta_i)^{p-2} \operatorname{Inf}\{ (\sum_i \mathrm{P}\,\delta_i)^2, v^2 \} v = \int_{\{\sum_i \mathrm{P}\,\delta_i \leqq |v|\}} (\sum_i \mathrm{P}\,\delta_i)^p |v| + \int_{\{\sum_i \mathrm{P}\,\delta_i \leqq |v|\}} (\sum_i \mathrm{P}\,\delta_i)^{p-2} |v|^3 = O(\|v\|^{p+1} + \|v\|^3). \quad (\text{B}.11)$$

Combining (B.1), . . ., (B.6), (B.8), . . ., (B.11) we obtain the fact that there exist two constants C_1 and C_2, which do not depend on ε such that:

$$\|v\| (\rho + \sum_i f_i - \mathrm{C}_1 \|v\|^{4/(\mathrm{N}-2)} - \mathrm{C}_2 \|v\|) \leqq \sum_i \mathrm{J}_i + O(\varepsilon_{i,j}^{(\mathrm{N}+2)/\mathrm{N}(\mathrm{N}-2)} |\operatorname{Log} \varepsilon_{i,j}|^{((\mathrm{N}+2)\,\theta)/2}),$$

and there exists $r > 0$ such that for ε small enough,

$$\rho + \sum_i f_i - \mathrm{C}_1 \|v\|^{4/(\mathrm{N}-2)} - \mathrm{C}_2 \|v\| \geqq r,$$

which concludes the proof of Lemma B.1.

In order to prove Proposition 11, we establish:

LEMMA B.2. — *Suppose* $5 \leqq \mathrm{N} \leqq 8$. *Then there exist a sequence* ε_n *which goes to zero as* $n \to +\infty$, *and an index* i_0 *such that* $d(a_{i_0, \varepsilon_n}, \partial\Omega_{\varepsilon_n})$ *goes to zero as* $n \to +\infty$.

Proof. – As in lemma B.1, we shall not write the index ε. We start with the equation satisfied by v, always using the same expansions:

$$-\Delta v = p \sum_i \varphi_i \delta_i^{p-1} + O\Big(|\varphi_i|^p + \varphi_i^2 \delta_i^{p-2} + |v|^p + \sum_{i \neq j} \mathrm{P}\,\delta_i^{p-1} \operatorname{Inf}(\mathrm{P}\,\delta_i, \mathrm{P}\,\delta_j)\Big) - O\Big(\sum_i \mathrm{P}\,\delta_i^{p-1} \operatorname{Inf}(v, \mathrm{P}\,\delta_i)\Big). \quad (\mathrm{B}.12)$$

Given an index k, we multiply (B.12) by $\dfrac{\partial \mathrm{P}\,\delta_k}{\partial \lambda}$ and integrate over Ω. Taking into account the relations of orthogonality, we get:

$$\left|\int \varphi_k \delta_k^{p-1} \frac{\partial \mathrm{P}\,\delta_k}{\partial \lambda}\right| \leqq \mathrm{C}\Bigg\{\left|\int \sum_{i \neq k} \theta_i \delta_i^{p-1} \frac{\partial \mathrm{P}\,\delta_k}{\partial \lambda}\right| + \left|\int \sum_i \varphi_i^p \frac{\partial \mathrm{P}\,\delta_k}{\partial \lambda}\right| + \left|\int \sum_{i \neq k} \varphi_i^2 \delta_i^{p-2} \frac{\partial \mathrm{P}\,\delta_k}{\partial \lambda}\right| + \left|\int |v|^p \frac{\partial \mathrm{P}\,\delta_k}{\partial \lambda}\right| + \sum_{i \neq j} \left|\int \mathrm{P}\,\delta_i^{p-1} \operatorname{Inf}(\mathrm{P}\,\delta_i, \mathrm{P}\,\delta_j) \frac{\partial \mathrm{P}\,\delta_k}{\partial \lambda}\right| + \sum_i \left|\int \mathrm{P}\,\delta_i^{p-1} \operatorname{Inf}(v, \mathrm{P}\,\delta_i) \frac{\partial \mathrm{P}\,\delta_k}{\partial \lambda}\right|\Bigg\}, \quad (\mathrm{B}.13)$$

where C is a positive constant. We recall some of the estimates of [R.L]:

$$\int \varphi_k \delta_k^{p-1} \frac{\partial \mathrm{P}\,\delta_k}{\partial \lambda} = \frac{\mathrm{C_N}}{\lambda^{\mathrm{N}-1}} \mathrm{H}(a_k, a_k) + O\left(\frac{\xi_k^{\mathrm{N}-1}}{\lambda_k}\right) + o\left(\frac{1}{\lambda_k^{\mathrm{N}-1}}\right), \quad (\mathrm{B}.14)$$

where $\mathrm{C_N}$ is a constant which just depends on N, H is the regular part of the Green's function, solution of the problem:

$$\left.\begin{aligned} &-\Delta_y \mathrm{H}(x,.) = 0 \quad \text{in } \Omega, \\ &\mathrm{H}(x,y) = \frac{1}{|x-y|^{n-2}} \quad \text{on } \partial\Omega. \\ &\int \varphi_i^2 \delta_i^{p-2} \frac{\partial \mathrm{P}\,\delta_i}{\partial \lambda} = O\left(\frac{\xi_i^{\mathrm{N}-1}}{\lambda_i}\right), \\ &\int v\, \delta_i^{p-1} \frac{\partial \mathrm{P}\,\delta_i}{\partial \lambda} = O\left(\|v\| \frac{\xi_i^{(\mathrm{N}-2)/2}}{\lambda_i}\right), \\ &\int |v|^p \left|\delta_i^{p-1} \frac{\partial \mathrm{P}\,\delta_i}{\partial \lambda}\right| = O\left(\|v\|^p \frac{\xi_i^{(\mathrm{N}-2)/2}}{\lambda_i}\right). \end{aligned}\right\} \quad (\mathrm{B}.15)$$

We now study the interaction terms. From [B], we have:

$$\int \varphi_i \delta_i^{p-1} \frac{\partial P\delta_k}{\partial\lambda}$$
$$= \frac{1}{\lambda_k} O\left\{ \frac{1}{\lambda_i^{(N-2)/2} \lambda_k^{(N-2)^2/2(N+2))}} \left(\varepsilon_{i,k}^{4/(N+2)} + \frac{1}{(\lambda_i \lambda_k)^{(2(N-2))/(N+2)}} \right) \right\}. \quad (B.15)$$

Both terms

$$\left| \int \varphi_i^2 \delta_i^{p-2} \frac{\partial P\delta_k}{\partial\lambda} \right| \quad \text{and} \quad \left| \int \varphi_i^p \frac{\partial P\delta_k}{\partial\lambda} \right|$$

are controled by $\int \varphi_i \delta_i^{p-1} \left| \frac{\partial P\delta_k}{\partial\lambda} \right|$.

We turn to $\int P\delta_i^{p-1} \operatorname{Inf}(P\delta_i, P\delta_j) \frac{\partial P\delta_k}{\partial\lambda}$. We have:

$$\int P\delta_i^{p-1} \operatorname{Inf}(P\delta_i, P\delta_j) \frac{\partial P\delta_k}{\partial\lambda} = O\left\{ \int \delta_i^{p-1} \delta_j \left(\frac{\partial\delta_k}{\partial\lambda} + \frac{\partial\varphi_k}{\partial\lambda} \right) \right\}.$$

We start with $\int \delta_i^{p-1} \delta_j \frac{\partial\varphi_k}{\partial\lambda}$. We may write $\Omega = B_i \cup B_j \cup (B_i^c \cap B_j^c)$. Then:

$$\begin{aligned}
\int \delta_i^{p-1} \delta_j \frac{\partial\varphi_k}{\partial\lambda} \leqq & \left\| \frac{\partial\varphi_k}{\partial\lambda} \right\|_{L^{p+1}} \| \delta_j \|_{L^\infty(\partial B_j)} \int_{B_i} \delta_i^{((p-1)(p+1))/p} \\
& + \| \delta_i^{p-1} \|_{L^\infty(\partial B_i)} \left\| \frac{\partial\varphi_k}{\partial\lambda} \right\|_{L^\infty} \int_{B_j} \delta_j \\
& + \| \delta_i^{p-1} \|_{L^\infty(\partial B_i)} \| \delta_j \|_{L^\infty(B_j)} \left\| \frac{\partial\varphi_k}{\partial\lambda} \right\|_{L^\infty} \\
= & O\left\{ \lambda_k^{-(N/2)} \lambda_i^{-(N/2)+1} o(1) + \lambda_k^{(-N/2)} \lambda_i^{-2} \lambda_j^{-(N/2)+1} \right\}. \quad (B.16)
\end{aligned}$$

In the same way, we write: $\Omega = B_i \cup B_j \cup B_k \cup (B_i^c \cap B_j^c \cap B_k^c)$. As in the previous estimate, we get:

$$\int \delta_i^{p-1} \delta_j \frac{\partial\delta_k}{\partial\lambda} = O\left\{ \lambda_k^{-(N/2)} \lambda_i^{-(N/2)+1} o(1) + \lambda_k^{(-N)/2} \lambda_i^{-2} \lambda_j^{-(N/2)+1} \right\}. \quad (B.17)$$

We are left with the term $\left| \int P\delta_i^{p-1} \operatorname{Inf}(v, P\delta_i) \frac{\partial P\delta_k}{\partial\lambda} \right|$, which is less than

$$C \int \delta^{p-1} |v| \left| \frac{\partial P\delta_k}{\partial\lambda} \right|,$$

C being a positive constant. By Hölder inequality:

$$\int \delta^{p-1} |v| \left|\frac{\partial \mathrm{P}\,\delta_k}{\partial\lambda}\right| \leqq \|v\| \left\{ \int \delta_i^{((p-1)(p+1))/p} \left|\frac{\partial \mathrm{P}\,\delta_k}{\partial\lambda}\right|^{(p+1)/p} \right\}^{p/(p+1)},$$

and

$$\int \delta_i^{((p-1)(p+1))/p} \left|\frac{\partial \varphi_k}{\partial\lambda}\right|^{(p+1)/p} = O\left\{ \lambda_k^{-(\mathrm{N}(p+1))/2p}\, o(1) + \lambda_k^{-(\mathrm{N}(p+1))/2p}\, \lambda_i^{-(2(p+1))/p} \right\}.$$

On the other hand,

$$\left|\frac{\partial \delta_k}{\partial\lambda}\right| \leqq \frac{\delta_k}{\lambda_k},$$

so that:

$$\int \delta_i^{((p-1)(p+1))/p} \left|\frac{\partial \delta_k}{\partial\lambda}\right|^{(p+1)/p} \leqq \frac{1}{\lambda_k^{(p+1)/p}} \int \delta_i^{((p-1)(p+1))/p}\, \delta_k^{(p+1)/p}.$$

Using again the estimates of [B], we obtain:

$$\int \mathrm{P}\,\delta_i^{p-1}\, \mathrm{Inf}(v, \mathrm{P}\,\delta_i) \frac{\partial \mathrm{P}\,\delta_k}{\partial\lambda} = O\left\{ \|v\| \left(\lambda_k^{\mathrm{N}/2}\, o(1) + \lambda_k^{\mathrm{N}/2}\, \lambda_i^{-2} + \frac{1}{\lambda_k} \varepsilon_{i,k}^{((\mathrm{N}+2)(\mathrm{N}-2)\,\theta)/4\mathrm{N}} \left|\mathrm{Log}\, \varepsilon_{i,k}\right|^{((\mathrm{N}+2)(\mathrm{N}-2)\,\theta)/4\mathrm{N}} \right) \right\}. \quad (\mathrm{B}.18)$$

Given ε small enough, there exists k_ε such that:

$$\lambda_{k_\varepsilon,\varepsilon} = \inf_{j=1,\ldots,k'} \lambda_{j,\varepsilon}.$$

Then there exist a fixed index k_0 and a sequence ε_n which goes to zero as n goes to infinity, such that $k_0 = k_{\varepsilon_n}$. Multiplying (B.13) by $\xi_{k_0}^{\mathrm{N}-1}$, combining Lemma B.1, (B.14), . . . ,(B.18), we get:

$$d_{k_0,\varepsilon_n}^{\mathrm{N}-1} |\mathrm{H}(a_{k_0,\varepsilon_n}, a_{k_0,\varepsilon_n})| \leqq o(1) + O(\|v\|\, \lambda_{k_0}^{\mathrm{N}-2}\, \varepsilon_{i,j}^{(p\,\theta(\mathrm{N}-2))/(2(p+1))} (\mathrm{Log}\,\varepsilon_{i,j})^{(p\,\theta(\mathrm{N}-2))/(2(p+1))} + \|v\|\, \lambda_{k_0}^{\mathrm{N}/2}\, o(1)),$$

for all $\mathrm{N} \geqq 5$. Taking into account the estimate of $\|v\|$, we shall need some restrictions over the dimension, in order to obtain a $o(1)$ in the second term of this last inequality. For $\mathrm{N}=5$, 6, it is easy. If $\mathrm{N}>6$, we must have:

$$\lambda_i^{-2}\, \lambda_j^{-2}\, \lambda_{k_0}^{\mathrm{N}/2} \leqq \mathrm{C},$$