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Generalized characteristics uniqueness and regularity of solutions in a hyperbolic system of conservation laws

by

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ABSTRACT. — Using the method of generalized characteristics, the authors investigate uniqueness and regularity of solutions of the Cauchy problem for a special system of hyperbolic conservation laws, with coinciding shock and rarefaction wave curves, arising in the theory of isotachophoresis.

Key words : Isotachophoresis, hyperbolic conservation laws, generalized characteristics.

RÉSUMÉ. — En utilisant la méthode des caractéristiques généralisées, on montre l'unicité et la régularité des solutions du problème de Cauchy pour un système hyperbolique de lois de conservation provenant de la théorie de l'électrophorèse.

To the memory of Ronald J. DiPerna

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1. INTRODUCTION

The hyperbolic system of conservation laws

$$\partial_t U_i + \partial_x \left(\frac{a_i U_i}{U_1 + U_2 + U_3} \right) = 0, \quad i = 1, 2, 3, \quad (1.1)$$

governs the process of separating three ionized chemical compounds by the electrophoretic method of *isotachopheresis*. The fields U_1, U_2, U_3 are proportional to the concentrations of the three species and the positive constants a_1, a_2, a_3 are the respective electrophoretic mobilities. A derivation of (1.1) is given in the thesis of Geng [9], where the reader may also find background information as well as references to the relevant chemical literature. The study of this system is warranted not only due to the intrinsic interest of the model but also because it provides an excellent vehicle for testing the effectiveness of newly developed techniques based on the theory of generalized characteristics [5].

Introducing new variables

$$\begin{aligned} u &= U_1 + U_2 + U_3, \\ v &= (a_2 + a_3) U_1 + (a_3 + a_1) U_2 + (a_1 + a_2) U_3, \\ \omega &= a_2 a_3 U_1 + a_3 a_1 U_2 + a_1 a_2 U_3, \end{aligned} \quad (1.2)$$

(1.1) takes the simpler form

$$\begin{aligned} \partial_t u - \partial_x \left(\frac{v}{u} \right) &= 0 \\ \partial_t v - \partial_x \left(\frac{\omega}{u} \right) &= 0 \\ \partial_t \omega &= 0. \end{aligned} \quad (1.3)$$

We consider the Cauchy problem for (1.1) assuming, for simplicity, that the initial data $U_i(x, 0)$, $i = 1, 2, 3$, render $\omega(x, 0) = 1$, $-\infty < x < \infty$, in which case (1.3) reduces to

$$\begin{aligned} \partial_t u - \partial_x \left(\frac{v}{u} \right) &= 0 \\ \partial_t v - \partial_x \left(\frac{1}{u} \right) &= 0. \end{aligned} \quad (1.4)$$

We shall study here (1.1) in its form (1.4).

One may also rewrite (1.1) as an equivalent system of two conservation laws by eliminating U_3 with the help of $\omega \equiv 1$. Upon renormalizing the remaining variables, U_1 and U_2 , (1.1) thus reduces to the system

$$\partial_t V_i + \partial_x \left(\frac{b_i V_i}{V_1 + V_2 + 1} \right) = 0, \quad i = 1, 2, \quad (1.5)$$

which arises in chromatography and has been considered earlier by Levesque and Temple [14] and by Serre [19].

As we shall see in Section 2, on the physically relevant range, (1.4) has positive characteristic speeds, $\lambda(u, v)$ and $\mu(u, v)$, which are distinct, $\lambda(u, v) < \mu(u, v)$, except at a single *umbilic* point. Both characteristic fields are genuinely nonlinear. Suitably constructed Riemann invariants z and w will provide a more convenient representation of the state vector than the original coordinates u and v .

The pivotal feature of (1.4) is that it belongs, together with (1.1) and (1.5), to the class of systems, identified by Temple [20], in which shock and rarefaction wave curves of each characteristic family coincide. As a consequence of this special property, Riemann invariants associated with each characteristic field do not jump across shocks of the opposite family. Furthermore, the interaction of waves of the same family never generates waves of the opposite family. In particular, centered rarefaction waves may originate only at the initial line $t=0$. In those respects, the behavior of solutions of these systems resembles closely the behavior of solutions of single, genuinely nonlinear conservation laws.

The coinciding of shock and rarefaction wave curves greatly facilitates the construction of solutions to the Cauchy problem, in the class BV of functions of bounded variation. In fact the effectiveness of a host of methods, including the Glimm scheme, the Godunov scheme, the Lax-Friedrichs scheme and the viscosity approach, has been documented in the literature ([14], [19], [9]).

Our program here is to study the properties of BV solutions of (1.4) directly, *i. e.*, without reference to any particular method of construction. To this end we will employ techniques developed in [1], [2], [3], [5], based on the theory of generalized characteristics.

A generalized characteristic associated with a BV solution of (1.4) is a trajectory of the standard characteristic ordinary differential equations, interpreted as differential inclusions, in the sense of Filippov [8]. Thus generalized characteristics are Lipschitz curves propagating with classical characteristic speed or with shock speed. The relevant parts of the theory are outlined in Section 3.

There are two families of generalized characteristics for our system. From any point (\bar{x}, \bar{t}) of the upper half-plane generally emanates a funnel of generalized backward characteristics of each family, confined between a minimal and a maximal one. These extremal backward characteristics always propagate with classical characteristic speed; moreover, the restriction on them of Riemann invariants of the corresponding family is constant. This last feature, which distinguishes (1.4) from the typical genuinely nonlinear system of two conservation laws (compare with [5]), is a reflection of the fact that Riemann invariants do not jump across shocks of

the opposite family. The above property will be established in Section 3 with the help of entropy inequalities.

One of the implications of the constancy of Riemann invariants along the extremal backward characteristics is that, once detached from the initial line $t=0$, the solution develops a one-sided Lipschitz condition. This in turn implies that the forward characteristic of each family emanating from any point of the upper half-plane is unique. In particular, in our system centered rarefaction waves may only originate from the initial line $t=0$.

In Section 4 we employ the theory of generalized characteristics to describe the regularity properties of solutions of (1.4). We follow closely the analysis in [1] for the single, genuinely nonlinear conservation law and the results are very similar: The set of shocks is at most countable. Each shock is a Lipschitz curve which is right-differentiable at every point and fails to be differentiable only at (the at most countable set of) points of interaction with other shocks or with centered compression waves of either family. The solution is continuous on the complement \mathcal{C} of the shock set and is in fact Lipschitz continuous on the (possibly empty) interior of \mathcal{C} . At any point where a shock is differentiable, classical one-sided limits of the solution exist and they satisfy the standard Rankine-Hugoniot jump conditions. The above are in agreement with DiPerna's description [6] of the structure of solutions of general genuinely nonlinear systems of two conservation laws constructed by the random choice method of Glimm [10].

Though no smoothness of solutions, beyond what was described above, is to be generally expected, if the single conservation law provides a good model for our system, it is conceivable that generically solutions with smooth initial data are piecewise smooth ([18], [1], [2], [3]). Indeed, in Section 5 we show that when the initial data are C^k smooth, then the shock set is closed and the solution is C^k smooth on the complement of it. In fact, generically, the number of shocks in any bounded region is finite. The idea of the proof, borrowed from [1], [2], is to monitor the onset of shocks and demonstrate that shock generation points cannot accumulate, unless the initial data satisfy a nongeneric degeneracy condition.

The one-sided Lipschitz bound on solutions will enable us to establish, in Section 6, uniqueness for the Cauchy problem, via Holmgren's method. The successful application of this approach to the single conservation law by Oleinik [16] is well-known. On the other hand, uniqueness theorems, for general or special systems, recorded in the literature ([17], [11], [15], [7]) impose at the outset restrictions of smoothness on solutions that have not been translated into conditions on the initial data. Though free from this shortcoming, our result here is not definitive, because it only covers

initial data in which generation points of centered rarefaction waves of the two families are strictly separated. It is not clear to us at this time whether the question of uniqueness is intrinsically harder when centered rarefaction waves of both families are generated at the same point or whether our failure to treat the general case is just technical.

As is well-known, in solutions of strictly hyperbolic systems with initial data of compact support the characteristic fields asymptotically decouple. Serre [19] realized that in systems with coinciding shock and rarefaction wave curves the two characteristic fields actually decouple completely in a finite time. In the final Section 7 we demonstrate that the method of generalized characteristics establishes that property, for our system (1.4), in a direct and simple manner.

It is expected that analogous results may be established, by the same methodology, for general systems of two conservation laws with coinciding shock and rarefaction wave curves [20] and even for systems of several equations endowed with special symmetry, for instance (1.1) with $i=1, \dots, n$.

2. RIEMANN INVARIANTS AND ENTROPIES

The Jacobian

$$D \begin{pmatrix} -\frac{v}{u} \\ \frac{1}{u} \end{pmatrix} = \begin{pmatrix} \frac{v}{u^2} & -\frac{1}{u} \\ \frac{1}{u^2} & 0 \end{pmatrix} \quad (2.1)$$

of the flux in the system (1.4) has eigenvalues λ , μ and associated eigenvectors

$$R = \begin{pmatrix} \lambda u^2 \\ 1 \end{pmatrix}, \quad S = \begin{pmatrix} \mu u^2 \\ 1 \end{pmatrix}. \quad (2.2)$$

Recalling the definition (1.2) of u , v , ω and our normalizing assumption $\omega=1$, it can be shown [9] that on the physically relevant range of (u, v) , induced by the natural restrictions $U_i \geq 0$, $i=1, 2, 3$, $U_1 + U_2 + U_3 > 0$, λ and μ are real and hence (1.4) is hyperbolic. In fact, assuming, for definiteness, that $a_1 < a_2 < a_3$, (1.4) is strictly hyperbolic, $\lambda < \mu$, except at the single umbilic point $u = a_2^{-2}$, $v = 2a_2^{-1}$, where $\lambda = \mu = a_2^3$.

We introduce the functions

$$z = \frac{1}{\lambda u}, \quad w = \frac{1}{\mu u}. \quad (2.3)$$

By account of

$$\lambda + \mu = \frac{v}{u^2}, \quad \lambda u = \frac{1}{u^3}, \quad (2.4)$$

we deduce the identities

$$z + w = v, \quad zw = u, \quad (2.5)$$

$$\lambda = \frac{1}{z^2 w}, \quad \mu = \frac{1}{zw^2}, \quad (2.6)$$

$$R = \begin{pmatrix} w \\ 1 \end{pmatrix}, \quad S = \begin{pmatrix} z \\ 1 \end{pmatrix}. \quad (2.7)$$

Using (2.5) we easily compute the differential of z and w , with respect to the variables (u, v) :

$$Dz = \begin{pmatrix} 1 & z \\ w-z & z-w \end{pmatrix}, \quad Dw = \begin{pmatrix} 1 & w \\ z-w & w-z \end{pmatrix} \quad (2.8)$$

In particular,

$$\left. \begin{aligned} (Dz)R &= 1, & (Dz)S &= 0, \\ (Dw)R &= 0, & (Dw)S &= 1, \end{aligned} \right\} \quad (2.9)$$

that is, z and w are properly normalized Riemann invariants of (1.4).

A simple calculation, using (2.8) and (2.7), yields

$$\left. \begin{aligned} R^T (DD^T z) R &= 0, & S^T (DD^T z) S &= 0, \\ R^T (DD^T w) R &= 0, & S^T (DD^T w) S &= 0. \end{aligned} \right\} \quad (2.10)$$

This degeneracy reflects the special structure of our system. Its implications will become clear shortly.

It will generally prove expedient to monitor the solution in the coordinate system (z, w) rather than through the original state variables (u, v) . In order to avoid cumbersome notation, we shall be using the same symbol to denote a particular field as a function of (u, v) or as a function of (z, w) .

By virtue of (2.6),

$$\lambda_z < 0, \quad \mu_w < 0, \quad (2.11)$$

so that both characteristic fields are genuinely nonlinear.

As is well known, a function η is an *entropy* for (1.4), associated with *entropy flux* q , if

$$q_z = \lambda \eta_z, \quad q_w = \mu \eta_w. \quad (2.12)$$

The entropy η will be convex in the variables (u, v) if

$$R^T (DD^T \eta) R \geq 0, \quad S^T (DD^T \eta) S \geq 0. \quad (2.13)$$

In [5], (2.13) are rewritten in an equivalent form, equations (3.13), (3.14), that involves derivatives of η with respect to (z, w) . By virtue of these

equations and (2. 10), (2. 13) here reduce to

$$\eta_{zz}(z, w) \geq 0, \quad \eta_{ww}(z, w) \geq 0. \tag{2. 14}$$

We now construct a family of entropies

$$\eta(z, w) = \exp\left(\frac{k}{z}\right) [1 + \Phi(z)w], \quad -\infty < k < \infty, \tag{2. 15}$$

of the type first considered by Lax [13]. Recalling (2. 6), the integrability condition

$$\left[\frac{1}{z^2 w} \eta_z(z, w) \right]_w - \left[\frac{1}{zw^2} \eta_w(z, w) \right]_z = 0, \tag{2. 16}$$

induced by (2. 12), reduces to the ordinary differential equation

$$z^3 \Phi' - z(z+k)\Phi - k = 0 \tag{2. 17}$$

with solution

$$\Phi(z) = -\frac{1}{z} + \frac{2}{k} - \frac{2}{k^2}z. \tag{2. 18}$$

Therefore, combining (2. 15), (2. 18) and (2. 12) we deduce

$$\eta(z, w) = \exp\left(\frac{k}{z}\right) \left[1 - \frac{w}{z} + \frac{2}{k}w - \frac{2}{k^2}zw \right], \tag{2. 19}$$

$$q(z, w) = \exp\left(\frac{k}{z}\right) \left[\frac{1}{z^2 w} \left(1 - \frac{w}{z} \right) - \frac{2}{k} \left(\frac{1}{zw} - \frac{2}{z^2} \right) + \frac{2}{k^2} \left(\frac{1}{w} - \frac{5}{z} \right) + \frac{12}{k^3} \right]. \tag{2. 20}$$

It follows from (2. 19) that

$$\eta(z, w) > 0, \tag{2. 21}$$

and (2. 14) hold for (z, w) away from the umbilic point $z = w = a_2^{-1}$, provided $|k|$ be sufficiently large. Thus η is convex in (u, v) when either k or $-k$ is large. This property differentiates (1. 4) from the typical genuinely nonlinear system, in which the Lax entropy is convex as k grows only in one direction. We shall see the implications of that in Section 3.

From (2. 19) and (2. 20) we infer

$$q(z, w) - \lambda(z, w) \eta(z, w) = -\frac{2}{k} \exp\left(\frac{k}{z}\right) \left[\frac{1}{zw} \left(1 - \frac{w}{z} \right) + O\left(\frac{1}{k}\right) \right] \tag{2. 22}$$

$$q(z, w) - [\lambda(z, w) + \varepsilon] \eta(z, w) = -\exp\left(\frac{k}{z}\right) \left[\varepsilon \left(1 - \frac{w}{z} \right) + O\left(\frac{1}{k}\right) \right]. \tag{2. 23}$$

It is clear that we may also construct entropy-entropy flux pairs analogous to (2. 19), (2. 20) but with the roles of z and w interchanged.

By virtue of (2.5), the Rankine-Hugoniot conditions for a shock joining the states (z_-, w_-) and (z_+, w_+) read

$$\left. \begin{aligned} \sigma(z_+ w_+ - z_- w_-) + z_+^{-1} + w_+^{-1} - z_-^{-1} - w_-^{-1} &= 0 \\ \sigma(z_+ + w_+ - z_- - w_-) + z_+^{-1} w_+^{-1} - z_-^{-1} w_-^{-1} &= 0. \end{aligned} \right\} \quad (2.24)$$

There are two families of shocks: 1-shocks in which $z_- \neq z_+$, $w_- = w_+$ and the speed is given by

$$\sigma = z_-^{-1} z_+^{-1} w_{\pm}^{-1} \quad (2.25)$$

and 2-shocks in which $z_- = z_+$, $w_- \neq w_+$ and the speed is given by

$$\sigma = z_{\pm}^{-1} w_-^{-1} w_+^{-1}. \quad (2.26)$$

In particular, since the Riemann invariants of each family do not jump across shocks of the opposite family, in our system (1.4) shock and rarefaction wave curves coincide.

3. GENERALIZED CHARACTERISTICS

We consider a BV (weak) solution $(u(x, t), v(x, t))$ of (1.4) on $(-\infty, \infty) \times [0, \infty)$ that induces Riemann invariant fields $(z(x, t), w(x, t))$ taking values in a small neighborhood of a fixed state (\hat{z}, \hat{w}) in the physically relevant range. The two characteristic fields are well separated, in the sense that $\mu(\hat{z}, \hat{w}) - \lambda(\hat{z}, \hat{w})$ is much larger than the oscillation of the solution.

We assume that for each $t \geq 0$ the functions $u(\cdot, t), v(\cdot, t)$, and thereby also the functions $z(\cdot, t), w(\cdot, t)$, have bounded variation on $(-\infty, \infty)$. In particular, one-sided limits $u(x \pm, t), v(x \pm, t), z(x \pm, t), w(x \pm, t)$ exist for all (x, t) on the upper half-plane. The solution is then normalized by

$$\begin{aligned} u(x, t) &= u(x-, t), & v(x, t) &= v(x-, t), \\ z(x, t) &= z(x-, t), & w(x, t) &= w(x-, t). \end{aligned}$$

The solution also satisfies the physically motivated condition

$$u(x-, t) \leq u(x+, t), \quad -\infty < x < \infty, \quad 0 < t < \infty, \quad (3.1)$$

which states that the concentration of the positively charged electrolyte, in which the three chemical compounds are immersed (cf. [9]), may jump upwards but not downwards. It turns out that this condition is equivalent to the standard Lax [12] shock admissibility criterion

$$\begin{aligned} \lambda(z(x-, t), w(x-, t)) &\geq \lambda(z(x+, t), w(x+, t)) \\ \mu(z(x-, t), w(x-, t)) &\geq \mu(z(x+, t), w(x+, t)) \\ -\infty < x < \infty, & \quad 0 < t < \infty \end{aligned} \quad (3.2)$$

or, equivalently, to

$$\left. \begin{aligned} z(x-, t) \leq z(x+, t), \quad w(x-, t) \leq w(x+, t), \\ -\infty < x < \infty, \quad 0 < t < \infty. \end{aligned} \right\} \quad (3.3)$$

In turn, these conditions imply that if η is any entropy which is convex in (u, v) and q is the associated entropy flux then

$$\partial_t \eta + \partial_x q \leq 0 \quad (3.4)$$

holds, in the sense of measures.

A generalized 1-characteristic, associated with the given solution, on an interval $[t_1, t_2]$, is a trajectory of the classical 1-characteristic equation

$$\frac{dx}{dt} = \lambda(z(x, t), w(x, t)) \quad (3.5)$$

in the sense of Filippov [8], i. e., a Lipschitz arc $\xi: [t_1, t_2] \rightarrow (-\infty, \infty)$ which satisfies the differential inclusion

$$\dot{\xi}(t) \in [\lambda(z(\xi(t)+, t), w(\xi(t)+, t)), \lambda(z(\xi(t)-, t), w(\xi(t)-, t))]. \quad (3.6)$$

Similarly, a generalized 2-characteristic on $[t_1, t_2]$ is a trajectory of the classical 2-characteristic equation

$$\frac{dx}{dt} = \mu(z(x, t), w(x, t)), \quad (3.7)$$

i. e., a Lipschitz arc $\zeta: [t_1, t_2] \rightarrow (-\infty, \infty)$ which satisfies

$$\dot{\zeta}(t) \in [\mu(z(\zeta(t)+, t), w(\zeta(t)+, t)), \mu(z(\zeta(t)-, t), w(\zeta(t)-, t))]. \quad (3.8)$$

We state a few elementary properties of generalized characteristics. The proofs may be found in [5]. Every generalized characteristic propagates with either classical characteristic speed or with shock speed of the corresponding family. From each point (\bar{x}, \bar{t}) of the upper half-plane emanates a funnel of generalized i -characteristics ($i=1, 2$), confined between a minimal and a maximal one (that may coincide). The extremal backward generalized characteristics always propagate with classical characteristic speed; specifically,

LEMMA 3.1. — Let $-\infty < \bar{x} < \infty, \bar{t} > 0$. If ξ is the minimal or the maximal backward 1-characteristic emanating from (\bar{x}, \bar{t}) , then

$$\left. \begin{aligned} z(\xi(t)-, t) = z(\xi(t)+, t), \\ w(\xi(t)-, t) = w(\xi(t)+, t) \quad \text{a. e. on } [0, \bar{t}], \end{aligned} \right\} \quad (3.9)$$

$$\dot{\xi}(t) = \lambda(z(\xi(t) \pm, t), w(\xi(t) \pm, t)), \quad \text{a. e. on } [0, \bar{t}]. \quad (3.10)$$

If ζ is the minimal or the maximal backward 2-characteristic emanating from (\bar{x}, \bar{t}) , then

$$\left. \begin{aligned} z(\zeta(t)-, t) &= z(\zeta(t)+, t), \\ w(\zeta(t)-, t) &= w(\zeta(t)+, t), \quad \text{a. e. on } [0, \bar{t}], \end{aligned} \right\} \quad (3.11)$$

$$\xi(t) = \mu(z(\zeta(t) \pm, t), w(\zeta(t) \pm, t)), \quad \text{a. e. on } [0, \bar{t}]. \quad (3.12)$$

In this section our objective is to investigate the propagation of Riemann invariants along the extremal backward characteristics.

LEMMA 3.2. — *Let ξ and ζ denote the minimal and maximal backward 1-characteristics emanating from any point (\bar{x}, \bar{t}) of the upper half-plane. Then $z(\xi(t)-, t)$ is a nonincreasing function on $[0, \bar{t}]$ while $z(\zeta(t)+, t)$ is a nondecreasing function on $[0, \bar{t}]$. Similarly, if ξ and ζ are the minimal and maximal backward 2-characteristics emanating from (\bar{x}, \bar{t}) , then $w(\xi(t)-, t)$ is a nonincreasing function on $[0, \bar{t}]$ while $w(\zeta(t)+, t)$ is a nondecreasing function on $[0, \bar{t}]$.*

Proof. — We shall establish only the first part of the theorem, where ξ and ζ are the extremal backward 1-characteristics emanating from (\bar{x}, \bar{t}) , because the proof of the second half is quite analogous. Our argument will follow closely the proof of Theorem 4.1 in [5] and so we feel free to be somewhat sketchy here, referring the reader to the above reference for further details.

We fix ε positive small and we let ξ_ε denote an integral curve, in the sense of Filippov, of the differential equation

$$\frac{dx}{dt} = \lambda(z(x, t), w(x, t)) + \varepsilon \quad (3.13)$$

emanating from the point $(\bar{x} - \varepsilon, \bar{t})$. Then

$$\dot{\xi}_\varepsilon(t) \geq \lambda(z(\xi_\varepsilon(t)+, t), w(\xi_\varepsilon(t)+, t)) + \varepsilon, \quad \text{a. e. on } [0, \bar{t}], \quad (3.14)$$

$\xi_\varepsilon(t) < \xi(t)$, $0 \leq t \leq \bar{t}$, and $\xi_\varepsilon(t) \rightarrow \xi(t)$, as $\varepsilon \rightarrow 0$, uniformly on $[0, \bar{t}]$.

We consider the inequality (3.4), where η and q are given by (2.19) and (2.20) with $-k$ very large, and we integrate it over the domain $\{(x, t) : s \leq t \leq \tau, \xi_\varepsilon(t) \leq x \leq \xi(t)\}$, for $0 \leq s < \tau \leq \bar{t}$, thus obtaining

$$\begin{aligned} & \int_{\xi_\varepsilon(s)}^{\xi(\tau)} \eta(z(x, \tau), w(x, \tau)) dx - \int_{\xi_\varepsilon(s)}^{\xi(s)} \eta(z(x, s), w(x, s)) dx \\ & \leq - \int_s^\tau \left\{ q(z(\xi(t)-, t), w(\xi(t)-, t)) \right. \\ & \quad \left. - \dot{\xi}(t) \eta(z(\xi(t)-, t), w(\xi(t)-, t)) \right\} dt \\ & \quad + \int_s^\tau \left\{ q(z(\xi_\varepsilon(t)+, t), w(\xi_\varepsilon(t)+, t)) \right. \\ & \quad \left. - \dot{\xi}_\varepsilon(t) \eta(z(\xi_\varepsilon(t)+, t), w(\xi_\varepsilon(t)+, t)) \right\} dt. \quad (3.15) \end{aligned}$$

On the right-hand side of (3.15), the first term is nonnegative by virtue of (3.10) and (2.22) and the second term is also nonnegative by account of (3.14) and (2.23). Hence

$$\int_{\xi_\varepsilon(\tau)}^{\xi(\tau)} \eta(z(x, \tau), w(x, \tau)) dx \leq \int_{\xi_\varepsilon(s)}^{\xi(s)} \eta(z(x, s), w(x, s)) dx. \tag{3.16}$$

We raise (3.16) to the power $-\frac{1}{k}$ and we let $k \rightarrow -\infty$. Finally, we let $\varepsilon \rightarrow 0$. This yields $z(\xi(\tau)-, \tau) \leq z(\xi(s)-, s)$, i. e., $z(\xi(t)-, t)$ is nonincreasing on $[0, \bar{t}]$.

To show that $z(\xi(t)+, t)$ is nondecreasing on $[0, \bar{t}]$, we employ a parallel argument: We consider an integral curve ζ_ε of the equation

$$\frac{dx}{dt} = \lambda(z(x, t), w(x, t)) - \varepsilon \tag{3.17}$$

emanating from the point $(\bar{x} + \varepsilon, \bar{t})$ and integrate over the domain $\{(x, t) : s \leq t \leq \tau, \zeta(t) \leq x \leq \zeta_\varepsilon(t)\}$ the inequality (3.4) where η and q are again given by (2.19) and (2.20) but now it is k (rather than $-k$) that is selected large. We omit the details.

The above result immediately induces invariant regions for solutions of (1.4):

COROLLARY 3.1. — *For any point (\bar{x}, \bar{t}) of the upper half plane*

$$\left. \begin{aligned} \inf z(x, 0) \leq z(\bar{x}, \bar{t}) \leq \sup z(x, 0) \\ \inf w(x, 0) \leq w(\bar{x}, \bar{t}) \leq \sup w(x, 0). \end{aligned} \right\} \tag{3.18}$$

Lemma 3.2 also yields bounds on the variation of the Riemann invariants along extremal characteristics of the opposite family. We will be using the symbols NV, PV and TV to denote negative variation, positive variation, and total variation, respectively.

LEMMA 3.3. — *Let ξ denote the minimal backward 1-characteristic and ζ denote the maximal backward 2-characteristic emanating from a point (\bar{x}, \bar{t}) of the upper half plane. Then $\bar{w}(t) := w(\xi(t)-, t)$ is a function of bounded variation on $[0, \bar{t})$ and $NV_{[0, \bar{t}]} \bar{w} \leq TV_{[\xi(0), \xi(0)]} w(\cdot, 0)$. Similarly, $\bar{z}(t) := z(\xi(t)+, t)$ is a function of bounded variation on $[0, \bar{t})$ and $PV_{[0, \bar{t}]} \bar{z} \leq TV_{[\zeta(0), \zeta(0)]} z(\cdot, 0)$.*

Proof. — To estimate the negative variation of \bar{w} on $[0, \bar{t}]$, we pick a mesh $0 \leq s_1 < \tau_1 < \dots < s_n < \tau_n < \bar{t}$ such that $\bar{w}(s_i) > \bar{w}(\tau_i)$, $i = 1, \dots, n$. By virtue of (3.9), it suffices to consider only meshes with $w(\xi(\tau_i)+, \tau_i) = w(\xi(\tau_i)-, \tau_i) = \bar{w}(\tau_i)$, $i = 1, \dots, n$.

For $i = 1, \dots, n$, we let ξ_i denote the minimal backward 2-characteristic emanating from $(\xi(s_i), s_i)$ and ζ_i denote the maximal backward 2-characteristic emanating from $(\xi(\tau_i), \tau_i)$. In particular, $\xi_{i+1}(0) \leq \zeta_i(0)$,

$i = 1, \dots, n - 1$. We claim that it is also $\zeta_i(0) \leq \xi_i(0)$, $i = 1, \dots, n$. Indeed, if for some $t \in (0, s_i)$ it were $\zeta_i(t) = \xi_i(t) = x$, then, by account of Lemma 3.2, $w(x-, t) \geq w(\xi_i(s_i) -, s_i) = \bar{w}(s_i) > \bar{w}(\tau_i) = w(\xi_i(\tau_i) +, \tau_i) \geq w(x +, t)$ (3.19)

in contradiction to (3.3). Another appeal to Lemma 3.2 yields

$$\bar{w}(s_i) - \bar{w}(\tau_i) = w(\xi(s_i) -, s_i) - w(\xi(\tau_i) +, \tau_i) \leq w(\xi_i(0) -, 0) - w(\zeta_i(0) +, 0), \quad (3.20)$$

whence we deduce the estimate $NV_{[0, \bar{t}]} \bar{w} \leq TV_{[\xi(0), \xi(0)]} w(\cdot, 0)$.

The proof of the second part of the theorem is quite similar and will be omitted.

The following proposition estimates the “widening” of extremal backward characteristic of one family effected by the characteristic field of the opposite family.

LEMMA 3.4. — *Let $-\infty < \bar{x} \leq \bar{y} < \infty$, $\bar{t} > 0$. Consider the minimal backward 1-characteristic ξ emanating from (\bar{x}, \bar{t}) and the maximal backward 1-characteristic ζ emanating from (\bar{y}, \bar{t}) . Then*

$$\left. \begin{aligned} \zeta(t) - \xi(t) &\leq K(\bar{y} - \bar{x}) + L[z(\bar{y} +, \bar{t}) - z(\bar{x} -, \bar{t})](\bar{t} - t), \\ 0 &\leq t \leq \bar{t}, \end{aligned} \right\} \quad (3.21)$$

where K and L are positive constants depending on $TV_{(-\infty, \infty)} w(\cdot, 0)$. Similarly, if ξ is the minimal backward 2-characteristic emanating from (\bar{x}, \bar{t}) and ζ is the maximal backward 2-characteristic emanating from (\bar{y}, \bar{t}) , then

$$\left. \begin{aligned} \zeta(t) - \xi(t) &\leq K(\bar{y} - \bar{x}) + L[w(\bar{y} +, \bar{t}) - w(\bar{x} -, \bar{t})](\bar{t} - t), \\ 0 &\leq t \leq \bar{t} \end{aligned} \right\} \quad (3.22)$$

where the positive constants K and L now depend on $TV_{(-\infty, \infty)} z(\cdot, 0)$.

Proof. — We need only discuss the case where ξ and ζ are 1-characteristics, since the proof for the other case is essentially identical.

By Lemma 3.1, for any $t \in [0, \bar{t}]$,

$$\zeta(t) - \xi(t) = \bar{y} - \bar{x} + \int_t^{\bar{t}} \left\{ \lambda(z(\xi(s) \pm, s), w(\xi(s) \pm, s)) - \lambda(z(\zeta(s) \pm, s), w(\zeta(s) \pm, s)) \right\} ds. \quad (3.23)$$

Recalling (2.6) and Lemma 3.2, (3.23) yields

$$\begin{aligned} \zeta(t) - \xi(t) &\leq \bar{y} - \bar{x} + [z^{-2}(\bar{x} -, \bar{t}) - z^{-2}(\bar{y} +, \bar{t})] \\ &\quad \times \int_t^{\bar{t}} w^{-1}(\zeta(s) -, s) ds \\ &\quad + z^{-2}(\bar{x}, \bar{t}) \int_t^{\bar{t}} \left\{ w^{-1}(\xi(s) -, s) - w^{-1}(\zeta(s) -, s) \right\} ds. \end{aligned} \quad (3.24)$$

We let \bar{s} denote the minimum of the set of $s \in [0, \bar{t}]$ with the property that the minimal backward 2-characteristic emanating from the point $(\zeta(s), s)$ is intercepted by ξ at $t = g(s) \geq 0$, before it terminates on the x -axis (if this set is empty, we shall not define \bar{s}). We note that g is a nondecreasing, right-continuous function on $[\bar{s}, \bar{t}]$ with the properties $g(s) \leq s$, $\bar{s} \leq s \leq \bar{t}$, and

$$\left| \frac{\zeta(s) - \zeta(g(s))}{s - g(s)} - \mu(\bar{z}, \bar{w}) \right| \ll 1, \quad \bar{s} \leq s \leq \bar{t}. \tag{3.25}$$

Furthermore, by Lemma 3.2,

$$w(\zeta(s) - , s) \leq w(\xi(g(s)) - , g(s)), \quad \bar{s} \leq s \leq \bar{t}. \tag{3.26}$$

For future reference we also define an “inverse” h of g by

$$h(\tau) := \max \{ s : g(s -) \leq \tau \leq g(s) \}, \quad g(\bar{s}) \leq \tau \leq g(\bar{t}). \tag{3.27}$$

We now fix t in $[0, \bar{t}]$ and define \hat{s} by $\hat{s} = \bar{s}$ if $0 \leq t \leq g(\bar{s})$, $\hat{s} = h(t)$ if $g(\bar{s}) < t \leq g(\bar{t})$ and $\hat{s} = \bar{t}$ if $g(\bar{t}) < t \leq \bar{t}$.

By account of (3.25) and the separation of the two characteristic speeds, we have $0 \leq \hat{s} - t \leq c[\zeta(t) - \xi(t)]$ and so

$$z^{-2}(\bar{x}, \bar{t}) \left| \int_t^{\hat{s}} \{ w^{-1}(\xi(s) - , s) - w^{-1}(\zeta(s) - , s) \} ds \right| \leq \delta[\zeta(t) - \xi(t)], \tag{3.28}$$

where $\delta \ll 1$, because the oscillation of the solution is small. This estimate also holds, with $\hat{s} = \bar{t}$, when \bar{s} is undefined.

We now use (2.6) and (3.26) to infer

$$\begin{aligned} z^{-2}(\bar{x}, \bar{t}) \int_{\bar{s}}^{\bar{t}} \{ w^{-1}(\xi(s) - , s) - w^{-1}(\zeta(s) - , s) \} ds \\ \leq \int_{\bar{s}}^{\bar{t}} [\Lambda(s) - \Lambda(g(s))] ds, \end{aligned} \tag{3.29}$$

where Λ denotes the positive variation function of $\lambda(z(\bar{x}, \bar{t}), w(\xi(\cdot) - , \cdot))$:

$$\Lambda(s) := z^{-2}(\bar{x}, \bar{t}) \text{PV}_{[0, s]} w^{-1}(\xi(\cdot) - , \cdot), \quad 0 \leq s \leq \bar{t}. \tag{3.30}$$

We construct a sequence $\{\Lambda_n\}$ of nondecreasing, absolutely continuous functions which converges to $\Lambda(s)$, pointwise on $[0, \bar{t}]$. Then

$$\begin{aligned} \int_{\bar{s}}^{\bar{t}} [\Lambda_n(s) - \Lambda_n(g(s))] ds &= \int_{\bar{s}}^{\bar{t}} \int_{g(s)}^s \dot{\Lambda}_n(\tau) d\tau ds \\ &= \int_{g(\bar{s})}^{\bar{s}} [h(\tau) - \hat{s}] \dot{\Lambda}_n(\tau) d\tau + \int_{\bar{s}}^{g(\bar{t})} [h(\tau) - \tau] \dot{\Lambda}_n(\tau) d\tau \\ &\quad + \int_{g(\bar{t})}^{\bar{t}} [\bar{t} - \tau] \dot{\Lambda}_n(\tau) d\tau. \end{aligned} \tag{3.31}$$

From (3.27), (3.25) and the separation of the two characteristic speeds we infer that

$$\begin{aligned} h(\tau) - \hat{s} &\leq \beta [\zeta(\tau) - \xi(\tau)] && \text{for } g(\hat{s}) \leq \tau \leq \hat{s}; \\ h(\tau) - \tau &\leq \beta [\zeta(\tau) - \xi(\tau)] && \text{for } \hat{s} < \tau \leq g(\bar{t}); \end{aligned}$$

and $\bar{t} - \tau \leq \beta [\zeta(\tau) - \xi(\tau)]$ for $g(\bar{t}) < \tau \leq \bar{t}$. Moreover, by the construction of \hat{s} it follows that $g(\hat{s}) \geq t$. Therefore (3.31) yields

$$\int_{\hat{s}}^{\bar{t}} [\Lambda_n(s) - \Lambda_n(g(s))] ds \leq \beta \int_t^{\bar{t}} [\zeta(\tau) - \xi(\tau)] d\Lambda_n(\tau). \tag{3.32}$$

Passing to the limit, $n \rightarrow \infty$, we obtain from (3.32),

$$\int_{\hat{s}}^{\bar{t}} [\Lambda(s) - \Lambda(g(s))] ds \leq \beta \int_t^{\bar{t}} [\zeta(\tau) - \xi(\tau)] d\Lambda(\tau). \tag{3.33}$$

Combining (3.24), (3.28), (3.29) and (3.33) we deduce the Gronwall-type inequality

$$\begin{aligned} \zeta(t) - \xi(t) &\leq \alpha(\bar{y} - \bar{x}) + \gamma[z(\bar{y} +, \bar{t}) - z(\bar{x} -, \bar{t})](\bar{t} - t) \\ &\quad + \theta \int_t^{\bar{t}} [\zeta(\tau) - \xi(\tau)] d\Lambda(\tau), \quad 0 \leq t \leq \bar{t}. \end{aligned} \tag{3.34}$$

We claim that this implies

$$\begin{aligned} \zeta(t) - \xi(t) &\leq \alpha(\bar{y} - \bar{x}) \exp \{ \theta [\Lambda(\bar{t}) - \Lambda(t)] \} \\ &\quad + \gamma[z(\bar{y} +, \bar{t}) - z(\bar{x} -, \bar{t})] \int_t^{\bar{t}} \exp \{ \theta [\Lambda(\tau) - \Lambda(t)] \} d\tau, \end{aligned} \tag{3.35}$$

$$0 \leq t \leq \bar{t}.$$

Indeed, if (3.35) were false, we could find $\varepsilon > 0$ and t^* in $(0, \bar{t})$ such that

$$\begin{aligned} \zeta(t) - \xi(t) &\leq [\alpha(\bar{y} - \bar{x}) + \varepsilon] \exp \{ \theta [\Lambda(\bar{t}) - \Lambda(t)] \} \\ &\quad + \gamma[z(\bar{y} +, \bar{t}) - z(\bar{x} -, \bar{t})] \int_t^{\bar{t}} \exp \{ \theta [\Lambda(\tau) - \Lambda(t)] \} d\tau, \end{aligned} \tag{3.36}$$

$$t^* \leq t \leq \bar{t},$$

and (3.36) holds as an equality at $t = t^*$. However, if we apply (3.34) for $t = t^*$ and majorize its right-hand side by use of (3.36), we obtain, after a short calculation,

$$\begin{aligned} \zeta(t^*) - \xi(t^*) &\leq -\varepsilon + [\alpha(\bar{y} - \bar{x}) + \varepsilon] \exp \{ \theta [\Lambda(\bar{t}) - \Lambda(t^*)] \} \\ &\quad + \gamma[z(\bar{y} +, \bar{t}) - z(\bar{x} -, \bar{t})] \int_{t^*}^{\bar{t}} \exp \{ \theta [\Lambda(\tau) - \Lambda(t^*)] \} d\tau \end{aligned} \tag{3.37}$$

which contradicts our assertion that (3.36) holds as an equality when $t = t^*$.

It is now clear that (3.35) yields the desired estimate (3.21) with constants K and L that depend on $\Lambda(\bar{t})$. Thus K, L may be estimated in

terms of the negative variation of $w(\xi(s)-, s)$ over $[0, \bar{t}]$ and thereby, on account of Lemma 3.3, in terms of $TV_{(-\infty, \infty)} w(\cdot, 0)$. The proof is complete.

An immediate corollary of Lemma 3.4 is the following

LEMMA 3.5. — *If (\bar{x}, \bar{t}) is any point of the upper half-plane with $z(\bar{x}-, \bar{t}) = z(\bar{x}+, \bar{t})$, then a unique backward 1-characteristic ξ emanates from (\bar{x}, \bar{t}) and*

$$z(\xi(t)-, t) = z(\xi(t)+, t) = z(\bar{x}, \bar{t}), \quad 0 < t \leq \bar{t}. \quad (3.38)$$

Similarly, if $w(\bar{x}-, t) = w(\bar{x}+, \bar{t})$, then a unique backward 2-characteristic ζ emanates from (\bar{x}, \bar{t}) and

$$w(\zeta(t)-, t) = w(\zeta(t)+, t) = w(\bar{x}, \bar{t}), \quad 0 < t \leq \bar{t}. \quad (3.39)$$

Proof. — When $z(\bar{x}-, \bar{t}) = z(\bar{x}+, \bar{t})$, (3.21) implies that the minimal and maximal (and thereby all other) backward 1-characteristics emanating from (\bar{x}, \bar{t}) collapse into a unique curve ξ . If t is any point in $(0, \bar{t})$, Lemma 3.2 implies $z(\xi(t)-, t) \geq z(\bar{x}-, \bar{t})$, $z(\xi(t)+, t) \leq z(\bar{x}+, \bar{t})$. However, by the admissibility condition (3.3), $z(\xi(t)-, t) \leq z(\xi(t)+, t)$. Thus (3.38) must hold.

The proof for the case $w(\bar{x}-, \bar{t}) = w(\bar{x}+, \bar{t})$ is quite similar and may thus be omitted.

Another implication of Lemma 3.4 is that the solution necessarily satisfies one-sided Lipschitz conditions:

THEOREM 3.1. — *There are positive constants A and B, depending on $TV_{(-\infty, \infty)} w(\cdot, 0)$ and $TV_{(-\infty, \infty)} z(\cdot, 0)$, respectively, such that*

$$\frac{z(\bar{y}, \bar{t}) - z(\bar{x}, \bar{t})}{\bar{y} - \bar{x}} \geq -\frac{A}{\bar{t}}, \quad -\infty < \bar{x} < \bar{y} < \infty, \quad \bar{t} > 0 \quad (3.40)$$

$$\frac{w(\bar{y}, \bar{t}) - w(\bar{x}, \bar{t})}{\bar{y} - \bar{x}} \geq -\frac{B}{\bar{t}}, \quad -\infty < \bar{x} < \bar{y} < \infty, \quad \bar{t} > 0. \quad (3.41)$$

Proof. — It suffices to establish (3.40) under the assumptions $z(\bar{y}-, \bar{t}) = z(\bar{y}+, \bar{t})$, $z(\bar{x}-, \bar{t}) = z(\bar{x}+, \bar{t})$, $z(\bar{x}, \bar{t}) > z(\bar{y}, \bar{t})$. By Lemma 3.5, there is a unique backward 1-characteristic ξ emanating from (\bar{x}, \bar{t}) and a unique backward 1-characteristic ζ emanating from (\bar{y}, \bar{t}) . We apply (3.21) with $t=0$ and since $\zeta(0) - \xi(0) \geq 0$ we arrive immediately at (3.40).

The proof of (3.41) is essentially identical and will thus be omitted.

The concluding proposition of this section collects and completes the properties of characteristics established above.

THEOREM 3.2. — *Let (\bar{x}, \bar{t}) be any point of the upper half-plane, with $\bar{t} > 0$. A unique forward 1-characteristic emanates from (\bar{x}, \bar{t}) . Furthermore, if ξ_- and ξ_+ are the minimal and maximal backward 1-characteristics*

emanating from (\bar{x}, \bar{t}) , then

$$\begin{aligned} z(\xi_{-}(0)+, 0) &\leq z(\xi_{-}(t)+, t) \\ &= z(\xi_{-}(t)-, t) = z(\bar{x}-, \bar{t}) \leq z(\xi_{-}(0)-, 0), \end{aligned} \tag{3.42}$$

$0 < t < \bar{t}$

$$\begin{aligned} z(\xi_{+}(0)+, 0) &\leq z(\xi_{+}(t)+, t) \\ &= z(\xi_{+}(t)-, t) = z(\bar{x}+, \bar{t}) \leq z(\xi_{+}(0)-, 0), \end{aligned} \tag{3.43}$$

$0 < t < \bar{t}$.

When $z(\bar{x}-, \bar{t}) = z(\bar{x}+, \bar{t})$, ξ_{-} and ξ_{+} coincide. Similarly, a unique forward 2-characteristic emanates from (\bar{x}, \bar{t}) . Moreover, if ζ_{-} and ζ_{+} are the minimal and maximal backward 2-characteristics emanating from (\bar{x}, \bar{t}) , then

$$\begin{aligned} w(\zeta_{-}(0)+, 0) &\leq w(\zeta_{-}(t)+, t) \\ &= w(\zeta_{-}(t)-, t) = w(\bar{x}-, \bar{t}) \leq w(\zeta_{-}(0)-, 0), \end{aligned} \tag{3.44}$$

$0 < t < \bar{t}$

$$\begin{aligned} w(\zeta_{+}(0)+, 0) &\leq w(\zeta_{+}(t)+, t) \\ &= w(\zeta_{+}(t)-, t) = w(\bar{x}+, \bar{t}) \leq w(\zeta_{+}(0)-, 0), \end{aligned} \tag{3.45}$$

$0 < t < \bar{t}$.

When $w(\bar{x}-, \bar{t}) = w(\bar{x}+, \bar{t})$, ζ_{-} and ζ_{+} coincide.

Proof. — From (2.6), (3.40) and (3.41) we infer

$$\frac{\lambda(z(y, t), w(y, t)) - \lambda(z(x, t), w(x, t))}{y - x} \leq \frac{C}{t}, \tag{3.46}$$

$-\infty < x < y < \infty, \quad t > 0$

$$\frac{\mu(z(y, t), w(y, t)) - \mu(z(x, t), w(x, t))}{y - x} \leq \frac{C}{t}, \tag{3.47}$$

$-\infty < x < y < \infty, \quad t > 0$.

It then follows from Filippov’s theory [8] that the initial value problem for (3.5) and (3.7) with data (\bar{x}, \bar{t}) , $\bar{t} > 0$, has a unique solution in the forward direction, *i.e.*, unique forward 1- and 2-characteristics emanate from (\bar{x}, \bar{t}) .

We now demonstrate (3.42). By Lemmas 3.1, 3.2 and 3.5, we already know that $z(\xi_{-}(0)+, 0) \leq z(\xi_{-}(t)+, t) = z(\xi_{-}(t)-, t) \leq z(\xi_{-}(0)-, 0)$, $0 < t < \bar{t}$. It remains to show

$$z(\xi_{-}(t)-, t) = z(\bar{x}-, \bar{t}), \quad 0 < t < \bar{t}. \tag{3.48}$$

We consider any increasing sequence $\{x_n\}$ such that $z(x_n-, \bar{t}) = z(x_n+, \bar{t})$, $n = 1, 2, \dots$, and $x_n \rightarrow \bar{x}$, as $n \rightarrow \infty$. Let ξ_n denote the unique backward 1-characteristic emanating from (x_n, \bar{t}) . By Lemma 3.5, it is

$$z(\xi_n(t)-, t) = z(\xi_n(t)+, t) = z(x_n, \bar{t}), \quad 0 < t \leq \bar{t}.$$

At the same time, as $n \rightarrow \infty$, $z(x_n, \bar{t}) \rightarrow z(\bar{x}^-, \bar{t})$, $\xi_n(t) \rightarrow \xi_-(t)$, $0 \leq t \leq \bar{t}$, and so $z(\xi_n(t) \pm, t) \rightarrow z(\xi_-(t) \pm, t)$, $0 < t \leq \bar{t}$. This establishes (3.48).

The proofs of (3.43), (3.44), (3.45) are quite similar and will thus be omitted.

4. STRUCTURE OF SOLUTIONS

The geometric structure of solutions of genuinely nonlinear, strictly hyperbolic systems of two conservation laws, constructed by the random choice method of Glimm [10], has been discussed by DiPerna [6]. Here we study the regularity of the solution of (1.4), considered in Section 3, by employing the properties of generalized characteristics. We follow closely the analysis in [1] for the single, genuinely nonlinear hyperbolic conservation law and the results are very similar.

THEOREM 4.1. — *Let (\bar{x}, \bar{t}) be any point on the upper half-plane with $\bar{t} > 0$. Consider the (unique) forward 1-characteristic χ and the (not necessarily distinct) minimal and maximal backward 1-characteristics ξ_- and ξ_+ emanating from (\bar{x}, \bar{t}) . Define the sets*

$$S_- := \{ (x, t) : 0 \leq t < \bar{t}, x \leq \xi_-(t) \text{ or } t \geq \bar{t}, x \leq \chi(t) \}$$

and

$$S_+ := \{ (x, t) : 0 \leq t < \bar{t}, x \geq \xi_+(t) \text{ or } t \geq \bar{t}, x \geq \chi(t) \}.$$

Then the restriction of $z(x-, t)$ to S_- and the restriction of $z(x+, t)$ to S_+ are continuous at (\bar{x}, \bar{t}) . In particular, z is continuous at (\bar{x}, \bar{t}) if and only if $z(\bar{x}^-, \bar{t}) = z(\bar{x}^+, \bar{t})$. Similarly, if χ is the forward 2-characteristic and ζ_- and ζ_+ the minimal and maximal backward 2-characteristics emanating from (\bar{x}, \bar{t}) , then the restriction of $w(x-, t)$ to the set

$$\Sigma_- := \{ (x, t) : 0 \leq t < \bar{t}, x \leq \zeta_-(t) \text{ or } t \geq \bar{t}, x \leq \chi(t) \}$$

and the restriction of $w(x+, t)$ to the set

$$\Sigma_+ := \{ (x, t) : 0 \leq t < \bar{t}, x \geq \zeta_+(t) \text{ or } t \geq \bar{t}, x \geq \chi(t) \}$$

are continuous at (\bar{x}, \bar{t}) . In particular, w is continuous at (\bar{x}, \bar{t}) if and only if $w(\bar{x}^-, \bar{t}) = w(\bar{x}^+, \bar{t})$.

Proof. — Take any sequence $\{(x_n, t_n)\}$ in S_- that converges to (\bar{x}, \bar{t}) , as $n \rightarrow \infty$, and consider the minimal backward 1-characteristic ξ_n emanating from (x_n, t_n) . Then $\xi_n(t) \leq \xi_-(t)$, $\xi_n(t) \rightarrow \xi_-(t)$, as $n \rightarrow \infty$. By Theorem 3.2,

$$z(\xi_n(t) -, t) = z(\bar{x}^-, \bar{t}), \quad 0 < t \leq \bar{t},$$

and $z(\xi_n(t) -, t) = z(x_n^-, t_n)$, $0 < t \leq t_n$. Therefore, $z(x_n^-, t_n) \rightarrow z(\bar{x}^-, \bar{t})$,

as $n \rightarrow \infty$. We have thus shown that the restriction of $z(x-, t)$ to S_- is continuous at (\bar{x}, \bar{t}) .

The proof of the remaining assertions of the theorem is similar and will be omitted.

The next proposition states that once a discontinuity develops it has to propagate all the way to infinity, as a shock.

THEOREM 4.2. — *If χ is the (unique) forward 1-characteristic emanating from a point (\bar{x}, \bar{t}) of the upper half-plane with $z(\bar{x}-, \bar{t}) < z(\bar{x}+, \bar{t})$ then $z(\chi(t)-, t) < z(\chi(t)+, t)$, for $\bar{t} < t < \infty$. Similarly, if ψ is the forward 2-characteristic emanating from (\bar{x}, \bar{t}) where $w(\bar{x}-, \bar{t}) < w(\bar{x}+, \bar{t})$, then $w(\psi(t)-, t) < w(\psi(t)+, t)$, for $\bar{t} < t < \infty$.*

Proof. — If for some $t > \bar{t}$ it were $z(\chi(t)-, t) = z(\chi(t)+, t)$, then, by Theorem 3.2, χ would be the unique backward 1-characteristic emanating from $(\chi(t), t)$ and $z(\chi(\bar{t})+, \bar{t}) \leq z(\chi(\bar{t})-, \bar{t})$, in contradiction to our hypothesis.

A similar argument rules out the possibility $w(\psi(t)-, t) = w(\psi(t)+, t)$ and completes the proof of the theorem.

A point (\bar{x}, \bar{t}) of the upper half-plane will be called a *1-shock generation point* if a forward 1-characteristic χ emanating from (\bar{x}, \bar{t}) satisfies $z(\chi(t)-, t) < z(\chi(t)+, t)$, for $\bar{t} < t < \infty$, while none of the backward 1-characteristics emanating from (\bar{x}, \bar{t}) contains any point of discontinuity of z . Similarly, (\bar{x}, \bar{t}) will be called a *2-shock generation point* if a forward 2-characteristic ψ emanating from (\bar{x}, \bar{t}) satisfies

$$w(\psi(t)-, t) < w(\psi(t)+, t), \text{ for } \bar{t} < t < \infty,$$

while none of the backward 2-characteristics emanating from (\bar{x}, \bar{t}) contains any point of discontinuity of w . By virtue of Theorem 4.2, it is easily seen that if (\bar{x}, \bar{t}) is a point of discontinuity of z (or w) then at least one backward 1-characteristic (or 2-characteristic) emanating from (\bar{x}, \bar{t}) must pass through a 1-shock (or a 2-shock) generation point.

When (\bar{x}, \bar{t}) is a 1-shock generation point, either $z(\bar{x}-, \bar{t}) = z(\bar{x}+, \bar{t})$ or $z(\bar{x}-, \bar{t}) < z(\bar{x}+, \bar{t})$. In the latter case (\bar{x}, \bar{t}) is the *focus* of a 1-compression wave. Similarly, 2-shock generation points (\bar{x}, \bar{t}) may be either points of continuity of w , $w(\bar{x}-, \bar{t}) = w(\bar{x}+, \bar{t})$, or focuses of 2-compression waves, when $w(\bar{x}-, \bar{t}) < w(\bar{x}+, \bar{t})$.

The following proposition describes the structure of shocks.

THEOREM 4.3. — *Let χ be a 1-shock generated at the point $(\chi(\bar{t}), \bar{t})$. Consider the four functions $z_{\pm}(t) := z(\chi(t) \pm, t)$, $w_{\pm}(t) := w(\chi(t) \pm, t)$, defined on $[\bar{t}, \infty)$. Then*

(i) z_{\pm} are right-continuous functions of bounded variation. For $t > \bar{t}$,

$$z_-(t) < z_+(t), \quad z_-(t-) \geq z_-(t+), \quad z_+(t-) \leq z_+(t+).$$

When $z_-(t-) = z_-(t+)$, $(\chi(t), t)$ is a point of continuity of the restriction

of z to the set $\{(x, \tau) : \tau > \bar{t}, x < \chi(\tau)\}$; otherwise, $(\chi(t), t)$ is a point of interaction of χ with another 1-shock or it is the focus of a 1-compression wave impinging from the left. When $z_+(t-) = z_+(t+)$, $(\chi(t), t)$ is a point of continuity of the restriction of z to the set $\{(x, \tau) : \tau > \bar{t}, x > \chi(\tau)\}$; otherwise, $(\chi(t), t)$ is a point of interaction of χ with another 1-shock or it is the focus of a 1-compression wave impinging from the right;

(ii) w_{\pm} are functions of bounded variation; w_- is right-continuous while w_+ is left-continuous. For $t > \bar{t}$, $w_-(t-) \geq w_-(t+)$, $w_+(t-) \geq w_+(t+)$, $w_-(t-) = w_+(t-)$, $w_-(t+) = w_+(t+)$. t is a point of discontinuity of w_{\pm} if $(\chi(t), t)$ is a point of interaction of χ with a 2-shock;

(iii) χ is right-differentiable at every $t \geq \bar{t}$ and

$$\frac{d^+}{dt} \chi(t) = \frac{1}{z_-(t)z_+(t)w_-(t)} \quad \bar{t} \leq t < \infty. \tag{4.1}$$

(iv) If t is a point of continuity of z_{\pm} and w_{\pm} then χ is differentiable at t .

A similar statement holds for 2-shocks ψ with the roles of z and w appropriately interchanged.

Proof. — To show that z_- has bounded variation, take any mesh $\bar{t} = t_1 < t_2 < \dots < t_n < \infty$ and consider the minimal backward 1-characteristic ξ_i emanating from $(\chi(t_i), t_i)$, $i = 1, \dots, n$. By Theorem 3.2, $\xi_1(0) \geq \xi_2(0) \geq \dots \geq \xi_n(0)$ and $z(\xi_i(0)+, 0) \leq z_-(t_i) \leq z(\xi_i(0)-, 0)$, $i = 1, \dots, n$. It follows easily that the total variation of z_- over $[\bar{t}, \infty)$ is majorized by the total variation of $z(\cdot, 0)$ over $(-\infty, \infty)$. The proof that z_+ , w_- and w_+ also have bounded variation is similar.

That z_{\pm} are right-continuous on $[\bar{t}, \infty)$, follows from Theorem 4.1. Let us fix $t > \bar{t}$ and let ξ denote the minimal backward 1-characteristic emanating from $(\chi(t), t)$. Take any increasing sequence $\{t_n\}$ in $[\bar{t}, t)$, such that $t_n \rightarrow t$ as $n \rightarrow \infty$, and let ξ_n be the minimal backward 1-characteristic emanating from $(\chi(t_n), t_n)$. As $n \rightarrow \infty$, $\xi_n \rightarrow \xi_0$, uniformly on $[0, t)$, where ξ_0 is a 1-characteristic emanating from $(\chi(t), t)$. On the one hand, it is $\xi(\tau) \leq \xi_0(\tau)$, $0 \leq \tau \leq t$. Also, by Theorem 3.2 it is

$$\begin{aligned} z(\xi(\tau) \pm, \tau) &= z_-(t) = z_-(t+), & 0 < \tau < t, \\ z(\xi_n(\tau) \pm, \tau) &= z_-(t_n), & 0 < \tau < t_n, \end{aligned}$$

and so $z(\xi_0(\tau) \pm, \tau) = z_-(t-)$, $0 < \tau < t$. If $z_-(t-) \leq z_-(t+)$, then, by Lemma 3.4, $\xi_0 \equiv \xi$ whence $z_-(t-) = z_-(t+)$. We have thus verified that $z_-(t-) \geq z_-(t+)$.

Let us assume first $z_-(t-) = z_-(t+)$, in which case, as shown above, $\xi_0 \equiv \xi$. If $\{(x_i, \tau_i)\}$ is any sequence, with $\tau_i > \bar{t}$, $x_i < \chi(\tau_i)$, which converges to $(\chi(t), t)$, as $i \rightarrow \infty$, and if ζ_i is the minimal backward characteristic emanating from (x_i, τ_i) , then, necessarily, $\zeta_i \rightarrow \xi$, as $i \rightarrow \infty$, uniformly on $[0, \tau)$. In particular, $z(x_i, \tau_i) \rightarrow z(\chi(t), t)$, $i \rightarrow \infty$, which shows that $(\chi(t), t)$

is a point of continuity of the restriction of z to the set $\{(x, \tau) : \tau > \bar{t}, x < \chi(t)\}$.

Assume now that $z_-(t-) > z_-(t+)$, in which case $\xi_0(\tau) > \xi(\tau)$, $0 < \tau < t$. The “funnel” between ξ and ξ_0 is filled with 1-characteristics emanating from $(\chi(t), t)$. If the collection of these characteristics includes no shocks, then $(\chi(t), t)$ is the focus of a 1-compression wave. Otherwise, $(\chi(t), t)$ is a point of interaction of 1-shocks.

The analogous properties of z_+ stated in the theorem are established by similar arguments.

We now turn to the functions w_{\pm} . By standard properties of BV solutions and the Rankine-Hugoniot conditions we have $w_-(t) = w_+(t)$ for almost all t in $[\bar{t}, \infty)$ and so $w_-(t-) = w_+(t-)$, $w_-(t+) = w_+(t+)$ for all t in (\bar{t}, ∞) . By virtue of Theorem 4.1, we deduce $w_-(t+) = w_-(t)$ and $w_+(t-) = w_+(t)$ for all t in (\bar{t}, ∞) . Since $w_-(t) \leq w_+(t)$, it follows that $w_+(t+) = w_-(t+) \leq w_+(t-) = w_-(t-)$. When t is a point of discontinuity of w_{\pm} , then $w(\chi(t)-, t) < w(\chi(t)+, t)$ and so, by Theorem 4.2, the forward 2-characteristic emanating from $(\chi(t), t)$ is a 2-shock.

As stated in Section 3, χ propagates almost everywhere with 1-shock speed, given by (2.25). Considering the continuity properties of the functions z_{\pm} and w_{\pm} established above, it follows that χ is right-differentiable at every $t \geq \bar{t}$ and (4.1) holds. In particular, if t is a point of continuity of z_{\pm} and w_{\pm} then χ is differentiable at t .

The proof of the corresponding properties of 2-shocks ψ is essentially identical and will thus be omitted.

Theorem 3.1 exhibits how genuine nonlinearity of our system induces one-sided Lipschitz bounds on the solution through spreading of waves in the forward time direction. The next proposition demonstrates the reverse action, *i.e.*, Lipschitz bounds on the opposite side induced by spreading of waves in the backward time direction.

THEOREM 4.4. — *Suppose that the set of points of continuity of z contains an open set \mathcal{O} . Then the restriction of z to \mathcal{O} is Lipschitz continuous. Similarly, when the set of points of continuity of w contains an open set \mathcal{P} , the restriction of w to \mathcal{P} is Lipschitz continuous.*

Proof. — We fix $(\bar{x}, \bar{t}) \in \mathcal{O}$ and consider the forward 1-characteristic ξ emanating from (\bar{x}, \bar{t}) . There is $\delta > 0$ such that $(\xi(t), t) \in \mathcal{O}$ for $\bar{t} \leq t \leq \bar{t} + \delta$. We take any $\bar{y} > \bar{x}$, so close to \bar{x} that if ζ is the forward 1-characteristic emanating from (\bar{y}, \bar{t}) then $(\zeta(t), t) \in \mathcal{O}$, $\bar{t} \leq t \leq \bar{t} + \delta$. By the results of Section 3, $\xi(t) < \zeta(t)$, $\bar{t} \leq t \leq \bar{t} + \delta$, and

$$z(\xi(t), t) = z(\bar{x}, \bar{t}), \quad z(\zeta(t), t) = z((\bar{y}, \bar{t})), \quad \bar{t} \leq t \leq \bar{t} + \delta. \quad (4.2)$$

From (2.6) and (4.2) we deduce, for almost all t in $[\bar{t}, \bar{t} + \delta]$

$$\begin{aligned} \zeta(t) - \xi(t) &= \lambda(z(\zeta(t), t), w(\zeta(t), t)) - \lambda(z(\xi(t), t), w(\xi(t), t))) \\ &= z^{-2}(\bar{y}, \bar{t}) w^{-1}(\zeta(t), t) - z^{-2}(\bar{x}, \bar{t}) w^{-1}(\xi(t), t) \\ &\leq -\alpha[z(\bar{y}, \bar{t}) - z(\bar{x}, \bar{t})] - \beta[w(\zeta(t), t) - w(\xi(t), t)], \end{aligned} \quad (4.3)$$

where α and β are positive constants. By virtue of Theorem 3.1, (4.3) yields

$$\frac{d}{dt} [\zeta(t) - \xi(t)] - \frac{\beta B}{t} [\zeta(t) - \xi(t)] \leq -\alpha[z(\bar{y}, \bar{t}) - z(\bar{x}, \bar{t})]. \quad (4.4)$$

Integrating the differential inequality (4.4) over the interval $[\bar{t}, \bar{t} + \delta]$ and after a short computation, using $\zeta(\bar{t}) - \xi(\bar{t}) = \bar{y} - \bar{x}$, $\zeta(\bar{t} + \delta) - \xi(\bar{t} + \delta) > 0$, we deduce

$$\frac{z(\bar{y}, \bar{t}) - z(\bar{x}, \bar{t})}{\bar{y} - \bar{x}} \leq \frac{\beta B}{\alpha \bar{t}} \left[1 - \exp\left(-\frac{\beta B \delta}{\bar{t}}\right) \right]^{-1}. \quad (4.5)$$

Since z is constant along 1-characteristics, by combining (3.40) with (4.5) we conclude that z is Lipschitz continuous at (\bar{x}, \bar{t}) .

The proof that w is Lipschitz continuous at every point of \mathcal{P} is similar and will thus be omitted.

The last proposition of this section should be compared with Theorem 3.1.

THEOREM 4.5. — Assume that, for some $a \geq 0$,

$$\frac{z(y, 0) - z(x, 0)}{y - x} \geq -a, \quad -\infty < x < y < \infty. \quad (4.6)$$

Then

$$\frac{z(\bar{y}, \bar{t}) - z(\bar{x}, \bar{t})}{\bar{y} - \bar{x}} \geq -K a, \quad -\infty < \bar{x} < \bar{y} < \infty, \quad \bar{t} \geq 0 \quad (4.7)$$

where K is the constant in (3.21). Similarly, if for some $b \geq 0$

$$\frac{w(y, 0) - w(x, 0)}{y - x} \geq -b, \quad -\infty < x < y < \infty, \quad (4.8)$$

then

$$\frac{w(\bar{y}, \bar{t}) - w(\bar{x}, \bar{t})}{\bar{y} - \bar{x}} \geq -K b, \quad -\infty < \bar{x} < \bar{y} < \infty, \quad \bar{t} \geq 0 \quad (4.9)$$

where K is the constant in (3.22).

Proof. — It suffices to establish (4.7) under the assumption $z(\bar{x} - , \bar{t}) = z(\bar{x} + , \bar{t})$, $z(\bar{y} - , \bar{t}) = z(\bar{y} + , \bar{t})$, $z(\bar{y}, \bar{t}) - z(\bar{x}, \bar{t}) \leq 0$. Let ξ and ζ denote the backward 1-characteristics emanating from (\bar{x}, \bar{t}) and (\bar{y}, \bar{t}) ,

respectively. Combining (3.42) with (3.21), we obtain

$$\frac{z(\bar{y}, \bar{t}) - z(\bar{x}, \bar{t})}{\bar{y} - \bar{x}} \geq \frac{z(\zeta(0)+, 0) - z(\xi(0)-, 0)}{\zeta(0) - \xi(0)} \cdot \frac{\zeta(0) - \xi(0)}{\bar{y} - \bar{x}} \geq -K a. \quad (4.10)$$

The proof of (4.9) is, of course, similar.

5. GENERIC SMOOTHNESS OF SOLUTIONS

In this section we assume that the initial data $(u_0(x), v_0(x))$ are C^k smooth, $k \geq 1$, and show that the solution $(u(x, t), v(x, t))$ is C^k in the complement of the shock set. We also prove that when $k \geq 4$ shock generation points cannot accumulate, unless the initial data satisfy a nongeneric degeneracy condition and therefore generically solutions are piecewise C^k smooth. As in earlier sections, it will be more convenient to minor the initial data and the solution through the induced Riemann invariant fields $(z_0(x), w_0(x))$ and $(z(x, t), w(x, t))$.

For each y in $(-\infty, \infty)$, we let $\chi(y, \cdot)$ and $\psi(y, \cdot)$ denote the (unique) forward 1-characteristic and 2-characteristic emanating from the point $(y, 0)$. In general, as shown in Section 4, $\chi(y, \cdot)$ passes through points of continuity of $z(x, t)$ on some maximal time interval $[0, \bar{t})$ and then follows a 1-shock over (\bar{t}, ∞) . This shock may have already been formed by the time \bar{t} χ impinges on it or it may be generated at the point $(\chi(y, \bar{t}), \bar{t})$ itself. In the latter case, $(\chi(y, \bar{t}), \bar{t})$ is either a point of continuity of $z(x, t)$ or the focus of a 1-compression wave. Of course $\psi(y, \cdot)$ has completely analogous properties. Our objective is to characterize the generation of shocks in terms of properties of the functions $\chi(y, t), \psi(y, t)$ that may in turn be translated into conditions on the initial data.

It is clear that $\chi(y, t)$ and $\psi(y, t)$ are well-defined on $(-\infty, \infty) \times (0, \infty)$, are Lipschitz in t , for fixed y , and are nondecreasing in y , for fixed t . In order to describe regularity properties of these functions with respect to y , we need the following preliminary

LEMMA 5.1. — *Let $-\infty < y_1 < y_2 < \infty$. Suppose that $\hat{t} > 0$ is such that $\chi(y_1, \cdot)$ and $\chi(y_2, \cdot)$ pass through points of continuity of $z(x, t)$ throughout the interval $[0, \hat{t})$. Setting $x_1 = \chi(y_1, \hat{t}), x_2 = \chi(y_2, \hat{t})$, we have*

$$\int_{x_1}^{x_2} z(x, \hat{t}) w(x, \hat{t}) dx - \int_{y_1}^{y_2} z_0(y) w_0(y) dy = 2[z_0^{-1}(y_2) - z_0^{-1}(y_1)] \hat{t} + z_0^2(y_2)[x_2 - y_2] - z_0^2(y_1)[x_1 - y_1]. \quad (5.1)$$

Similarly, if $\hat{t} > 0$ is such that $\psi(y_1, \cdot)$ and $\psi(y_2, \cdot)$ pass through points of continuity of $w(x, t)$ throughout the interval $[0, \hat{t})$ and upon setting

$x_1 = \psi(y_1, \hat{t}), x_2 = \psi(y_2, \hat{t})$, we have

$$\int_{x_1}^{x_2} z(x, \hat{t}) w(x, \hat{t}) dx - \int_{y_1}^{y_2} z_0(y) w_0(y) dy = 2[w_0^{-1}(y_2) - w_0^{-1}(y_1)] \hat{t} + w_0^2(y_2)[x_2 - y_2] - w_0^2(y_1)[x_1 - y_1]. \tag{5.2}$$

Proof. — By virtue of (2.5), (2.3) and (2.6), we obtain

$$\frac{v}{u} + \lambda u = \frac{2}{z} + z^2 \lambda. \tag{5.3}$$

We integrate equation (1.4)₁ over the domain

$$\{(x, t) : 0 < t < \hat{t}, \chi(y_1, t) < x < \chi(y_2, t)\}$$

and use (2.5), Theorem 3.1 and (5.3) to arrive at (5.1).

The proof of (5.2) is, of course, similar.

Let (\hat{x}, \hat{t}) be any point of the upper half-plane. Throughout this section, $\xi_-(\cdot; \hat{x}, \hat{t})$ and $\xi_+(\cdot; \hat{x}, \hat{t})$ will denote the minimal and maximal backward 1-characteristics emanating from (\hat{x}, \hat{t}) ; also $\zeta_-(\cdot; \hat{x}, \hat{t})$ and $\zeta_+(\cdot; \hat{x}, \hat{t})$ will denote the minimal and maximal backward 2-characteristics emanating from (\hat{x}, \hat{t}) . Note that $\hat{x} = \chi(\hat{y}, \hat{t})$ if and only if

$$\xi_-(0; \hat{x}, \hat{t}) \leq \hat{y} \leq \xi_+(0; \hat{x}, \hat{t});$$

also $\hat{x} = \psi(\hat{y}, \hat{t})$ if and only if $\zeta_-(0; \hat{x}, \hat{t}) \leq \hat{y} \leq \zeta_+(0; \hat{x}, \hat{t})$.

We now introduce the functions

$$Z(y, x, t) := w_0(y) - z_0(y) + 2z'_0(y)[x - y] - 2z_0^{-3}(y)z'_0(y)t, \tag{5.4}$$

$$W(y, x, t) := z_0(y) - w_0(y) + 2w'_0(y)[x - y] - 2w_0^{-3}(y)w'_0(y)t \tag{5.5}$$

defined for $-\infty < y < \infty, -\infty < x < \infty, 0 \leq t < \infty$, and state

LEMMA 5.2. — For any $\hat{t} \geq 0$, the functions $\chi(\cdot, \hat{t}), \psi(\cdot, \hat{t})$ are left and right-differentiable at every \hat{y} in $(-\infty, \infty)$. Specifically, upon setting $\hat{x} = \chi(\hat{y}, \hat{t})$, it is

$$[w(\hat{x}^-, \hat{t}) - z(\hat{x}^-, \hat{t})] \partial_y^- \chi(\hat{y}, \hat{t}) = Z(\hat{y}, \hat{x}, \hat{t}) \quad \text{if } \hat{y} = \xi_-(0; \hat{x}, \hat{t}) \tag{5.6}$$

$$\partial_y^- \chi(\hat{y}, \hat{t}) = 0 \quad \text{if } \hat{y} > \xi_-(0; \hat{x}, \hat{t}) \tag{5.7}$$

$$[w(\hat{x}^+, \hat{t}) - z(\hat{x}^+, \hat{t})] \partial_y^+ \chi(\hat{y}, \hat{t}) = Z(\hat{y}, \hat{x}, \hat{t}) \quad \text{if } \hat{y} = \xi_+(0; \hat{x}, \hat{t}) \tag{5.8}$$

$$\partial_y^+ \chi(\hat{y}, \hat{t}) = 0 \quad \text{if } \hat{y} < \xi_+(0; \hat{x}, \hat{t}). \tag{5.9}$$

Similarly, setting $\hat{x} = \psi(\hat{y}, \hat{t})$, it is

$$[z(\hat{x}^-, \hat{t}) - w(\hat{x}^-, \hat{t})] \partial_y^- \psi(\hat{y}, \hat{t}) = W(\hat{y}, \hat{x}, \hat{t}) \quad \text{if } \hat{y} = \zeta_-(0; \hat{x}, \hat{t}) \tag{5.10}$$

$$\partial_y^- \psi(\hat{y}, \hat{t}) = 0 \quad \text{if } \hat{y} > \zeta_-(0; \hat{x}, \hat{t}) \tag{5.11}$$

$$[z(\hat{x}^+, \hat{t}) - w(\hat{x}^+, \hat{t})] \partial_y^+ \psi(\hat{y}, \hat{t}) = W(\hat{y}, \hat{x}, \hat{t}) \quad \text{if } \hat{y} = \zeta_+(0; \hat{x}, \hat{t}) \tag{5.12}$$

$$\partial_y^+ \psi(\hat{y}, \hat{t}) = 0 \quad \text{if } \hat{y} < \zeta_+(0; \hat{x}, \hat{t}). \tag{5.13}$$

Proof. — To show (5.7), it suffices to notice that for any $\xi_-(0; \hat{x}, \hat{t}) < y < \hat{y}$ it is $\chi(y, \hat{t}) = \hat{x} = \chi(\hat{y}, \hat{t})$. The proof of (5.9), (5.11) and (5.13) is, of course, similar.

We now assume $\hat{y} = \xi_-(0; \hat{x}, \hat{t})$, we fix $\bar{y} < \hat{y}$ and set $\bar{x} = \chi(\bar{y}, \hat{t})$, $y_- = \xi_-(0; \bar{x}, \hat{t})$, $y_+ = \xi_+(0; \bar{x}, \hat{t})$. We note that

$$\frac{\hat{x} - \bar{x}}{\hat{y} - y_-} \leq \frac{\chi(\hat{y}, \hat{t}) - \chi(\bar{y}, \hat{t})}{\hat{y} - \bar{y}} \leq \frac{\hat{x} - \bar{x}}{\hat{y} - y_+}. \tag{5.14}$$

Applying (5.1), we get

$$\int_{\bar{x}}^{\hat{x}} z(x, \hat{t}) w(x, \hat{t}) dx - \int_{y_{\pm}}^{\hat{y}} z_0(y) w_0(y) dy = 2[z_0^{-1}(\hat{y}) - z_0^{-1}(y_{\pm})] \hat{t} + [z_0^2(\hat{y}) - z_0^2(y_{\pm})][\hat{x} - \hat{y}] + z_0^2(y_{\pm})[\hat{x} - \bar{x} - \hat{y} + y_{\pm}]. \tag{5.15}$$

Dividing (5.15) by $\hat{y} - y_{\pm}$ and letting $\bar{y} \uparrow \hat{y}$ we arrive at (5.6).

A similar argument yields (5.8), (5.10) and (5.12).

LEMMA 5.3. — Assume (\hat{x}, \hat{t}) is a point of continuity of $z(x, t)$ and let $\hat{y} = \xi_{\pm}(0; \hat{x}, \hat{t})$. If $\partial_y^{\pm} \chi(\hat{y}, \hat{t}) > 0$ then $z(\cdot, \hat{t})$ is left and right-differentiable at \hat{x} and

$$\partial_x^{\pm} z(\hat{x}, \hat{t}) = z'_0(\hat{y}) / \partial_y^{\pm} \chi(\hat{y}, \hat{t}). \tag{5.16}$$

Similarly assume (\hat{x}, \hat{t}) is a point of continuity of $w(x, t)$ and let $\hat{y} = \zeta_{\pm}(0; \hat{x}, \hat{t})$. If $\partial_y^{\pm} \psi(\hat{y}, \hat{t}) > 0$ then $w(\cdot, \hat{t})$ is left and right-differentiable at \hat{x} and

$$\partial_x^{\pm} w(\hat{x}, \hat{t}) = w'_0(\hat{y}) / \partial_y^{\pm} \psi(\hat{y}, \hat{t}). \tag{5.17}$$

Proof. — Assuming $\partial_y^- \chi(\hat{y}, \hat{t}) > 0$, take any increasing sequence $\{x_n\}$, $x_n \uparrow \hat{x}$ as $n \rightarrow \infty$, and set $y_n = \xi_-(0; x_n, \hat{t})$.

By virtue of Theorem 3.2,

$$\frac{z(x_n, \hat{t}) - z(\hat{x}, \hat{t})}{x_n - \hat{x}} = \frac{z_0(y_n) - z_0(\hat{y})}{y_n - \hat{y}} \cdot \frac{y_n - \hat{y}}{x_n - \hat{x}}. \tag{5.18}$$

Since $y_n \uparrow \hat{y}$, $n \rightarrow \infty$, and $x_n = \chi(y_n, \hat{t})$, letting $n \rightarrow \infty$ in (5.18) we arrive at (5.16), with the $(-)$ sign.

The proof of the remaining assertions of the lemma is quite similar and need not be recorded here.

We have now laid the preparation for characterizing shock generation points:

LEMMA 5.4. — A point (\hat{x}, \hat{t}) of continuity of $z(x, t)$, with $\hat{y} = \zeta_{\pm}(0; \hat{x}, \hat{t})$ is a 1-shock generation point if and only if $\partial_y^{\pm} \chi(\hat{y}, \hat{t}) = 0$, i. e.,

$$Z(\hat{y}, \hat{x}, \hat{t}) = 0. \tag{5.19}$$

Proof. — Combining Theorem 4.1 with (5.4) and (2.6) we deduce that if $(\chi(\hat{y}, t), t)$ is a point of continuity of $z(x, t)$ then

$$\begin{aligned} \partial_t^+ Z(\hat{y}, \chi(\hat{y}, t), t) &= 2z'_0(\hat{y})\lambda(z_0(\hat{y}), w(\chi(\hat{y}, t)-, t)) - 2z_0^{-3}(\hat{y})z'_0(\hat{y}) \\ &= 2z'_0(\hat{y})z_0^{-2}(\hat{y})[w^{-1}(\chi(\hat{y}, t)-, t) - z_0^{-1}(\hat{y})]. \end{aligned} \quad (5.21)$$

Therefore, if (5.19) holds [in particular $z'_0(\hat{y}) \neq 0$] it follows that $Z(\hat{y}, \chi(\hat{y}, t), t) > 0$ for $t - \hat{t}$ positive small in which case $(\chi(\hat{y}, t), t)$ is not a point of continuity of $z(x, t)$ and so (\hat{x}, \hat{t}) is a 1-shock generation point.

Conversely, assume (\hat{x}, \hat{t}) is a 1-shock generation point. We fix $\bar{t} > \hat{t}$ and set $\bar{x} = \chi(\hat{y}, \bar{t})$. We note that $z(\bar{x}-, \bar{t}) < z(\bar{x}+, \bar{t})$. We define $p_- = \xi_-(\bar{t}; \bar{x}, \bar{t})$, $p_+ = \xi_+(\bar{t}; \bar{x}, \bar{t})$. Applying (3.21) with $\bar{y} = \bar{x}$, $t = \bar{t}$, we deduce

$$p_+ - p_- \leq L[z(\bar{x}+, \bar{t}) - z(\bar{x}-, \bar{t})](\bar{t} - \hat{t}). \quad (5.22)$$

Since $z(p_-, \hat{t}) = z(\bar{x}-, \bar{t})$ and $z(p_+, \hat{t}) = z(\bar{x}+, \bar{t})$, (5.22) implies

$$\frac{z(p_+, \hat{t}) - z(p_-, \hat{t})}{p_+ - p_-} \rightarrow \infty, \quad \text{as } \bar{t} \downarrow \hat{t}. \quad (5.23)$$

On the other hand

$$\frac{z(p_+, \hat{t}) - z(p_-, \hat{t})}{p_+ - p_-} \leq \left| \frac{z(p_+, \hat{t}) - z(\hat{x}, \hat{t})}{p_+ - \hat{x}} \right| + \left| \frac{z(p_-, \hat{t}) - z(\hat{x}, \hat{t})}{p_- - \hat{x}} \right|. \quad (5.24)$$

If (5.19) were false, i.e., $\partial_y^\pm \chi(\hat{y}, \hat{t}) > 0$, then Lemma 5.3 and (5.24) would contradict (5.23).

The proof of the remaining assertions of the lemma is quite similar.

LEMMA 5.5. — Assume (\hat{x}, \hat{t}) is the focus of a 1-compression wave and let $y_- = \xi_-(0; \hat{x}, \hat{t})$, $y_+ = \xi_+(0; \hat{x}, \hat{t})$. Then

$$Z(\hat{y}, \hat{x}, \hat{t}) = 0, \quad y_- \leq \hat{y} \leq y_+. \quad (5.25)$$

Similarly, assume (\hat{x}, \hat{t}) is the focus of a 2-compression wave and let $y_- = \zeta_-(0; \hat{x}, \hat{t})$, $y_+ = \zeta_+(0; \hat{x}, \hat{t})$. Then

$$W(\hat{y}, \hat{x}, \hat{t}) = 0, \quad y_- \leq \hat{y} \leq y_+. \quad (5.26)$$

Proof. — Fix $y_- \leq \bar{y} < \hat{y} \leq y_+$ and apply (5.1), noting that

$$\chi(\bar{y}, \hat{t}) = \hat{x} = \chi(\hat{y}, \hat{t}).$$

This yields

$$-\int_{\bar{y}}^{\hat{y}} z_0(y) w_0(y) dy = 2[z_0^{-1}(\hat{y}) - z_0^{-1}(\bar{y})] \hat{t} + [z_0^2(\hat{y}) - z_0^2(\bar{y})][\hat{x} - \hat{y}] - z_0^2(\bar{y})[\hat{y} - \bar{y}]. \quad (5.27)$$

Dividing (5.27) by $\hat{y} - \bar{y}$ and letting $\bar{y} \uparrow \hat{y}$ we obtain (5.25).

The proof of (5.26) is essentially identical.

The 1-shock set (or 2-shock set) is the union of the set of points of discontinuity of $z(x, t)$ [or $w(x, t)$] and the set of 1-shock (or 2-shock) generation points. The union of the 1-shock set and the 2-shock set forms the shock set.

LEMMA 5.6. — *The 1-shock set and the 2-shock set are closed.*

Proof. — Assume (\bar{x}, \bar{t}) is the limit of a sequence $\{(x_n, t_n)\}$ of points of the 1-shock set. As stated in Section 4, from each point (x_n, t_n) emanates at least one backward 1-characteristic, say $\xi_n(\cdot)$, which passes through a 1-shock generation point, i. e., $(\xi_n(\tau_n), \tau_n)$ is a 1-shock generation point for some τ_n in $(0, t_n]$. In particular, by Lemmas 5.4 and 5.5 it is

$$Z(\xi_n(0), \xi_n(\tau_n), \tau_n) = 0. \quad (5.28)$$

We may assume (\bar{x}, \bar{t}) is a point of continuity of $z(x, t)$ since otherwise (\bar{x}, \bar{t}) obviously lies on the 1-shock set. Then $\{\xi_n(\cdot)\}$ converges to the (unique) backward 1-characteristic $\xi_{\pm}(\cdot)$ emanating from (\bar{x}, \bar{t}) , uniformly on $[0, \bar{t}]$. In particular, $\{(\xi_n(\tau_n), \tau_n)\}$, or a subsequence thereof, converges to a point $(\xi_{\pm}(\hat{t}), \hat{t})$, $\hat{t} \in (0, \bar{t}]$. Upon setting $\hat{x} = \xi_{\pm}(\hat{t})$, $\hat{y} = \xi_{\pm}(0)$, (5.28) implies that $Z(\hat{y}, \hat{x}, \hat{t}) = 0$, in which case, by Lemma 5.4, (\hat{x}, \hat{t}) is a 1-shock generation point. It follows that $\hat{t} = \bar{t}$, $\hat{x} = \bar{x}$ and (\bar{x}, \bar{t}) lies on the 1-shock set.

The proof that the 2-shock set is also closed is essentially the same.

The following proposition describes the regularity of the solution:

THEOREM 5.1. — *If the initial data are C^k smooth, $k \geq 1$, then the shock set is closed and on its (open) complement the solution is C^k smooth.*

Proof. — The shock set is closed by Lemma 5.6. Any point (\hat{x}, \hat{t}) in the complement \mathcal{O} of this set is a point of continuity of both $z(x, t)$ and $w(x, t)$. Moreover, if $\hat{y} = \xi_{\pm}(0; \hat{x}, \hat{t})$ and $\bar{y} = \zeta_{\pm}(0; \hat{x}, \hat{t})$, Lemma 5.4 implies $\chi_y(\hat{y}, \hat{t}) > 0$, $\psi_y(\bar{y}, \hat{t}) > 0$. It follows that $x = \chi(y, \hat{t})$ and $x = \psi(y, \hat{t})$ may be inverted on the relevant range: $y = f(x, \hat{t})$ and $y = g(x, \hat{t})$; in particular, $\hat{y} = f(\hat{x}, \hat{t})$ and $\bar{y} = g(\hat{x}, \hat{t})$.

At the same time, (5.6) and (5.10) yield

$$\chi_y(\hat{y}, \hat{t}) = [w(\chi(\hat{y}, \hat{t}), \hat{t}) - z_0(\hat{y})]^{-1} Z(\hat{y}, \chi(\hat{y}, \hat{t}), \hat{t}) \quad (5.29)$$

$$\psi_y(\bar{y}, \hat{t}) = [z(\psi(\bar{y}, \hat{t}), \hat{t}) - w_0(\bar{y})]^{-1} W(\bar{y}, \psi(\bar{y}, \hat{t}), \hat{t}) \quad (5.30)$$

whence it follows, by induction, that if $z(x, t)$ and $w(x, t)$ are C^l on \mathcal{O} , for some $l=0, \dots, k-1$, then $\chi(y, t)$, $\psi(y, t)$, and thereby also $f(x, t)$, $g(x, t)$, are C^{l+1} . However, since

$$z(\hat{x}, \hat{t}) = z_0(f(\hat{x}, \hat{t})), \quad w(\hat{x}, \hat{t}) = w_0(g(\hat{x}, \hat{t})),$$

we infer that $f(x, t), g(x, t)$ in $C^{l+1}, l=0, \dots, k-1$, implies that $z(x, t), w(x, t)$ are in C^{l+1} . Therefore, the restriction of the solution on \mathcal{O} is in C^k and the proof of the theorem is complete.

In general, the shock set may be quite sizeable. We now proceed to show that generically, for initial data in $C^k, k \geq 4$, the set of shock generation points is locally finite.

LEMMA 5.7. — Suppose (\hat{x}, \hat{t}) is a 1-shock generation point and let $y_- = \xi_-(0; \hat{x}, \hat{t}), y_+ = \xi_+(0; \hat{x}, \hat{t})$. Then

$$Z_y(\hat{y}, \hat{x}, \hat{t}) = 0, \quad y_- \leq \hat{y} \leq y_+. \tag{5.31}$$

Similarly, if (\hat{x}, \hat{t}) is a 2-shock generation point and $y_- = \zeta_-(0; \hat{x}, \hat{t}), y_+ = \zeta_+(0; \hat{x}, \hat{t})$, then

$$W_y(\hat{y}, \hat{x}, \hat{t}) = 0, \quad y_- \leq \hat{y} \leq y_+. \tag{5.32}$$

Proof. — When (\hat{x}, \hat{t}) is the focus of a 1-compression wave, (5.31) follows immediately from (5.25). We thus assume (\hat{x}, \hat{t}) is a point of continuity of $z(x, t)$ and we set $\hat{y} = \xi_{\pm}(0; \hat{x}, \hat{t})$. Then (5.19) holds and may be written in the form

$$Z(\hat{y}, \chi(\hat{y}, \hat{t}), \hat{t}) = 0. \tag{5.33}$$

Consider any sequence $\{x_n\}, x_n \rightarrow \hat{x}$ as $n \rightarrow \infty$, and set $y_n = \xi_-(0; x_n, \hat{t})$.

In particular, $y_n \rightarrow \hat{y}$ as $n \rightarrow \infty$. By Lemma 5.2 we deduce

$$Z(y_n, \chi(y_n, \hat{t}), \hat{t}) \leq 0, \quad n = 1, 2, \dots \tag{5.34}$$

Lemma 5.4 implies that $\chi(\cdot, \hat{t})$ is differentiable at $y = \hat{y}$ and $\chi_y(\hat{y}, \hat{t}) = 0$. Therefore, $Z(\cdot, \chi(\cdot, \hat{t}), \hat{t})$ is differentiable at $y = \hat{y}$ and

$$\partial_y Z(\hat{y}, \chi(\hat{y}, \hat{t}), \hat{t}) = Z_y(\hat{y}, \hat{x}, \hat{t}). \tag{5.35}$$

By virtue of (5.33), (5.34), the left-hand side of (5.35) vanishes and this establishes (5.31).

The proof of (5.32) is essentially identical.

THEOREM 5.2. — Solutions of (1.4) with initial data in $C^k, k \geq 4$, are generically piecewise C^k smooth and do not contain centered compression waves. Solutions of (1.4) with (real) analytic initial data are always piecewise smooth.

Proof. — Suppose the solution contains a bounded sequence $\{(x_n, t_n)\}$ of distinct 1-shock generation points. Upon setting $y_n = \xi_-(0; x_n, t_n)$,

(5.19) and (5.31) imply

$$Z(y_n, x_n, t_n) = 0, \quad Z_y(y_n, x_n, t_n) = 0, \quad n = 1, 2, \dots \quad (5.36)$$

By account of (5.25) and (5.31), (5.36) also holds when the solution contains a 1-compression wave, with focus at, say (\hat{x}, \hat{t}) , where now $x_n = \hat{x}$, $t_n = \hat{t}$ and $\{y_n\}$ is any sequence in the interval $[\xi_-(0; \hat{x}, \hat{t}), \xi_+(0; \hat{x}, \hat{t})]$. In either case, $x_n = \chi(y_n, t_n)$. Also, passing if necessary to a subsequence, we may assume that $\{y_n\}$ converges to some \hat{y} . Because of (5.36), $z'_0(y_n)$ is bounded away from zero, uniformly in n , and so $z'_0(\hat{y}) \neq 0$.

For any y near \hat{y} we define

$$T(y) := \frac{1}{6} z_0^4(y) [z'_0(y)]^{-3} \\ \times \{ [w_0(y) - z_0(y)] z''_0(y) + [3z'_0(y) - w'_0(y)] z'_0(y) \}, \quad (5.37)$$

$$X(y) := z_0^{-3}(y) T(y) + y - \frac{1}{2} [z'_0(y)]^{-1} [w_0(y) - z_0(y)]. \quad (5.38)$$

Using (5.4) it is easy to verify that $Z(y, x, t) = 0$, $Z_y(y, x, t) = 0$ hold simultaneously if and only if $x = X(y)$, $t = T(y)$. In particular, $x_n = X(y_n)$, $t_n = T(y_n)$. The construction of these functions also implies

$$X'(y) = z_0^{-3}(y) T'(y). \quad (5.39)$$

We now consider the Lipschitz function

$$F(y) := \chi(y, T(y)) - X(y) \quad (5.40)$$

and we note that

$$F(y_n) = \chi(y_n, T(y_n)) - X(y_n) = \chi(y_n, t_n) - x_n = 0, \quad \left. \vphantom{F(y_n)} \right\} \\ n = 1, 2, \dots \quad (5.41)$$

By virtue of Theorems 4.1 and 4.3, $\chi(y, t)$ is right-differentiable with respect to t and its right derivative is given by

$$\partial_t^+ \chi(y, t) = z^{-1}(\chi(y, t)-, t) z^{-1}(\chi(y, t)+, t) w^{-1}(\chi(y, t)-, t). \quad (5.42)$$

This together with Lemma 5.2 imply that the function $\chi(y, T(y))$ is right-differentiable at any y with $T'(y) \geq 0$, left-differentiable at any y with $T'(y) \leq 0$ and in either case

$$\frac{d^\pm}{dy} \chi(y, T(y)) = \partial_y^\pm \chi(y, t) \Big|_{t=T(y)} + M(y) T'(y) \quad (5.43)$$

where

$$M(y) := z^{-1}(\chi(y, T(y))-, T(y)) \\ \times z^{-1}(\chi(y, T(y))+, T(y)) w^{-1}(\chi(y, T(y))-, T(y)). \quad (5.44)$$

From (5.40), (5.43) and (5.39) it follows that almost everywhere the derivative of $F(y)$ is given by

$$F'(y) = \chi_y(y, T(y)) + [M(y) - z_0^{-3}(y)] T'(y). \tag{5.45}$$

By Lemma 5.2, $\chi_y(y, T(y))$ is either zero or proportional to $Z(y, \chi(y, T(y)), T(y))$. Since $Z(y, X(y), T(y)) = 0$, we infer, by account of (5.40),

$$\chi_y(y, T(y)) = O(|F(y)|). \tag{5.46}$$

We also note that $M(y) - z_0^{-3}(y)$ is bounded away from zero and $T'(y)$ is C^{k-3} smooth.

Recalling that $\{y_n\}$ converges to \hat{y} , we deduce from (5.41) that in the vicinity of \hat{y} $F'(y)$ changes sign infinitely many times or vanishes identically. This being the case, it follows from (5.45), (5.46) that if $F(y) = O(|y - \hat{y}|^l)$, for some $l = 1, \dots, k-3$, then $T'(y)$, and thereby $F'(y)$ itself, should also be $O(|y - \hat{y}|^l)$; hence $F(y) = O(|y - \hat{y}|^{l+1})$. Thus, starting out from $F(y) = O(|y - \hat{y}|)$, which follows directly from (5.41), we show by induction that $F(y) = O(|y - \hat{y}|^{k-2})$. Returning to the right-hand side of (5.45), we now have $\chi_y(y, T(y)) = O(|y - \hat{y}|^{k-2})$ and this implies $T'(y) = o(|y - \hat{y}|^{k-3})$, *i. e.*,

$$T^{(l)}(\hat{y}) = 0, \quad l = 1, \dots, k-2. \tag{5.47}$$

We have thus shown that 1-compression waves cannot appear and 1-shock generation points cannot accumulate, unless the initial data satisfy the conditions (5.47) at some point \hat{y} in $(-\infty, \infty)$. Recalling the definition (5.37) of $T(y)$, we easily verify that the set of initial data $(z_0(y), w_0(y))$ which satisfy (5.47) at some point \hat{y} in any fixed compact interval $[a, b]$ is closed and nowhere dense in the Banach space $C^k(-\infty, \infty)$, $k \geq 4$. Therefore, the set of initial data that satisfy (5.47) at some point \hat{y} in $(-\infty, \infty)$ is of the first category.

Assume now the initial data are analytic on $(-\infty, \infty)$. If (5.47) is to hold for $l = 1, 2, \dots$ then $T(y)$ has to be constant, say $\hat{\ell}$, over $(-\infty, \infty)$ and, by virtue of (5.39), $X(y)$ also has to be constant, say \hat{x} , over $(-\infty, \infty)$. Under these conditions at most one 1-shock generation point may appear, namely at $(\hat{x}, \hat{\ell})$. Thus in every case the number of 1-shocks is locally finite.

A similar argument shows that the set of initial data that may generate 2-compression waves and/or infinitely many 2-shock generation points in a bounded region of the upper half-plane is of the first category in $C^k(-\infty, \infty)$, $k \geq 4$. Similarly, it is shown that when the initial data are (real) analytic the number of 2-shocks is locally finite. The proof of the theorem is complete.

6. UNIQUENESS OF SOLUTIONS

In this section we show uniqueness of solutions for the Cauchy problem of (1.4) via Holmgren's method, which works here mainly due to the one-sided Lipschitz estimates on solutions established in Theorems 3.1 and 4.5. Our result applies only when on a neighborhood of each point the initial data for at least one of the Riemann invariants satisfies a one-sided Lipschitz condition that rules out centered rarefaction waves of the corresponding family.

THEOREM 6.1. — *Let $(u_0(x), v_0(x))$ be functions with bounded variation and small oscillation defined on $(-\infty, \infty)$ and taking values in the physically relevant range. Assume that the induced Riemann invariant field $z_0(x)$ satisfies a one-sided Lipschitz condition*

$$\frac{z_0(y) - z_0(x)}{y - x} \geq -a, \quad -\infty < x < y < \infty. \quad (6.1)$$

Then there is a unique admissible BV solution $(u(x, t), v(x, t))$ for the Cauchy problem for (1.4) on $(-\infty, \infty) \times [0, \infty)$ with initial data $u(x, 0) = u_0(x), v(x, 0) = v_0(x), -\infty < x < \infty$.

Proof. — Suppose $(\bar{u}(x, t), \bar{v}(x, t))$ is another admissible BV solution of (1.4) on a strip $(-\infty, \infty) \times [0, T]$ with $\bar{u}(x, 0) = u_0(x), \bar{v}(x, t) = v_0(x), -\infty < x < \infty$. We have to show that the column-vector field $V := (\bar{u} - u \quad \bar{v} - v)^T$ vanishes identically or, equivalently, that

$$\int_0^T \int_{-\infty}^{\infty} GV \, dx \, dt = 0 \quad (6.2)$$

holds for any fixed C^1 "test" row-vector field $G(x, t)$ with compact support in $(-\infty, \infty) \times (0, T)$.

Upon noting the identity

$$-\begin{bmatrix} \frac{\bar{v}}{u} - \frac{v}{u} \\ \frac{1}{\bar{u}} - \frac{1}{u} \end{bmatrix} = \mathcal{A}V, \quad (6.3)$$

with

$$\mathcal{A} := \begin{bmatrix} \frac{v}{u} & -\frac{1}{u} \\ \frac{u}{u} & \frac{1}{u^2} \\ \frac{1}{\bar{u}} & 0 \\ \frac{1}{u} & \frac{1}{u^2} \end{bmatrix}, \quad (6.4)$$

we deduce

$$\partial_t V + \partial_x (\mathcal{A}V) = 0, \quad -\infty < x < \infty, \quad 0 < t < \infty \tag{6.5}$$

$$V(x, 0) = 0, \quad -\infty < x < \infty. \tag{6.6}$$

In order to apply Holmgren's method, we approximate \mathcal{A} with a family \mathcal{A}_ε , $\varepsilon > 0$, of smooth matrix fields constructed by the following procedure: We fix any nonnegative C^∞ kernel $\chi(y, s)$ with support contained in the unit circle and total mass 1. For $\varepsilon > 0$ we set $\chi_\varepsilon(y, s) = \varepsilon^{-2} \chi(y/\varepsilon, s/\varepsilon)$ and mollify, in the usual way, the Riemann invariant fields $(z(x, t), w(x, t))$ and $(\bar{z}(x, t), \bar{w}(x, t))$ associated with $(u(x, t), v(x, t))$ and $(\bar{u}(x, t), \bar{v}(x, t))$:

$$z_\varepsilon := z * \chi_\varepsilon, \quad w_\varepsilon := w * \chi_\varepsilon, \quad \bar{z}_\varepsilon := \bar{z} * \chi_\varepsilon, \quad \bar{w}_\varepsilon := \bar{w} * \chi_\varepsilon, \tag{6.7}$$

defined for $-\infty < x < \infty$, $\varepsilon \leq t < \infty$. Finally we set

$$u_\varepsilon := z_\varepsilon w_\varepsilon, \quad v_\varepsilon := z_\varepsilon + w_\varepsilon, \quad \bar{u}_\varepsilon := \bar{z}_\varepsilon \bar{w}_\varepsilon, \quad \bar{v}_\varepsilon := \bar{z}_\varepsilon + \bar{w}_\varepsilon \tag{6.8}$$

and then define

$$\mathcal{A}_\varepsilon := \frac{u_\varepsilon}{\bar{u}_\varepsilon} \begin{bmatrix} \frac{v_\varepsilon}{u_\varepsilon} & -\frac{1}{u_\varepsilon} \\ \frac{1}{\bar{u}_\varepsilon} & 0 \end{bmatrix}. \tag{6.9}$$

We now solve the Cauchy problem

$$\partial_t P_\varepsilon + \partial_x P_\varepsilon \mathcal{A}_\varepsilon = G, \quad -\infty < x < \infty, \quad \varepsilon \leq t \leq T, \tag{6.10}$$

$$P_\varepsilon(x, T) = 0, \quad -\infty < x < \infty, \tag{6.11}$$

and proceed to derive bounds for $\partial_x P_\varepsilon$, independent of ε .

We view $\partial_x P_\varepsilon$ as solutions of the equation

$$\partial_t \partial_x P_\varepsilon + \partial_x (\partial_x P_\varepsilon \mathcal{A}_\varepsilon) = \partial_x G, \quad -\infty < x < \infty, \quad \varepsilon \leq t \leq T \tag{6.12}$$

which obtains by differentiating (6.10) with respect to x . We must rewrite (6.12) in characteristic form. Comparing (6.9) with (2.1) and recalling (2.5) to (2.9) and (6.8) we deduce that the eigenvalues of \mathcal{A}_ε are $(z_\varepsilon \bar{z}_\varepsilon \bar{w}_\varepsilon)^{-1}$, $(w_\varepsilon z_\varepsilon \bar{z}_\varepsilon)^{-1}$; the corresponding right (column) eigenvectors are

$$R = \begin{pmatrix} w_\varepsilon \\ 1 \end{pmatrix}, \quad S = \begin{pmatrix} z_\varepsilon \\ 1 \end{pmatrix}; \tag{6.13}$$

and the corresponding left (row) eigenvectors are

$$L = \begin{pmatrix} 1 & z_\varepsilon \\ w_\varepsilon - z_\varepsilon & z_\varepsilon - w_\varepsilon \end{pmatrix}, \quad M = \begin{pmatrix} 1 & w_\varepsilon \\ z_\varepsilon - w_\varepsilon & w_\varepsilon - z_\varepsilon \end{pmatrix}. \tag{6.14}$$

We write

$$\partial_x P_\varepsilon = \varphi L + \psi M \tag{6.15}$$

where

$$\varphi := \partial_x P_\varepsilon R, \quad \psi := \partial_x P_\varepsilon S. \tag{6.16}$$

We also set

$$g := \partial_x GR, \quad h := \partial_x GS. \tag{6.17}$$

We multiply (6.12) first by R and then by S . Using (6.15), (6.16), (6.17), (6.13), (6.14) we deduce, after a lengthy but straightforward calculation:

$$\begin{aligned} \partial_t \varphi + (z_\varepsilon \bar{z}_\varepsilon \bar{w}_\varepsilon)^{-1} \partial_x \varphi &= \frac{1}{w_\varepsilon - z_\varepsilon} [\partial_t w_\varepsilon + (z_\varepsilon \bar{z}_\varepsilon \bar{w}_\varepsilon)^{-1} \partial_x w_\varepsilon] \varphi - \partial_x (z_\varepsilon \bar{z}_\varepsilon \bar{w}_\varepsilon)^{-1} \varphi \\ &\quad + \frac{1}{z_\varepsilon - w_\varepsilon} [\partial_t w_\varepsilon + (\bar{z}_\varepsilon \bar{w}_\varepsilon w_\varepsilon)^{-1} \partial_x w_\varepsilon] \psi + g, \end{aligned} \tag{6.18}$$

$$\begin{aligned} \partial_t \psi + (\bar{z}_\varepsilon \bar{w}_\varepsilon w_\varepsilon)^{-1} \partial_x \psi &= \frac{1}{z_\varepsilon - w_\varepsilon} [\partial_t z_\varepsilon + (\bar{z}_\varepsilon \bar{w}_\varepsilon w_\varepsilon)^{-1} \partial_x z_\varepsilon] \psi - \partial_x (\bar{z}_\varepsilon \bar{w}_\varepsilon w_\varepsilon)^{-1} \psi \\ &\quad + \frac{1}{w_\varepsilon - z_\varepsilon} [\partial_t z_\varepsilon + (z_\varepsilon \bar{z}_\varepsilon \bar{w}_\varepsilon)^{-1} \partial_x z_\varepsilon] \varphi + h. \end{aligned} \tag{6.19}$$

We have to estimate the coefficients of φ and ψ on the right-hand side of (6.18), (6.19). We claim that

$$\begin{aligned} -\lambda_+ \partial_x z_\varepsilon(x, t) - \frac{A(\lambda_+ - \lambda_-)}{t - \varepsilon} &\leq \partial_t z_\varepsilon(x, t) \leq -\lambda_- \partial_x z_\varepsilon(x, t) + \frac{A(\lambda_+ - \lambda_-)}{t - \varepsilon} \end{aligned} \tag{6.20}$$

where λ_- and λ_+ denote, respectively, the infimum and the supremum of $\lambda(u(x, t), v(x, t))$ over the upper half-plane and A is the constant appearing in (3.40). To verify (6.20) we fix x in $(-\infty, \infty)$, $t \geq \varepsilon$, (y, s) in the support of χ_ε and $\tau > 0$. Let ξ denote the minimal backward 1-characteristic emanating from the point $(x - y, t - s + \tau)$. Then

$$-\lambda_+ \tau \leq \xi(t - s) - x + y \leq -\lambda_- \tau. \tag{6.21}$$

Furthermore, by virtue of Theorem 3.2,

$$z(x - y, t - s + \tau) = z(\xi(t - s), t - s). \tag{6.22}$$

Combining (6.22), (6.21) and (3.40) we deduce

$$z(x - y, t - s + \tau) \leq z(x - y - \lambda_- \tau, t - s) + \frac{A(\lambda_+ - \lambda_-)}{t - \varepsilon} \tau, \tag{6.23}$$

$$z(x - y, t - s + \tau) \geq z(x - y - \lambda_+ \tau, t - s) - \frac{A(\lambda_+ - \lambda_-)}{t - \varepsilon} \tau. \tag{6.24}$$

Since

$$\partial_t z_\varepsilon(x, t) = \lim_{\tau \rightarrow 0} \frac{1}{\tau} \iint [z(x-y, t-s+\tau) - z(x-y, t-s)] \chi_\varepsilon(y, s) dy ds, \quad (6.25)$$

(6.20) follows easily from (6.23), (6.24).

A similar argument shows

$$-\mu_+ \partial_x w_\varepsilon(x, t) - \frac{B(\mu_+ - \mu_-)}{t - \varepsilon} \leq \partial_t w_\varepsilon(x, t) \leq -\mu_- \partial_x w_\varepsilon(x, t) + \frac{B(\mu_+ - \mu_-)}{t - \varepsilon} \quad (6.26)$$

where μ_- and μ_+ denote, respectively, the infimum and the supremum of $\mu(u(x, t), v(x, t))$ over the upper half-plane and B is the constant appearing in (3.41).

Next we observe that by virtue of (6.7), (6.1) and (4.7) we have

$$\partial_x z_\varepsilon(x, t) \geq -K a, \quad \partial_x \bar{z}_\varepsilon(x, t) \geq -K a \quad (6.27)$$

while (3.41) yields

$$\partial_x w_\varepsilon(x, t) \geq -\frac{B}{t - \varepsilon}, \quad \partial_x \bar{w}_\varepsilon(x, t) \geq -\frac{B}{t - \varepsilon}. \quad (6.28)$$

We are now ready to estimate φ and ψ through (6.18) and (6.19). To that end we will use a method introduced in [4]: We construct a Lipschitz function Φ on $[\varepsilon, T]$ by

$$\Phi(t) := \max_{(-\infty, \infty)} \max \{ |\varphi(\cdot, t)|, |\psi(\cdot, t)| \}. \quad (6.29)$$

Fixing a point t of differentiability of Φ , we proceed to estimate $\dot{\Phi}(t)$. Assume first that $\Phi(t) = \varphi(\bar{x}, t)$ for some \bar{x} in $(-\infty, \infty)$. Then $\partial_x \varphi(\bar{x}, t) = 0$, $\partial_t \varphi(\bar{x}, t) = \dot{\Phi}(t)$ and so (6.18) yields

$$\begin{aligned} \dot{\Phi}(t) = & -\frac{1}{z_\varepsilon - w_\varepsilon} [\partial_t w_\varepsilon + (z_\varepsilon \bar{z}_\varepsilon \bar{w}_\varepsilon)^{-1} \partial_x w_\varepsilon] \Phi(t) \\ & - \partial_x (z_\varepsilon \bar{z}_\varepsilon \bar{w}_\varepsilon)^{-1} \Phi(t) \\ & + \frac{1}{z_\varepsilon - w_\varepsilon} [\partial_t w_\varepsilon + (\bar{z}_\varepsilon \bar{w}_\varepsilon w_\varepsilon)^{-1} \partial_x w_\varepsilon] \psi(\bar{x}, t) + g(\bar{x}, t). \end{aligned} \quad (6.30)$$

with all coefficients evaluated at the point (\bar{x}, t) . By virtue of (6.26) we get

$$\begin{aligned}
 & -[\partial_t w_\varepsilon + (z_\varepsilon \bar{z}_\varepsilon \bar{w}_\varepsilon)^{-1} \partial_x w_\varepsilon] \Phi(t) \\
 & \quad + [\partial_t w_\varepsilon + (\bar{z}_\varepsilon \bar{w}_\varepsilon w_\varepsilon)^{-1} \partial_x w_\varepsilon] \psi(\bar{x}, t) \\
 & \geq \{ [\mu_- - (z_\varepsilon \bar{z}_\varepsilon \bar{w}_\varepsilon)^{-1}] \Phi(t) - [\mu_0 - (\bar{z}_\varepsilon \bar{w}_\varepsilon w_\varepsilon)^{-1}] \psi(\bar{x}, t) \} \partial_x w_\varepsilon \\
 & \quad - \frac{\mathbf{B}(\mu_+ - \mu_-)}{t - \varepsilon} [\Phi(t) + |\psi(\bar{x}, t)|] \tag{6.31}
 \end{aligned}$$

where μ_0 stands for μ_+ when $\psi(\bar{x}, t) \geq 0$ and for μ_- when $\psi(\bar{x}, t) < 0$. By (6.29), $|\psi(\bar{x}, t)| \leq \Phi(t)$. Furthermore, since our solutions take values in the strictly hyperbolic regime and have small oscillation, it follows from (2.6) that

$$|\mu_0 - (\bar{z}_\varepsilon \bar{w}_\varepsilon w_\varepsilon)^{-1}| \ll \mu_- - (z_\varepsilon \bar{z}_\varepsilon \bar{w}_\varepsilon)^{-1}. \tag{6.32}$$

Therefore, the coefficient of $\partial_x w_\varepsilon$ on the right-hand side of (6.31) is positive and thus (6.28) may be invoked to yield a lower bound for that term. In what follows, δ will denote a generic constant that may be made arbitrarily small by taking T and the oscillation of the initial data sufficiently small. A careful review of the proof of Theorem 3.1 and Lemma 3.4 reveals that

$$\mathbf{A} \leq \frac{1}{2} \hat{z}^3 \hat{w} + \delta, \quad \mathbf{B} \leq \frac{1}{2} \hat{z} \hat{w}^3 + \delta, \tag{6.33}$$

where (\hat{z}, \hat{w}) is any fixed state in the range of the initial data (z_0, w_0) for the Riemann invariants. It then follows from (6.30), (6.31), (6.27) and (6.28) that

$$\dot{\Phi}(t) \geq -\frac{k}{t - \varepsilon} \Phi(t) - c \tag{6.34}$$

where k and c are positive constants independent of ε and $k < \hat{w}/\hat{z} + \delta$. The same inequality (6.34) obtains, with $k < \hat{w}/\hat{z} + \delta$, when $\Phi(t) = -\varphi(\bar{x}, t)$. On the other hand, when $\Phi(t) = \pm \psi(\bar{x}, t)$, the above procedure yields (6.34) with $k < 1 + \delta$. Since $\hat{w} < \hat{z}$, we conclude that (6.34) holds almost everywhere on $[\varepsilon, T]$ with

$$k < 1 + \delta. \tag{6.35}$$

Integrating the differential inequality (6.34), starting out from $\Phi(T) = 0$, we deduce

$$\Phi(t) \leq \frac{c T^{k+1}}{(k+1)(t - \varepsilon)^k}, \quad \varepsilon < t \leq T. \tag{6.36}$$

Recalling (6.29) and (6.15), (6.36) gives

$$|\partial_x P_\varepsilon(x, t)| \leq \frac{c_1}{(t-\varepsilon)^k}, \quad -\infty < x < \infty, \quad \varepsilon < t \leq T. \quad (6.37)$$

In turn, (6.10) and (6.37) imply

$$|\partial_t P_\varepsilon(x, t)| \leq \frac{c_2}{(t-\varepsilon)^k}, \quad -\infty < x < \infty, \quad \varepsilon < t \leq T, \quad (6.38)$$

whence, assuming $k > 1$,

$$|P_\varepsilon(x, t)| \leq \frac{c_3}{(t-\varepsilon)^{k-1}}, \quad -\infty < x < \infty, \quad \varepsilon < t \leq T. \quad (6.39)$$

In particular, it follows from (6.37), (6.38), (6.39) that we may extract a sequence $\{\varepsilon_n\}$, $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$, such that

$$P_{\varepsilon_n}(x, t) \rightarrow P(x, t), \quad n \rightarrow \infty, \quad (6.40)$$

uniformly on compact subsets of $(-\infty, \infty) \times (0, T]$, where P is a locally Lipschitz function satisfying

$$|P(x, t)| \leq \frac{c_3}{t^{k-1}}, \quad -\infty < x < \infty, \quad 0 < t \leq T. \quad (6.41)$$

We have now laid the preparation for verifying (6.2). We fix $\tau \in (0, T)$. For any ε in $(0, \tau)$, we combine (6.10), (6.11) and (6.5) to deduce, after a short calculation,

$$\begin{aligned} \int_\tau^T \int_{-\infty}^\infty GV \, dx \, dt &= \int_\tau^T \int_{-\infty}^\infty [\partial_t P_\varepsilon + \partial_x P_\varepsilon \mathcal{A}_\varepsilon] V \, dx \, dt \\ &= - \int_{-\infty}^\infty P_\varepsilon(x, \tau) V(x, \tau) \, dx \\ &\quad + \int_\tau^T \int_{-\infty}^\infty \partial_x P_\varepsilon (\mathcal{A}_\varepsilon - \mathcal{A}) V \, dx \, dt. \end{aligned} \quad (6.42)$$

Recalling (6.37), (6.40) and since, as $\varepsilon \rightarrow 0$, $\mathcal{A}_\varepsilon \rightarrow \mathcal{A}$, boundedly almost everywhere on $(-\infty, \infty) \times (0, T)$, (6.42) yields

$$\int_\tau^T \int_{-\infty}^\infty GV \, dx \, dt = - \int_{-\infty}^\infty P(x, \tau) V(x, \tau) \, dx. \quad (6.43)$$

From (6.6) and the general properties of BV functions we get

$$\int_{-\infty}^\infty |V(x, \tau)| \, dx \leq \beta\tau, \quad \tau > 0. \quad (6.44)$$

Therefore, if T and the oscillation of the initial data are sufficiently small to render $\delta < 1$ in (6.35), *i.e.* $k < 2$, it follows from (6.41), (6.43) that the

right-hand side of (6.43) tends to zero, as $\tau \rightarrow 0$, and this establishes (6.2). The proof of the theorem is now complete.

Uniqueness of solutions also holds when the one-sided Lipschitz condition for the Riemann invariant z in Theorem 6.1 is replaced with a one-sided Lipschitz condition for the other Riemann invariant w . However, in that case an additional restriction on the initial data becomes necessary.

THEOREM 6.2. — *Let $(u_0(x), v_0(x))$ be functions with bounded variation and small oscillation defined on $(-\infty, \infty)$ and taking values in the physically relevant range. Assume that the induced Riemann invariant fields $(z_0(x), w_0(x))$ take values in a small neighborhood of a state (\hat{z}, \hat{w}) with $\hat{z} < 2\hat{w}$ and w_0 satisfies a one-sided Lipschitz condition*

$$\frac{w_0(y) - w_0(x)}{y - x} \geq -b, \quad -\infty < x < y < \infty. \tag{6.45}$$

Then there is a unique admissible BV solution $(u(x, t), v(x, t))$ for the Cauchy problem for (1.4) on $(-\infty, \infty) \times [0, \infty)$ with initial data $u(x, 0) = u_0(x), v(x, 0) = v_0(x), -\infty < x < \infty$.

Proof. — We retrace the steps of the proof of Theorem 6.1. Estimates (6.27), (6.28) should now be replaced with

$$\partial_x z_\varepsilon(x, t) \geq -\frac{A}{t - \varepsilon}, \quad \partial_x \bar{z}_\varepsilon(x, t) \geq -\frac{A}{t - \varepsilon}, \tag{6.46}$$

$$\partial_x w_\varepsilon(x, t) \geq -Kb, \quad \partial_x \bar{w}_\varepsilon(x, t) \geq -Kb. \tag{6.47}$$

This will lead us again to (6.34), almost everywhere on $[\varepsilon, T]$, where, however, (6.35) is now replaced with

$$k < \frac{\hat{z}}{\hat{w}} + \delta. \tag{6.48}$$

Since we are assuming $\hat{z} < 2\hat{w}$, we may still choose T and the oscillation of the initial data so small that $k < 2$. This is exactly what we need in order to infer, via (6.41) and (6.44), that the right-hand side of (6.43) tends to zero as $\tau \rightarrow 0$. The proof of the theorem is complete.

REMARK 6.1. — Suppose we drop both assumptions (6.1) and (6.45) and still attempt to establish uniqueness by the argument used in the proof of Theorems 6.1 and 6.2. We now have to estimate $\partial_x z_\varepsilon$ and $\partial_x w_\varepsilon$ via (6.46) and (6.28), respectively. This again leads to (6.34), almost everywhere on $[\varepsilon, T]$, but now with (6.35) or (6.48) replaced with

$$k < 1 + \frac{\hat{z}}{\hat{w}} + \delta. \tag{6.49}$$

Thus, we can no longer guarantee that $k < 2$. Perhaps sharper analysis of (6.18), (6.19) would yield estimate (6.37) with $k < 2$ or a direct treatment

of (6.10) would render estimate (6.39) with $k < 2$. However, we have been unable to improve our estimates so it is not clear to us at this time whether the simultaneous presence in the solution of centered rarefaction waves of both families renders the issue of uniqueness intrinsically harder or whether our failure is purely technical.

Because of the localized range of influence in hyperbolic systems, one may combine Theorems 6.1 and 6.2 into the following, more general statement.

THEOREM 6.3. — *Let $(u_0(x), v_0(x))$ be functions with bounded variation and small oscillation defined on $(-\infty, \infty)$ and taking values in the physically relevant range. Assume that the induced Riemann invariant fields $(z_0(x), w_0(x))$ take values in a small neighborhood of a state (\hat{z}, \hat{w}) with $\hat{z} < 2\hat{w}$. Furthermore, suppose that $(-\infty, \infty)$ is the union of two open subsets Z and W such that (6.1) holds for all x, y , in Z and (6.45) holds for all x, y in W . Then there is unique admissible BV solution $(u(x, t), v(x, t))$ for the Cauchy problem for (1.4) on $(-\infty, \infty) \times (0, \infty)$ with initial data $u(x, 0) = u_0(x), v(x, 0) = v_0(x), -\infty < x < \infty$.*

Proof. — Without loss of generality, Z and W may be assumed to be locally finite unions of open intervals. Now if (α, β) is an interval contained in Z and ε is a very small positive number, repeating the proof of Theorem 6.1 with a test vector field G of compact support in $(\alpha + \varepsilon, \beta - \varepsilon) \times (0, T)$, we conclude that any two solutions of the initial value problem must coincide on the rectangle $(\alpha + \varepsilon, \beta - \varepsilon) \times (0, T)$, for T sufficiently small. Similarly if (γ, δ) is contained in W , repeating the proof of Theorem 6.2 we infer that any two solutions coincide on the rectangle $(\gamma + \varepsilon, \delta - \varepsilon) \times (0, T)$. The proof is then easily completed with the help of Theorems 3.1 and 6.1.

7. DECOUPLING OF CHARACTERISTIC FIELDS

Here we employ the results of Section 3 to demonstrate, in a direct and simple manner, that in solutions of (1.4) with initial data that are constant outside a bounded interval the two characteristic fields decouple completely in a finite time.

THEOREM 7.1. — *Let $(z(x, t), w(x, t))$ be the Riemann invariant fields induced by an admissible weak solution $(u(x, t), v(x, t))$ of (1.4) with initial data that are constant outside a bounded interval, say*

$$(z(x, 0), w(x, 0)) = \begin{cases} (z_-, w_-), & -\infty < x \leq x_- \\ (z_+, w_+), & x_+ \leq x < \infty \end{cases} \quad (7.1)$$

where (z_-, w_-) , (z_+, w_+) are neighboring states and $-\infty < x_- \leq x_+ < \infty$. Let ψ be the minimal forward 2-characteristic emanating from $(x_-, 0)$ and χ be the maximal forward 1-characteristic emanating from $(x_+, 0)$. If \bar{t} is the time at which χ and ψ intersect, then

(a) for $t > \bar{t}$ and $\chi(t) < x < \psi(t)$ it is $z(x, t) = z_+$, $w(x, t) = w_-$;

(b) for $t > \bar{t}$ and $x < \chi(t)$ it is $w(x, t) = w_-$ while z satisfies the single, genuinely nonlinear hyperbolic conservation law

$$\partial_t z - \partial_x \left(\frac{1}{w_- z} \right) = 0; \quad (7.2)$$

(c) for $t > \bar{t}$ and $x > \psi(t)$ it is $z(x, t) = z_+$ while w satisfies the single, genuinely nonlinear hyperbolic conservation law

$$\partial_t w - \partial_x \left(\frac{1}{z_+ w} \right) = 0. \quad (7.3)$$

Proof. — The minimal backward 2-characteristic ζ emanating from any point (x, t) with $t > 0$, $x < \psi(t)$ has to stay strictly to the left of ψ on $[0, t]$, because ψ is minimal, and hence it is intercepted by the x -axis inside the interval $(-\infty, x_-)$. Then Theorem 3.2 yields $w(x, t) = w_-$. From (2.5), $v = z + w_-$, $u = w_- z$ and so (1.4)₂ reduces to (7.2).

Similarly, the minimal backward 1-characteristic ξ emanating from any point (x, t) with $t > 0$, $x > \chi(t)$ has to stay strictly to the right of χ on $[0, t]$, because χ is maximal, and hence it is intercepted by the x -axis inside the interval (x_+, ∞) . Then Theorem 3.2 yields $z(x, t) = z_+$. From (2.5), $v = z_+ + w$, $u = z_+ w$ and so (1.4)₂ reduces to (7.3).

In particular, if $t > \bar{t}$ and $\chi(t) < x < \psi(t)$ then it is $z(x, t) = z_+$, $w(x, t) = w_-$. This completes the proof of the theorem.

By virtue of Theorem 7.1, one may determine the large time behavior of solutions of our system (1.4) by studying the large time behavior of solutions of the two single, genuinely nonlinear conservation laws (7.2) and (7.3), under initial data that are constant outside a bounded interval. A detailed treatment of this problem, based on the theory of generalized characteristics, may be found in [1].

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