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On a superlinear elliptic equation

by

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ABSTRACT. — In this note we establish multiple solutions for a semilinear elliptic equation with superlinear nonlinearility without assuming any symmetry.

Key words : Elliptic equation, superlinear nonlinearility, linking method, Morse theory.

RÉSUMÉ. — Dans cet article, nous montrons l'existence de plusieurs solutions pour une équation semi-linéaire elliptique avec une condition de non-linéarité superlinéaire, sans hypothèse de symmétrie.

1. INTRODUCTION

In this note we consider the existence of multiple solutions of a class of semilinear elliptic equations with superlinear nonlinearity. A simple model equations is

$$\left.\begin{aligned} -\Delta u = f(u) \quad & in\ \Omega \\ u = 0 \quad & on\ \partial\Omega \end{aligned}\right\} \tag{1}$$

Classification A.M.S. : 35 J 20, 35 J 60, 58 E 050.

where $\Omega \subset R^n$ a bounded domain with regular boundary. We assume f satisfies the following conditions:

(f_1) $f \in C^1(R, R), f(0) = f'(0) = 0$.

(f_2) there are constants C_1, C_2 s. t.

$$|f(t)| \leqq C_1 + C_2 |t|^\alpha, \qquad 1 < \alpha < \frac{n+2}{n-2}.$$

(f) there exist constants $\mu > 2$ and $M > 0$, s. t.

$$0 < \mu F(t) \leqq t f(t), \qquad for \quad |t| \geqq M.$$

where $F(t) = \int_0^t f(\tau)\, d\tau$.

Our main result is

THEOREM. — *Assume f satisfies (f_1), (f_2) and (f_3), then equation* (1) *possesses at least three nontrivial solutions.*

Remark 1.1. — There have been many results studying superlinear elliptic equations like (1). In [1], under essentially the same conditions as here Ambrosetti and Rabinowitz obtained two nontrivial solutions. They also obtained infinitely many solutions in the case of an odd nonlinearity f. After that many results were devoted to the studies of multiple solutions for equations like (1), mainly for the perturbations of odd nonlinearity (*see* [2], [3], [13], [15] and so on). On the other hand, without any symmetrical assumptions infinitely many solutions were obtained in the case of $n = 1$ for both Dirichlet and periodic boundary contitions (*see* [10], [12]). Our results establish multiple solutions of (1) in dimension two or greater without assuming any symmetry. Under much stronger assumptions on the nonlinearity than ours a similar result was obtained in [14] by a different method, but with an error. After finishing this paper we received a correction of [14] from Struwe in a private communication.

Remark 1.2. — The idea here is quite simple, but might be useful to obtain more solutions of equation (1) without assuming any symmetry. Our approach is to construct a link, which gives a new critical point, from known critical points. We shall look for unknown critical points from two "Mountain Pass" critical points, just as from two local minimal critical points one may obtain a "Mountain Pass" critical point.

The paper is organized as follows. In section 2 we give the proof of the main theorem in three steps. 1. Recall the existence of two nontrivial solutions obtained in [1]. 2. Analyze the local behaviours of the functional corresponding to equation (1) near these two solutions. 3. The existence of the third solution. In section 3, based on the step 1 and 2 in section 2, we give a different proof of the existence of the third solution by using Morse theory for isolated critical points developed in [5].

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2. THE PROOF OF THE MAIN THEOREM

It is well known (*see* [1]) that the classical solutions of equation (1) correspond to the critical points of the following functional defined on $H = H_0^1(\Omega)$

$$J(u) = \int_\Omega \left\{ \frac{1}{2} |\nabla u|^2 - F(u) \right\} dx, \qquad u \in H_0^1(\Omega)$$

and that under the assumptions of the theorem $J \in C^2$ and satisfies (P.S.) condition. The proof of the theorem is divided into the following three steps.

Step 1. – The existence of two nontrivial solutions.

As in [1], we define

$$\tilde{f}(t) = \begin{cases} f(t), & t \geqq 0 \\ 0, & t < 0 \end{cases} \tag{2}$$

and consider the modified functional

$$\tilde{J}(u) = \int_\Omega \left\{ \frac{1}{2} |\nabla u|^2 - \tilde{F}(u) \right\} dx, \qquad u \in H_0^1(\Omega) \tag{3}$$

where $\tilde{F}(t) = \int_0^t \tilde{f}(\tau)\, d\tau$.

One may check that $\tilde{J} \in C^2(H, R)$ and satisfies (P.S.) condition (*see* [1]). By f_3 we can choose a $r > 0$ large enough,

$$\tilde{J}(re_1) \leqq 0$$

where e_1 is the first eigenfunction of $(-\Delta)$ and $\| e_1 \| = 1$.

Set

$$\Gamma = \{ \gamma \in C^0([0, 1], X) \mid \gamma(0) = \theta,\ \gamma(1) = re_1 \}$$

and

$$c_1 = \inf_{\gamma \in \Gamma} \sup_{t \in I} \tilde{J}(\gamma(t)) \tag{4}$$

By f_1, $c_1>0$ and applying Mountain Pass lemma (*see* [1]) we get that c_1 is a critical value of $\tilde{J}$. That is, there exists $u_1 \in H_0^1(\Omega)$ satisfying $\tilde{J}(u_1)=c_1$ and

$$\begin{cases} -\Delta u_1 = \tilde{f}(u_1) & \textit{in } \Omega \\ u_1 = 0 & \textit{on } \partial\Omega \end{cases}$$

By means of the maximum principle we get $u_1>0$ and then is a solution of equation (1).

By the similar procedure we can get the other solution $u_2<0$. Without loss of generality we assume $c_1 = J(u_1) \geqq J(u_2) = c_2$.

Step 2. – The local behaviour of J near u_1.

To get an additional nontrivial solution we need the local information of J near u_1, which looks like a "local link". In the next step, by using this "local link" we construct a global link which gives a new critical value $c>c_1$. We use the following notations. For a real $c \in R$,

$$J_c = \{ u \in H \mid J(u) \leqq c \},$$
$$K_c(J) = \{ u \in H \mid J(u) = c,\ J'(u) = 0 \}.$$

$\| . \|$ denotes the $H_0^1(\Omega)$ norm. For the simplicity, assume $K_{c_1}(J) = \{ u_1 \}$ though our approach works in more general situation, for example K_{c_1} finite.

Since $u_1 \geqq 0$, it follows from a direct calculation

$$J(u_1) = \tilde{J}(u_1) = c_1$$
$$J'(u_1) = \tilde{J}'(u_1) = 0$$
$$J''(u_1) = \tilde{J}''(u_1)$$

Hence if u_1 is a nondegenerate critical point of $\tilde{J}$, it is a nondegenerate critical point of J too. And then $\tilde{J}$ and J have essentially the same local behaviours near u_1. However the situation might be degenerate. The main tool we use is the generalized Morse lemma (*see* [5], [7]).

LEMMA 2.1 (Generalized Morse Lemma). – *Suppose that* U *is a neighbourhood of the origin* θ *in a Hilbert space* H *and that* $g \in C^2(U, R)$. *Assume that* θ *is the only critical point of* g, *and that* $A = g''(\theta)$ *with kernel* N. *If* 0 *is at most an isolated point of the spectrum* $\sigma(A)$, *then there exists a ball* B_δ, *centered at* θ, *an origin preserving local homeomorphism* Φ, *defined on* B_δ, *and a* C^1 *mapping* $h: B_\delta \cap N \to N^\perp$ *such that*

$$g \circ \Phi(z+y) = \frac{1}{2}(Az, z) + g(h(y)+y), \qquad \forall u \in B_\delta \tag{5}$$

where $y = P_N u$, $z = P_{N^\perp} u$ *and* P_N *is the orthogonal projection onto the subspace* N.

Now let us denote the kernel of $J''(u_1)$ by N and apply lemma 2.1 to functional J in a neighbourhood of u_1, say $B_\delta(u_1) = \{ u \in H \mid \| u - u_1 \| < \delta \}$, then we have

$$\begin{aligned} J\circ\Phi(u_1+z+y) &= \frac{1}{2}(J''(u_1)z,z)+J(u_1+h(y)+y) \\ \forall (z,y)\in N^\perp\oplus N, &\quad \text{and} \quad \|z\|+\|y\|<\delta \end{aligned} \tag{6}$$

where $\Phi: B_\delta(u_1)\to H$ a homeomorphism preserving u_1, and $h: N\cap B_\delta(\theta)\to N^\perp$ a C^1 mapping with $h(\theta)=\theta$.

The main conclusion of this step is the following proposition which will be used in the next step.

PROPOSITION 2.1. – *In the expression* (9), *there is a* $0<\delta_1\leqq\delta$ *s. t.*

(i) *when the Morse index* $m_-(u_1)=0$, *then*

$$J(u_1+h(y)+y)<J(u_1)=c_1 \qquad \text{for} \quad 0<\|y\|\leqq\delta_1$$

(ii) *when the Morse index* $m_-(u_1)=1$, *then*

$$J(u_1+h(y)+y)>J(u_1)=c_1 \qquad \text{for} \quad 0<\|y\|\leqq\delta_1$$

where $m_-(u_1)$ *is the Morse index of* u_1 *defined as the dimension of the maximal negative definite subspace of* $J''(u_1)$.

Because of the lack of minimax characterization of u_1 in terms of J we can not give precise information of J directly. The proof of proposition depends on the following two lemmas which give the comparison of J and $\tilde{J}$ as well as the local behaviour of $\tilde{J}$ near u_1. Consequently we'll have the local behaviour of J near u_1.

LEMMA 2.2. – *In the expression* (9) *of* J *near* u_1, *we have*

(i) $$h(y)\in C^1(\bar{\Omega}) \qquad \text{for} \quad \|y\|\leqq\delta$$

(ii) $$\|h(y)\|_{C^1(\bar{\Omega})}\to 0 \quad \text{as} \; \|y\|\to 0$$

LEMMA 2.3. – *If applying Lemma* 2.1 *to* $\tilde{J}$ *in a neighbourhood of* u_1, *say* $B_{\tilde{\delta}}(u_1)$, *and similarly we have* $\tilde{\Phi}$, $\tilde{h}$ *and*

$$\begin{aligned} \tilde{J}\circ\tilde{\Phi}(u_1+z+y) &= \frac{1}{2}(\tilde{J}''(u_1)z,z)+\tilde{J}(u_1+\tilde{h}(y)+y) \\ \forall (z,y)\in N^\perp\oplus N, &\quad \text{and} \quad \|z\|+\|y\|<\tilde{\delta} \end{aligned} \tag{7}$$

there exists a $\tilde{\delta}_1$ *with* $0<\tilde{\delta}_1<\tilde{\delta}$, *s. t.*

(i) *when* $m_-(u_1)=0$, *then*

$$\tilde{J}(u_1+\tilde{h}(y)+y)<\tilde{J}(u_1)=c_1 \qquad \text{for} \quad 0<\|y\|\leqq\tilde{\delta}_1$$

(ii) *when* $m_-(u_1)=1$, *then*

$$\tilde{J}(u_1+\tilde{h}(y)+y)>\tilde{J}(u_1)=c_1 \qquad \text{for} \quad 0<\|y\|\leqq\tilde{\delta}_1.$$

Before proving these two lemmas we use them to give

The Proof of Proposition 2.1. – Firstly we claim that there exists a $\delta_2>0$ s.t.

$$\tilde{h}(y)=h(y) \qquad \text{for} \quad \|y\| \leqq \delta_2. \tag{8}$$

From the definition of h (*see* [5] for the proof of Lemma 2.1), h is the unique solution of equation

$$L(z)=\frac{\partial J}{\partial z}(u_1+z+y)=0 \qquad \text{for} \quad \|y\| \leqq \delta$$

i. e.

$$\frac{\partial J}{\partial z}(u_1+h(y)+y)=0$$

Let $N=\text{span}\{\omega_1, \ldots, \omega_k\}$, ω_i smooth function orthogonal each other in H_0^1 and $\|\omega_i\|=1$. If write $y=\Sigma\,\alpha_i\,\omega_i$ then $\|y\|^2=\Sigma\,\alpha_i^2$.

From (ii) of Lemma 2.2, there exists a $\delta_2>0$ s. t.

$$u_1+h(y)+y \geqq 0, \quad \text{in } \Omega, \qquad \textit{for} \quad \|y\| \leqq \delta_2 \tag{9}$$

here we have used the fact that

$$\frac{\partial u_1}{\partial n}<0 \quad \text{on } \partial\Omega$$

Hence by the definition of $\tilde{J}$

$$\tilde{J}'_z(u_1+h(y)+y)=J'_z(u_1+h(y)+y)=0.$$

and then it follows from the uniqueness of $\tilde{h}$

$$\tilde{h}(y)=h(y) \qquad \text{for} \quad \|y\| \leqq \delta_2$$

That is (8).

Now, for $0<\|y\| \leqq \delta_2$,

$$\begin{aligned} & J(u_1+h(y)+y) \quad \text{by (9)} \\ =\; & \tilde{J}(u_1+h(y)+y) \quad \text{by (8)} \\ =\; & \tilde{J}(u_1+\tilde{h}(y)+y) \quad \text{by Lemma 2.3} \\ <\; & \tilde{J}(u_1)=c_1 \quad \text{if } m_-(u_1)=0 \\ (>\; & \tilde{J}(u_1)=c_1 \quad \text{if } m_-(u_1)=1) \end{aligned}$$

Proof of Lemma 2.2. – We recall that $h(y)$ is the unique solution of the following equation

$$L(z)=\frac{\partial J}{\partial z}(u_1+z+y)=0, \qquad y \in N \quad \text{and} \quad \|y\| \leqq \delta$$

That is,

$$\int_\Omega \{\nabla(u_1+h(y)+y)\nabla z - f(u_1+h(y)+y)\,z\}\,dx=0, \qquad \forall z \in N^\perp \tag{10}$$

If we write $N = \text{span}\{\omega_1, \ldots, \omega_k\}$, ω_i smooth function orthogonal each other in $L^2(\Omega)$ and $\int \omega_i = 1$, (10) implies that there are $\beta_i(y) \in R$ s. t. $h(y)$ satisfies

$$- \Delta h(y) = f(u_1 + h(y) + y) - f(u_1) - f'(u_1) y + \sum_{i=1}^{k} \beta_i(y) \omega_i \qquad (11)$$
$$\forall y \in N, \quad \|y\| \leqq \delta.$$

Therefore, we get

$$\beta_i(y) = - \int_\Omega \{ f(u_1 + h(y) + y) - f(u_1) - f'(u_1) y - f'(u_1) h(y) \} \omega_i \, dx \qquad \textit{for} \quad i = 1, \ldots, k.$$

By the assumption of f and the property of h, $\beta_i \in C^1(B_\delta(\theta) \cap N, R)$ and $\beta_i(\theta) = 0$. By f_2 and L^p estimate, for any $p > 2$ there exists constant $C_p > 0$ s. t.

$$\| h(y) \|_{W_2^p(\Omega)} \leqq C_p (\| h(y) \| + \| y \| + \sum \max | \beta_i | + 1)$$

and then there exists $C > 0$, s. t.

$$\| h(y) \|_{C^1(\bar{\Omega})} \leqq C, \qquad \text{for} \quad \| y \| \leqq \delta.$$

From this, one may find constant C

$$| f(u_1 + h(y) + y) - f(u_1) | \leqq C | h(y) + y |, \qquad \forall \| y \| \leqq \delta.$$

Combining the above formula and L^p estimate again we get

$$\| h(y) \|_{C^1(\bar{\Omega})} \leqq C (\| h(y) \| + \| y \| + \sum | \beta_i |)$$

which gives the required estimate (ii) of Lemma 2.2.

Proposition 2.1 and lemma 2.3 look the same. The difference is that there is a variational characterization of c_1 in terms of $\tilde{J}$, that is, u_1 is obtained by applying Mountain-Pass lemma to $\tilde{J}$. In this spirit, much more information has been obtained by Hofer, Tian (*see* [5], [8], [16]). Lemma 2.3 essentially is a consequence of their results. A short proof of lemma 2.3 is given in the following. Firstly, we need a version of deformation lemma (*see* [8]).

LEMMA 2.4. – *Assume that* $J \in C^1(H, R)$ *satisfies the* (P.S.) *condition. Assume that* c *is a real number and that*

$$N_\delta(K_c(J)) = \{ u \in H \mid \text{dist}(u, K_c(J)) \leqq \delta \}$$

is a closed neighbourhood of $K_c(J)$. *Then there is a continuous map* $\eta : [0, 1] \times H \to H$ *as well as real numbers* $\bar{\varepsilon} > \varepsilon > 0$ *such that*

(1) $\eta(t, u) = u$, $\forall u \notin J^{-1}[c - \bar{\varepsilon}, c + \bar{\varepsilon}]$;

(2) $\eta(0, u) = u$, $\forall u \in H$;

(3) $\eta(1, J_{c+\varepsilon} \setminus N_{\delta/2}) \subset J_{c-\varepsilon}$;
(4) $\eta(1, N_{\delta/2}) \subset N_\delta$;
(5) $\forall t \in [0,1]$, $\eta(t, .)$ is a homeomorphism.

LEMMA 2.5. – *For any neighbourhood* U *of* u_1,

$$\{ \tilde{J}_{c_1} \setminus \{ u_1 \} \} \cap U \neq \varnothing,$$

and is not path connected.

Proof. – Take a $\delta > 0$ s. t. $B_{2\delta} \subset U$ and choose a path $\gamma \in \Gamma$ s. t. $\{ \gamma(t) \} \cap B_{\delta/2} \neq \varnothing$. By Lemma 2.4, we may assume

$$\tilde{J}(\gamma(t)) < c_1 \qquad \text{if} \quad \gamma(t) \notin \bar{B}_\delta$$

This implies $\{ \tilde{J}_{c_1} \setminus \{ u_1 \} \} \cap U \neq \varnothing$. Set $t_1 = \inf \{ t \mid \gamma(t) \in \bar{B}_\delta \}$ and $t_2 = \sup \{ t \mid \gamma(t) \in \bar{B} \}$, then $t_1 < t_2$ and $\gamma(t_i) \in \{ \tilde{J}_{c_1} \setminus \{ u_1 \} \} \cap U$, $i = 1, 2$. If $\{ \tilde{J}_{c_1} \setminus \{ u_1 \} \} \cap U$ is path connected one may connect $\gamma(t_1)$ and $\gamma(t_2)$ in $\{ \tilde{J}_{c_1} \setminus \{ u_1 \} \} \cap U$ and get a new path γ_1 s. t.

$$\tilde{J}(\gamma_1(t)) \leqq c_1 \quad \text{and} \quad \gamma_1(t) \neq u_1, \qquad \forall t \in I$$

So we can flow it down further and get a contradiction with the definition (4) of c_1.

Proof of Lemma 2.3. – *Case* (i). – In this case, 0 is the smallest eigenvalue of $-\Delta - f'(u_1)$ and by the result of [9], $\dim N = 1$. Take a neighbourhood $B_{\delta_1}(u_1)$ of u_1, by Lemma 2.1 module a homeomorphism

$$\tilde{J}(u_1 + z + y) = \| z \|^2 + \tilde{J}(u_1 + \tilde{h}(y) + y)$$

One may find $0 < \delta_1 \leqq \delta$ such that $y = 0$ is the unique critical point of $a(y) = \tilde{J}(u_1 + \tilde{h}(y) + y)$. And then there are only three possibilities: $(a)\, y = \theta$ is a local minimal of a; $(b)\, y = \theta$ is a saddle point of a; $(c)\, y = \theta$ is a local maximal of a.

We can easily see that in case (a), $\{ \tilde{J}_{c_1} \setminus \{ u_1 \} \} \cap B_{\delta_1}(u_1)$ *is empty and in case* (b), $\{ \tilde{J}_{c_1} \setminus \{ u_1 \} \} \cap B_{\delta_1}(u_1)$ is path connected, respectively. So from lemma 2.5 case (c) is the only possible case which gives the conclusion of (i) in Lemma 1.3.

Case (ii). – In this case we write $N^\perp = V \oplus X$, V and X correspond to the positive and negative eigenspaces of $\tilde{J}''(u_1)$, respectively. We know also $\dim X = 1$. Take a neighbourhood of u_1 as follows

$$U = \{ u = (u_1 + v + x + y) \mid \| v \| \leqq \delta, \| x \| \leqq \delta, \| \leqq \delta' \}$$

where δ' is chosen small such that

$$| \tilde{J}(u_1 + \tilde{h}(y) + y) - c_1 | \leqq \frac{\delta^2}{2} \qquad \textit{for} \quad \| y \| \leqq \delta'$$

By Lemma 2.1 module a homeomorphism

$$\tilde{J}(u_1 + v + x + y) = \| v \|^2 - \| x \|^2 + \tilde{J}(u_1 + \tilde{h}(y) + y) \tag{12}$$

We claim that any point $(u_1+v+x+y)\in\{\tilde{J}_{c_1}\setminus\{u_1\}\}\cap U$ can be connected in $\{\tilde{J}_{c_1}\setminus\{u_1\}\}\cap U$ to one of the following two points: $(u_1+\theta+\delta+\theta)$, $(u_1+\theta-\delta+\theta)$. This can be done in the following way: $(u_1+v+x+y)$

can be connected to $(u_1+\theta+x+y)$ by

$$\gamma(t)=u_1+tv+x+y; \qquad (u_1+\theta+x+y)$$

can be connected to

$$(u_1+\theta+\delta+y) \quad \text{if } x\geqq 0 \quad [\text{or } (u_1+\theta+\delta+y) \quad \text{if } x\leqq 0]$$

by

$$\gamma(t)=u_1+\theta+(1-t)x+\delta+y \quad [\text{or } \gamma(t)=u_1+\theta+(1-t)x-t\delta+y];$$

and finally

$(u_1+\theta\pm\delta+y)$ can be connected to $(u_1+\theta\pm\delta+\theta)$ by $\gamma(t)=u_1+\theta\pm\delta+ty$.

Now we are going to prove the conclusion (ii). If it is not true there exists y_0 with $0<\|y_0\|\leqq\delta_1$ s. t.

$$\tilde{J}(u_1+\tilde{h}(y_0)+y_0)\leqq c_1$$

By (12) this means $(u_1+\theta+x+y_0)\in\tilde{J}_{c_1}\setminus\{u_1\}$, $\forall\|x\|\leqq\delta$ and then gives a path connecting $(u_1+\theta+\delta+y_0)$ and $(u_1+\theta-\delta+y_0)$ in $\{\tilde{J}_{c_1}\setminus\{u_1\}\}\cap U$. Notice our claim before, we find $(u_1+\theta+\delta+\theta)$ and $(u_1+\theta-\delta+\theta)$ can be connected by a path in $\tilde{J}_{c_1}\setminus\{u_1\}\cap U$, which means that $\tilde{J}_{c_1}\setminus\{u_1\}\cap U$ is path connected and contradicts Lemma 2.5.

Step 3. – The existence of the third nontrivial solution.

In this step, by using the local structure of J near u_1 we construct a link. And the minimax method will give a new critical value of J bigger than c_1 (similar idea was used in [11]). We just consider the case of $m_-(u_1)=0$ and the other one can be handled similarly.

Let $N=\text{span}\{\omega\}$. By Proposition 1.1 there exists $\delta>0$ s. t.

$$J(u_1+h(t\omega)+t\omega)<c_1 \qquad \textit{for} \quad 0<|t|\leqq\delta$$

and also there exist $\rho>0$, $\varepsilon>0$ s. t.

$$\left.\begin{array}{ll} J(u_1+z)>c_1 & \textit{for} \quad 0<\|z\|\leqq\rho \\ J(u_1+z)\geqq c_1+\varepsilon & \textit{for} \quad z\in S_\rho=\{z\,|\,\|z\|=\rho\} \end{array}\right\} \tag{13}$$

Let

$$U=\{u=(u_1+z+t\omega)\,|\,|t|\leqq\delta, \|z\|\leqq\rho\}$$

By Proposition 2.1 there are exactly two connected components of $\{\tilde{J}_{c_1}\setminus\{u_1\}\}\cap U$ containing $(u_1+h(-\delta\omega)-\delta\omega)$ and $(u_1+h(\delta\omega)+\delta\omega)$ respectively. It is a consequence of Lemma 2.5 that θ and re_1 can be connected by pathes in $\tilde{J}_{c_1}$ to one of the two connected components.

Without loss of generality we assume there are pathes γ_1, γ_2 s. t.

$$\gamma_1(0)=\theta, \qquad \gamma_1(1)=u_1+h(-\delta\omega)-\delta\omega;$$
$$\gamma_2(0)=u_1+h(\delta\omega)+\delta\omega, \qquad \gamma_2(1)=re_1,$$

and $\tilde{J}(\gamma_1)t))<c_1$, $\tilde{J}(\gamma_2(t))<c_1$. By connecting γ_1, γ_0 and γ_2 together we get a path γ_+ in $\tilde{J}_{c_1}$ which connects θ and re_1. Since

$$\tilde{J}(u_+ + u_-)\geqq\tilde{J}(u_+)$$

where $u=u_+ + u_-$ and $u_+=\max\{u, 0\}$, $u_-=u-u_+$, we may assume $\gamma_+(t)\geqq 0$ and then $J(\gamma_+(t))=\tilde{J}(\gamma_+(t))\leqq c_1$.

By the same method one may find another path γ_- connecting θ and $-re_1$ in J_{c_1} with $\gamma_-(t)\leqq 0$. So $\gamma_-(t)\neq u_1$.

By f_2,

$$J(u)\to-\infty \quad as \ \|u\|\to\infty$$

uniformly in any finite dimensional subspace of H. So we may choose a path $\bar{\gamma}$ connecting re_1 and $-re_1$ s. t. $J(\bar{\gamma}(t))<c_1$, $\forall t\in I$.

Now by connecting γ_+, γ_-, $\bar{\gamma}$ we get a mapping

$$\varphi:\ S^1\to H, \qquad \varphi(S^1)=\gamma_+(I)\cup\gamma_-(I)\cup\bar{\gamma}(I).$$

where S^1 is one dimensional sphere and

$$\gamma_+(I)=\gamma_1(I)\cup\gamma_0([-\delta, \delta])\cup\gamma_2(I).$$

Define

$$\Psi=\{\psi\in C(B^2, H)\mid \psi|_{\partial B^2}=\varphi\}$$

and

$$c_3=\inf_{\psi\in\Psi}\ \sup_{s\in B^2} J(\Psi(s)) \tag{14}$$

where B^2 is the two dimensional unit ball. If $c_3>c_1$ the minimax principle gives that c_3 is a critical value of J. To prove $c_3>c_1$ it suffices to prove the following Lemma 2.6.

LEMMA 2.6. – *For any* $\psi\in\Psi$,

$$\psi(B^2)\cap\{u_1+S_\rho\}\neq\varnothing \tag{15}$$

where $\{u_1+S_\rho\}=\{u=u_1+z+t\omega \mid |t|=0, \|z\|=\rho\}$, *and* S_ρ *is given in* (13).

Proof. – If (15) is not true, there is a $\psi_0\in\Psi$ such that

$$\psi_0(B^2)\cap\{u_1+S_\rho\}=\varnothing \tag{16}$$

We use a degree argument to give a contradiction. Define a homotopy mapping $F_t: B_\rho\times S^1\to H$ by

$$F_t(z, s)=u_1+z-\psi_0(ts), \qquad \forall(t, z, s)\in[0, 1]\times B_\rho\times S^1$$

where $B_\rho = \{ z \mid \| z \| \leqq \rho \}$, $\partial B_\rho = S_\rho$.

Firstly, by the assumption (16) for $t \in [0, 1]$ and

$$(z, s) \in \partial (B_\rho \times S^1) = S_\rho \times S^1$$

$$F_t(z, s) \neq \theta$$

Hence,

$$\deg (F_t, B_\rho \times S^1, \theta) = \text{constant} \quad \textit{for} \quad t \in [0, 1].$$

However,

$$F_0(z, s) = u_1 + z - \psi_0(\theta)$$

where θ is the center of B^2. Without loss of generality we may assume $\psi_0(\theta) \notin \{ u_1 + \bar{B}^{\rho} \}$ (if necessary make a reparametrization of B^2), and then

$$\deg (F_0, B_\rho \times S^1, \theta) = 0 \tag{17}$$

On the other hand,

$$F_1(z, s) = u_1 + z - \psi_0(s) = u_1 + z - \varphi(s)$$

Since $\varphi(S^1) = \gamma_+(I) \cup \gamma_-(I) \cup \bar{\gamma}(I)$, and

$$\gamma_+(I) = \gamma_1(I) \cup \gamma_0([-\delta, \delta]) \cup \gamma_2(I),$$

it is easy to check that

$$\{ u_1 + B_\rho \} \cap \{ \varphi(S^1) \setminus \gamma_0([-\delta, \delta]) \} = \varnothing$$

By excision property,

$$\deg (F_1, B_\rho \times S^1, \theta) = \deg (F_1, B_\rho \times (-\delta, \delta), \theta)$$

where on $B_\rho \times (-\delta, \delta)$,

$$F_1(z, s) = u_1 + z - \gamma_0(s) = u_1 + z - (u_1 + h(s\omega) + s\omega) = z - h(s\omega) - s\omega$$

Define a homotopy mapping

$$E_t(z, s) = z - th(s\omega) - s\omega, \qquad (t, z, s) \in [0, 1] \times B_\rho \times (-\delta, \delta)$$

If for some (t, z, s), $E_t(z, s) = \theta$, then

$$s\omega = \theta, \qquad \text{and} \qquad z - th(s\omega) = \theta$$

that is, $s = 0$, $z = \theta$. So for $(t, z, s) \in [0, 1] \times \partial (B_\rho \times (-\delta, \delta))$, $E_t(z, s) \neq \theta$. Then we have

$$\begin{aligned} \deg (F_1, B_\rho \times (-\delta, \delta), \theta) &= \deg (E_1, B_\rho \times (-\delta, \delta), \theta) \\ &= \deg (E_0, B_\rho \times (-\delta, \delta), \theta) \\ &= -1 \end{aligned}$$

a contradiction with (17).

3. AN APPROACH VIA MORSE THEORY

In this section we use Morse theory for isolated critical points to give some homological characterizations of the solutions of equation (2.1). A different proof of the existence of the third solution of equation (2.1) will be given. Now we recall the definition of the concept of critical groups for a smooth function at an isolated critical point, which was given and studied in [5].

DEFINITION 3.1 (*see* [5]). — Suppose that u is an isolated critical point of J, the critical groups of J at u are defined by,

$$C_q(J, u) = H_q(J_c \cap U, \{ J_c \setminus \{ u \} \} \cap U, \mathscr{F})$$

where $J(u) = c_1$, U is a neighbourhood of u, H_q is the singular homology group with the coefficients groups $\mathscr{F}$, say Z_2, R.

By the excision property we know the definition does not depend on the choice of U. The following result was proved in [5].

LEMMA 3.1. — *Suppose that* c *is an isolated critical value of* J *and that* K_c *contains only finite number of critical points,* $\{ u_1, \ldots, u_k \}$, *then for any* $\varepsilon > 0$ *small,*

$$H_q(J_{c+\varepsilon}, J_{c-\varepsilon}) \cong \bigoplus_{i=1}^{k} C_q(J, u_i), \qquad q = 0, 1 \ldots$$

The following theorems gives a different proof for the existence of the third solution and then recovers our main theorem.

THEOREM 3.1. — *Assume that* f *satisfies* (f_1), (f_2) *and* (f_3), *then* J *possesses at least three nontrivial critical points.*

To prove the above theorem we need the following lemma (*see* [4] for similar result).

LEMMA 3.2. — *Assume* f *satisfies* (f_1), (f_2) *and* (f_3) *then there exists a* $K_0 > 0$ *s. t.* $\forall K \geqq K_0$

$$J_{-K} \cong S^\infty \quad (\textit{homotopy equivalent})$$

where $S^\infty = \{ u \in H \mid \|u\| = 1 \}$.

Proof. — At first by (f_3) it is easy to see for any $u \in S^\infty$

$$J(tu) \to -\infty \quad \text{as } t \to +\infty \tag{18}$$

Secondly we claim that there exists a $K_0 > 0$ s. t. $\forall K \geqq K_0$ if $J(tu) \leqq -K$ for some $(t, u) \in (0, +\infty) \times S^\infty$, then

$$\frac{dJ}{dt}(tu) < 0 \tag{19}$$

To see this, set

$$K_0 = 2M|\Omega| \max_{|t| \leq M} |f(t)| + 1$$

where M appeared in (f_3). Then for $K \geqq K_0$, if

$$J(tu) = \frac{t^2}{2} - \int F(tu) \leqq -K$$

then

$$\begin{aligned}\frac{d}{dt} J(tu) &= t - \int f(tu)\, u \\ &\leqq \frac{2}{t}\left\{ \int \left\{ F(tu) - \frac{1}{2} f(tu)\, tu \right\} - K \right\} \\ &\leqq \frac{2}{t}\left\{ \left(\frac{1}{\mu} - \frac{1}{2}\right) \int_{|tu| \geqq M} f(tu)\, tu \right. \\ &\quad \left. + \int_{|tu| \leqq M} \left\{ F(tu) - \frac{1}{2} f(tu)\, tu \right\} - K \right\} \\ &< \frac{-2}{t} \\ &< 0\end{aligned}$$

Combining (18) and (19), we find that for any fixed $K \geqq K_0$ there exists unique $T(u) > 0$ s. t.

$$J(T(u)\, u) = -K \qquad \textit{for} \quad u \in S^\infty$$

From this formula and implicit function theorem $T(u) \in C(S^\infty, R)$. Without loss of generality, assume $T(u) \geqq 1$, define a deformation retract $\tau : [0, 1] \times (H \setminus B^\infty) \to H \setminus B^\infty$ by

$$\tau(s, u) = (1 - s)\, u + s\, T(u)\, u$$

which satisfies $\tau(0, u) = u$ and $\tau(1, u) \in J_{-K}$, $\forall u \in H \setminus B^\infty$, *i. e.* $H \setminus B^\infty \cong J_{-K}$. Obviously, $H \setminus B^\infty \cong S^\infty$. So we proved Lemma 3.2.

Now we give

Proof of Theorem 3.1. – Firstly, we known, by (f_1), θ is a local minimal critical point of J, then we have (*see* [5]),

$$C_q(J, \theta) = \begin{cases} \mathscr{F}, & q = 0 \\ 0, & q \neq 0 \end{cases} \tag{20}$$

[This is because if we take $U = B_\delta(\theta)$ a neighbourhood of θ s. t. $J(u) > 0$, $u \in B_\delta(\theta)$, $u \neq 0$, then $J_0 \cap B_\delta(\theta) = B_\delta(\theta)$, $J_0 \setminus \{\theta\} = \varnothing$, and (20) follows immediately.]

For u_1 by Proposition 2.1 [we just consider case (i), case (ii) is similar], module a homeomorphism

$$\mathrm{J}(u_1+z+y)=\|z\|+\mathrm{J}(u_1+h(y)+y),$$
$$\|z\|+\|y\|\leqq\delta$$

and

$$\mathrm{J}(u_1+h(y)+y)<c_1, \qquad 0<\|y\|\leqq\delta$$

Define a deformation retract σ: $[0, 1]\times\{\mathrm{J}_{c_1}\cap\mathrm{B}_\delta(u_1)\}\to\mathrm{J}_{c_1}\cap\mathrm{B}_\delta(u_1)$ by

$$\sigma(s, u_1+z+y)=u_1+(1-s)z+y$$

And then we have

$$\mathrm{J}_{c_1}\cap\mathrm{B}_\delta(u_1)\cong\mathrm{J}_{c_1}\cap\{\mathrm{B}_\delta(u_1)\cap\mathrm{N}\}=\mathrm{B}_\delta(u_1)\cap\mathrm{N}.$$

Hence

$$\begin{aligned}\mathrm{C}_q(\mathrm{J}, u_1)&\cong\mathrm{H}_q(\mathrm{J}_{c_1}\cap\mathrm{B}_\delta(u_1), \{\mathrm{J}_{c_1}\setminus\{u_1\}\}\cap\mathrm{B}_\delta(u_1))\\&\cong\mathrm{H}_q(\mathrm{B}_\delta(u_1)\cap\mathrm{N}, \{\mathrm{B}_\delta(u_1)\setminus\{u_1\}\}\cap\mathrm{N})\\&\cong\begin{Bmatrix}\mathscr{F}, & q=1\\ 0, & q\neq 1\end{Bmatrix}\end{aligned} \tag{21}$$

Similarly we can have

$$\mathrm{C}_q(\mathrm{J}, u_2)\cong\begin{Bmatrix}\mathscr{F}, & q=1\\ 0, & q\neq 1\end{Bmatrix} \tag{22}$$

Now if J possesses only three critical points θ, u_1, u_2, we reduce a contradiction as follows. Take b_1, b_2, b_3 satisfying

$$b_1<-\mathrm{K}_0<0<b_2<c_2\leqq c_1<b_3.$$

Then by deformation and Lemma 3.1

$$\mathrm{H}_q(\mathrm{J}_{b_2}, \mathrm{J}_{b_1})\cong\mathrm{C}_q(\mathrm{J}, \theta)$$

and

$$\mathrm{H}_q(\mathrm{J}_{b_3}, \mathrm{J}_{b_2})\cong\mathrm{C}_q(\mathrm{J}, u_1)\oplus\mathrm{C}_q(\mathrm{J}, u_2)$$

Take a exact triad $(\mathrm{J}_{b_1}\ \mathrm{J}_{b_2}\ \mathrm{J}_{b_3})$, then we get an exact sequence (*see* [6])

$$\ldots\to\mathrm{H}_q(\mathrm{J}_{b_2}, \mathrm{J}_{b_1})\to\mathrm{H}_q(\mathrm{J}_{b_3}, \mathrm{J}_{b_1})\to\mathrm{H}_q(\mathrm{J}_{b_3}, \mathrm{J}_{b_2})\to\ldots$$

By Lemma 3.2 it is easy to see we have

$$\mathrm{H}_q(\mathrm{J}_{b_3}, \mathrm{J}_{b_1})\cong 0, \qquad q=0, 1, \ldots$$

Therefore we get

$$\mathrm{C}_q(\mathrm{J}, \theta)\cong\mathrm{H}_q(\mathrm{J}_{b_2}, \mathrm{J}_{b_1})\cong\mathrm{H}_{q+1}(\mathrm{J}_{b_3}, \mathrm{J}_{b_2})\cong\mathrm{C}_{q+1}(\mathrm{J}, u_1)\oplus\mathrm{C}_{q+1}(\mathrm{J}, u_2)$$

Take $q=0$, we get a contradiction with (20), (21) and (22).

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