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by

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ABSTRACT. — We show that there are no non-trivial (potential) energy stable minimal cones in $\mathbb{R}^n \times \mathbb{R}^+ \times \mathbb{R}^+$ with singularity at 0, if $2 \leq n \leq 5$. The sharpness of this result is demonstrated by proving that a certain six dimensional cone in $\mathbb{R}^7$ is stable. Moreover, we extend all results to the more general $\alpha$-energy functional.

Key words : Stable cones.

A well known result due to J. Simons [SJ] states that there are no non-trivial $n$-dimensional stable minimal cones in $\mathbb{R}^{n+1}$ (with singularity at zero), provided $n \leq 6$. One of the crucial ingredients in his proof is an important identity for the Laplacian of the second fundamental form for minimal hypersurfaces. Using sharper estimates than had previously been

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realized, Schoen-Simon-Yau [SSY] gave a considerably simpler proof of Simons’ result.

Simons also proved that the seven dimensional cone $x_1^2 + \ldots + x_4^2 = x_5^2 + \ldots + x_8^2$ in $\mathbb{R}^8$ is stable and, in fact, it was proved by Bombieri-De Giorgi-Giusti [BDG] that it even minimizes area in $\mathbb{R}^8$. This result dashes the hope for general interior regularity of codimension one solutions to the problem of least area in $\mathbb{R}^8$.

In two papers [D 1] and [D 2] the author has investigated the cones

$$C_\alpha := \left\{ x = (x_1, \ldots, x_{n+1}); \ 0 \leq x_{n+1} \leq \sqrt{\frac{\alpha}{n-1}} \left( x_1^2 + \ldots + x_n^2 \right)^{1/2} \right\} \subset \mathbb{R}^{n+1}$$

which have boundaries of least $\alpha$-energy

$$\mathcal{E}_\alpha = \int_{\partial \varphi_U} x_{n+1}^2 \left| D \varphi_U \right| \text{ in } \mathbb{R}^n \times \mathbb{R}^+,$$

provided one of the following conditions holds:

(i) $\alpha + p \geq 6$, where $\alpha \geq 2$ and $p := n-1 \geq 2$,

or

(ii) $\alpha + p \geq 7$, for $\alpha \geq 1$ and $p \geq 1$.

Here $\mathbb{R}^+ = \{ t \geq 0 \}$, $U \subset \mathbb{R}^n \times \mathbb{R}^+$, and $\left| D \varphi_U \right|$ is the $n$-dimensional Hausdorff measure restricted to the reduced boundary of $U$. Also a set $C \subset \mathbb{R}^n \times \mathbb{R}^+$ with characteristic function $\varphi_C$ has a boundary of least $\alpha$-energy in $\mathbb{R}^n \times \mathbb{R}^+$, if and only if for each $g \in L^1_{\text{loc}}(\mathbb{R}^n \times \mathbb{R}^+)$ with compact support $K \subset \mathbb{R}^n \times \mathbb{R}^+$ we have

$$\int_K x_{n+1}^2 \left| D \varphi_C \right| \leq \int_K x_{n+1}^2 \left| D (\varphi_C + g) \right| .$$

Furthermore it could be shown in [D 2] that the six dimensional boundary of the cone $C_6$ does not minimize the energy $\mathcal{E}_1$ in $\mathbb{R}^6 \times \mathbb{R}^+$. Similarly, the cone $C_5$ does not minimize $\mathcal{E}_5$ in $\mathbb{R}^2 \times \mathbb{R}^+$.

We wish to emphasize the physical relevance of the problem. Namely if we regard the boundary $M = \partial U$ of $U$ as a material surface of constant mass density, then $\mathcal{E}_1$ corresponds to the potential energy of $M$ under gravitational forces. Here we have of course assumed that the gravitational force acts in the $-x_{n+1}$ direction. Therefore, we refer to $\mathcal{E}_\alpha$ as the $\alpha$-energy, and, in particular, if $\alpha = 1$ we shall simply omit the addition "$\alpha$".

In this paper we will employ the method of Schoen-Simon-Yau [SSY] to obtain a result on the non-existence of non-trivial $\alpha$-stable minimal cones in $\mathbb{R}^n \times \mathbb{R}^+$ i.e., cones which are stable with respect to the $\alpha$-energy $\mathcal{E}_\alpha$. We in fact prove (Theorem 2) that such a result holds true provided that

$$\alpha + p < 3 + \sqrt{8}, \quad p = n-1.$$
On the other hand we show in Theorem 1 that the cones
\[ x_{n+1} = + \sqrt{\frac{\alpha}{p}} \left[ x_1^2 + \ldots + x_n^2 \right]^{1/2} \]
are \( \alpha \)-stable, if
\[ \alpha + p \geq 3 + \sqrt{8}. \]

Note that this in particular implies stability, if \( \alpha = 1, n = 6 \), or \( \alpha = 5, n = 2 \), but because of [D 2] the boundaries of the set \( C^1_6 \) or \( C^5_2 \) do not minimize the corresponding \( \alpha \)-energy in \( \mathbb{R}^n \times \mathbb{R}^+ \). In fact, we might even obtain a field of \( \alpha \)-stable and non-minimizing minimal cones, e.g. the two-dimensional cones \( x_3 = \sqrt{\alpha} [x_1^2 + x_2^2]^{1/2} \) in \( \mathbb{R}^3 \) where \( 2 + \sqrt{8} = \alpha \leq 5 \).

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1. NOTATIONS AND RESULTS

In this section we set up our terminology and, in particular, we give simple expressions for the first and second variation of the \( \alpha \)-energy. Finally, we formulate our main results.

Let \( M \) be an \( n \)-dimensional submanifold of class \( C^2 \) contained in the open half-space \( \mathbb{R}^n + \mathbb{R}^+ \subset \mathbb{R}^{n+1} \), \( \mathbb{R}^+ = \{ t > 0 \} \), and let \( U \subset \mathbb{R}^n \times \mathbb{R}^+ \) be open with \( U \cap M \neq \emptyset \), \( (\text{cl} \ M - M) \cap U = \emptyset \), \( \mathcal{H}^n (\mathbb{M} \cap K) < \infty \) for each compact set \( K \subset U \); here \( \mathcal{H}^t \), \( t \geq 0 \), denotes \( t \)-dimensional Hausdorff measure. We consider one parameter families \( \{ \Phi_t \}, -1 \leq t \leq 1 \), of diffeomorphisms from \( U \) into \( U \), with the following properties:

\[ \Phi(t, x) = \Phi_t(x) \in C^2((-1,1) \times U, U), \]
\[ \Phi_0(x) = x \quad \text{for all } x \in U, \]
\[ \Phi_t(x) = x \quad \text{for all } t \in (-1, 1) \]
\[ \text{and all } x \in U - K \text{ for some compact set } K \subset U. \]

Put
\[ X(x) = \frac{\partial \Phi}{\partial t}(t, x)|_{t=0}, \]
and
\[ Z(x) = \frac{\partial^2 \Phi}{\partial t^2}(t, x)|_{t=0}, \]
to denote the initial velocity and acceleration vectors of \( \Phi_t \) respectively. Then, because of (3), \( X \) and \( Z \) have compact support \( K \subset U \), and furthermore
\[ \Phi_t(x) = x + t X(x) + \frac{t^2}{2} Z(x) + o(t^2). \]
Let $M := \Phi_t(M)$ denote the image of $M = M_0$ under $\Phi_t$; then we are interested in the first and second variation of the $\alpha$-energy functional

$$\delta \alpha(M) = \int_M \alpha^{n+1} d\mathcal{H}^n,$$

where $x = (x_1, \ldots, x_n, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R}^+$, $\alpha > 0$, i.e. we want to compute

$$\delta \delta \alpha(M, X) = \frac{d}{dt} \left. \frac{d}{dt} \int_{M_t} \alpha^{n+1} d\mathcal{H}^n \right|_{t=0},$$

and

$$\delta^2 \alpha(M, X, Z) = \frac{d^2}{dt^2} \left. \int_{M_t} \alpha^{n+1} d\mathcal{H}^n \right|_{t=0}.$$

Choose a local field of orthonormal frames $\tau^1, \ldots, \tau^n, \nu$ such that $\tau^1, \ldots, \tau^n \in T_x M$ are tangent to $M$. For a given vectorfield $Y$ on $M$ (not necessarily tangential) we denote by $D_{\tau^i} Y$ the directional derivative of $Y$ in the direction $\tau^i$. Also

$$\text{div} \ Y = \sum_{i=1}^n (D_{\tau^i} Y) \tau^i$$

stands for the divergence on $M$, and

$$\nabla f = \sum_{i=1}^n (D_{\tau^i} f) \tau^i$$

denotes the gradient of the function $f \in C^1(M, \mathbb{R})$ respectively. We shall also employ the symbol $\Delta$ to denote the Laplacian on $M$, i.e. $\Delta = \nabla_i \nabla_i$ where $\nabla_i = D_{\tau^i}$, and $Y^\perp = Y - \sum_{j=1}^n (Y \cdot \tau^j) \tau^j$ stands for the normal part of $Y$.

**Lemma 1.** Let $M, \Phi_t : U \to U$ and

$$X(x) = (X_1(x), \ldots, X_{n+1}(x)),$$
$$Z(x) = (Z_1(x), \ldots, Z_{n+1}(x))$$

be defined as above. Then

$$\delta \alpha(M, X) = \int_M \left\{ \alpha^{n+1} \text{div} X + \alpha x_{n+1}^2 \right\} d\mathcal{H}^n$$

(4)
and
\[
\delta^2 \mathcal{E}_\alpha(M, X, Z) = \int_M \left\{ \alpha (\alpha - 1) x_{n+1}^{a-2} X_{n+1}^2 + \alpha x_{n+1}^{a-1} Z_{n+1} + 2 \alpha x_{n+1}^{a-1} X_{n+1} \right. \\
+ 2 \alpha x_{n+1}^{a-1} X_{n+1} \operatorname{div} X + x_{n+1}^{a-1} \left[ \operatorname{div} Z + (\operatorname{div} X)^2 \\
+ \sum_{i=1}^n \left( (D_i X)^i \right)^2 - \sum_{i, j=1}^n (\tau^i D_j X)(\tau^j D_i X) \right] \right\} d\mathcal{H}_n. \tag{5}
\]

**Proof.** - From the general area formula (see e.g. [SL], § 8, or [FH] 3.2.20 Cor.), we infer that
\[
\mathcal{E}_\alpha(\Phi_t(M \cap K)) = \int_{M \cap K} \left( \psi_t \right)^a_{n+1} J \psi_t d\mathcal{H}_n,
\]
where \( \psi_t = \Phi_t|_{M \cap K} \) and \( J \psi_t \) denotes the Jacobian of \( \psi_t \) and \( (\psi_t)^a_{n+1} \) is the \((n+1)\)-th component of \( \psi_t \) to the power \( \alpha \). The Jacobian \( J \psi_t \) may be computed as in [SL], p. 50,
\[
J \psi_t = 1 + t \cdot \operatorname{div} X + \frac{t^2}{2} \left\{ \operatorname{div} Z + (\operatorname{div} X)^2 + \sum_{i=1}^n \left( (D_i X)^i \right)^2 - \sum_{i, j=1}^n (\tau^i D_j X)(\tau^j D_i X) \right\} + o(t^2).
\]
Similarly we find
\[
(\psi_t)^a_{n+1} = x_{n+1}^{a-1} + t \alpha x_{n+1}^{a-1} X_{n+1} + \frac{t^2}{2} \left[ \alpha (\alpha - 1) x_{n+1}^{a-2} X_{n+1}^2 + \alpha x_{n+1}^{a-1} Z_{n+1} \right] + o(t^2).
\]
Now the result follows immediately by computing the coefficients of \( t \) and \( \frac{t^2}{2} \) in the product \((\psi_t)^a_{n+1} J \psi_t \). \( \square \)

**Definition 1.** - A \( C^1 \)-submanifold \( M \subset \mathbb{R}^n \times \mathbb{R}^+ \) is called \( \alpha \)-stationary in \( U \subset \mathbb{R}^n \times \mathbb{R}^+ \), if \( \mathcal{E}_\alpha(M \cap K) < \infty \) for all compact sets \( K \subset U \), and
\[
\int_M \left\{ x_{n+1}^{a} \operatorname{div} X + \alpha x_{n+1}^{a-1} X_{n+1} \right\} d\mathcal{H}_n = 0, \tag{6}
\]
for all vector fields \( X \in C^\infty_c(U, \mathbb{R}^{n+1}) \).

**Lemma 2.** - Suppose \( M \) is \( \alpha \)-stationary in \( U \) and of class \( C^2 \). Then the mean curvature \( H \) of \( M \) with respect to the unit normal \( v = (v_1, \ldots, v_{n+1}) \) is given by
\[
H(x) = \alpha x_{n+1}^{-1} v_{n+1} \quad \text{for all } x \in M \cap U.
\]

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Proof. – Take some arbitrary function \( \xi \in C^1_{\text{c}}(M, \mathbb{R}) \) with compact support in \( U \) and put \( X = \xi \cdot v \). Then we infer from (6)

\[
0 = \int_M \left\{ x_{n+1}^2 \text{div}(\xi \cdot v) + \alpha x_{n+1}^{-1} \xi \cdot v_{n+1} \right\} d\mathcal{H}_n
\]

\[
= \int_M \left\{ \text{div}(x_{n+1}^2 \xi \cdot v) + \alpha x_{n+1}^{-1} \xi \cdot v_{n+1} \right\} d\mathcal{H}_n,
\]

\[
= -\int_M \left\{ x_{n+1}^2 \xi \cdot v \cdot H - \alpha x_{n+1}^{-1} \xi \cdot v_{n+1} \right\} d\mathcal{H}_n,
\]

where \( H = v \cdot H \) is the mean curvature vector of \( M \). The lemma follows by applying the fundamental lemma in the calculus of variations. \( \square \)

We take again the special variation \( X \equiv \xi \cdot v \in C^1_{\text{c}}(M, \mathbb{R}) \) and find successively,

\[
\text{div} X = -X \cdot H = -\alpha v_{n+1} x_{n+1}^{-1} \xi
\]

\[
\sum_{i=1}^{n} |(D_{\xi} X)^i|^2 = \sum_{i=1}^{n} |v \cdot D_{\xi} \xi|^2 = |V \xi|^2,
\]

and

\[
\sum_{i, j=1}^{n} (\tau^i D_{\xi} X)(\tau^j D_{\xi} X) = \xi^2 |A|^2,
\]

where \( |A| \) denotes the length of the second fundamental form \( A = h_{ij} \tau^i \otimes \tau^j \), i.e.

\[
|A|^2 = \sum_{i, j=1}^{n} h_{ij}^2.
\]

Thus we have proved

**Lemma 3.** — Suppose \( M \subset \mathbb{R}^n \times \mathbb{R}^+ \) is a submanifold of class \( C^2 \) which is \( \alpha \)-stationary in \( U \subset \mathbb{R}^n \times \mathbb{R}^+ \), (clos \( M-M \)) \( \cap U = \emptyset \). If \( X \equiv \xi \cdot v \) for some function \( \xi \in C^1_{\text{c}}(M, \mathbb{R}) \) with compact support in \( U \), then the second variation is given by

\[
\delta^2 \mathcal{E}(M, \xi) = \int_M \left\{ x_{n+1}^2 |V \xi|^2 - \alpha x_{n+1}^{-2} \xi v_{n+1}^2 \xi^2 - x_{n+1}^2 |A|^2 \xi^2 \right\} d\mathcal{H}_n.
\]

Hence it is reasonable to define stability as follows.

**Definition 2.** — Suppose \( M \subset \mathbb{R}^n \times \mathbb{R}^+ \) is an \( n \)-dimensional submanifold of class \( C^2 \) which is \( \alpha \)-stationary in \( U \subset \mathbb{R}^n \times \mathbb{R}^+ \), (clos \( M-M \)) \( \cap U = \emptyset \). Then \( M \) is called \( \alpha \)-stable in \( U \), if

\[
\int_M \left\{ x_{n+1}^2 |V \xi|^2 - \alpha x_{n+1}^{-2} \xi v_{n+1}^2 \xi^2 - x_{n+1}^2 |A|^2 \xi^2 \right\} d\mathcal{H}_n \geq 0 \tag{7}
\]

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for each \( \xi \in C_1^1(M, \mathbb{R}) \) with compact support in \( U \). In particular, if \( \mathcal{C} = \text{clos} \, M \) is a cone in \( \mathbb{R}^n \times \mathbb{R}^+ \) with singularity at \( \{0\} \), and if \( M = \mathcal{C} - \{0\} \subset \mathbb{R}^n \times \mathbb{R}^+ \) is \( \alpha \)-stationary in \( \mathbb{R}^n \times \mathbb{R}^+ \), then \( \mathcal{C} \) is called \( \alpha \)-stable if (7) holds for all \( \xi \in C_1^1(M, \mathbb{R}) \).

Put \( c_n^\alpha(y) = \sqrt{\frac{\alpha}{p}} [y_1^2 + \ldots + y_n^2]^{1/2}, \ y \in \mathbb{R}^n, \ \alpha > 0, \ p = n - 1 \) and define the cones

\[
\mathcal{C}_n^\alpha = \{(y, c_n^\alpha(y)) : y \in \mathbb{R}^n\},
\]
then we have

**Theorem 1.** - The cones \( \mathcal{C}_n^\alpha \) are \( \alpha \)-stable, if \( \alpha + p \geq 3 + \sqrt{8} \).

Observe that the critical number \( 3 + \sqrt{8} \) also enters the discussion of the ordinary differential system \([11]\) in \([D1]\). Here, it appears as a necessary, though not sufficient condition for the construction of a minimal foliation about the cone \( \mathcal{C}_n^\alpha \).

**Theorem 2.** - Suppose \( \mathcal{C} \subset \mathbb{R}^n \times \mathbb{R}^+ \) is an \( \alpha \)-stable \( n \)-dimensional cone with singularity at \( \{0\} \). If \( \alpha + p < 3 + \sqrt{8} \) then \( \mathcal{C} \) is a hyperplane \( \mathcal{P} \). Furthermore, \( \mathcal{P} \) must be perpendicular to the plane \( \{x_{n+1} = 0\} \).

**Corollary.** - In particular, if \( 2 \leq n \leq 5 \) there are no non-trivial (potential-) energy stable cones in \( \mathbb{R}^n \times \mathbb{R}^+ \) with singularity at \( \{0\} \).

### 2. Proofs

Let \( \xi \in C_1^1(\mathbb{R}_n^+ - \{0\}, \mathbb{R}) \) be arbitrary and put \( X(x) = x \cdot |x|^{-2} \xi^2 \) for \( x \in \mathbb{R}^n \times \mathbb{R}^+ \) where \( |x|^2 = (x_1^2 + \ldots + x_{n+1}^2) \). A standard calculation yields (see \([SL]\), § 17)

\[
\text{div } X = \sum_{i=1}^n (D_i X) \tau_i = 2 |x|^{-2} (x \nabla \xi) \xi + (n-2) \xi^2 |x|^{-2} + 2 |x|^{-2} \xi^2 |(D_i |x|)^+|^2.
\]

Since \( \mathbb{R}_n^+ - \{0\} \) is \( \alpha \)-stationary in \( \mathbb{R}^n \times \mathbb{R}^+ \), we conclude from (6) that

\[
\int_{\mathbb{R}_n^+ \setminus \{0\}} x_{n+1}^\alpha \left\{ 2 |x|^{-2} (x \nabla \xi) \xi + (n-2+\alpha) |x|^{-2} \xi^2 \right\} d\mathcal{H}_n \leq 0.
\]

We apply Schwarz inequality and obtain

\[
\left( \frac{n-2+\alpha}{2} \right)^2 \int_{\mathbb{R}_n^+ \setminus \{0\}} x_{n+1}^\alpha |x|^{-2} \xi^2 d\mathcal{H}_n \leq \int_{\mathbb{R}_n^+ \setminus \{0\}} x_{n+1}^\alpha |\nabla \xi|^2 d\mathcal{H}_n.
\]
Therefore \( \mathcal{C}_n^\alpha \) is \( \alpha \)-stable, if
\[
\left( \frac{n - 2 + \alpha}{2} \right)^2 \geq |x|^2 |A|^2 + \alpha x_{n+1}^{-2} |x|^2 v_{n+1}^2. \tag{8}
\]

An elementary calculation shows that for the cone \( \mathcal{C}_n^\alpha \) the length of the second fundamental form is given by
\[
|A|^2 = \frac{\alpha p}{\alpha + p} r^{-2} = \alpha |x|^{-2} \quad \text{for all } x \in \mathcal{C}_n^\alpha - \{0\},
\]
where we have put \( r^2 = (x_1^2 + \ldots + x_n^2) \). Then along \( \mathcal{C}_n^\alpha, x_{n+1} = \sqrt{\frac{\alpha}{p}} r \) and we infer from (8) that \( \mathcal{C}_n^\alpha \) is stable, if
\[
\left( \frac{n - 2 + \alpha}{2} \right)^2 \geq \alpha + \alpha x_{n+1}^{-2} |x|^2 v_{n+1}^2
\]
\[
= \alpha + \frac{\alpha p}{\alpha + p} \left[ 1 + \frac{r^2}{x_{n+1}^2} \right] = \alpha + p.
\]
This is true in case that \( \alpha + p \geq 3 + \sqrt{8} \). Theorem 1 follows.

Proof of Theorem 2. - In the following we shall always assume that \( M = \mathcal{C} - \{0\} \) is an \( \alpha \)-stable cone in \( \mathbb{R}^n \times [0, \infty) \), so that in particular (7) holds true. Replacing \( \xi \) by \( |A| \xi \) in (7) we get
\[
\int_M \left\{ x_{n+1}^\alpha |A|^4 \xi^2 + \alpha x_{n+1}^{-2} v_{n+1}^2 |A|^2 \xi^2 \right\} dH_n
\]
\[
\leq \int_M x_{n+1}^2 \left\{ |A|^2 |\nabla \xi|^2 + |\nabla A||A||^2 \xi^2 + 2 \xi |A| (\nabla \xi \nabla |A|) \right\} dH_n. \tag{9}
\]
Now
\[
2 \int_M x_{n+1}^2 |A| \xi (\nabla \xi \nabla |A|) dH_n = \int_M x_{n+1}^2 (\nabla \xi^2) \left( \frac{1}{2} |A|^2 \right) dH_n
\]
\[
= - \int_M x_{n+1}^2 \xi^2 \Delta \left( \frac{1}{2} |A|^2 \right) dH_n - \int_M \xi^2 (\nabla x_{n+1}^2) \left( \frac{1}{2} |A|^2 \right) dH_n. \tag{10}
\]

In order to conclude further we need a sharp estimate for the Laplacian of \( |A|^2 \). This will be provided by the following

Lemma 4 ([SSY], [SL, appendix B]). - If \( M = \mathcal{C} - \{0\} \) is a cone, then
\[
- \frac{1}{2} \Delta |A|^2 \leq |A|^4 - 2 |x|^{-2} |A|^2 - |\nabla A|^2 - h_{ij} H, ij - H h_{mi} h_{mj} h_{ij} \quad (1)
\]

(1) The summation convention is used freely here!
Here $H_{ij}$ denote the second covariant derivatives of the mean curvature $H$ with respect to $\tau^i$ and $\tau^j$, and, as above, $h_{ij}$ are the coefficients of $A$.

**Proof of Lemma 4.** — B.8 Lemma and B.9 Lemma in [SL] yield the relations

$$\Delta \left( \frac{1}{2} |A|^2 \right) = \sum_{i, j, k} h_{ij, k}^2 = |A|^2 + h_{ij} H_{ij} + H h_{mi} h_{mj} h_{ij},$$

here $H = h_{kk} = \text{trace } A$ and $h_{ij, k}$ denotes the covariant derivative of $A$ with respect to $\tau^k$; also

$$\sum_{i, j, k} h_{ij, k}^2 - |\nabla |A||^2 \geq 2 |x|^{-2} |A|^2 \quad \text{for all } x \in M.$$

Both relations imply Lemma 4. □

From (9), (10) and Lemma 4 we conclude that

$$\int_M \xi^2 \left\{ 2 x_{n+1}^a |x|^{-2} |A|^2 + x_{n+1}^a x_{n+1}^2 |A|^2 \right. $$

$$+ \nabla (x_{n+1}^a) \nabla \left( \frac{1}{2} |A|^2 \right) + x_{n+1}^a h_{ij} H_{ij} + x_{n+1}^a H h_{mi} h_{mj} h_{ij} \right\} d\mathcal{H}^n_n$$

$$\leq \int_M x_{n+1}^a |A|^2 |\nabla \xi|^2 d\mathcal{H}^n_n. \quad (11)$$

Relation (11) will be of crucial importance in what follows.

To begin, select an orthonormal frame $\tau^1, \ldots, \tau^n \in T_x M$ so that $\tau^a = \frac{x^a}{|x|}$ and $\tau^1, \ldots, \tau^n$ are constant along the ray through $x$. Also we can assume that $\tau_{n+1}^1 = \tau_{n+1}^2 = \ldots = \tau_{n+1}^{n-1} = 0$. Then $h_{in} = h_{ni} = 0$ for $i \in \{1, \ldots, n\}$ and, since $h_{ij}(\lambda x) = \lambda^{-1} h_{ij}(x)$, $\lambda > 0$, we have $h_{ij, n} = - |x|^{-1} h_{ij}$.

We first compute the expression

$$(\nabla x_{n+1}^a) \left( \nabla \frac{1}{2} |A|^2 \right) = \alpha x_{n+1}^{a-1} (D_k x_{n+1}^a) \left( D_k \left( \frac{1}{2} |A|^2 \right) \right)$$

$$= \alpha x_{n+1}^{a-1} h_{ij, k} h_{ij} t_{n+1}^k = - \alpha x_{n+1}^a |x|^{-2} |A|^2, \quad (12)$$

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and then
\[ \frac{1}{\alpha} H_{,ij} = \frac{1}{\alpha} \nabla_i \nabla_j H = \nabla_i \nabla_j \left( \frac{V_{n+1}}{x_{n+1}} \right) \]
\[ = \nabla_i \left\{ -x_{n+1}^{-2} \left( \nabla_j x_{n+1} \right) v_{n+1} + x_{n+1}^{-1} \partial_j v_{n+1} \right\} \]
\[ = 2x_{n+1}^{-3} \nabla_i x_{n+1} \nabla_j x_{n+1} v_{n+1} - x_{n+1}^{-2} \nabla_i \nabla_j x_{n+1} v_{n+1} \]
\[ - x_{n+1}^{-2} \nabla_i x_{n+1} \nabla_i v_{n+1} + x_{n+1}^{-1} \partial_i v_{n+1} \]
\[ = 2x_{n+1}^{-3} \tau_{n+1}^i \tau_{n+1}^j \nabla_i v_{n+1} \]
\[ - x_{n+1}^{-2} \nabla_i \tau_{n+1}^i v_{n+1} - x_{n+1}^{-1} \tau_{n+1}^i \partial_i v_{n+1} \]
\[ - x_{n+1}^{-2} \tau_{n+1}^i \nabla_j v_{n+1} + x_{n+1}^{-1} \nabla_i \nabla_j v_{n+1}. \]

By virtue of
\[ \nabla_i v = -h_{il} \tau^l \]
and
\[ \nabla_i \tau^l = h_{ij} v \]
we obtain
\[ \frac{1}{\alpha} H_{,ij} = 2x_{n+1}^{-3} \tau_{n+1}^i \tau_{n+1}^j v_{n+1} \]
\[ - x_{n+1}^{-2} h_{ij} v_{n+1} + x_{n+1}^{-2} \tau_{n+1}^j h_{il} \tau_{n+1}^l \]
\[ + x_{n+1}^{-2} \tau_{n+1}^i h_{jl} \tau_{n+1}^l - x_{n+1}^{-1} \nabla_i \left[ h_{jl} \tau_{n+1}^l \right]. \]

Using the Codazzi equations we conclude
\[ \nabla_i \left[ h_{jl} \tau_{n+1}^l \right] = h_{jl, i} \tau_{n+1}^l + h_{jl} \nabla_i \tau_{n+1}^l \]
\[ = h_{ij, l} \tau_{n+1}^l + h_{jl} \nabla_i v_{n+1}, \]
whence
\[ \frac{1}{\alpha} h_{ij} H_{,ij} = 2x_{n+1}^{-3} \tau_{n+1}^i \tau_{n+1}^j h_{ij} v_{n+1} - x_{n+1}^{-2} |A|^2 v_{n+1}^2 \]
\[ + x_{n+1}^{-2} h_{ij} h_{il} \tau_{n+1}^l + x_{n+1}^{-2} h_{ij} h_{jl} \tau_{n+1}^l \tau_{n+1}^l \]
\[ - x_{n+1}^{-1} h_{ij} h_{ij, l} \tau_{n+1}^l - x_{n+1}^{-1} h_{ij} h_{jl} h_{il} v_{n+1}. \]
Thus
\[ \frac{1}{\alpha} h_{ij} H_{,ij} = -x_{n+1}^{-2} |A|^2 v_{n+1}^2 + |x|^{-2} |A|^2 - x_{n+1}^{-1} h_{ij} h_{jl} h_{il} v_{n+1}, \]
and finally
\[ h_{ij} H_{,ij} = -\alpha x_{n+1}^{-2} |A|^2 v_{n+1}^2 + \alpha |x|^{-2} |A|^2 - H_{ij} h_{jl} h_{il}. \] (13)
(12), (13), and (11) yield the relation
\[ 2 \int_M x_{n+1}^2 |x|^{-2} |A|^2 \xi^2 \, dH_n \leq \int_M x_{n+1}^2 |A|^2 \nabla \xi^2 \, d\mathcal{H}_n \]
for all \( \xi \in C^1_c (M, \mathbb{R}) \).

If \( \xi \) does not have compact support in \( M = \mathbb{R}^n - \{0\} \) then (14) continues to hold, if only
\[ \int_M x_{n+1}^2 |x|^{-2} |A|^2 \xi^2 \, d\mathcal{H}_n < \infty. \]

In fact, replace \( \xi \) by \( \xi \cdot \gamma_\varepsilon \) where \( \gamma_\varepsilon \) is a suitable cut off function with
\[ \gamma_\varepsilon = \begin{cases} 1 & \text{for } |x| < \varepsilon^{-1} \\ 0 & \text{for } |x| > \varepsilon^{-1} \end{cases} \]
and \( 0 \leq \gamma_\varepsilon \leq 1, \ |\nabla \gamma_\varepsilon (x)| \leq 3 |x|^{-1} \) in all of \( \mathbb{R}^n \times \mathbb{R}^+ \). Then \( \xi \cdot \gamma_\varepsilon \) is admissible in (14) and the assertion follows by letting \( \varepsilon \to 0^+ \) and using (15).

Note that (15) is satisfied, if
\[ \int_M |x|^{n-2} |A|^2 \xi^2 \, d\mathcal{H}_n < \infty. \]
From the coarea formula we infer that
\[ \int_M \phi (x) \, d\mathcal{H}_n (x) = \int_0^\infty r^{n-1} \int_\Sigma \phi (r \omega) \, d\mathcal{H}_{n-1} \, dr \]
for all non-negative \( \phi \in C^0 (M) \), where \( \Sigma = M \cap S^n \), and \( S^n \subset \mathbb{R}^{n+1} \) denotes the unit \( n \)-sphere. Also, since \( M \) is a cone, we find
\[ |A (x)|^2 = |x|^{-2} |A (x/|x|)|^2 \quad \text{for all } x \in M. \]
Hence, we readily infer from (17) and (16) that
\[ \xi = |x|^{1+\varepsilon-\alpha}. |x|^{1+\alpha-(n/2)-2 \varepsilon}, \]
where
\[ |x|_1 = \max (1, |x|), \]
is admissible in (14), if \( \varepsilon > \frac{\alpha}{2} \) (where we have of course assumed that \( n \geq 2 \)).

Furthermore we find
\[ |\nabla \xi|^2 \leq \begin{cases} (1+\varepsilon-\alpha)^2 |x|^2 \varepsilon^{-2} & \text{in } M \cap B_1 (0), \\
(2-n-2 \varepsilon)^2 |x|^{2-n-2 \varepsilon} & \text{in } (\mathbb{R}^{n+1} - B_1 (0)) \cap M \end{cases}, \]
where \( B_1 (0) = \{ |x| < 1 \} \).
and (14) implies

\[ 2 \int_{M \cap B_1} x_{n+1}^\alpha |A|^2 |x|^2 \varepsilon^{-2} \alpha \, d\mathcal{H}_n \]

\[ + 2 \int_{M \cap (\mathbb{R}^{n+1} - B_1)} x_{n+1}^\alpha |A|^2 |x|^{2-n-2} \varepsilon \, d\mathcal{H}_n \]

\[ \leq (1 + \varepsilon - \alpha)^2 \int_{M \cap B_1} x_{n+1}^\alpha |A|^2 |x|^2 \varepsilon^{-2} \alpha \, d\mathcal{H}_n \]

\[ + \left( 2 - \frac{n}{2} - \varepsilon \right)^2 \int_{M \cap (\mathbb{R}^{n+1} - B_1)} x_{n+1}^\alpha |A|^2 |x|^{2-n-2} \varepsilon \, d\mathcal{H}_n. \]

We would like to choose \( n, \varepsilon, \alpha \) so that

\[ \varepsilon > \frac{\alpha}{2}, \quad (1 + \varepsilon - \alpha)^2 < 2 \quad \text{and} \quad \left( \frac{n}{2} + \varepsilon - 2 \right)^2 < 2. \]  

(18) is equivalent to

\[-1 - \sqrt{2} + \alpha < \varepsilon < \sqrt{2} + \alpha - 1 \quad \text{and} \quad \frac{\alpha}{2} < \varepsilon < 2 + \sqrt{2} - \frac{n}{2}. \]

If \( \alpha + n < 4 + 2 \sqrt{2} \) then a suitable choice of \( \varepsilon \) is

\[ \varepsilon = \frac{\alpha}{2} + \delta, \]

where

\[ \delta = N^{-1} \left[ 2 + \sqrt{2} - \frac{n}{2} - \frac{\alpha}{2} \right] > 0 \]

with \( N \in \mathbb{N} \) large. Thus we conclude that \( |A|^2 = 0 \) i.e. \( M \) is a hyper-plane \( \mathcal{P} \). Because of \( 0 = H = \alpha \frac{v_{n+1}}{x_{n+1}} \) we must have \( v_{n+1} = 0 \) as required.

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