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Existence of geodesics for the Lorentz metric of a stationary gravitational field (*)

by

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ABSTRACT. — Let $g = g(z)$ ($z = (z_0, \dots, z_3) \in \mathbb{R}^4$) be a Lorentz metric (with signature $+$, $-$, $-$, $-$) on the space-time manifold \mathbb{R}^4 . Suppose that g is stationary, *i. e.* g does not depend on z_0 . Then we prove, under some other mild assumptions on g , that for any two points $a, b \in \mathbb{R}^4$ there exists a geodesic, with respect to g , joining a and b .

Key words : Lorentz metric, geodesic, critical point.

RÉSUMÉ. — Soit $g = g(z)$ ($z = (z_0, \dots, z_3) \in \mathbb{R}^4$) une métrique de Lorentz (avec signature $+$, $-$, $-$, $-$) sur l'espace-temps \mathbb{R}^4 . On suppose que g soit stationnaire, c'est-à-dire indépendante de z_0 . Nous démontrons, sous des autres convenable hypothèses sur g , l'existence d'arcs de géodésique joignant deux points a, b arbitrairement donné dans \mathbb{R}^4 .

Classification A.M.S. : 58 E 10, 49 B 40, 53 B 30, 83 C 99.

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0. INTRODUCTION AND STATEMENT OF THE RESULTS

In General Relativity a gravitational field is described by a symmetric, second order tensor

$$g \equiv g(z)[\dots], \quad z = (z_0, \dots, z_3) \in \mathbb{R}^4$$

on the space-time manifold \mathbb{R}^4 . The tensor g is assumed to have the signature $+, -, -, -$; namely for all $z \in \mathbb{R}^4$ the bilinear form $g(z)[\dots]$ possesses one positive and three negative eigenvalues. The "pseudometric" induced by g is called Lorentz-metric.

In this paper we study the existence of geodesics, with respect to g , connecting two points $a, b \in \mathbb{R}^4$.

To this end we consider the "action" functional related to g , *i. e.*

$$f(z) = \frac{1}{2} \int_0^1 g(z(s)) [\dot{z}(s), \dot{z}(s)] ds = \frac{1}{2} \int_0^1 \sum_{i,j=0}^3 g_{ij}(z(s)) \dot{z}_i(s) \dot{z}_j(s) ds \quad (0.1)$$

where $g_{ij}(i, j=0, \dots, 3)$ denote the components of g and $z = z(s)$ belongs to the Sobolev space

$$H^1 \equiv H^1((0, 1), \mathbb{R}^4)$$

of the curves $z: (0,1) \rightarrow \mathbb{R}^4$ which are square integrable with their first derivative $\dot{z} = \frac{dz}{ds}$. If g is smooth, f defined in (0.1) is Fréchet differentiable in H^1 . Let $a, b \in \mathbb{R}^4$, then a geodesic joining a and b is a critical point of f on the manifold

$$M = \{z \in H^1 \mid z(0) = a, z(1) = b\}. \quad (0.2)$$

Due to the indefiniteness of the metric g it is easy to see that the functional (0.1) is unbounded both from below and from above even modulo submanifolds of finite dimension or codimension. Then the Morse index of a geodesic is $+\infty$, in contrast with the situation for positive definite Riemannian spaces. This fact causes difficulties in the research of a geodesic connecting a and b and actually such a geodesic, in general, does not exist (*cf.* [3], § 5.2 or [5], remark 1.14).

However the above difficulties can be overcome if the events a, b are causally related, namely if a, b can be joined by a smooth curve $z = z(s)$ such that

$$g(z(s)) [\dot{z}(s), \dot{z}(s)] \geq 0 \quad \text{for all } s \in (0,1). \quad (0.3)$$

Such a curve is called causal.

In this case, under mild assumptions on g , the existence of a causal geodesic joining a, b can be achieved just maximizing the functional

$$f^*(z) = \int_0^1 \sqrt{g(z)(s)[\dot{z}(s), \dot{z}(s)]} ds$$

over all the causal curves in M (cf. [1], [8] or [3], chapt. 6).

Here we are interested to find sufficient conditions on the metric tensor g which guarantee the existence of geodesics connecting any two given points $a, b \in \mathbb{R}^4$.

We shall prove the following result.

THEOREM 0.1. — *Let $g_{ij}(i, j=0, \dots, 3)$ denote the components of the metric tensor g . We assume that:*

- (g_1) $g_{ij} \in C^1(\mathbb{R}^4, \mathbb{R})$ ($i, j=0, \dots, 3$).
- (g_2) $g_{00}(z) \geq \nu > 0$ for all $z \in \mathbb{R}^4$.
- (g_3) There exists $\mu > 0$ s. t.

$$- \sum_{i, j=1}^3 g_{ij}(z) \xi_i \xi_j \geq \mu |\xi|^2 \quad \text{for all } z \in \mathbb{R}^4$$

and all

$$\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3.$$

(g_4) The functions $g_{0i}(i=0, \dots, 3)$ are bounded.

(g_5) $\frac{\partial g_{ij}}{\partial z_0}(z) = 0$ for all $z \in \mathbb{R}^4$.

Then for any two points $a, b \in \mathbb{R}^4$ there exists a geodesic, with respect to the metric g , joining a and b .

The assumptions (g_1), \dots , (g_4) are reasonably mild.

The most restrictive assumption is (g_5) which means that the gravitational field is stationary (cf. [4], §88). The proof of theorem 0.1 is attained by using some minimax arguments which have been recently developed in the study of nonlinear differential equations having a variational structure (cf. e. g. [7] for a review on these topics).

1. PROOF OF THEOREM 0.1

The manifold M in H^1 defined in (0.2) can be written as follows

$$M = \bar{z} + H_0^1$$

where

$$\bar{z} = a + (b - a)s, \quad s \in (0, 1)$$

and

$$H_0^1 = \{ z \in H^1 \mid z(0) = z(1) = 0 \}.$$

In order to prove theorem 0.1 we shall first carry out a finite dimensional approximation.

Let $n \in \mathbb{N}$ and set

$$M_n = \bar{z} + H_n \quad (1.1)$$

where

$$H_n = \text{span} \{ \varphi_j \sin \pi l s : j = 0, \dots, 3; l = 1, \dots, n \}$$

$\varphi_j (j = 0, \dots, 3)$ being the canonical base in \mathbb{R}^4 .

Moreover we set

$$\begin{aligned} V_n &= \text{span} \{ \varphi_0 \sin \pi l s : l = 1, \dots, n \} \\ W_n &= \text{span} \{ \varphi_j \sin \pi l s : j = 1, 2, 3; l = 1, \dots, n \} \\ S_n &= \bar{z} + V_n \\ Q_n(\mathbb{R}) &= \bar{z} + W_n \cap B_{\mathbb{R}} \end{aligned} \quad (1.2)$$

where

$$B_{\mathbb{R}} = \{ z \in H_0^1 \mid \|z\| \leq \mathbb{R} \}, \quad \mathbb{R} > 0$$

and $\|\cdot\|$ denotes the standard norm in the Sobolev space H^1 . Finally we set

$$f_n = f|_{M_n} \quad (1.3)$$

where f denotes the functional defined in (0.1). First we prove the existence of a critical point of f_n , that is to say of a point $z_n \in M_n$ such that

$$\langle f'(z_n), \zeta \rangle = 0 \quad \text{for all } \zeta \in H_n$$

where f' is the Fréchet-differential of f and $\langle \cdot, \cdot \rangle$ denotes the pairing between H^1 and its dual. More precisely the following theorem holds.

THEOREM 1.1. — *Suppose that g satisfies the assumptions of theorem 0.1. Then there exists a critical point $z_n \in M_n$ of f_n such that*

$$c' \leq f(z_n) \leq c'' \quad (1.4)$$

where c' and c'' are two constants independent on n .

The proof of theorem 1.1 is based on a variant of the “saddle point theorem” of P. H. Rabinowitz [cf. [6] or propositions 2.1 and 2.2 in [2]]. We need some lemmas.

LEMMA 1.2. — *Fix $n \in \mathbb{N}$ and $\mathbb{R} > 0$. Then S_n and the boundary $\partial Q_n(\mathbb{R})$ of $Q_n(\mathbb{R})$ link, namely for any continuous map $h : M_n \rightarrow M_n$ s. t. $h(z) = z$ for all $z \in \partial Q_n(\mathbb{R})$, we have*

$$h(Q_n(\mathbb{R})) \cap S_n \neq \emptyset$$

Proof. — Let $h: M_n \rightarrow M_n$ s. t. $h(z) = z$ for all $z \in \partial Q_n(\mathbb{R})$ and define

$$\tilde{h}: H_n \rightarrow H_n \quad \text{s. t.} \quad \forall \gamma \in H_n: \tilde{h}(\gamma) = h(\gamma + \bar{z}) - \bar{z}$$

It is easy to see that

$$\tilde{h}(\gamma) = \gamma, \quad \forall \gamma \in \partial(B_R \cap W_n)$$

Then by using the Brouwer degree (cf. [2], prop. 2.1 or [6]) it can be shown that there exists $\gamma \in \tilde{h}(W_n \cap B_R) \cap V_n$ and therefore $\bar{z} + \gamma \in h(Q_n(\mathbb{R})) \cap S_n$. \square

We denote by $f'|_{M_n}$ the Fréchet differential of f on the manifold M_n and by $\|\cdot\|$ the standard norm in H^1 . Moreover we set

$$t = z_0 \quad \text{and} \quad x = (z_1, z_2, z_3)$$

Now we prove that $f|_{M_n}$ satisfies the Palais-Smale condition. More precisely the following lemma holds.

LEMMA 1.3. — *Let g satisfy the assumptions of Theorem 0.1. Let $\{z_k\}$ be a sequence in M_n such that*

$$f'|_{M_n}(z_k) \rightarrow 0 \quad \text{as } k \rightarrow \infty \tag{1.5}$$

and

$$\{f(z_k)\} \text{ is bounded} \tag{1.6}$$

Then $\{z_k\}$ is bounded in the H^1 norm and consequently it is precompact.

Proof. — Since $z_k \in M_n$, we can set

$$z_k = (t_k, x_k) = \bar{z} + (\tau_k, \xi_k)$$

with $\tau_k \in V_n$ and $\xi_k \in W_n$ [cf. (1.1), (1.2)].

By (1.5) we deduce that

$$\langle f'(z_k), \zeta \rangle = \varepsilon_k \|\zeta\| \quad \text{for all } \zeta \in H_n \tag{1.7}$$

where $\varepsilon_k \rightarrow 0$ as $k \rightarrow \infty$.

Then for all $\zeta = (\tau, \xi)$, with $\tau \in V_n$ and $\xi = (\xi_1, \xi_2, \xi_3) \in W_n$, we have

$$\int_0^1 g(x_k)[\dot{z}_k, \dot{\zeta}] ds + \frac{1}{2} \int_0^1 \sum_{i,j=0}^3 \sum_{l=1}^3 \frac{\partial g_{ij}}{\partial x_l}(x_k) \cdot \xi_l (\dot{z}_k)_i \cdot (\dot{z}_k)_j ds = \varepsilon_k \|\zeta\|. \tag{1.8}$$

And, if we take $\zeta = (\tau_k, 0) = \tau_k$, we get

$$\int_0^1 g(x_k)[\dot{z}_k, \dot{\tau}_k] ds = \varepsilon_k \|\tau_k\| \tag{1.9}$$

Now set

$$T = \begin{pmatrix} +1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}. \quad (1.10)$$

Then

$$\tau_k = \frac{1}{2} [z_k - \bar{z} + T(z_k - \bar{z})]$$

and from (1.9) we get

$$\frac{1}{2} \int_0^1 g(x_k) [\dot{z}_k, \dot{z}_k - \dot{\bar{z}}] ds - \varepsilon_k \|\tau_k\| = -\frac{1}{2} \int_0^1 g(x_k) [\dot{z}_k, T(\dot{z}_k - \dot{\bar{z}})] ds. \quad (1.11)$$

By (1.6) there exists $c_1 > 0$ such that for all $k \in \mathbb{N}$

$$|f(z_k)| = \frac{1}{2} \left| \int_0^1 g(x_k) [\dot{z}_k, \dot{z}_k] ds \right| \leq c_1.$$

From (1.11) we get

$$\begin{aligned} & \frac{1}{2} \int_0^1 g(x_k) [\dot{z}_k, T \dot{z}_k] ds \\ & \leq c_1 + \frac{1}{2} \int_0^1 g(x_k) [\dot{z}_k, \dot{\bar{z}}] ds + \frac{1}{2} \int_0^1 g(x_k) [\dot{z}_k, T \dot{\bar{z}}] ds + \varepsilon_k \|\tau_k\| \\ & = c_1 + \int_0^1 (g_{00}(x_k) \dot{t}_k + \sum_{i=1}^3 g_{0i}(x_k) (\dot{x}_k)_i) \dot{\bar{t}} ds + \varepsilon_k \|\tau_k\| \end{aligned} \quad (1.12)$$

where $(\bar{t}, \bar{x}) = \bar{z}$.

Since g_{0i} ($i=0, 1, 2, 3$) are bounded, from (1.12) we easily get

$$\int_0^1 g(x_k) [\dot{z}_k, T \dot{z}_k] ds \leq 2c_1 + c_2 \|z_k\| + 2\varepsilon_k \|\tau_k\| \quad (1.13)$$

where c_2 is a positive constant depending on \bar{t} and g_{0i} ($i=0, \dots, 3$).

Now it can be easily verified that

$$g(x_k) [\dot{z}_k, T \dot{z}_k] = g_{00}(x_k) \dot{t}_k^2 - \sum_{i,j=1}^3 g_{ij}(x_k) (\dot{x}_k)_i \cdot (\dot{x}_k)_j. \quad (1.14)$$

From (1.13) and (1.14) and by using (g_2) , (g_3) we get

$$c_3 \|z_k\|^2 \leq 2c_1 + c_2 \|z_k\| + 2\varepsilon_k \|\tau_k\| \quad (1.15)$$

where c_3 is a positive constant.

From (1.15) we deduce that

$$\{z_k\} \text{ is bounded in } H^1. \quad \square$$

Proof of Theorem 1.1. – Set

$$W = \overline{\sum_{n \in \mathbb{N}} W_n}, \quad V = \overline{\sum_{n \in \mathbb{N}} V_n}$$

(the closures are taken in the H_0^1 -norm)

$$S = \bar{z} + V, \quad Q = Q(R) = \bar{z} + W \cap B_R.$$

It is easy to see that

$$f(z) \rightarrow -\infty \text{ as } \|z\| \rightarrow \infty, \quad z \in \bar{z} + W$$

and

$$\inf f(S) > -\infty.$$

Then if R is large enough we get

$$\sup f(\partial Q(R)) < \inf f(S).$$

Let $n \in \mathbb{N}$ and set

$$c_n = \inf_{h \in \mathcal{H}_n} \sup f(h(Q_n)) \tag{1.16}$$

where

$$\mathcal{H}_n = \{h : M_n \rightarrow M_n, h \text{ continuous and s. t. } h(u) = u, \forall u \in \partial Q_n\}$$

and Q_n is defined in (1.2).

By Lemma 1.2 c_n is well defined and

$$c' = \inf f(S) \leq c_n \leq \sup f(Q) = c''.$$

Moreover by lemma 1.3 $f|_{M_n}$ satisfies the Palais-Smale condition; then, by the saddle point theorem (cf. [6] or Theorem 2.3 in [2]), c_n defined by (1.16) is a critical value of $f|_{M_n}$. \square

We are now ready to prove Theorem 0.1.

Proof of Theorem 0.1. – Consider the sequence $\{z_n\}$ of the critical points of $f|_{M_n}$ found in Theorem 1.1.

The same arguments used in proving lemma 1.3 show that $\{z_n\}$ is bounded in H^1 , then there exists a subsequence, which we continue to call $\{z_n\}$ such that

$$z_n \rightarrow z^* \text{ weakly in } H^1. \tag{1.17}$$

We shall prove that

$$z_n \rightarrow z^* \text{ strongly in } H^1. \tag{1.18}$$

We set

$$z_n = \bar{z} + \zeta_n, \quad \zeta_n = (\tau_n, \xi_n)$$

$$z^* = \bar{z} + \zeta^*, \quad \zeta^* = (\tau^*, \xi^*)$$

and $(\tau_n^*, \xi_n^*) = \zeta_n^* = P_n \zeta^*$

P_n being the projection on H_n .

Since z_n are critical points of $f|_{M_n}$ we have

$$\begin{aligned} \langle f'(z_n), T(\zeta_n - \zeta_n^*) \rangle &= \int_0^1 g(x_n) [\dot{z}_n, T(\dot{\zeta}_n - \dot{\zeta}_n^*)] ds \\ &\quad - \frac{1}{2} \int_0^1 \sum_{i,j=0}^3 \sum_{l=1}^3 \frac{\partial g_{ij}}{\partial x_l}(x_n) \cdot (\xi_n - \xi_n^*)_l \cdot (\dot{z}_n)_i \cdot (\dot{z}_n)_j ds = 0 \end{aligned} \quad (1.19)$$

where T is defined in (1.10) and $(t_n, x_n) = z_n$.

H^1 is compactly embedded into L^∞ , then by (1.17), $\xi_n \rightarrow \xi^*$ in L^∞ and $\{z_n\}$ is bounded in L^∞ . Therefore

$$\begin{aligned} \frac{\partial g_{ij}}{\partial x_l}(x_n) (\xi_n - \xi_n^*)_l &\rightarrow 0 \quad \text{in } L^\infty \\ (i, j = 0, \dots, 3 \text{ and } l = 1, 2, 3) \end{aligned} \quad (1.20)$$

Then from (1.19), (1.20), (1.17) we deduce that

$$\int_0^1 g(x_n) [\dot{z}_n, T(\dot{\zeta}_n - \dot{\zeta}_n^*)] ds = O(1). \quad (1.21)$$

In (1.21) and in the sequel $O(1)$ denotes a sequence converging to zero.

Since $z_n = \bar{z} + \zeta_n$ we have

$$\int_0^1 g(x_n) [\dot{\bar{z}}, T(\dot{\zeta}_n - \dot{\zeta}_n^*)] ds + \int_0^1 g(x_n) [\dot{\zeta}_n, T(\dot{\zeta}_n - \dot{\zeta}_n^*)] ds = O(1)$$

Then, since

$$T(\dot{\zeta}_n - \dot{\zeta}_n^*) \rightarrow 0 \quad \text{weakly in } L^2 \quad (1.22)$$

and $g_{ij}(x_n) \dot{\bar{z}}_i$ converges (strongly) in L^∞ , we get

$$\int_0^1 g(x_n) [\dot{\zeta}_n, T(\dot{\zeta}_n - \dot{\zeta}_n^*)] ds = O(1) \quad (1.23)$$

which can also be written as

$$\int_0^1 g(x_n) [(\dot{\zeta}_n - \dot{\zeta}_n^*), T(\dot{\zeta}_n - \dot{\zeta}_n^*)] ds + \int_0^1 g(x_n) [\dot{\zeta}_n^*, T(\dot{\zeta}_n - \dot{\zeta}_n^*)] ds = O(1)$$

by (1.22) and since $g_{ij}(x_n) (\dot{\zeta}_n^*)_i$ converges in L^2 , we get

$$\int_0^1 g(x_n) [(\dot{\zeta}_n - \dot{\zeta}_n^*), T(\dot{\zeta}_n - \dot{\zeta}_n^*)] ds = O(1). \quad (1.24)$$

On the other hand,

$$\int_0^1 g(x_n)[(\dot{\zeta}_n - \dot{\zeta}_n^*), T(\dot{\zeta}_n - \dot{\zeta}_n^*)] ds \geq \text{const.} \|\dot{\zeta}_n - \dot{\zeta}_n^*\|_{L^2}^2. \quad (1.25)$$

From (1.24) and (1.25) and since $\zeta_n \rightarrow \zeta^*$ in H_0^1 we get

$$z_n \rightarrow z^* \quad \text{in } H^1. \quad (1.26)$$

Let us finally show that z^* is a critical point of $f|_M$. By (1.26) we have

$$\forall \zeta \in H_0^1, \quad \langle f'(z_n), \zeta \rangle \rightarrow \langle f'(z^*), \zeta \rangle \quad \text{as } n \rightarrow \infty. \quad (1.27)$$

On the other hand

$$\langle f'(z_n), \zeta \rangle = \langle f'(z_n), \zeta_n \rangle + \langle f'(z_n), \zeta - \zeta_n \rangle \quad (1.28)$$

where $\zeta_n = P_n \zeta$.

Since z_n is a critical point of $f|_{M_n}$ and $\zeta - \zeta_n \rightarrow 0$ as $n \rightarrow \infty$, from (1.28) we deduce that

$$\langle f'(z_n), \zeta \rangle = O(1). \quad (1.29)$$

Finally from (1.27) and (1.29) we deduce that

$$\forall \zeta \in H_0^1, \quad \langle f'(z^*), \zeta \rangle = 0$$

and therefore z^* is a critical point of $f|_M$. \square

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