ANNALES DE L'I. H. P., SECTION C

KLAUS ECKER
GERHARD HUISKEN

Interior curvature estimates for hypersurfaces of prescribed mean curvature

Annales de l'I. H. P., section C, tome 6, nº 4 (1989), p. 251-260
<http://www.numdam.org/item?id=AIHPC_1989__6_4_251_0>

Ann. Inst. Henri Poincaré,

Vol. 6, n° 4, 1989, p. 251-260. *Analyse non linéaire*

Interior curvature estimates for hypersurfaces of prescribed mean curvature

by

Klaus ECKER

Department of Mathematics,
University of Melbourne,
Parkville Vic. 3052, Australia

and

Gerhard HUISKEN

Department of Mathematics, R. S. Phys. S.,
Australian National University,
G.P.O. Box 4, Canberra, A.C.T. 2601, Australia

ABSTRACT. — We prove that a smooth solution of the prescribed mean curvature equation $\operatorname{div}(v^{-1}\,\mathrm{D}u)=\mathrm{H}$, $v=(1+|\mathrm{D}u|^2)^{1/2}$ satisfies an interior curvature estimate of the form

$$|\mathrm{A}|\,v(0)\leqq c\,\mathrm{R}^{-1}\sup_{\mathrm{B_R}(0)} v.$$

Key words : Mean curvature, second fundamental form, capillary surfaces.

RÉSUMÉ. — On démontre qu'une solution de l'équation $\operatorname{div}(v^{-1}\,\mathrm{D}u)=\mathrm{H}$, $v=(1+|\mathrm{D}u|^2)^{1/2}$ satisfait l'estimation intérieure pour la courbure

$$|\mathrm{A}|\,v(0)\leqq c\,\mathrm{R}^{-1}\sup_{\mathrm{B_R}(0)} v.$$

Interior gradient estimates for solutions of the prescribed mean curvature equation

$$\operatorname{div}\left(\frac{\mathrm{D}u}{\sqrt{1+|\mathrm{D}u|^2}}\right)=\mathrm{H}(x,\,u)\quad\text{in }\mathbf{R}^n \tag{1}$$

Classification A.M.S. : 53 A 10.

Annales de l'Institut Henri Poincaré - Analyse non linéaire - 0294-1449
Vol. 6/89/04/251/10/$ 3,00/© Gauthier-Villars

were derived by Finn [6] and Bombieri, De Giorgi and Miranda [1] in the case $H \equiv 0$, and by Ladyzhenskaya and Ural'tseva [11], Heinz [8], Trudinger ([16], [17]) and Korevaar [10] in the general case, assuming that $\frac{\partial H}{\partial u} \geqq 0$. The exponential form of the estimate

$$v(x_0) \leqq c_1 \exp\left(c_2 R^{-1} \sup_{B_R(x_0)} (u - u(x_0))\right), \tag{2}$$

where $x_0 \in \mathbf{R}^n$, $v = \sqrt{1 + |Du|^2}$ and c_1, c_2 depend on n, $R \sup_{B_R(x_0)} |H|$ and $R^2 \sup_{B_R(x_0)} \left|\frac{\partial H}{\partial x}\right|$ cannot be improved as was shown by Finn [6] in the case $H \equiv 0$.

From here one can obtain interior second derivative estimates by employing standard linear elliptic theory. However, the estimates thus obtained are nowhere near optimal in terms of their dependence on the gradient.

As far as we know Heinz [9] was the first to obtain interior curvature estimates for minimal hypersurfaces (not necessarily graphs) in two dimensions, which were later generalized to the case $n \leqq 5$ by Schoen, Simon and Yau [14].

Interior curvature estimates in all dimensions for solutions of (1) were recently established by Caffarelli, Nirenberg and Spruck [2]. They proved an estimate of the form

$$|A|(x_0) \leqq c(n) R^{-1} \sup_{B_R(x_0)} v^2,$$

where $|A|$ denotes the norm of the second fundamental form of $M = \operatorname{graph} u$, assuming $H = H(u) \geqq 0$, $\frac{\partial H}{\partial u} \geqq 0$ and $\frac{\partial^2 H}{\partial u^2} \geqq 0$. Their estimate holds in fact for a general class of nonlinear elliptic equations.

However, for solutions of (1) the dependence on the gradient in the above estimate can be significantly improved by exploiting the strong geometric information contained in the Codazzi equations. We prove the curvature estimate

$$(|A| v)(x_0) \leqq c R^{-1} \sup_{\mathscr{K}_R(x_0)} v \tag{3}$$

where $\mathscr{K}_R(x_0) = \{ x \in \mathbb{R}^n / |x - x_0|^2 + |u(x) - u(x_0)|^2 \leqq R^2 \}$ and the constant depends on n, H and the first two covariant derivatives of H, *see* Theorem 2.1. This estimate generalizes an estimate for minimal graphs obtained in [5], which led to a new Bernstein type result for entire minimal graphs. Another important case we have in mind are capillary surfaces

(Corollary 2.2), *i. e.* hypersurfaces satisfying

$$H(u)=\kappa u, \qquad \kappa>0.$$

In this case estimate (3) for solutions in $B_R(x_0)$, $R \leqq 1$ reduces to

$$(|A| v)(x_0) \leqq c(n, \kappa) R^{-3}, \tag{4}$$

which seems to be natural in view of Concus' and Finn's interior height estimate in [3]

$$|u|(x_0) \leqq c(n, x) R^{-1}$$

and the interior gradient estimate

$$v(x_0) \leqq c(n, \kappa) R^{-2}$$

obtained in [4].

1. PRELIMINARIES

Let M be a hypersurface in $\mathbb{R}^{n+1}$ represented as a graph over $\mathbb{R}^n$ with position vector $\mathbf{x}$ and upward unit normal ν. We define the height of M by

$$u=\langle \mathbf{x}, \mathbf{e}_{n+1} \rangle$$

and the gradient function by

$$v=\langle \nu, \mathbf{e}_{n+1} \rangle^{-1},$$

where $\mathbf{e}_{n+1}$ denotes the $(n+1)$st coordinate vector in $\mathbb{R}^{n+1}$. Note that $\mathbf{x}(x)=(x, u(x))$, $v(x)=\sqrt{1+|Du(x)|^2}$ and $\nu(x)=v^{-1}(x).(-Du(x), 1)$ for $x \in \mathbb{R}^n$. The second fundamental form of M is given by

$$h_{ij}=\langle \nabla_{\tau_i} \tau_j, \nu \rangle = v \nabla_i \nabla_j u, \qquad 1 \leqq i, j \leqq n, \tag{5}$$

where ∇ denotes covariant differentiation in M and $\{\tau_i\}_{1 \leqq i \leqq n}$ is an orthonormal frame for M.

It is well-known that the gradient function then satisfies the equation

$$\Delta v=|A|^2 v+2 v^{-1} |\nabla v|^2+v^2 \langle \nabla H, \mathbf{e}_{n+1} \rangle, \tag{6}$$

where Δ, $|A|$ and H denote Laplace-Beltrami operator, norm of the second fundamental form mean curvature of M respectively.

The following lemma gives a generalization of inequality (1.34) in [14]:

1.1. Lemma

$$\Delta |A|^2 \geqq 2 h_{ij} \nabla_i \nabla_j H - 2|A|^4 + 2 H h_{ij} h_{ik} h_{jk} + 2\left(1+\frac{2}{n+1}\right) |\nabla |A||^2 - c(n) |\nabla H|^2. \tag{7}$$

Proof. — From (1.20), (1.29)-(1.31) in [14] we infer the relations

$$h_{ij}\Delta h_{ij}=h_{ij}\nabla_i\nabla_j \mathrm{H}-|\mathrm{A}|^4+\mathrm{H}\,h_{ij}h_{ik}h_{jk},$$

$$|\nabla \mathrm{A}|^2-|\nabla|\mathrm{A}||^2\geqq 2\sum_{\substack{i,j\\ i\neq j}}^{n} h_{iij}^2$$

and

$$|\nabla|\mathrm{A}||^2\leqq \sum_{\substack{i,k\\ i\neq k}}^{n} h_{iik}^2+\sum_{i=1}^{n} h_{iii}^2,$$

where the totally symmetric tensor $\nabla \mathrm{A}$ is given by $(h_{ijk})_{1\leqq i,j,k\leqq n}$.

Since for fixed i we have

$$h_{iii}^2=\Big(\sum_{\substack{j=1\\ j\neq i}}^{n} h_{jji}\Big)^2+(\nabla_i \mathrm{H})^2-2\sum_{\substack{j=1\\ j\neq i}}^{n} h_{jji}\nabla_i \mathrm{H},$$

the result follows in view of Young's inequality.

1.2. *Remark.* — It is worth noting that for $n=2$ the inequality

$$\mathrm{H}\,h_{ij}h_{ik}h_{jk}\geqq \frac{1}{2}\mathrm{H}^2|\mathrm{A}|^2 \tag{8}$$

holds.

The next lemma was proved in the case $\mathrm{H}=0$ in [5].

1.3. LEMMA. — *For $p,\ q\geqq 2$ we have the inequality*

$$\begin{aligned}\Delta|\mathrm{A}|^p v^q\geqq{}&(q-p)|\mathrm{A}|^{p+2}v^q\\ &+q\left[q+1-\frac{pq}{p-1+(2/(n+1))}\right]v^{-2}|\nabla v|^2|\mathrm{A}|^p v^q\\ &+p\left(\mathrm{H}\,h_{ij}h_{ik}h_{jk}-\frac{c(n)}{2}|\nabla \mathrm{H}|^2+h_{ij}\nabla_i\nabla_j\mathrm{H}\right)|\mathrm{A}|^{p-2}v^q\\ &+qv\langle\nabla \mathrm{H},\,\mathbf{e}_{n+1}\rangle|\mathrm{A}|^p v^q.\end{aligned} \tag{9}$$

Proof. — Combining (6) and (7) we infer

$$\begin{aligned}\Delta|\mathrm{A}|^p v^q\geqq{}&(q-p)|\mathrm{A}|^{p+2}v^q+p\left(p-1+\frac{2}{n+1}\right)|\mathrm{A}|^{-2}|\nabla|\mathrm{A}||^2|\mathrm{A}|^p v^q\\ &+p\left(\mathrm{H}\,h_{ij}h_{ik}h_{jk}-\frac{c(n)}{2}|\nabla \mathrm{H}|^2+h_{ij}\nabla_i\nabla_j\mathrm{H}\right)|\mathrm{A}|^{p-2}v^q\\ &+q(q+1)v^{-2}|\nabla v|^2|\mathrm{A}|^p v^q+2pq|\mathrm{A}|^{-1}v^{-1}\nabla|\mathrm{A}|\cdot\nabla v|\mathrm{A}|^p v^q\\ &+qv\langle\nabla \mathrm{H},\,\mathbf{e}_{n+1}\rangle|\mathrm{A}|^p v^q.\end{aligned}$$

Inequality (9) then follows from Young's inequality.

1.4. *Remark.* — By choosing p, q s. t. $p \leqq q \leqq \left(\frac{n+1}{n-1}\right)(p-1)-1$ we can achieve

$$q\left[q+1-\frac{pq}{p-1+(2/(n+1))}\right] \geqq 1. \tag{10}$$

2. THE MAIN RESULT

Let the quantities H_1 and H_2 be defined by

$$H_1 = -(v\langle \nabla H, e_{n+1}\rangle)^-$$

$$H_2 = \begin{cases} -\left(\dfrac{h_{ij}}{|A|}\nabla_i \nabla_j H\right)^-, & \text{if } |A| \neq 0 \\ 0, & \text{if } |A| = 0 \end{cases}$$

where f^- denotes the negative part of a function f. Then we have the following interior curvature estimates:

2.1. THEOREM. — *Let* $M = \text{graph } u$ *be a hypersurface in* $\mathbb{R}^{n+1}$ *defined over the ball* $B_R(x_0) \subset \mathbb{R}^n$. *Then the estimate*

$$(|A|v)(x_0) \leqq c R^{-1} \sup_{\mathscr{K}_R(x_0)} v \tag{11}$$

holds, where $\mathscr{K}_R(x_0) = \{x \in \mathbb{R}^n / |x - x_0|^2 + |u(x) - u(x_0)|^2 \leqq R^2\}$ *and the constant depends on* n, $R \sup_{B_R(x_0)} |H|$, $R^2 \sup_{B_R(x_0)} (|\nabla H| + H_1)$ *and* $R^3 \sup_{B_R(x_0)} H_2$.

In the special case where M is a capillary surface, this leads to

2.2. COROLLARY (Capillary surfaces). — *Let* $M = \text{graph}\, u$ *be a hypersurface in* $\mathbb{R}^{n+1}$ *satisfying* $H(u) = \kappa u$ for $\kappa > 0$ *in the ball* $B_{2R}(x_0) \subset \mathbb{R}^n$. *Then for* $R \leqq 1$ *we have the estimate*

$$(|A|v)(x_0) \leqq c(n, \kappa) R^{-3}. \tag{12}$$

Proof. — Note that in this case we have the identities

$$\langle \nabla H, e_{n+1}\rangle = \kappa v^{-2} |Du|^2$$
$$\nabla_i \nabla_j H = \kappa v^{-1} h_{ij}$$

by virtue of (5). Since $\kappa > 0$ this implies $H_1 \equiv H_2 \equiv 0$ and as $|\nabla u| \leqq 1$ we also have

$$|\nabla H| \leqq \kappa.$$

The result now follows in view of Concus' and Finn's estimate [3]

$$\sup_{B_R(x_0)} |u| \leqq c(n, \kappa) R^{-1}, \qquad R \leqq 1 \tag{13}$$

and the interior gradient estimate [4]

$$\sup_{\mathscr{K}_R(x_0)} v \leqq c(n, \kappa) R^{-2}, \qquad R \leqq 1. \tag{14}$$

2.3. *Remark.* — (i) (11) generalizes an estimate for minimal graphs proved in [5].

(ii) Estimate (12) with a constant depending on n, κ, $R \sup_{B_R(x_0)} |H|$, $R^2 \sup_{B_R(x_0)} \left|\frac{\partial H}{\partial u}\right|$ and $R^3 \sup_{B_R(x_0)} \left|\frac{\partial^2 H}{\partial u^2}\right|$ is in fact valid for any hypersurface satisfying $H = H(u)$ with $\frac{\partial H}{\partial u} \geqq \kappa > 0$, as (13) and (14) continue to hold in this case.

If $H = H(u)$ satisfies $\frac{\partial H}{\partial u} \geqq 0$, i. e. in the case where the interior gradient estimate (21) holds, the constant in (1) depends on n, $R \sup_{B_R(x_0)} |H|$, $R^2 \sup_{B_R(x_0)} \left|\frac{\partial H}{\partial u}\right|$ and $R^3 \sup_{B_R(x_0)} \left|\frac{\partial^2 H}{\partial u^2}\right|$ in view of (5).

(iii) If H does not change sign we have

$$\int_{B_R(x_0)} |H| = \left| \int_{B_R(x_0)} H \right| \leqq n \omega_n R^{n-1}$$

by the divergence theorem, which enables us to estimate the n-dimensional Hausdorff measure of $\mathscr{K}_R(x_0)$ by

$$\mathscr{H}^n(\mathscr{K}_R(x_0)) \leqq c(n) R^n$$

as in [7].

In the two dimensional case this and Remark 1.2 are sufficient to eradicate the $R \sup_{B_R(x_0)} |H|$-dependence of the constant in (11), as can be seen from the proof of Theorem 2.1. This implies in particular

$$(|A| v)(x_0) \leqq c R^{-1} \sup_{B_R(x_0)} v$$

with c independent of R in the case $H \equiv \text{const.}$, $n = 2$.

(iv) The constant in (11) is given by

$$c(n, \delta)\Big(1 + R \sup_{B_R(x_0)} |H| + R^2 \sup_{B_R(x_0)} (|\nabla H| + H_1) + R^3 \sup_{B_R(x_0)} H_2\Big)^{1+\delta}$$

for any $\delta > 0$.

In order to prove Theorem 2.1 we have to establish an L^p-estimate for the curvatures.

2.4. LEMMA. — *If* $p \geqq \max(3, n)$ *the estimate*

$$\int_{\mathscr{K}_{R/2}(x_0)} |A|^{2p} v^{2p} \, d\mathscr{H}^n \leqq c \, R^{n-2p} \sup_{\mathscr{K}_R(x_0)} v^{2p} \tag{15}$$

holds, where c depends on p, n, $R \sup_{B_R(x_0)} |H|$, $R^2 \sup_{B_R(x_0)} (|\nabla H| + H_1)$ *and* $R^3 \sup_{B_R(x_0)} H_2$.

Proof. — If $p \geqq \max(3, n)$ inequality (9) reduces to

$$\begin{aligned}\Delta |A|^{p-1} v^p \geqq |A|^{p+1} v^p + pv \langle \nabla H, \mathbf{e}_{n+1} \rangle |A|^{p-1} v^p \\ + (p-1)\left(H h_{ij} h_{ik} h_{jk} - \frac{c(n)}{2} |\nabla H|^2 + h_{ij} \nabla_i \nabla_j H\right) |A|^{p-3} v^p.\end{aligned}$$

Using Young's inequality and the definition of H_1 and H_2 we infer

$$\begin{aligned}\Delta |A|^{p-1} v^p \geqq \frac{1}{2} |A|^{p+1} v^p - cp^2 H^2 |A|^{p-1} v^p \\ - cp(|\nabla H|^2 |A|^{p-3} v^p + H_1 |A|^{p-1} v^p + H_2 |A|^{p-2} v^p).\end{aligned}$$

Multiplying by $|A|^{p-1} v^p \eta^{2p}$, where η is a test-function with compact support, and integrating by parts we obtain in view of Young's inequality

$$\begin{aligned}\int_M |A|^{2p} v^{2p} \eta^{2p} \, d\mathscr{H}^n \\ \leqq c(p)\Bigg(\Big(\int_M |A|^{2(p-1)} v^{2p} (\eta^{2(p-1)} |\nabla\eta|^2 + (H^2 + H_1)\eta^{2p}) d\mathscr{H}^n \\ + \int_M |A|^{2(p-2)} v^{2p} |\nabla H|^2 \eta^{2p} \, d\mathscr{H}^n + \int_M |A|^{2p-3} v^{2p} H_2 \eta^{2p} \, d\mathscr{H}^n\Bigg).\end{aligned}$$

By means of the inequality $ab \leqq \varepsilon a^{p/p-1} + \varepsilon^{1-p} b^p$ we arrive at

$$\begin{aligned}\int_M |A|^{2p} v^{2p} \eta^{2p} \, d\mathscr{H}^n \leqq c(p) \int_M v^{2p} (|\nabla\eta|^{2p} \\ + (H^{2p} + |\nabla H|^p + H_1^p + H_2^{2p/3}) \eta^{2p}) \, d\mathscr{H}^n.\end{aligned}$$

The result now follows by letting η be the standard cut-off function for $\mathscr{K}_{R/2}(x_0)$ and recalling the estimate [7]

$$\mathscr{H}^n(\mathscr{K}_R(x_0)) \leqq c(n) R^n (1 + R \sup_{B_R(x_0)} |H|). \tag{16}$$

Proof of Theorem 2.1. — Let $\beta = q-p>0$. Then for $p \geqq \max\left(3, n+\dfrac{\beta(n-1)}{2}\right)$ the conditions of Remark 1.4 and of Lemma 2.4 are satisfied.

Define $f=|A|^p v^p$. From (9), (10) and

$$H\, h_{ij} h_{ik} h_{jk} \leqq \beta |A|^4 + \frac{c(n)}{4\beta} H^2 |A|^2$$

we then derive

$$\begin{aligned}\Delta f v^\beta \geqq\, & v^{-2} |\nabla v|^2 f v^\beta \\ & - cp^2 (H^2 f + |\nabla H|^2 |A|^{p-2} v^p + H_2 |A|^{p-1} v^p) v^\beta \\ & - (p+\beta) H_1 f v^\beta.\end{aligned}$$

Let us multiply this inequality by $f v^{-\beta} \eta^2$, where η is a test function with compact support and integrate by parts. Note that

$$\nabla(f v^\beta)\cdot\nabla(f v^{-\beta}) = |\nabla f|^2 - \beta^2 v^{-2} |\nabla v|^2 f^2$$

and

$$|2\eta f v^{-\beta} \nabla\eta\cdot\nabla(f v^\beta)| \leqq \beta^2 v^{-2} |\nabla v|^2 f^2 \eta^2 + \frac{1}{2} |\nabla f|^2 \eta^2 + 3 f^2 |\nabla\eta|^2.$$

In order to control the $|\nabla v|^2$-term we choose β s.t. $2\beta^2 \leqq 1$. Furthermore we employ Young's inequality in the form

$$\begin{aligned}|\nabla H|^2 |A|^{2(p-1)} v^{2p} &\leqq \varepsilon_1 |\nabla H|^2 f^2 + \varepsilon_1^{1-p} |\nabla H|^2 v^{2p} \\ H_2 |A|^{2p-1} v^{2p} &\leqq \varepsilon_2 H_2 f^2 + \varepsilon_2^{1-2p} H_2 v^{2p}.\end{aligned}$$

Altogether we obtain

$$\begin{aligned}\int_M |\nabla f|^2 \eta^2\, d\mathscr{H}^n \leqq\, & cp^2 \int_M f^2 [|\nabla\eta|^2 \\ & + (H^2 + H_1 + \varepsilon_1 |\nabla H|^2 + \varepsilon_2 H_2)\eta^2]\, d\mathscr{H}^n \\ & + cp^2 \int_M (\varepsilon_1^{1-p} |\nabla H|^2 + \varepsilon_2^{1-2p} H_2) v^{2p} \eta^2\, d\mathscr{H}^n. \quad (17)\end{aligned}$$

We now apply the Sobolev inequality [12] in the following way

$$\begin{aligned}\left(\int_M (f^2\eta^2)^{n/n-1}\, d\mathscr{H}^n\right)^{n-1/n} & \leqq c(n) \int_M (|\nabla(f^2\eta^2)| + |H| f^2 \eta^2)\, d\mathscr{H}^n \\ \leqq \varepsilon_0 \Bigg(\int_M |\nabla f|^2 \eta^2\, d\mathscr{H}^n & + \int_M f^2 |\nabla\eta|^2\, d\mathscr{H}^n + \int_M H^2 f^2 \eta^2\, d\mathscr{H}^n\Bigg) \\ & + c\,\varepsilon_0^{-1} \int_M f^2 \eta^2\, d\mathscr{H}^n. \quad (18)\end{aligned}$$

Let η be the cut-off function defined by $\eta \equiv 1$ on $\mathscr{K}_{\rho-\sigma} = \mathscr{K}_{\rho-\sigma}(x_0)$, $\eta \equiv 0$ in $M \sim \mathscr{K}$, $\mathscr{K}_\rho = \mathscr{K}_\rho(x_0)$, $|\nabla\eta| \leqq c\,\sigma^{-1}$, where $0<\sigma<\rho<R$ and set $\varepsilon_0 = \sigma$.

$\varepsilon_2 = R$ and $\varepsilon_1 = R^2$. Combining (17) and (18) we then obtain with $\lambda = n/n-1$

$$\left(\int_{\mathscr{X}_{\rho-\sigma}} f^{2\lambda}\, d\mathscr{H}^n\right)^{1/\lambda} \leqq cp^2\sigma^{-1}\left(\int_{\mathscr{X}_\rho} f^2\, d\mathscr{H}^n + R^{-2p}\int_{\mathscr{X}_\rho} v^{2p}\, d\mathscr{H}^n\right), \quad (19)$$

where c depends on n, $R \sup_{B_R(x_0)} |H|$, $R^2 \sup_{B_R(x_0)} (|\nabla H| + H_1)$ and $R^3 \sup_{B_R(x_0)} H_2$.

We intend to employ an iteration scheme due to Moser [13] in a similar way as in [15] or [4]. To this end let $\gamma \geqq \max\left(3,\, n + \beta\dfrac{n-1}{2}\right)$ be fixed and set $\alpha = \lambda^{r-1}$ for $r \geqq 1$ and $p = \alpha \cdot \gamma$ such that $f^2 = (|A|v)^{2\gamma\alpha}$. We then define

$$g = (|A|v)^{2\gamma}.$$

Now let $\rho_0 = R/2$, $\sigma_r = R/2^{r+2}$, $\rho_{r+1} = \rho_r - \sigma_{r+1}$ and replace ρ by ρ_{r-1} and σ by σ_r for $r \geqq 1$ in (19). In view of the area-estimate (16) this leads to

$$\int_{\mathscr{X}_{\rho_r}} g^{\lambda^r}\, d\mathscr{H}^n \leqq c^r R^{-\lambda}\left(\int_{\mathscr{X}_{\rho_{r-1}}} g^{\lambda^{r-1}}\, d\mathscr{H}^n + R^n k^{\lambda^{r-1}}\right)^{\lambda}, \quad (20)$$

where $k = \left(R^{-1} \sup_{\mathscr{X}_R(x_0)} v\right)^{2\gamma}$ and c depends on n, $R \sup_{B_R(x_0)} |H|$, $R^2 \sup_{B_R(x_0)} (|\nabla H| + H_1)$, $R^2 \sup_{B_R(x_0)} (|\nabla H| + H_1)$ and $R^3 \sup_{B_R(x_0)} H_2$.

For $r \geqq 0$ we define

$$I_r = \left(R^{-n}\int_{\mathscr{X}_{\rho_r}} g^{\lambda^r}\, d\mathscr{H}^n + k^{\lambda^r}\right)^{\lambda^{-r}}.$$

We now multiply (20) by R^{-n}, raise it to the power λ^{-r} and use the inequality $(a+b)^m \leqq 2^m(a^m + b^m)$ and the definition of λ to conclude for $r \geqq 1$

$$I_r \leqq c^{r\lambda^{-r}} I_{r-1} \quad (21)$$

with a fixed, but slightly larger constant c. Iterating (21) we arrive at

$$I_r \leqq c^{\sum_{s=1}^{r} s\lambda^{-s}} I_0 \quad (22)$$

for $r \geqq 0$. By letting $r \to \infty$ we finally obtain

$$\sup_{\mathscr{X}_{R/4}(x_0)} (|A|v)^{2\gamma} \leqq c\left(R^{-n}\int_{\mathscr{X}_{R/2}(x_0)} (|A|v)^{2\gamma}\, d\mathscr{H}^n + R^{-2\gamma} \sup_{\mathscr{X}_R(x_0)} v^{2\gamma}\right),$$

where c depends on the quantities listed above. Since $\gamma \geqq \max(3, n)$ we may apply Lemma 2.4 with $p = \gamma$ to complete the proof.

Remark (2.3) follows by choosing γ large enough depending on δ.

REFERENCES

[1] E. BOMBIERI, E. DE GIORGI and M. MIRANDA, Una maggiorazione apriori relative alle ipersuperfici minimali non parametriche, *Arch. Rat. Mech. Anal.*, Vol. **32**, 1969, pp. 255-269.

[2] L. CAFFARELLI, L. NIRENBERG and J. SPRUCK, *On a Form of Bernstein's Theorem*, preprint (to appear).

[3] P. CONCUS and R. FINN, On Capillary Free Surfaces in a Gravitational Field, *Acta Math.*, Vol. **132**, 1974, pp. 207-223.

[4] K. ECKER, An Interior Gradient Bound for Solutions of Equations of Capillary Type, *Arch. Rat. Mech. Anal.*, Vol. **92**, 1986, pp. 137-151.

[5] K. ECKER and G. HUISKEN, A Bernstein Result for Minimal Graphs of Controlled Growth, *J. Diff. Geom.* (to appear).

[6] R. FINN, On Equations of Minimal Surface Type, *Ann. of Math.* (2), Vol. **60**, 1954, pp. 397-415.

[7] D. GILBARG and N. S. TRUDINGER, *Elliptic Partial Differential Equations of Second Order* (2nd edition), Springer-Verlag, Berlin-Heidelberg, New York, 1983.

[8] E. HEINZ, Über die Lösungen der Minimalflächengleichung, *Nachr. Akad. Wiss. Göttingen Math. Phys. Kl*, Vol. **II**, 1952, pp. 51-56.

[9] E. HEINZ, Interior Gradient Estimates for Surfaces $z=F(x, y)$ with Prescribed Mean Curvature, *J. Diff. Geom.*, Vol. **5**, 1971, pp. 149-157.

[10] N. KOREVAAR, An Easy Proof of the Interior Gradient Bound for Solutions to the Prescribed Mean Curvature Equation, *Proc. Symp. Pure Math.*, Vol. **45**, 1986, Part 2.

[11] O. A. LADYZHENSKAYA and N. N. URAL'TSEVA, Local Estimates for Gradients of Solutions of Nonuniformly Elliptic and Parabolic Equations, *Comm. Pure Appl. Math.*, Vol. **23**, 1970, pp. 677-703.

[12] J. MICHAEL and L. SIMON, Sobolev and Mean Value Inequalities on Generalized Submanifolds of $\mathbf{R}^n$, *Comm. Pure Appl. Math.*, Vol. **26**, 1973, pp. 361-379.

[13] J. MOSER, A New Proof of De Giorgi's Theorem Concerning the Regularity Problem for Elliptic Differential Equations, *Comm. Pure Appl. Math.*, Vol. **13**, 1960, pp. 457-468.

[14] R. SCHOEN, L. SIMON and S. T. YAU, Curvature Estimates for Minimal Hypersurfaces, *Acta Math.*, Vol. **134**, 1975, pp. 275-288.

[15] L. SIMON and J. SPRUCK, Existence and Regularity of a Capillary Surface with Prescribed Contact Angle, *Arch. Rat. Mech. Anal.*, Vol. **61**, 1976, pp. 19-34.

[16] N. S. TRUDINGER, A New Proof of the Interior Gradient Bound for the Minimal Surface Equation in n-Dimensions, *Proc. Nat. Acad. Sci. U.S.A.*, Vol. **B69B**, 1972, pp. 821-823.

[17] N. S. TRUDINGER, Gradient Estimates and Mean Curvature, *Math. Z.*, Vol. **131**, 1973, pp. 165-175.

[18] J. C. C. NITSCHE, *Lectures on Minimal Surfaces*, Vol. **I** (to appear).

(*Manuscrit reçu le* .)