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On the local solution of the tangential
Cauchy-Riemann equations (*)

by

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ABSTRACT. — We study the solution operators $P$ and homotopy formula introduced by G. M. Henkin for the tangential Cauchy-Riemann complex of a suitable small domain $D$ on a strictly pseudoconvex real hypersurface in complex $n$-space. The main difficulties stem from the fact that $P$ is an integral operator with a rather complicated kernel. For $U \subset D$, we derive a $C^k$-norm estimate of the form $\|P\phi\|_{U, k} \leq K \|\phi\|_{D, k'}$, where the constant $K$ blows up as $U$ increases to $D$. We obtain careful control of the rate of this blow-up and of the dependence of $K$ on the derivatives of the function defining the real hypersurface. Our estimates are sufficient for application to the local CR embedding problem.

RÉSUMÉ. — Nous étudions les opérateurs intégraux $P$ dans la formule d'homotopie de G. M. Henkin pour le complexe tangentiel de Cauchy-Riemann sur un petit domaine d'une hypersurface réelle strictement pseudo-convexe dans l'espace $\mathbb{C}^n$. Avec les $C^k$-normes pour les domaines $U \subset D$ nous dérivons une borne, $|P\phi|_{U, k} \leq K |\phi|_{D, k'}$, dans laquelle le constant $K$ tend vers $+\infty$ lorsque $U$ tend vers $D$. Nous constatons cette croissance de $K$ et la dépendance de $K$ sur les dérivées de la fonction qui définit l'hypersurface.

Mots clés : Hypersurface réelle, complexe tangentiel, noyau intégral.

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This paper is concerned with the $\overline{\partial}_{b}$ or tangential Cauchy-Riemann, complex on a small portion $M_{r}$ of a strictly pseudo-convex real hypersurface $M^{n-1}$ in complex space $\mathbb{C}^{n}$. Under suitable restrictions on $M_{r}$, there exist solution operators $P$ and $Q$ satisfying the homotopy formula

$$\varphi = \overline{\partial}_{b} P\varphi + Q \overline{\partial}_{b} \varphi,$$

for $(0, s)$-forms $\varphi$, $1 \leq s \leq n-3$, restricted to $M_{r}$. We shall study the regularity properties of certain of these operators.

Various aspects of the equation (0.1) have been studied by a number of people since the early works of H. Lewy [6], Kohn-Rossi [5], and Andreotti-Hill [1]. We should mention the works of Henkin [4], Romanov [8], and Skoda [9], in particular. We shall work with the explicit operators constructed by Henkin in [4], in the formulation given by Harvey and Polking [3]. As shown in [4] (0.1) holds on the compact manifold-with-boundary

$$M_{r} = \{ z \in M : r^{0}(z) < \rho \},$$

where $r^{0}$ is a (suitable) pluriharmonic function. In addition to the results of [4] and [3], the higher differentiability properties of similar such $P$ and $Q$ were studied by Boggess [2].

For $M$ as above of differentiability class $C^{l}$, we take $M_{r}$ as in (0.2) with $r^{0}$ a real function of one of the holomorphic coordinates, both suitably chosen. Our results yield estimates of the form

$$\| P\varphi \|_{p(1-\sigma), k} \leq c_{k} \delta^{-s} \| \varphi \|_{p, k}.$$

Here, $0 < \sigma < 1$, $\delta = \text{dist}(\partial M_{r}, \partial M_{p})$, $0 \leq k \leq l-3$, $s = s(k) > 0$, and $\| \varphi \|_{p, k}$ is the usual sup norm, taken over $M_{r}$ of the derivatives up to order $k$ of the coefficients of the form $\varphi$. The same estimate holds for $Q$. A much more precise result is stated in theorem (4.1) below.

The formula (0.1) and the estimates (0.3) for $s=1$ form a major element of our proof [11] of the local embedding theorem for formally integrable, strictly pseudoconvex CR structures of dimension $2n-1$. The restriction $1 \leq s \leq n-3$ limits it to $2n-1$ $\geq 7$. To be sure there is a "weak" homotopy formula (0.1) for the case $1 \leq s = n-2$, as we shall indicate below. However, in this degree the operator $Q$ inherits an additional term for which we have no estimate. The argument of [11] is based on the methods of Nash and Moser, with (0.1), (0.3) being used in solving the "linearized problem".

Hopefully, our estimates in theorem (4.1) will eventually be improved. This would probably decrease the derivative loss in the main result of [11]. For $k=0$, Henkin [4] has obtained (0.3) with $s=0$. For $k>0$, it seems difficult to avoid $s>0$. Major difficulties stem from the boundary...
integrals occurring in \( P \) (and \( Q \)). Estimates similar to (0.3) for the \( \overline{\partial} \)-complex were used in [10] to give a proof of the sharp form of the Newlander-Nirenberg theorem. The paper [10] may serve as a useful introduction to the methods of the present work and of [11].

In section 1 we recall the construction of Henkin's \( \overline{\partial}_b \)-homotopy formula. We take the first derivatives of \( P \varphi \) in sections 2 and 3, and estimate the higher derivatives in section 4.

1. THE HENKIN \( \overline{\partial}_b \)-HOMOTOPY FORMULA

We begin by sketching the particular results needed from [4], making use of the exterior calculus developed in [3]. Let \( w \in \mathbb{C}^n, \xi \in \mathbb{C}^m \), and \( w = g(\xi) \) be a sufficiently smooth map. Using a dot product notation, we define a (1,0)-form

\[
\omega = \omega^\varphi = \frac{g \cdot dw}{g \cdot w}, \quad g \cdot dw = \sum_{j=1}^{n} g_j dw_j, \quad \text{etc.},
\]

on the set of \((\xi, w) \in \mathbb{C}^m \times \mathbb{C}^n\) for which \( g \cdot w \neq 0 \). This may be considered a generalization of the Cauchy kernel, since

\[
\omega^\varphi = \frac{dw}{w_n}, \quad \text{if} \quad g_\alpha = 0, \quad 1 \leq \alpha \leq n-1, \quad g_n \neq 0.
\]

Given \( l \) such maps, \( g^{(j)}, 1 \leq j \leq l \), from \( \mathbb{C}^m \) to \( \mathbb{C}^n \) and the corresponding (1,0)-forms \( \omega^j \), we define, on the set where all denominators are non-zero, the \((n, n-l)\)-form

\[
\Omega^1 \cdots l = \omega^1 \wedge \ldots \wedge \omega^l \wedge \sum (\overline{\partial} \omega^1)^{\alpha_1} \wedge \ldots \wedge (\overline{\partial} \omega^l)^{\alpha_l}.
\]

Here the sum is over all \( l \)-tuples of non-negative integers \( (\alpha_1, \ldots, \alpha_l) \) for which \( \alpha_1 + \ldots + \alpha_l = n-l \).

We introduce the vector field \( v \),

\[
v = w \cdot \overline{\partial}_w = \sum_{j=1}^{n} w_j \frac{\partial}{\partial w_j},
\]

and the interior product \( t_v \). One readily verifies

\[
t_v \omega^\varphi = 1, \quad t_v (\overline{\partial} \omega^\varphi) = 0.
\]

If \( \beta_1 + \ldots + \beta_l = n - l + 1, \beta_j \geq 0 \), then

\[
0 = \omega^1 \wedge \ldots \wedge \omega^l \wedge (\overline{\partial} \omega^1)^{\beta_1} \wedge \ldots \wedge (\overline{\partial} \omega^l)^{\beta_l},
\]

since each term has a wedge product of \( n+1 \) of the differentials \( dw_j \), \( 1 \leq j \leq n \). If we take the interior product of equation (1.6) with (1.4) and...
use (1.5), we get

\[ 0 = \sum_{j=1}^{l} (-1)^{j+1} \omega^1 \wedge \ldots \wedge \omega^j \wedge \ldots \wedge \omega^l \wedge (\partial \omega^1)^{\beta_1} \wedge \ldots \wedge (\partial \omega^l)^{\beta_l}. \] (1.7)

This formula is used to derive the generalized Koppelman lemma,

\[ \partial \Omega^1 \ldots l = \sum_{j=1}^{l} (-1)^{j} \Omega^1 \ldots j \ldots l. \] (1.8)

For this we write

\[ \partial \Omega^1 \ldots l = \sum_{j=1}^{l} (-1)^{j+1} \omega^1 \wedge \ldots \wedge \omega^j \wedge \ldots \wedge \omega^l \wedge \Sigma \]

\[ = \sum_{j=1}^{l} (-1)^{j+1} \omega^1 \wedge \ldots \wedge \omega^j \wedge \ldots \wedge \omega^l \wedge \{ \Sigma_{(j, 1)} + \Sigma_{(j, 0)} \}
\]

\[ + \sum_{j=1}^{l} (-1)^{j} \omega^1 \wedge \ldots \wedge \omega^j \wedge \ldots \wedge \omega^l \wedge \Sigma_{(j, 0)}, \]

Here \( \Sigma \) denotes the sum in (1.3), \( \Sigma_{(j, 1)} \) denotes the similar sum with \( \alpha_1 + \ldots + \alpha_l = n-l+1, \quad \alpha_j \geq 1, \) and \( \Sigma_{(j, 0)} \) the sum with \( \alpha_1 + \ldots + \alpha_l = n-l+1, \quad \alpha_j = 0. \) The expression \( \Sigma_{(j, 1)} + \Sigma_{(j, 0)} \) is independent of \( j, \) so the first alternating sum vanishes by (1.7). The second alternating sum is precisely the right hand side of (1.8). Only the cases

\[ \partial \Omega^1 = 0, \] (1.9)

\[ \partial \Omega^{12} = \Omega^1 - \Omega^2, \] (1.10)

\[ \partial \Omega^{123} = -\Omega^{23} + \Omega^{31} - \Omega^{12}, \] (1.11)

are used for our construction. For this we take \( \xi = (\zeta, z), \quad \zeta, \ z \in \mathbb{C}^r, \) and make the substitution \( w = \zeta - z. \) Decomposition according to \( z \)-type gives

\[ \Omega^1 \ldots l (\zeta, z) = \sum_{i=0}^{n} \sum_{s=0}^{n-i} \Omega_{i, s} \ldots l (\zeta, z), \] (1.12)

where the subscript \( (i, s) \) indicates that the “double” form is of type \( (i, s) \) in \( z \) and type \( (n-i, n-l-s) \) in \( \zeta. \)

We shall work with a real hypersurface \( M \) which is a graph over the \( (z_\alpha, x_\alpha) \)-coordinate hyperplane \( y_\alpha = 0, \ z_\alpha = x_\alpha + iy_\alpha. \) We assume that the defining function \( r \) is at least three times continuously differentiable, and

\[ M: r = 0, \quad R = O (|z|^3), \]

\[ r (z) = -y_\alpha + \sum_{\alpha, \beta = 1}^{n} g_{\alpha \beta} z_\alpha \overline{z}_\beta + R (z_\alpha, x_\alpha). \] (1.13)
Here, the hermitian matrix $g_{a\bar{b}}$, the Levi form of $M$ at 0, is assumed to be a small perturbation of the identity matrix $\delta_{a\bar{b}}$. We define

$$M_p = \{ z \in M : r^0 (z_n) < \rho \},$$

where $r^0$ is a sufficiently smooth real valued function of the last holomorphic coordinate $z_n$ only. This is a slight departure from [4], where $r^0$ is assumed to be pluri-harmonic. In either case a most natural choice would be $r^0 = \text{Re} \log z_n$, for a suitable branch of log, so that $M_p = \{ z \in M : |z_n| < \rho \}$. (A different choice of $r^0$ turns out to be more appropriate in [11].) We further define

$$g^+ (\zeta, z) = g^+ (\zeta, z) = r_z = (r_{z_1} (\zeta), \ldots, r_{z_n} (\zeta)),
\quad g^- (\zeta, z) = g^- (z) = (r_{z_1} (z), \ldots, r_{z_n} (z)),
\quad g^0 (\zeta, z) = g^0 (\zeta) = r^0 = (0, \ldots, 0, r^0 (\zeta)),
$$

and denote the corresponding forms $\omega$ by $\omega^+$, $\omega^-$, and $\omega^0$. For $\zeta, z \in M_p$ and $w = \zeta - z$, one shows (e.g. see sec. 4 of [11]) that $g^+ \cdot w$ and $g^- \cdot w$ vanish only for $\zeta = z$, if $\rho$ is sufficiently small. (We assume $M_p$ shrinks to 0 as $\rho \to 0$.)

From (1.10), (1.12), and the decomposition $\bar{\partial} = \bar{\partial}_t + \bar{\partial}_z$, we get

$$\bar{\partial}_t \Omega^0_{0,s} + \bar{\partial}_z \Omega^+_{0,s-1} = \Omega^+_{0,s} - \Omega^-_{0,s}.
$$

Since $\omega^+$ is holomorphic in $z$, $\Omega^+$ contains no differentials $d\bar{z}_j$. Hence, $\Omega^+_0 = 0$ for $s \geq 1$. Since $\omega^-$ is holomorphic in $\zeta$, $\Omega^-_{0,s} = 0$ for $n - 1 - s \geq 1$, or $s \leq n - 2$. Thus,

$$\bar{\partial}_t \Omega^+_{0,s} + \bar{\partial}_z \Omega^+_{0,s-1} = 0, \quad 1 \leq s \leq n - 2.
$$

For a form $\phi (\zeta)$ of type $(0, s)$ in $\zeta$, $1 \leq s \leq n - 2$, $\phi (\zeta) \wedge \Omega^+_{0,s} (\zeta, z)$ is of type $(n, n - 1)$ in $z$, and (1.16) gives

$$d_z (\phi \wedge \Omega^+_0) = \bar{\partial}_t (\phi \wedge \Omega^+_0) = \bar{\partial}_t \phi \wedge \Omega^+_{0,s} - \bar{\partial}_z (\phi \wedge \Omega^+_{0,s-1}).$$

We apply Stokes' theorem on the manifold-with-boundary $\{ \zeta \in M_p : |\zeta - z| \geq \varepsilon \}$, $\varepsilon > 0$, for a fixed $z$ in $M_p$, and let $\varepsilon$ tend to zero. The resulting residue at $z$ is a non-zero constant multiple of $\phi (z)$. Moving the exterior derivative $d_z$ past the integral sign, we obtain formally

$$\int_{\partial M_p} \phi - B \phi = \int_{\partial M_p} \phi + Q_0 \phi.
$$

Here,

$$P_0 \phi (z) = c_1 \int_{M_p} \phi (\zeta) \wedge \Omega^+_{0,s-1} (\zeta, z),
$$

$$Q_0 \psi^{0,s+1} (z) = c_2 \int_{M_p} \psi (\zeta) \wedge \Omega^+_{0,s} (\zeta, z),
$$

$$B \phi (z) = c_3 \int_{\partial M_p} \phi (\zeta) \wedge \Omega^+_{0,s} (\zeta, z).
$$

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The preceding argument is rigorous if \( \varphi \) vanishes in a neighborhood of \( z \). For the general case we may assume that \( \varphi \) has compact support in \( M_p \) and apply either theorem (3.2) of [4] or theorem (9.13) of [3]. In these theorems (1.17) is verified in the sense of currents of type \((n, n-1-q)\) along \( M \), which results in equality only mod \( \delta r \). This is to be understood in (0.1) or (1.27) below. Only the tangential part of the homotopy formula, which gives equality, is used in [11].

To transform the boundary integral (1.20), we use (1.11), which gives

\[
\partial_z \omega_{0, s}^0 + \partial_z \omega_{0, s-1}^0 = -\Omega_{0, s}^0 + \Omega_{0, s-1}^0. \tag{1.21}
\]

We note that \( \omega^0 \) is given by (1.2), and \( w_n = \zeta_n - z_n \). Also, \( \partial \omega^0 = 0 \), \( \partial^+ \omega^+ = \partial_z \omega^+ \), and \( \partial \omega^- = \partial_z \omega^- \), so that

\[
\begin{align*}
\Omega_{0, s}^0 = & \omega^0 \wedge \omega^- \wedge (\partial_z \omega^-)^{n-2}, \\
\Omega_{0, s}^0 = & \omega^0 \wedge \omega^- \wedge (\partial_z \omega^-)^{n-2}, \\
& (dw \rightarrow d\zeta).
\end{align*}
\tag{1.22}
\]

Thus,

\[
\partial_z \omega_{0, s}^0 + \partial_z \omega_{0, s-1}^0 = -\Omega_{0, s}^0, \quad 1 \leq s \leq n-3. \tag{1.23}
\]

We insert (1.23) into (1.20), use Stokes’ theorem over \( \partial M_p \) to throw \( \partial_z \) onto \( \varphi(\zeta) \) in the first integral and take \( \partial_z \) outside the second integral to get

\[
B \varphi = \partial P_1 \varphi + Q_1 \partial \varphi, \tag{1.24}
\]

where

\[
P_1 \varphi(z) = c_4 \int_{\partial M_p} \varphi(\zeta) \wedge \omega_{0, s-1}^0(z, \zeta), \tag{1.25}
\]

\[
Q_1 \psi(0, s+1)(\zeta) = c_5 \int_{\partial M_p} \psi(\zeta) \wedge \omega_{0, s}^0(z, \zeta). \tag{1.26}
\]

Thus, since our forms are restricted to \( M, \partial \zeta = \partial_b \), and we have

\[
\begin{align*}
\varphi = & \partial_b P \varphi + Q \partial_b \varphi, \\
P = & P_0 + P_1, \\
Q = & Q_0 + Q_1.
\end{align*}
\tag{1.27}
\]

We briefly consider the case \( 1 \leq s = n-2 \), in which we have (1.27) with the boundary integral of \( \varphi \wedge \omega_{0, n-2}^0 \) added to the right hand side. Following Henkin [4], we approximate \( \omega^0 = (\zeta_n - z_n)^{-1} d\zeta_n \) by \( P_j(\zeta_n, z_n) d\zeta_n \), where \( P_j \) is a sequence of polynomials converging uniformly for \( \zeta_n \) on the arc \( r^{\theta}(\zeta_n) = 0, \operatorname{Im} \zeta_n > 0 \), and \( r^{\theta}(\zeta_n) < 0, \) \( z_n \) fixed. We also approximate \( r^2(\zeta_n) = (\zeta_n - \zeta_n) \) by \( r(\zeta_n) \cdot (\zeta_n - \zeta_n) \). Denote the resulting form by \( \Omega_{0, n}(\zeta_n, \zeta_n) \).
Then
\[ \int_{\partial M_\rho} \phi(\zeta) \wedge \Omega^{0,-}_{0,n-2}(\zeta, z) = \lim_{\epsilon \to 0} \int_{\partial M_\rho} \phi(\zeta) \wedge \Omega_{(\zeta, \epsilon)}(\zeta, z) \]
\[ = \lim_{j \to \infty} \int_{\partial M_\rho} \partial \phi(\zeta) \wedge \Omega_{(\zeta, \epsilon)}(\zeta, z) = Q_2(\partial \phi)(z), \quad (1.28) \]

since \( \Omega_{(\zeta, \epsilon)} \) is holomorphic in \( \zeta \) and without singularity on \( M_\rho' \). Thus, (1.27) holds with \( Q = Q_0 + Q_1 + Q_2 \). However, we are not able to obtain any useful bounds for the operator \( Q_2 \).

Returning to (1.27) with \( 1 \leq s \leq n - 3 \), we introduce the notation
\[ p(\zeta, z) = r_{\zeta} \cdot (\zeta - z), \quad q(\zeta, z) = r_{\zeta} \cdot (\zeta - z), \quad w_n = \zeta - z, \quad (1.29) \]
in order to write out the kernels more explicitly. Then,
\[ \Omega^{0,+}_{1,s} = \frac{\partial_{\zeta} r \wedge (r_z \cdot d\zeta) \wedge (\partial_{\zeta} r)^{n-2-s} \wedge (\partial_z r_z \wedge d\zeta)^s}{p(\zeta, z)^{n-1-s} q(\zeta, z)^{s+1}}, \quad (1.30) \]
and
\[ \Omega^{0,+}_{0,s} = \frac{d\zeta \wedge (r_z \cdot (r_z \cdot d\zeta) \wedge (\partial_{\zeta} r)^{n-3-s} \wedge (\partial_z r_z \wedge d\zeta)^s}{w_n p(\zeta, z)^{n-2-s} q(\zeta, z)^{s+1}}. \quad (1.31) \]
These expressions make evident the following property of the four operators \( P_0, Q_0, P_1, Q_1 \). Each annihilates the ideal of forms generated by \( \partial_{\zeta} r \).

This is because each integrand contains the factor \( \partial_{\zeta} r \), and restricting to \( r = 0 \) i.e. to \( M \), \( d_{\zeta} r = \partial_{\zeta} r + \partial_{\zeta} r = 0 \). Thus any term in \( \phi(\zeta) \) or \( \psi(\zeta) \) containing \( \partial_{\zeta} r \) is annihilated by the wedge product.

We need to determine the nature of the operators \( P \) and \( Q \) as acting on the coefficients of the form \( \phi^{(0,s)}(\zeta) \) or \( \psi^{(0,s+1)}(\zeta) \) relative to the differentials \( d\zeta \). For this let \( D_\rho \) be the projection of \( M_\rho \) onto \( y_n = 0 \), so that by (1.13) \( M_\rho \) is a graph over \( D_\rho \). If \( f(\zeta) \) is a typical such coefficient, then \( P_0 \) and \( Q_0 \) are (sums of) operators of the form
\[ K f(z) = \int_{D_\rho} f(\zeta) k(\zeta, z) dV(\zeta), \quad (1.32) \]
while \( P_1 \) and \( Q_1 \) are operators of the form
\[ L f(z) = \int_{\partial D_\rho} f(\zeta) l(\zeta, z) dS(\zeta). \quad (1.33) \]
Here, \( dV \) and \( dS \) are the Euclidean volume and surface measures in \( \mathbb{R}^{n-1} \). Occuring in each numerator in (1.30), (1.31) is
\[ \partial_{\zeta} r \wedge (r_z \cdot d\zeta) = r_{\zeta} \cdot d\zeta \wedge (r_z - r_{\zeta}) \cdot d\zeta. \]
We use $\zeta(t) = z + t(\zeta - z)$ to write

\begin{equation}
 r_s(\zeta) - r_s(z) = \int_{t=0}^1 \{ r_{zzz}(\zeta(t)) \cdot (\zeta - z) + r_{zz}(\zeta(t)) \cdot (\zeta - z) \}.
\end{equation}

We then have an expression of each $d\bar{z}$ component of the $(n, n-1)\zeta$-form $\varphi^{(0,1)}(\zeta) \wedge \Omega^{+}_{\zeta, z-1}(\zeta, z)$ as a linear combination of the differentials

\begin{equation}
d\zeta_1 \wedge \ldots \wedge d\zeta_n \wedge d\bar{\zeta}_1 \wedge \ldots \wedge d\bar{\zeta}_m, \quad 1 \leq i \leq n,
\end{equation}

having coefficients which are rational combinations of $p$, $q$, the first and second derivatives $\partial^1 r$, $\partial^2 r$ evaluated at $\zeta$ or $z$ or integrated as in (1.34), and of $\zeta_i - z_p$, $\bar{\zeta}_i - \bar{z}_p$. Further, we express (1.35) as $a_j(\zeta) dV(\zeta)$, where each $a_j(\zeta)$ is an easily computed expression in $\partial^j r(\zeta)$. It follows from (1.30) that $k(\zeta, z)$ can be put into the form

\begin{equation}
k(\zeta, z) = A(\zeta, z) B^U p^{-\alpha} q^{-\beta},
\end{equation}

where $\alpha \geq 1$, $\beta \geq 1$, and $I$, $J$ are non-negative multi-indices. (Initially $|I| + |J| = 1$.) $A$ is constructed from $\zeta$, $z$ and up to a certain number (initially 2) of derivatives $\partial^j r$ of the defining function as described. Similarly, the kernel $l(\zeta, z)$ has the form

\begin{equation}
l(\zeta, z) = A(\zeta, z) B^U p^{-\alpha} a^{-\beta} w^{-\gamma}.
\end{equation}

As shown in [4], [3], $k(\zeta, z)$ is then absolutely integrable in $\zeta$, uniformly in $z$. Thus $Kf$ as well as $Lf$ are continuous over the interior of $M$.

We denote by $\mu$, $\mu(k)$, or $\mu(l)$ an upper bound.

\begin{equation}
\mu \geq 2(\alpha + \beta) + \gamma - |I| - |J|.
\end{equation}

For integrals like (1.32) over $M_p$ we shall always have $\gamma = 0$ and $\mu = 2n - 1$. As shown in [4], [3], $k(\zeta, z)$ is then absolutely integrable in $\zeta$, uniformly in $z$. Thus $Kf$ as well as $Lf$ are continuous over the interior of $M$.

We denote by $\delta_z$ a vector field in $\mathbb{C}^n$ tangent to $M$,

\begin{equation}
\delta_z = v(z) \cdot \partial_z + \bar{v}(z) \cdot \partial_{\bar{z}},
\end{equation}

and by $\delta_\zeta$ the corresponding operator in $\zeta$-coordinates. As a fixed basis of such fields $\delta_z$, we shall take the real and imaginary parts of

\begin{equation}
r_n(z) \partial_{z^\alpha} - r_n(z) \partial_{\bar{z}^\alpha}, \quad 1 \leq \alpha \leq n - 1,
\end{equation}

\begin{equation}
ir_n(z) \partial_{z^\alpha} - ir_n(z) \partial_{\bar{z}^\alpha}.
\end{equation}

In particular, the coefficients $v$ are constructed from the first derivatives of $r$. Any of the $(z_\alpha = x_\alpha + iy_\alpha, x_\alpha)$-coordinate partial derivatives is a linear combination, with function coefficients, of the fields of this basis, and conversely. Thus, to measure the $C^j$-norm of $Kf$ or $Lf$, it suffices to apply up to $j$ of these vector fields $\delta_z$. 

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2. FIRST DERIVATIVES OF THE KERNEL $k$

We proceed to differentiate (1.32) using (1.39) and prepare to throw the derivative onto $f$ via integration by parts. The nature of the kernel (1.36) complicates the process. For similar arguments involving the $\bar{\partial}$ complex, one may consult [3] or [7], for example. We shall make use of the operator

$$T_\zeta = i(r_\zeta \cdot \partial_\zeta - r_\zeta \cdot \partial_\zeta),$$

(2.1)

which is tangent to $M$ but transverse to the holomorphic tangent planes to $M$. It has been used in [7] and by a number of other people. From (1.29) and $|r_\zeta| \geq \frac{1}{4}$, we may assume that

$$|T_\zeta p| \geq c, \quad |T_\zeta q| \geq c,$$

(2.2)

for all $\zeta, \zeta \in M$, and a constant $c > 0$, by taking $\rho$ sufficiently small.

To compute

$$(\delta_\zeta + \delta_\bar{\zeta}) k = [(\delta_\zeta + \delta_\zeta) A \cdot B^U + A(\delta_\zeta + \delta_\zeta) B^U] p^{-\alpha} q^{-\beta} + \alpha B^U (\delta_\zeta + \delta_\zeta) [p^{-\alpha} q^{-\beta}],$$

(2.3)

we note that

$$(\delta_\zeta + \delta_\bar{\zeta}) (\zeta_i - z_j) = v_i(\zeta) - v_i(z) = \sum v_{im} B^U,$$

$$\left( |I| + |J| = 1 \right),$$

where by (1.34) the coefficients $v_{im}$ involve $\partial^j r, j \leq 2$. Hence,

$$(\delta_\zeta + \delta_\bar{\zeta}) B^U = \sum A^U_{KL} B^K,$$

(2.4)

the coefficients $A^U_{KL}$ depending through $v$ on $\partial^j r, j \leq 2$. Also, we have

$$(\delta_\zeta + \delta_\bar{\zeta}) [p^{-\alpha}] = F T_\zeta [p^{-\alpha}], \quad F = \frac{(\delta_\zeta + \delta_\bar{\zeta}) p}{T_\zeta p},$$

$$ (\delta_\zeta + \delta_\bar{\zeta}) [q^{-\beta}] = E T_\zeta [q^{-\beta}], \quad E = \frac{(\delta_\zeta + \delta_\bar{\zeta}) q}{T_\zeta q};$$

(2.5)

Hence,

$$(\delta_\zeta + \delta_\bar{\zeta}) [p^{-\alpha} q^{-\beta}] = E p^{-\alpha} T_\zeta [q^{-\beta}] + F T_\zeta [p^{-\alpha}] q^{-\beta} + \alpha G T_\zeta [p^{-\alpha} q^{-\beta} - - \alpha G T_\zeta [p^{-\alpha} q^{-\beta} - - \alpha G (T_\zeta p) p^{-\alpha - 1} q^{-\beta}],$$

(2.6)

$$G \equiv F - E.$$
From (2.5) 
\[ E(\zeta, z) = (T_\zeta q)^{-1} \{ (\delta_z r_z \cdot (\zeta - z) + r_z \cdot (v(\zeta) - v(z)) \} \]
\[ = \sum E_{ij} B^u \quad (|I| + |J| = 1), \tag{2.7} \]
where \(E_{ij}\) has denominator \(T_\zeta q\) and numerator involving \(\partial^j r, j \leq 2\). From (2.6)

\[ G(\zeta, z) = \left( \frac{1}{T_\zeta p} - \frac{1}{T_\zeta q} \right) (\delta_z + \delta_\zeta) p + \frac{1}{T_\zeta q} \{ (\delta_z + \delta_\zeta) p - (\delta_z + \delta_\zeta) q \}, \]
\[ = (T_\zeta p T_\zeta q)^{-1} \{ i(r_z - r_\zeta) \cdot r_\zeta - (T_\zeta r_\zeta) \cdot (\zeta - z) \}, \tag{2.8} \]
\[ (\delta_z + \delta_\zeta) p - (\delta_z + \delta_\zeta) q = (\delta_z r_\zeta - \delta_\zeta r_z) \cdot (\zeta - z) + (r_\zeta - r_z) \cdot (v(\zeta) - v(z)). \]

It follows that
\[ G(\zeta, z) = \sum G_{ij} B^u \quad (|I| + |J| = 2), \tag{2.9} \]
where \(G_{ij}\) has denominator \(T_\zeta p T_\zeta q\) and numerator involving \(\partial^j r, j \leq 3\). Since one has (see e.g. section 4 of [11])
\[ |p(\zeta, z)| \geq c |\zeta - z|^2, \quad |q(\zeta, z)| \geq c |\zeta - z|^2, \quad c > 0, \tag{2.10} \]
the additional factor \(p^{-1}\) in (2.6) is nullified by the factor \(G\). If we write
\[ AB^u T_\zeta [E p^{-\alpha} q^{-\beta}] = T_\zeta [E k] - (T_\zeta A) EB^u p^{-\alpha} q^{-\beta} - AET_\zeta [B^u] p^{-\alpha} q^{-\beta}, \tag{2.11} \]
then from (2.3), (2.6) we get
\[ (\delta_z + \delta_\zeta) k = T_\zeta k^1 + k^0, \]
\[ \mu(k^0) = \mu(k), \quad \mu(k^1) = \mu(k) - 1. \tag{2.12} \]

More explicitly,
\[ k^0 = \sum A^0_{KL, po} B^{KL} p^{-\rho} q^{-\sigma}, \quad 2(\rho + \sigma) - |K| - |L| \leq \mu, \]
\[ k^1 = \sum A^1_{KL, po} B^{KL} p^{-\rho} q^{-\sigma}, \quad 2(\rho + \sigma) - |K| - |L| \leq \mu - 1, \tag{2.13} \]
where
\[ A^0_{KL, po} = w_1 [A], \quad A^1_{KL, po} = w_0 [A], \]
\[ w_0 A = S_0 A, \quad w_1 A = S_1 \partial^1 A + S_2 A. \tag{2.14} \]

This means that \(w_0\) is the zeroth order operator multiplication by \(E = S_0 (\partial^j r, j \leq 2)\), and \(w_1\) is a first order operator with coefficients \(S_1 = S_1 (\partial^j r, j \leq 1)\), \(S_2 (\partial^j r, j \leq 3)\). In each case \(S_0, S_1, S_2\) have denominators
3. FIRST DERIVATIVES OF Kf AND Lf

We compute \( \delta_z Kf \) in the sense of distributions. Let \( g \) be a smooth function with compact support in \( D_p \), then

\[
\int_{D_p} Kf(z) \delta_z g(z) dV(z) = \lim_{\varepsilon \to 0} I_\varepsilon
\]

\[
I_\varepsilon = \int_{|z-z_\varepsilon| > \varepsilon} f(\zeta) k(\zeta, z) \delta_z g(z) dV(\zeta) dV(z).
\]

Using (2.12) we rewrite the integrand as

\[
f k \delta_z g = f \delta_z [kg] - f \delta_z [k] g
\]

\[
= f \delta_z [kg] + f (\delta_z k - T_\zeta k^1 - k^0) g
\]

\[
= f \delta_z [kg] + \delta_z [f k] g - (\delta_z f) k g
\]

\[- T_\zeta [f k^1] g + (T_\zeta f) k^1 g - f k^0 g.
\]

Hence, \( I_\varepsilon = I_1 + I_2 + I_3 + I_4 + I_5 \),

\[
I_1 = - \int_{\zeta \in |z-z_\varepsilon| > \varepsilon} \{ \delta_z f k - T_\zeta f k^1 \} dV(\zeta) g(z) dV(z)
\]

\[
I_2 = - \int_{\zeta \in |z-z_\varepsilon| > \varepsilon} f k^0 dV(\zeta) g(z) dV(z)
\]

\[
I_3 = \int_{\zeta} f(\zeta) \int_{|z-z_\varepsilon| > \varepsilon} \delta_z [kg] dV(z) dV(\zeta)
\]

\[
I_4 = \int_{\zeta} \int_{|z-z_\varepsilon| > \varepsilon} \delta_z [f k] dV(\zeta) g(z) dV(z)
\]

\[
I_5 = - \int_{\zeta} \int_{|z-z_\varepsilon| > \varepsilon} T_\zeta [f k^1] dV(\zeta) g(z) dV(z)
\]

We shall transform \( I_3, I_4 \) and \( I_5 \) using Stokes' theorem on the inner integrals. For this we consider the integrals as over \( M \subset \mathbb{C}^* \), and denote by \( N(\zeta), N_z(\zeta), \) and \( N_\zeta(z) \), respectively, the outward unit normals tangent to \( M \) for the domains \( M_p \), \( \{ \zeta : |z-z| < \varepsilon \} \), and \( \{ z : |\zeta-z| < \varepsilon \} \) which lie on \( M \). Since \( M \) is of class at least \( C^3 \), we have

\[
N_\zeta(z) + N_z(\zeta) = O(|\zeta-z|).
\]

We denote by \( \langle , \rangle \) and \( \text{div} \), the real Euclidean inner product and divergence relative to \( M \). The resulting integrals over the interior of \( M_p \)
for the three terms give

\[ I_6 = \int \int_{|\zeta - z| > \varepsilon} f(\zeta) k^2(\zeta, z) g(z) dV(\zeta) dV(z), \]

\[ k^2 \equiv (\text{div} \delta_z + \text{div} \delta_\zeta) k - \text{div} T_\zeta k, \]

For the boundary terms we get \( I_7 + I_8 \),

\[ I_7 = \int \int_{\zeta \in \partial M_p} f(\zeta) l^1(\zeta, z) dS(\zeta) g(z) dV(z), \]

\[ I_8 = \int \int_{|\zeta - z| = \varepsilon} f(\zeta) l^2(\zeta, z) dS(\zeta) g(z) dV(z), \]

where

\[ l^1 = \langle N(\zeta), \delta_\zeta \rangle k - \langle N(\zeta), T_\zeta \rangle k^1 \]

\[ l^2 = \left\{ \langle N_\zeta(z), \delta_z \rangle + \langle N_z(\zeta), \delta_\zeta \rangle \right\} k + \langle N_z(\zeta), T_\zeta \rangle k^1. \]

Using (3.1) we see that the coefficient of \( k \) in \( l^2 \) is \( O(|\delta - z|) \) so that

\[ \mu(l^2) = \mu - 1 = 2n - 2. \]

Therefore, \( I_8 \to 0 \) as \( \varepsilon \to 0 \) by essentially the same argument as for formula (3.18) in lemma 3.3 of [4]. Since all the integrals

\[
\lim_{\varepsilon \to 0} I_\varepsilon = - \int \int_{\zeta \in M_p} \left\{ \delta_{\zeta} f k - T_{\zeta} f k^1 \right\} dV(\zeta) g(z) dV(z)
\]

\[
- \int \int_{\zeta} f(k^0 + k^2) dV(\zeta) g(z) dV(z)
\]

\[
+ \int \int_{\zeta \in \partial M_p} f l^1 dS(\zeta) g(z) dV(z).
\]

This gives \( \delta^* Kf = - (\delta_z + \text{div} \delta_z) Kf \) in the sense of distributions. Since it and \( Kf \) are both continuous, it is a derivative in the ordinary sense, and we have

**Lemma (3.1).** *If \( f \in C^1(M) \) and \( \mu(\mu) = 2n - 1 \), then*

\[ \delta Kf = K(\delta f) + K_\delta(T_{\zeta} f) + K_\delta(f) + L_\delta(f), \]

*where the new operators have kernels* \( -k^1, \ k^3 \equiv k^0 + k^2 - (\text{div} \delta_z) k, \ -l^1, \)

*with* \( \mu(k^1) = \mu - 1, \ \mu(k^3) = \mu(l^1) = \mu. \)

In order to state lemma (3.1) a little more precisely, we must introduce

**Lemma (3.2).** *For an integral over \( M_\rho \) (or \( D_\rho \)) of the form (1.32), (1.36) we use round brackets,***

\[ Kf = (f, A)_{\text{round}} \equiv (f, A)_{\text{p}}, \]

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with the latter notation usually sufficing. Similarly, for a boundary integral of the form (1.33), (1.37) we get

$$\mathbf{L} f = \langle f, A \rangle_{\mathcal{I} \mathcal{A} \beta} = \langle f, A \rangle_\mu. \quad (3.6)$$

We also use the same notation for a finite sum of such integrals with the same value for $\mu$ (1.38). We also let $\partial$ represent any, and all, our first order operators $\delta$. Then, with the notation (2.14) and $\mu = 2n - 1$, we have

$$\partial (f, A)_\mu = (\partial f, w_0 A)_\mu + (f, w_1 A)_\mu + \langle f, w_0 A \rangle_\mu. \quad (3.7)$$

Our analysis of the derivatives of $\mathbf{L} f$ is much simpler. We simply let the operator $\delta$ fall on the kernel $I(\zeta, z)$, worsening the singularity at the boundary. From (1.37) we have

$$\delta \mathbf{z} = (\delta \mathbf{z}) B^\mu p^{-\alpha} q^{-\beta} w_n^{-\gamma} + A \delta \mathbf{z} [B^\mu p^{-\alpha} q^{-\beta} w_n^{-\gamma}],$$

which leads to

$$\partial \langle f, A \rangle = \langle f, w_1 A \rangle_\mu + \langle f, w_0 A \rangle_{\mu + 2} = \langle f, w_1 A \rangle_{\mu + 2}, \quad (3.8)$$

by regarding $w_0$ as a first order operator $w_1$ and replacing $\mu$ by $\mu + 2$ in the first term.

4. HIGHER ORDER DERIVATIVES

From (1.30) and (1.31) we see that $P$ and $Q$ have the character

$$\mathbf{P} f = (f, A)_\mu + \langle f, A \rangle_{\mu - 1}, \quad \mu = 2n - 1, \quad (4.1)$$

as operators on the coefficients $f$ of the differential form $\varphi$. The two $A$'s are constructed from $\zeta, z$, and $\partial^j r$, $j \leq 2$. From (3.8) we have, taking $b$ derivatives

$$\partial^b \langle f, A \rangle_{\mu - 1} = \langle f, w_1^b A \rangle_{\mu - 1 + 2b}. \quad (4.2)$$

Taking a second derivative of $(f, A)_\mu$, using (3.7) and (3.8), and combining several terms gives

$$\partial^2 (f, A)_\mu = (\partial^2 f, w_0^2 A)_\mu + (\partial f, w_0 w_1 A)_\mu + (f, w_1^2 A)_\mu + \langle f, w_0 w_1 A \rangle_{\mu + 2}.$$

The notation $w_0^2 w_1^2 A$ indicates that $\alpha$ zeroth order and $\beta$ first order operators of the form (2.14) are applied to $A$ in some order. After taking
\begin{equation}
\partial^b \langle f, A \rangle = \sum_{j=0}^{b} (\partial^j f, w_0^j w_1^{b-j} A)_{\mu} \\
+ \sum_{j=0}^{b-1} \langle \partial^j f, w_0^{j+1} w_1^{b-1-j} A \rangle_{\mu+2 (b-1-j)} \tag{4.3}
\end{equation}

We define norms according to
\begin{align*}
\|f\|_p = \sup \{ |f(\zeta)| : \zeta \in M_p \}, \\
\|A\|_p = \sup \{ |A(\zeta, z)| : \zeta, z \in M_p \}, \\
\|\partial^b f\|_p = \sup \{ \|\partial^K f\|_p : |K| = b \}, \\
\|f\|_{p, b} = \max_{0 \leq j \leq b} \|\partial^j f\|_p.
\end{align*}

For \( \mu \leq 2n-1 \) and \( \rho \leq \rho_0, \rho_0 \) fixed, we have
\begin{equation}
\|\langle f, A \rangle_{\mu} (z) \|_p \leq \|f\|_p \|A\|_p \int_{M_\rho} |B^u p^{-a} q^{-b}| dV(\zeta),
\end{equation}
so that
\begin{equation}
\|\langle f, A \rangle_{\mu}\|_p \leq c \|f\|_p \|A\|_p. \tag{4.4}
\end{equation}

Also,
\begin{equation}
\|\langle f, A \rangle_{\mu} (z) \|_p \leq \|f\|_p \|A\|_p \int_{\partial M_\rho} |B^u p^{-a} q^{-b} w^{-\gamma}| dS(\zeta).
\end{equation}

For \( \zeta \in \partial M_\rho \), \( z \in M_{p (1-\sigma)} \), we have
\begin{align}
|\zeta - z| \geq \delta, \\
|w_\alpha| \geq c \delta, \\
|p(\zeta, z)| \geq c \delta^2, \\
|q(\zeta, z)| \geq c \delta^2, \\
|B^u p^{-a} q^{-b} w^{-\gamma}| \leq c \delta^{-\mu}, \tag{4.5}
\end{align}
where, as in (0.3),
\begin{equation}
\delta \leq \text{dist}(M_{p (1-\sigma)}, \partial M_\rho).
\end{equation}

Thus
\begin{equation}
\|\langle f, A \rangle_{\mu}\|_{p (1-\sigma)} \leq c \delta^{-\mu} \|A\|_p \|f\|_p. \tag{4.6}
\end{equation}

Applied to (4.2), these remarks give
\begin{equation}
\|\partial^b \langle f, A \rangle_{\mu-1}\|_{p (1-\sigma)} \leq c_b \|f\|_p \|A\|_p \delta^{-\mu-2 b + 1} \\
\leq c_b \|f\|_p \|A\|_p \delta^{-2 (n+b-1)}, \tag{4.7}
\end{equation}

since \( \mu = 2n-1 \).
In (4.3) we may replace $w_0^{b+1} w_1^{b-1-j}$ in the second sum by $w_0^j w_1^{b-j}$, again by regarding $w_0$ as a $w_1$ (2.14). Then

$$
\| \partial^b (f, A) \|_{\mathcal{P} (1-o)} \leq c_b \left\{ \| \partial^b f \|_\mathcal{P} \| w_0^b A \|_\mathcal{P} + \sum_{j=0}^{b-1} \| \partial^j f \|_\mathcal{P} \| w_0^j w_1^{b-j} A \|_\mathcal{P} \delta^{-\mu - 2 (b-1-j)} \right\}
$$

$$\leq c_b \delta^{-2 (n+b-1)+1} \sum_{j=0}^{b} \| \partial^j f \|_\mathcal{P} \| w_0^j w_1^{b-j} A \|_\mathcal{P} \quad (4.8)$$

where we have used $\mu = 2n - 1$ and assumed $\delta \leq 1$. Combining (4.1), (4.7), and (4.8) gives

$$
\| \partial^b P f \|_{\mathcal{P} (1-o)} \leq c \delta^{-2 (n+b-1)} \sum_{j=0}^{b} \| \partial^j f \|_\mathcal{P} \| w_0^j w_1^{b-j} A \|_\mathcal{P} \quad (4.9)
$$

In a term $w_0^j w_1^{b-j} A$, $A$ and the coefficients of $w_0$ and $w_1$ involve $\partial^i r$, $1 \leq i \leq 3$. Also there are $b-j$ further differentiations. Such a term is therefore a sum of terms of the form

$$
F(\partial^i r) \partial^{\alpha_1} (\partial^i r) \ldots \partial^{\alpha_s} (\partial^i r), \quad \alpha_1 + \ldots + \alpha_s = b-j, \quad 1 \leq i \leq 3. \quad (4.10)
$$

Here $F$ is a certain rational function (modulo the operation (1.34)) in the derivatives $\partial^i r$. By (2.2) its denominators are bounded away from 0 by a positive constant depending on $b$. The construction of $A(\zeta, z)$ and related expressions involves $r$ and its derivatives on the line segment in $\mathbb{C}^n$ from $\zeta$ to $z$ (1.34) for all points $\zeta, z \in M_\rho$. Therefore we denote

$$
M_\rho = \text{convex hull of } M_\rho,
\quad \| f \|_\rho = \sup \{ |f(z)| : z \in M_\rho \}, \quad \text{etc.}
$$

It follows that

$$
\| w_0^j w_1^{b-j} A \|_\rho \leq c_b (1 + \| r \|_{\mathcal{P}, b-j+3})^\gamma (b) \quad (4.11)
$$

for some positive constants $c_b$, $\gamma (b)$ depending on $b$.

Combining the above gives the following more precise form of (0.3).

**Theorem (4.1).** — *Let the real hypersurface $M$ in (1.13) be of class $C^1$ and the $(0, s)$ form $\varphi$, $1 \leq s \leq n-3$, be of class $C^k$ on the closure of $M_\rho$.*
\( k \leq l - 3. \) Then \( P \varphi \) is of class \( C^k \) on \( M_\rho \), and for \( 0 < \sigma < 1 \)
\[
\| P \varphi \|_{(1-\sigma),k} \leq K \| \varphi \|_{\rho, k^*},
\]
\[
K = c_k (1 + \| r \|_{\rho, k+3} \gamma (k)^{\delta - 2 (s+k-1)}),
\]
where \( c_k \) and \( \gamma (k) \) are positive constants depending on \( k \).

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