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Critical points of embeddings of $H_0^{1,n}$ into Orlicz spaces

by

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ABSTRACT. — For a domain $\Omega \subset \mathbb{R}^n$ embeddings $u \to \exp(\alpha(|u|/\|u\|_{1,n})^{n/n-1})$ of $H_0^{1,n}(\Omega)$ into Orlicz spaces are considered. At the critical exponent $\alpha = \alpha_n$ a loss of compactness reminiscent of the Yamabe problem is encountered; however by a result of Carlesson and Chang, if Ω is a ball the best constant for the above embedding is attained.

In dimension $n = 2$ we identify the "limiting problem" responsible for the lack of compactness at the critical exponent $\alpha_2 = 4\pi$ in the radially symmetric case and establish the existence of extremal functions also for nonsymmetric domains Ω. Moreover, we establish the existence of two "branches" of critical points of this embedding beyond the critical exponent $\alpha_2 = 4\pi$.

Key words: Sobolev embedding, variational methods, loss of compactness, limiting exponent, limiting problem, local compactness.

RÉSUMÉ. — Étant donné un domaine $\Omega \subset \mathbb{R}^n$, on considère des immersions de $H_0^{1,n}(\Omega)$ dans des espaces d'Orlicz, du type $u \to \exp(\alpha(|u|/\|u\|_{1,n})^{n/n-1})$. Pour l'exposant critique $\alpha = \alpha_n$, se produit une perte de compacité. Toutefois, grâce à un résultat de Carleson et Chang, si Ω est une boule, la meilleure constante pour l'immersion est atteinte.

Classification A.M.S. : 35 J 60, 58 E 15.

Dans le cas $n=2$, le problème limite responsable de la perte de compacité à l'exposant critique $\alpha_2=4\pi$ est identifié dans le cas radialement symétrique. Dans le cas non symétrique, on démontre encore l'existence de fonctions extrêmales. En outre, on montre l'existence de deux branches de points critiques d'immersion au-delà de l'exposant critique $\alpha_2=4\pi$.

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1. Let Ω be a bounded domain in $\mathbb{R}^n$, and let $H_0^{1,p}(\Omega)$ denote the completion of $C_0^\infty(\Omega)$ in the norm

$$\|u\|_{1,p}^p=\int_\Omega |\nabla u|^p\,dx.$$

For $p<n$ there are continuous embeddings

$$H_0^{1,p}(\Omega)\to L^{p^*}(\Omega),\qquad \frac{1}{p^*}=\frac{1}{p}-\frac{1}{n}. \tag{1.1}$$

Moreover, the Sobolev constant

$$S(p)=\sup\left\{\int_\Omega |u|^{p^*}\,dx\,\middle|\,u\in H_0^{1,p}(\Omega):\ \|u\|_{1,p}=1\right\}$$

is independent of Ω, due to invariance of the $H_0^{1,p}$- and L^{p^*}-norms under scaling

$$u\to u_R(x)\equiv R^{(n-p)/p}\,u(R\,x),$$

and (therefore) for $1<p<n$ is never achieved on a bounded domain.

In the border-line case $p=n$, since Ω is bounded, there are continuous embeddings

$$H_0^{1,n}(\Omega)\to L^q(\Omega),\qquad \forall\, q<\infty;$$

however, functions in $H_0^{1,n}(\Omega)$ need not be (essentially) bounded.

Instead the limit case for Sobolev's embedding of $H_0^{1,n}(\Omega)$ occurs for embeddings into Orlicz spaces: For any $\alpha<\infty$ the map

$$H_0^{1,n}(\Omega)\ni u\to U=\exp(|u|^{n/(n-1)})\in L^\alpha(\Omega) \tag{1.2}$$

is well-defined and smooth locally; however, there is a limiting exponent $\alpha_0 = \alpha_0(n)$ such that the unit ball in $H_0^{1,n}(\Omega)$ is mapped to a bounded set in $L^{\alpha_0}(\Omega)$ under this map.

This result is due to Moser [6], sharpening and extending an earlier result by Trudinger [15].

For a domain Ω let

$$⨍_\Omega \varphi\, dx = \frac{1}{\text{meas}(\Omega)} \int_\Omega \varphi\, dx$$

denote the mean value of a function φ over Ω.

Then we may state Moser's result as follows:

THEOREM 1.1. — *There exist constants α_n, c_n depending only on the dimension n such that for all $\alpha \leqq \alpha_n$ there holds*

$$⨍_\Omega \exp(\alpha\, |u|^{n/(n-1)})\, dx \leqq c_n,$$

uniformly for all $u \in H_0^{1,n}(\Omega)$ with $\|u\|_{1,n} \leqq 1$. The constant α_n is given by

$$\alpha_n = n \cdot (\omega_{n-1})^{n/(n-1)},$$

where ω_{n-1} denotes the $(n-1)$-dimensional measure of the unit sphere $S^{n-1} \subset \mathbb{R}^n$.

Moreover, for $\alpha > \alpha_n$

$$\sup ⨍_\Omega \exp(\alpha\, |u|^{n/(n-1)})\, dx = \infty$$

where the supremum is taken with respect to all $u \in H_0^{1,n}(\Omega)$ such that $\|u\|_{1,p} \leqq 1$.

Complementing this result, and in striking contrast with the case of Sobolev's embedding (1.1) for $1 < p < n$, Carleson and Chang [2] have observed:

THEOREM 1.2. — *Let Ω be a ball in $\mathbb{R}^n$. Then*

$$\sup_{\substack{u \in H_0^{1,n}(\Omega) \\ \|u\|_{1,n} \leqq 1}} ⨍_\Omega \exp(\alpha_n\, |u|^{n/(n-1)})\, dx \tag{1.3}$$

is achieved at some function $u_0 \in H_0^{1,n}(\Omega)$ (1).

In order to interpret this result in the general context of the calculus of variations we regard (1.3) as a constrained maximization problem.

Denote

$$\Sigma = \Sigma(\Omega) = \{u \in H_0^{1,n}(\Omega) \mid \|u\|_{1,n} = 1\}$$

the unit sphere in $H_0^{1,n}(\Omega)$ and introduce the functional $E_\alpha : H_0^{1,n}(\Omega) \to \mathbb{R}$

$$E_\alpha(u) = \fint_\Omega \exp(\alpha |u|^{n/(n-1)})\, dx. \tag{1.4}$$

Then Theorem 1.2 is equivalent to the assertion that E_α for $\alpha = \alpha_n$ achieves its supremum on Σ; i. e. that a suitable maximizing sequence $\{u_m\}$ for E_{α_n} in Σ is convergent.

Arguing indirectly, Carleson and Chang first estimate

$$\gamma = \limsup_{m \to \infty} E_{\alpha_n}(u_m),$$

assuming $\{u_m\}$ to be divergent.

Then they succeed in constructing a comparison function $u \in \Sigma$ with $E_{\alpha_n}(u) > \gamma$, and Theorem 1.2 follows.

This local compactness property of the functional E_{α_n} bears some resemblance with properties enjoyed by other functionals involving limiting cases of Sobolev embeddings. In many cases one can show compactness e. g. of a maximizing sequence $\{u_m\}$ unless energy "concentrates" at (finitely many) "singular" points in the domain. Often a close-up view of the behaviour of $\{u_m\}$ near such a point reveals a uniform pattern: Properly rescaled, the sequence $\{u_m\}$ converges to a solution of some "limiting problem" associated with the original problem, one of whose characteristic properties is the invariance with respect to a non-compact group action. In the original problem this symmetry may be "hidden" by perturbations, and it becomes "manifest" only in the limit where the influence of the perturbation is eliminated by the action.

In many cases this group action is the action of the conformal group, in particular dilatations of $\mathbb{R}^n$, i. e. the action of the multiplicative group

(1) Note that for a *fixed* domain Ω the integral $\fint_\Omega \exp(\alpha |u|^{n/(n-1)})\, dx$ is *not* invariant under scaling $u \to u_R(x) \equiv u(R\,x)$.

$\mathbb{R}_+$ via $(\lambda, x) \to x/\lambda$ for all $\lambda > 0$, $x \in \mathbb{R}^n$. Then the limiting problem is found in the limit $\lambda \to \infty$, and hence may be referred to as "problem at infinity". Moreover, this problem is posed on $\mathbb{R}^n$, which is conformally equivalent to the sphere $S^n - \{p\}$ via stereographic projection from $p \in S^n$. Hence the resolution of singularities by rescaling also reveals a topological degeneration near the points of concentration, and divergence e. g. of maximizing sequences may be attributed to the "separation of spheres".

This phenomenon was first observed for harmonic maps of surfaces [9], resp. surfaces of constant mean curvature [16]; cp. also [1], [12], [13]. Here the term "separation of spheres" has a clear geometric meaning. Subsequently, related phenomena were found in numerous other problems as well, cp. [3], [4], [10], [11], [14].

For the functional E_{α_n} the following result was obtained by P. L. Lions [4], Theorem I.6:

THEOREM 1.3. — *Let Ω be a bounded domain in $\mathbb{R}^n$, and let $\{u_m\}$ be a sequence in $H_0^{1,n}(\Omega)$ such that $\|u_m\|_{1,n} \leqq 1$ for all m.*

We may suppose that $u_m \to u$ weakly in $H_0^{1,n}(\Omega)$, $|\nabla u_m|^n dx \to \mu$ weakly in measure. Then either: (i) *$\mu = \delta_{x_0}$, the Dirac measure of mass 1 concentrated at $x_0 \in \Omega$, and $u \equiv 0$, or* (ii) *there exists $\alpha > \alpha_n$ such that the family $U_m = \exp(|u_m|^{n/(n-1)})$ is uniformly bounded in $L^\alpha(\Omega)$ and thus $E_{\alpha_n}(u_m) \to E_{\alpha_n}(u)$ as $m \to \infty$. In particular, this is the case if $u \not\equiv 0$.*

This "concentration-compactness principle" describes the behaviour of divergent maximizing sequences for E_{α_n} in Σ on a macroscopic scale.

In our first result in this paper we gain a close-up view of possible singularities and identify the limiting problem associated with E_{α_n} in the case of radial symmetry. Moreover, for simplicity we restrict ourselves to the two-dimensional case: $n=2$, $\alpha_2 = 4\pi$.

THEOREM 1.4. — *Let $\Omega = B_R(0)$ be a ball in $\mathbb{R}^2$, and suppose $\{u_m\}$ is a sequence of radially symmetric functions $u_m(x) = u_m(|x|) \in H_0^{1,n}(\Omega)$ with norm $\|u_m\|_{1,2} \leqq 1$. Scaling $x \mapsto R x$ we may assume that $R = 1$.*

Assume that $E_{4\pi}(u_m) \to \beta > 1$ while $|\nabla u_m|^2 dx \to \delta_0$ weakly in measure. Moreover, suppose that

$$\sup |\langle dE_{4\pi}(u_m), \varphi \rangle| \leqq C \tag{1.5}$$

where the supremum is taken with respect to $\varphi \in H_0^{1,n}(\Omega)$, such that $\|\varphi\|_{1,2} \leqq 1$ and $\int_\Omega \nabla u_m \nabla \varphi \, dx = 0$.

Then $\beta \leqq e+1$, *and there exists a sequence* $R_m \to 0$, *a constant* $\bar{\kappa} \in \mathbb{R}$ *and a function* $\bar{w} \in H^{1,2}_{\mathrm{loc}}(\mathbb{R}^2)$ *such that the rescaled functions*

$$\bar{w}_m(x) \equiv 4\pi u_m^2(R_m x) + 2\log R_m x) + 2\log R_m + \bar{\kappa} \to 2\bar{w} \text{ in } H^{1,2}_{\mathrm{loc}}(\mathbb{R}^2), \tag{1.6}$$

where $\bar{w}$ *solves the limiting problem*

$$-\Delta \bar{w} = \exp(2\bar{w}) \quad \text{in } \mathbb{R}^2 \tag{1.7}$$

with asymptotic behavior

$$\bar{w}(x) + 2\log|x| = O(1). \tag{1.8}$$

$\bar{w}$ hence corresponds to a conformal change of metric on $\mathbb{R}^2$ from the standard metric to a metric of constant Gaussian curvature $K=1$. Thus $\bar{w}$ is induced by a stereographic projection

$$\bar{\Phi}: \mathbb{R}^2 \ni x = (\xi, \eta) \to \frac{2}{1+|x|^2}(\xi, \eta, 1) \in S^2 \subset \mathbb{R}^3$$

and is in fact given by

$$\bar{w}(x) = \frac{1}{2}\log(\det(d\bar{\Phi})) = \log\frac{2}{1+|x|^2}. \tag{1.9}$$

From Theorem 1.4, by using symmetrization techniques, we deduce a local compactness property for the functional $E_{4\pi}$ on any bounded domain $\Omega \subset \mathbb{R}^2$.

THEOREM 1.5. — *Suppose* Ω *is a bounded domain in* $\mathbb{R}^2$, *and let* $\{u_m\}$ *be a sequence in* $H_0^{1,2}(\Omega)$ *such that* $\|u_m\|_{1,2} \leqq 1$. *We may assume that* $u_m \to u$ *weakly in* $H_0^{1,2}(\Omega)$.

Suppose that $E_{4\pi}(u_m) \to \beta > e+1$. *Then there exists* $\alpha > 4\pi$ *such that the functions* $U_m = \exp(u_m^2)$ *are uniformly bounded in* $L^\alpha(\Omega)$. *In particular,* $E_{4\pi}(u_m) \to E_{4\pi}(u) = \beta$, *and* $u \not\equiv 0$.

Remark that the number $e+1$ agrees with the number computed by Carleson and Chang for the maximal limit of the energies $E_{4\pi}(u_m)$ of a diverging sequence $\{u_m\}$, and hence is best possible.

Theorem 1.5 in particular applies to a maximizing sequence $\{u_m\}$ for $E_{4\pi}$ in $\Sigma(\Omega)$

$$E_{4\pi}(u_m) \to \sup_{u \in \Sigma(\Omega)} E_{4\pi}(u) = \beta_\Omega^{(\max)}.$$

For a ball $\Omega = B_R(0)$, Carleson and Chang exhibit a comparison function u with $\|u\|_{1,2}=1$, $E_{4\pi}(u)=\beta_0 > e+1$. Hence in this case $\beta_\Omega^{(\max)} > e+1$, and by Theorem 1.5 a maximizing sequence has to contain a strongly convergent subsequence.

Actually, the same argument is applicable on domains close to a ball in measure:

COROLLARY 1.6. — *Suppose Ω is a bounded domain in $\mathbb{R}^2$ and let $B_R(\bar{x})$ be a ball contained in Ω. Assume that*

$$\pi R^2 (\beta_{B_R}^{(\max)} - 1) > \text{meas}(\Omega) \cdot e,$$

then $E_{4\pi}$ achieves its supremum in $\Sigma(\Omega)$.

Moreover, numerical evidence suggests that for $\Omega = B_R(0) \subset \mathbb{R}^2$ also for small $\alpha > 4\pi$ a branch of radially symmetric local maximizers u_α of E_α on Σ exists, emanating from a solution $u_{4\pi}$ of the constrained maximization problem for $E_{4\pi}$ on Σ. This branch persists until apparently it meets a branch of "unstable" critical points of the restricted functional $E_\alpha | \Sigma$, bifurcating from infinity at $\alpha = 4\pi$; cp. [5].

The existence of a "branch" of relative maximizers for E_α beyond $\alpha = 4\pi$ is established in the following:

THEOREM 1.7. — *Let Ω be a bounded domain in $\mathbb{R}^2$ and suppose that for $\alpha = 4\pi$ there holds: $\sup_\Sigma E_{4\pi} > e+1$.*

Then there exists a number $\alpha^* > 4\pi$ such that for any $\alpha \in]0, \alpha^*[$ there exists a function $u_\alpha \in \Sigma$ which locally maximizes E_α on Σ.

Actually, we will construct a set $C \subset \mathbb{R} \times \Sigma$ such that for any $(\alpha, u) \in C$ the function u locally maximizes E_α in Σ and such that the projection of C to the first component covers the interval $]0, \alpha^*[$. Moreover, α^* is characterized by the condition that either: (i) there exist pairs $(\alpha, u_\alpha) \in C$, $\alpha \leqq \alpha^*$, and functions $v_\alpha \in \Sigma$ such that $\alpha \to \alpha^*$ while

$$E_\alpha(v_\alpha) > E_\alpha(u_\alpha), \qquad \|u_\alpha - v_\alpha\|_{1,2} \to 0 \quad \text{as } \alpha \to \alpha^*,$$

(i. e. the functions u_α "loose their stability" as $\alpha \to \alpha^*$), or (ii)

$$\operatorname*{ess\,sup}_{x \in \Omega} |u_\alpha(x)| \to \infty \quad \text{as } \alpha \to \alpha^*,$$

(i. e. u_α "becomes unbounded").

The existence of saddle-point-type solutions for $\alpha \in]4\pi, \alpha^*[$ is established rigourously only for a dense set of values α in a right neighborhood of 4π:

THEOREM 1.8. — *Suppose the conditions of Theorem* 1.7 *are satisfied and let* α^*, u_α *be defined as in Theorem* 1.7. *There exists a constant* $\alpha_* \in]4\pi, \alpha^*]$ *such that for almost every* $\alpha \in]4\pi, \alpha_*]$ *in the sense of Lebesgue measure there exists a second critical point* $u^\alpha \in \Sigma$ *of* E_α, $u^\alpha \neq u_\alpha$.

For the proof of Theorem 1.8 we employ critical point methods beyond the compactness range. To overcome the resulting technical difficulties we use a device from [13] to obtain suitable *a priori* bounds on comparison functions by varying the parameter α. Our method works at the dense set of points of differentiability of a certain monotone function, defined by a minimax-scheme.

In order to close the "gaps" one would need to have *a priori* estimates for the L^∞-norms of critical points of E_α in terms of their energies and e. g. $\log(\alpha - 4\pi)$, which seem to be unknown.

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2. PRELIMINARIES

We briefly collect some well-known facts about the functional E_α and introduce some concepts from critical point theory that we shall use later on.

First recall the following regularity properties of E_α:

LEMMA 2.1. — *Let* Ω *be a bounded domain in* $\mathbb{R}^n$. *The functional* $E_\alpha : H_0^{1,n}(\Omega) \to \mathbb{R}$, *given by* (1.4), *is continuously Fréchet differentiable on* $H_0^{1,n}(\Omega)$, *for any* $\alpha > 0$, *and* dE_α *is locally uniformly continuous and bounded.*

Proof. — Using Theorem 1.1 and Vitali's convergence theorem for uniformly absolutely continuous integrals we easily verify that E_α is differentiable and its differential is uniformly bounded locally near any $u \in H_0^{1,n}(\Omega)$.

Indeed, for u, $\varphi \in H_0^{1,n}(\Omega)$, $\varepsilon > 0$:

$$E_\alpha(u+\varepsilon\varphi) - E_\alpha(u)$$
$$= \alpha \frac{n}{n-1} \int_0^\varepsilon \fint_\Omega \varphi\,(u+t\,\varphi)^{1/(n-1)} \exp(\alpha\,|u+t\,\varphi|^{n/(n-1)})\,dx\,dt, \quad (2.1)$$

where we denote $s^p := s\,|s|^{p-1}$ for all $s \in \mathbb{R}$, $p > 0$.

Now estimate for any $t \in [0, \varepsilon]$:

$$|\varphi\,(u+t\,\varphi)^{1/(n-1)} \exp(\alpha\,|u+t\,\varphi|^{n/(n-1)})$$
$$\leqq |\varphi| \exp(c\cdot|u+t\,\varphi|^{n/(n-1)})$$
$$\leqq |\varphi| \exp(c\,|u|^{n/(n-1)}) \cdot \exp(c\,\varepsilon^{n/(n-1)}\,|\varphi|^{n/(n-1)}) \quad (2.2)$$

with constants c depending on n and α. Note that for any $u \in H_0^{1,n}(\Omega)$ the function

$$U = \exp(|u|^{n/(n-1)}) \in L^q, \qquad \forall q < \infty.$$

Similarly, $\varphi \in L^q$, $\forall q < \infty$. Finally, by Theorem 1.1, if $\varepsilon > 0$ is sufficiently small also $\exp(c\,\varepsilon^{n/(n-1)}\,|\varphi|^{n/(n-1)})$ is uniformly bounded e. g. in $L^2(\Omega)$ for all $\varphi \in H_0^{1,n}(\Omega)$ with $\|\varphi\|_{1,n} \leqq 1$. By Hölder's inequality then the term

$$\varphi\,(u+t\,\varphi)^{n/(n-1)} \exp(\alpha\,|u+t\,\varphi|^{n/(n-1)}) \in L^{2-\delta}(\Omega)$$

for any $\delta > 0$, and the integral of this expression is uniformly absolutely continuous for $t \in [0, \varepsilon]$.

Hence, we may divide by ε and pass to the limit $\varepsilon \to 0$ to obtain that for all u, $\varphi \in H_0^{1,n}(\Omega)$ the partial derivative

$$\langle dE_\alpha(u), \varphi \rangle = \frac{d}{d\varepsilon} E_\alpha(u+\varepsilon\varphi)\big|_{\varepsilon=0}$$
$$= \alpha \frac{n}{n-1} \fint_\Omega \varphi\, u^{1/(n-1)} \exp(\alpha\,|u|^{n/(n-1)})\,dx \quad (2.3)$$

exists.

Estimate (2.2) and the discussion following it moreover show locally uniform continuity and boundedness of all partial derivatives. In particular, E_α is Fréchet differentiable and dE_α: $H_0^{1,n}(\Omega) \to (H_0^{1,n}(\Omega))^*$ is locally uniformly continuous and bounded.

Q.E.D.

In order to restrict E_α to the unit sphere Σ in $H_0^{1,n}(\Omega)$ we compose E_α with the radial projection $\pi: u \to u/\|u\|_{1,n}$. The composed map $E_\alpha \circ \pi$ again will be differentiable in a neighborhood of Σ; moreover, at any $u \in \Sigma$ the differential in direction of an arbitrary function $\varphi \in H_0^{1,n}(\Omega)$ is given by

$$\langle d(E_\alpha \circ \pi)(u), \varphi \rangle = \alpha \frac{n}{n-1} \left(\fint_\Omega \varphi u^{1/(n-1)} \exp(\alpha |u|^{n/(n-1)})\, dx \right.$$
$$\left. - \lambda \fint_\Omega \nabla \varphi \nabla u |\nabla u|^{n-2}\, dx \right), \quad (2.4)$$

with some constant $\lambda \in \mathbb{R}$. Since clearly $\langle d(E_\alpha \circ \pi)(u), u \rangle = 0$, λ may be easily computed

$$\lambda = \lambda(u) = \int_\Omega |u|^{n/(n-1)} \exp(\alpha |u|^{n/(n-1)})\, dx \sim \frac{\partial}{\partial \alpha} E_\alpha(u). \quad (2.5)$$

By definition, $u \in \Sigma$ is a critical point for E_α on Σ if $d(E_\alpha \circ \pi)(u) = 0$, or equivalently if for some number $\lambda > 0$ given by (2.5) u weakly solves the differential equation

$$-\operatorname{div}(\nabla u |\nabla u|^{n-2}) = \lambda^{-1} u^{1/(n-1)} \exp(\alpha |u|^{n/(n-1)}) \quad \text{in } \Omega \quad (2.6)$$

with boundary data

$$u|_{\partial\Omega} = 0. \quad (2.7)$$

To give the proof of Theorem 1.4 it will be convenient to reduce the variational problem for E_α on $H_0^{1,n}(\Omega)$ to a one-dimensional variational problem. This may be achievied by substituting a radial function $u(x) = u(|x|) \in H_0^{1,n}(B_R(0))$ by a function

$$v(t) = \alpha_n^{(n-1)/n} u(R e^{-t/n}) \in H_0^{1,n}([0, \infty[), \quad (2.8)$$

where $H_0^{1,n}([0, \infty[)$ denotes the completion of $C_0^\infty([0, \infty[)$ in the norm

$$\|v\|_{1,n}^n = \int_0^\infty |\dot{v}|^n\, dt.$$

For convenience, we denote $\frac{d}{dt} v = \dot{v}$, etc. α_n was defined in Theorem 1.1.

In fact, under (2.8) the norms in $H_0^{1,n}(B_R(0))$ and $H_0^{1,n}([0, \infty[)$ are related by

$$\| u \|_{1, n; B_R(0)} = \| v \|_{1, n; [0, \infty[}, \tag{2.9}$$

while E_α transforms

$$E_\alpha(u) = n R^{-n} \int_0^R \exp(\alpha | u |^{n/(n-1)}) r^{n-1} dr$$

$$= \int_0^\infty \exp(\alpha \alpha_n^{-1} | v |^{n/(n-1)} - t) \, dt =: I_{\alpha/\alpha_n}(v).$$

In particular, under (2.8) the critical exponents α_n transform into $\beta_n = 1$.

Similarly, the derivatives transform: Let v and u, resp. ψ and φ be related by (2.8). Then

$$\langle dE_\alpha(u), \varphi \rangle = \langle dI_{\alpha/\alpha_n}(v), \psi \rangle.$$

In particular, for $\alpha = \alpha_n$, $u \in H_0^{1,n}(B_R(0))$ with $\| u \|_{1,n} = 1$:

$$\| dE_\alpha(u) \| = \sup_{\varphi \neq 0} | \langle dE_{\alpha_n}(u), \varphi \rangle / \| \varphi \|_{1,n}$$

$$= \sup_{\psi \neq 0} | \langle dI_1(v), \psi \rangle | / \| \psi \|_{1,n} = \| dI_1(v) \|.$$

Similarly, if we also denote $\rho: v \to v/\| v \|_{1,n}$ the radial projection in $H_0^{1,n}([0, \infty[)$ we have

$$(E_{\alpha_n} \circ \pi)(u) = (I_1 \circ \rho)(v), \quad \| d(E_{\alpha_n} \circ \pi)(u) \| = \| d(I_1 \circ \rho)(v) \|; \tag{2.10}$$

Analogous to (2.4)-(2.5) we also have the explicit expression for $v \in \Sigma \subset H_0^{1,n}([0, \infty[)$:

$$\langle d(I_1 \circ \rho)(v), \psi \rangle$$

$$= \frac{n}{n-1} \left(\int_0^\infty \psi v^{1/(n-1)} \exp(| v |^{n/(n-1)} - t) \, dt \right.$$

$$\left. - \lambda \int_0^\infty \dot{\psi} \dot{v} | \dot{v} |^{n-2} \, dt \right) \tag{2.11}$$

with

$$\lambda=\int_0^\infty |v|^{n/(n-1)}\exp(|v|^{n/(n-1)}-t)\,dt, \tag{2.12}$$

and $v\in\Sigma$ is critical for I_ρ on Σ iff for some number $\lambda>0$ given by (2.12) v satisfies the equation

$$-(\dot v\,|\dot v|^{n-2})^{\cdot}=\lambda^{-1}v^{1/(n-1)}\exp(|v|^{n/(n-1)}-t)\quad\text{in } [0,\infty[, \tag{2.13}$$

with initial and asymptotic data

$$v(0)=0,\qquad \dot v(t)\to 0\quad (t\to\infty). \tag{2.14}$$

Moreover, for the analysis of the functional I_1 on Σ Hölder's inequality for $v\in\Sigma$

$$v^n(t)=\left(\int_0^t \dot v\,dt\right)^n \leqq t^{n-1}\int_0^t |\dot v|^n\,dt \leqq t^{n-1} \tag{2.15}$$

and invariance of the $H_0^{1,n}$-norm with respect to scaling

$$v\to v_\lambda(t)=\lambda^{(1-n)/n}\cdot v(\lambda t) \tag{2.16}$$

will be used repeatedly.

NOTATIONS. — The letter c denotes a generic constant. For simplicity we often write $\|u\|=\|u\|_{1,2}$ if no confusion is possible.

3. PROOF OF THEOREM 1.4

In terms of the functional $I_1: H_0^{1,2}([0,\infty[)\to\mathbb{R}$ Theorem 1.4 may be rephrased as follows:

THEOREM 3.1. — *Suppose $n=2$ and let $\{v_m\}$ be a sequence of functions $v_m\in H_0^{1,2}([0,\infty[)$ with $\|v_m\|_{1,2}=1$. Assume that $I_1(v_m)\to\beta>1$ while $v_m\to 0$ in $H_{loc}^{1,2}([0,\infty[)$. Moreover, suppose that $\|d(I\circ\rho)(v_m)\|\leqq C$ uniformly.*

Then there exists a sequence $\tau_m\to\infty$, and a function $w\in H_{loc}^{1,2}(\mathbb{R})$ such that the shifted functions

$$w_m(t):\equiv v_m^2(\tau_m+t)-v_m^2(\tau_m)-t\to w\quad \textit{in } H_{loc}^{1,2}(\mathbb{R}), \tag{3.1}$$

where w *solves the* limiting problem *associated with* I_1*:*

$$2\ddot{w}+e^{w}=0 \quad in\ ]-\infty, \infty[, \tag{3.2}$$

$$w(0)=\dot{w}(0)=0. \tag{3.3}$$

Moreover, necessarily $\beta \leqq 1+e$.

Theorem 1.4 will be a consequence of Theorem 3.1 and the explicit form of the solution to (3.2)-(3.3). The constant $\bar{\kappa}$ will be obtained as $\bar{\kappa}=\lim_{m\to\infty}(\tau_m-v_m^2(\tau_m))$.

We now give the *proof of Theorem* 3.1.

Step 1. — Definition of τ_m.

Using (2.15) we may choose $t_m \geqq 1$ such that

$$\frac{v_m^2(t_m)}{t_m}=\sup_{t\geqq 1}\frac{v_m^2(t)}{t}.$$

LEMMA 3.2. — $v_m^2(t_m)/t_m \to 1$ *as* $m\to\infty$.

Proof. — By (2.15) clearly $v_m^2(t)/t \leqq 1$ for all $t>0$, m $\mathbb{N}$. Suppose by contradiction that for some $\varepsilon>0$ and all $t\geqq 1$ there holds the estimate

$$v_m^2(t)-t\leqq -\varepsilon t$$

uniformly in m. Then by (2.15) and since $v_m \to 0$ in $H_{loc}^{1,2}$, we may estimate for arbitrary $\Lambda \geqq 1$:

$$\begin{aligned} I_1(v_m) &= \int_0^\Lambda \exp(v_m^2-t)\,dt+\int_\Lambda^\infty \exp(v_m^2-t)\,dt \\ &\leqq \sup_{t\leqq\Lambda}\exp(v_m^2(t))\int_0^\Lambda \exp(-t)\,dt+\int_\Lambda^\infty \exp(-\varepsilon t)\,dt \\ &\leqq \sup_{t\leqq\Lambda}\exp(v_m^2(t))+\varepsilon^{-1}\exp(-\varepsilon\Lambda)\to 1+\varepsilon^{-1}\exp(-\varepsilon\Lambda). \end{aligned}$$

Since Λ was arbitrary we conclude that $I_1(v_m)\to\beta=1$, contrary to hypothesis.

Q.E.D.

Remark that since $v_m\to 0$ in $H_{loc}^{1,2}$, by Lemma 3.2 necessarily $t_m\to\infty$ as $m\to\infty$.

Conversely to Lemma 3.2 there holds

LEMMA 3.3. — *Given $\delta > 0$ there exists $\varepsilon > 0$ such that*

$$v_m^2(t)/t \leqq 1-\varepsilon$$

whenever $t \geqq 1$ and $|t-t_m| \geqq \delta t_m$, provided m is sufficiently large.

Proof. — Suppose that for sequences s_m, $t_m \geqq 1$ we have $v_m^2(s_m)/s_m$, $v_m^2(t_m)/t_m \to 1$ while $r_m = s_m/t_m \not\to 1$.

Possibly exchanging $\{s_m\}$ and $\{t_m\}$ we may assume that $s_m < t_m$. Scaling with (2.16), we introduce $w_m = (v_m)_{t_m}$. $\{w_m\}$ satisfies $\|w_m\|_{1,2} = 1$ while by (2.15)

$$\int_0^{r_m} |\dot{w}_m|^2\,dt = \int_0^{s_m} |\dot{v}_m|^2\,dt \geqq v_m^2(s_m)/s_m \to 1. \tag{3.4}$$

We may assume that $w_m \to w$ weakly in $H_0^{1,2}([0, \infty[)$ and locally uniformly, while $r_m \to r < 1$. By (3.4)

$$\int_r^{\infty} |\dot{w}|^2\,dt \leqq 1 - \liminf_{m \to \infty} \int_0^{r_m} |\dot{w}_m|^2\,dt = 0,$$

and $w(t) \equiv w(r)$ is constant for $t \geqq r$. By (2.15) again $w^2(r) \leqq r < 1$. But on the other hand

$$w^2(r) = w^2(1) = \lim_{m \to \infty} w_m^2(1) = \lim_{m \to \infty} v_m^2(t_m)/t_m = 1.$$

The contradiction proves the lemma.

Q.E.D.

For arbitrary $\delta \in]0, 1[$ now let $\tau_m \geqq \delta t_m$, κ_m be defined as follows:

$$\tau_m - v_m^2(\tau_m) = \inf_{t \geqq \delta t_m} t - v_m^2(t) =: \kappa_m.$$

Note that since $t_m \to \infty$ also $\tau_m \to \infty$. Moreover, by Lemma 3.2

$$\frac{\kappa_m}{\tau_m} \leqq \frac{t_m - v_m^2(t_m)}{\delta t_m} \to 0 \tag{3.5}$$

and hence also

$$\frac{v_m^2(\tau_m)}{\tau_m} = 1 - \frac{\kappa_m}{\tau_m} \to 1$$

as $m \to \infty$. In particular, by Lemma 3.3 now also $\tau_m/t_m \to 1$, and we note that the definition of τ_m is in fact independent of δ, for sufficiently large m.

Step 2. — Boundedness of the sequence $\{\kappa_m\}$.

LEMMA 3.4. — *For any* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for sufficiently large m there holds*

$$\int_{t_m - \delta t_m}^{t_m + \delta t_m} |\dot{v}_m|^2 \, dt \leqq \varepsilon.$$

Proof. — By Hölder's inequality (2.15)

$$\int_{t_m - \delta t_m}^{t_m + \delta t_m} |\dot{v}_m|^2 \, dt \leqq 1 - \int_0^{t_m - \delta t_m} |\dot{v}_m|^2 \, dt$$

$$\leqq 1 - \frac{v_m^2(t_m - \delta t_m)}{t_m - \delta t_m} = 1 - \frac{v_m^2(t_m)}{t_m - \delta t_m} + \frac{v_m^2(t_m) - v_m^2(t_m - \delta t_m)}{t_m - \delta t_m}.$$

Again using Hölder's inequality and the definition of t_m:

$$v_m^2(t_m) - v_m^2(t_m - \delta t_m)$$

$$= (v_m(t_m) + v_m(t_m - \delta t_m))(v_m(t_m) - v_m(t_m - \delta t_m))$$

$$\leqq 2 v_m(t_m) \int_{t_m - \delta t_m}^{t_m} |\dot{v}_m| \, dt \leqq 2 v_m(t_m) \sqrt{\delta t_m} \leqq 2 \sqrt{\delta} \, t_m.$$

Hence from Lemma 3.2 we infer that for any $\varepsilon > 0$ we may choose $\delta > 0$ such that

$$\int_{t_m - \delta t_m}^{t_m + \delta t_m} |\dot{v}_m|^2 \, dt \leqq 1 - \frac{1}{1-\delta} + \frac{2\sqrt{\delta}}{1-\delta} + o(1) < \varepsilon, \quad (^2)$$

if m is sufficiently large.

Q.E.D.

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(2) $o(1)$ denotes error terms $o(1) \to 0 \, (m \to \infty)$.

The next lemma relates κ_m and the asymptotic growth of the functions w_m given by (3.1):

LEMMA 3.5. — *Given any* $\varepsilon > 0$, $\Lambda \geqq 1$, *there holds the inequality*

$$\int_{-\Lambda\varkappa_m}^{0} |\dot{w}_m - 1|^2 \, dt \leqq (4+\varepsilon)\, \kappa_m,$$

provided m is sufficiently large.

Proof. — Compute

$$\int_{-\Lambda\varkappa_m}^{0} |\dot{w}_m - 1|^2 \, dt = 4 \int_{\tau_m - \Lambda\varkappa_m}^{\tau_m} |v_m \dot{v}_m - 1|^2 \, dt$$

$$= 4 \int_{\tau_m - \Lambda\varkappa_m}^{\tau_m} |v_m(\tau_m)\dot{v}_m - 1|^2 \, dt + 8 \int_{\tau_m - \Lambda\varkappa_m}^{\tau_m} (v_m(\tau_m) - v_m)\, \dot{v}_m \, dt$$

$$- 4 \int_{\tau_m - \Lambda\varkappa_m}^{\tau_m} (v_m^2(\tau_m) - v_m^2)\, |\dot{v}_m|^2 \, dt.$$

By definition of τ_m and (3.5) for any $t \in [\tau_m - \Lambda\kappa_m, \tau_m]$ there holds $v_m^2(t) \leqq v_m^2(\tau_m)$. Hence the last term is non-negative.

Moreover,

$$\int_{\tau_m - \Lambda\varkappa_m}^{\tau_m} |v_m(\tau_m)\dot{v}_m - 1|^2 \, dt \leqq \int_{0}^{\tau_m} |v_m(\tau_m)\dot{v}_m - 1|^2 \, dt$$

$$= v_m^2(\tau_m)\left(\int_0^{\tau_m} |\dot{v}_m|^2 \, dt - 2\right) + \tau_m \leqq \tau_m - v_m^2(\tau_m) = \kappa_m.$$

Finally, by (3.5) and Lemma 3.4

$$\int_{\tau_m - \Lambda\varkappa_m}^{\tau_m} (v_m(\tau_m) - v_m)\, \dot{v}_m \, dt$$

$$\leqq \sup_{0 \leqq t \leqq \Lambda\varkappa_m} |v_m(\tau_m) - v_m(\tau_m - t)| \, . \int_{\tau_m - \Lambda\varkappa_m}^{\tau_m} |\dot{v}_m| \, dt$$

$$\leqq \left(\int_{\tau_m - \Lambda\varkappa_m}^{\tau_m} |\dot{v}_m| \, dt\right)^2 \leqq \Lambda\kappa_m \int_{\tau_m - \Lambda\varkappa_m}^{\tau_m} |\dot{v}_m|^2 \, dt \leqq \varepsilon\kappa_m,$$

if m is sufficiently large.

Q.E.D.

Let

$$\lambda_m = \int_0^\infty v_m^2 \exp(v_m^2 - t)\, dt$$

denote the Lagrange-multipliers associated with v_m by (2.12). We establish:

LEMMA 3.6. — *$\lambda_m / v_m^2(t_m) \to \beta - 1$, as $m \to \infty$.*

Proof. — Since $v_m \to 0$ locally uniformly for any $\Lambda > 0$ we have

$$\int_0^\Lambda v_m^2 \exp(v_m^2 - t)\, dt \to 0, \qquad \int_0^\Lambda \exp(v_m^2 - t)\, dt \to 1 - \exp(-\Lambda).$$

Moreover, by Lemma 3.3 for any $\delta > 0$ there is $\varepsilon > 0$ such that by (2.15)

$$\int_{\{t \mid t \geqq \Lambda,\ |t - t_m| \geqq \delta t_m\}} (1 + v_m^2) \exp(v_m^2 - t)\, dt \leqq \int_\Lambda^\infty (1 + t) \exp(-\varepsilon t)\, dt + o(1),$$

and the last integral can be made arbitrarily small by choosing Λ sufficiently large.

Now choose a sequence of numbers $\delta_k > 0$, $\delta_k \to 0$.

By the above we obtain that for any fixed k, as $m \to \infty$:

$$\int_{\{t \mid |t - t_m| \geqq \delta_k t_m\}} v_m^2 \exp(v_m^2 - t)\, dt \to 0, \tag{3.6}$$

$$\int_{\{t \mid |t - t_m| \geqq \delta_k t_m\}} \exp(v_m^2 - t)\, dt \to 1, \tag{3.7}$$

while by Lemma 3.4 as $k \to \infty$

$$\limsup_{m \to \infty} \int_{t_m - \delta_k t_m}^{t_m + \delta_k t_m} |\dot{v}_m|^2\, dt \to 0. \tag{3.8}$$

Relabelling the sequence $\{\delta_k\}$, moreover, we may assume that (3.6)-(3.8) hold with δ_m.

Next, by Hölder's inequality (2.15), for $|t-t_m| \leqq \delta_m t_m$:

$$v_m^2(t)-v_m^2(t_m)=(v_m(t)+v_m(t_m))(v_m(t)-v_m(t_m))$$

$$\leqq 2(t_m+\delta_m t_m)^{1/2}\left(\int_{t_m-\delta_m t_m}^{t_m+\delta_m t_m} |\dot v_m|\,dt\right)$$

$$\leqq 2(t_m+\delta_m t_m)^{1/2}(2\delta_m t_m)^{1/2}\left(\int_{t_m-\delta_m t_m}^{t_m+\delta_m t_m} |\dot v_m|^2\,dt\right)^{1/2} \leqq 4\sqrt{\delta_m}\, t_m.$$

By Lemma 3.2 therefore for such t:

$$v_m^2(t)/v_m^2(t_m) \to 1, \tag{3.9}$$

as $m \to \infty$.

Hence by (3.6)-(3.7) we obtain:

$$\lambda_m/v_m^2(t_m)=\int_{t_m-\delta_m t_m}^{t_m+\delta_m t_m} \exp(v_m^2-t)+o(1)$$

$$=\beta-\int_{\{t\ |\ |t-t_m|\geqq \delta_m t_m\}} \exp(v_m^2-t)\,dt+o(1)\to\beta-1,$$

and the proof is complete.

Q.E.D.

Now we are ready to prove:

LEMMA 3.7. — *The sequence* $\{\kappa_m\}$ *is uniformly bounded.*

Proof. — Suppose by contradiction that $\kappa_m \to \infty$. For some $\Lambda > 0$ to be determined in the sequel let η_m be functions $\eta_m \in H_0^{1,2}(\mathbb{R})$ with support in $[-\Lambda\kappa_m, 1]$ such that $\|\eta_m\|_{1,2} \leqq c$ and $|\eta_m(t)| \leqq c$ uniformly. Define testing functions $\varphi_m(t) \equiv 2v_m(t)\eta_m(t-\tau_m)/v_m^2(t_m) \in H^{1,2}([0,\infty[)$. Note that $\|\varphi_m\|_{1,2} \to 0$ as $m \to \infty$. Hence, boundedness of $\|d(I_1 \circ \rho)(v_m)\|$ implies that

$$\lambda_m\int_0^\infty \dot\varphi_m \dot v_m\,dt-\int_0^\infty \varphi_m v_m \exp(v_m^2-t)\,dt \to 0.$$

By Lemma 3.6 and using (3.5), Lemma 3.2, and (3.9) this implies

$$2\int_0^\infty |\dot{v}_m|^2 \eta_m(.-\tau_m)\,dt + \int_0^\infty (v_m^2)^{\cdot}\,\dot{\eta}_m(.-\tau_m)\,dt - \frac{2}{\beta-1}\int_0^\infty \eta_m(.-\tau_m)\exp(v_m^2-t)\,dt \to 0.$$

Hence, using Lemma 3.4 we may conclude that

$$\int_{-\infty}^\infty \dot{w}_m\,\dot{\eta}_m\,dt - \frac{2}{\beta-1}\int_{-\infty}^\infty \eta_m \exp(w_m-\kappa_m)\,dt \to 0. \tag{3.10}$$

Incidentally, the same reasoning will yield (3.10) for a sequence $\eta_m \in H_0^{1,2}(\mathbb{R})$ with support in any fixed finite interval and having $\|\eta_m\|_{1,2} \leqq c$, $|\eta_m(t)| \leqq c$, uniformly. We will return to this later.

Now we use Lemma 3.5. Choose $\Lambda = 40$, $\varepsilon = 1$. Then for suitable numbers $s_m \in [-\Lambda\kappa_m, -1]$ and sufficiently large m:

$$\left|\int_{s_m}^{s_m+1} \dot{w}_m\,dt - 1\right|^2 \leqq \int_{s_m}^{s_m+1} |\dot{w}_m - 1|^2\,dt \leqq \frac{2}{\Lambda\kappa_m}\int_{-\Lambda\kappa_m}^0 |\dot{w}_m - 1|^2\,dt \leqq \frac{10}{\Lambda} \leqq \frac{1}{4}.$$

Now let

$$\eta_m(t) = \begin{cases} 1-t, & 0 \leqq t \leqq 1 \\ 1, & s_m+1 \leqq t \leqq 0 \\ t-s_m, & s_m \leqq t \leqq s_m+1 \\ 0, & \text{else} \end{cases}$$

Then from (3.10) we obtain that

$$w_m(0) - w_m(1) + \int_{s_m}^{s_m+1} \dot{w}_m\,dt \leqq \frac{2}{\beta-1} e^{-\kappa_m}\int_{-\Lambda\kappa_m}^1 e^{w_m}\,dt + o(1).$$

But by definition of τ_m and choice of s_m:

$$w_m(t) \leqq w_m(0) = 0, \qquad \int_{s_m}^{s_m+1} \dot{w}_m\,dt \geqq \frac{1}{2}.$$

Hence the above estimate implies that

$$\frac{1}{2} \leqq \frac{2}{\beta-1}(\Lambda \kappa_m + 1) e^{-\varkappa_m} + o(1) \to 0$$

as $m \to \infty$. The contradiction implies the claim.

Q.E.D.

Step 3. – A subsequence $\{w_m\}$ converges locally in $H^{1,2}_{\mathrm{loc}}(\mathbb{R})$ to a solution w of (3.2)-(3.3).

First we establish that $\int |\dot{w}_m|^2 \, dt$ is locally bounded. We use boundedness of $\{\kappa_m\}$ to sharpen Lemma 3.5 as follows:

LEMMA 3.8. – *For any* $\Lambda > 0$ *there holds the estimate*

$$\int_{-\Lambda}^{\Lambda} |\dot{w}_m + \operatorname{sgn}(t)|^2 \, dt \leqq 4\kappa_m + o(1).$$

Proof. – As in the proof of Claim 3.5 we write

$$\begin{aligned}
A := \int_{-\Lambda}^{\Lambda} |\dot{w}_m + \operatorname{sgn}(t)|^2 \, dt
&= 4 \int_{\tau_m - \Lambda}^{\tau_m} |v_m(\tau_m)\dot{v}_m - 1|^2 \, dt + 4 \int_{\tau_m}^{\tau_m + \Lambda} |v_m(\tau_m)\dot{v}_m|^2 \, dt \\
&\quad - 4 \int_{\tau_m - \Lambda}^{\tau_m} [(v_m(\tau_m) + v_m)\dot{v}_m - 2][(v_m(\tau_m) - v_m)\dot{v}_m] \, dt \\
&\quad - 4 \int_{\tau_m}^{\tau_m + \Lambda} [(v_m(\tau_m) + v_m)\dot{v}_m][(v_m(\tau_m) - v_m)\dot{v}_m] \, dt \\
&\leqq 4 \left[v_m^2(\tau_m) \left(\int_0^{\infty} |\dot{v}_m|^2 \, dt - 2 \right) + \tau_m \right] \\
&\quad + 8 \sup_{\tau_m - \Lambda \leqq t \leqq \tau_m} |v_m(\tau_m) - v_m(t)| \int_{\tau_m - \Lambda}^{\tau_m} |\dot{v}_m| \, dt \\
&\quad + 8 \sup_{\tau_m \leqq t \leqq \tau_m + \Lambda} |v_m(\tau_m) - v_m(t)| \\
&\qquad \times \sup_{\tau_m \leqq t \leqq \tau_m + \Lambda} |v_m(t)| \int_{\tau_m}^{\tau_m + \Lambda} |\dot{v}_m|^2 \, dt.
\end{aligned}$$

Estimating

$$|v_m(\tau_m)-v_m(t)| \leqq \int_{\tau_m-\Lambda}^{\tau_m} |\dot{v}_m|\,dt \qquad \text{if} \quad t\in[\tau_m-\Lambda,\,\tau_m]$$

resp.

$$|v_m(\tau_m)-v_m(t) \leqq \int_{\tau_m}^{\tau_m+\Lambda} |\dot{v}_m|\,dt \leqq \left(\Lambda\int_0^\infty |\dot{v}_m|^2\,dt\right)^{1/2} = \sqrt{\Lambda},$$

$$|v_m(t)| \leqq \int_0^t |\dot{v}_m|\,dt \leqq \left((\tau_m+\Lambda)\int_0^\infty |\dot{v}_m|^2\,dt\right)^{1/2} = \sqrt{\tau_m+\Lambda} \leqq 2\sqrt{\tau_m}$$

if $t_m\in[\tau_m,\,\tau_m+\Lambda]$, $m \geqq m_0(\Lambda)$, we may bound

$$A \leqq 4\kappa_m + 8\int_{\tau_m-\Lambda}^{\tau_m} |\dot{v}_m|^2\,dt + 16\sqrt{\Lambda\tau_m}\int_{\tau_m}^{\tau_m+\Lambda} |\dot{v}_m|^2\,dt.$$

But by (3.5)

$$\int_{\tau_m}^{\tau_m+\Lambda} |\dot{v}_m|^2\,dt \leqq 1-\int_0^{\tau_m} |\dot{v}_m|^2\,dt \leqq 1-\frac{v_m^2(\tau_m)}{\tau_m} = \frac{\kappa_m}{\tau_m},$$

whence the claim follows from Lemmas 3.4 and 3.7.

Q.E.D.

LEMMA 3.9. — *A subsequence* $\{w_m\}$ *converges strongly locally in* $H_{loc}^{1,2}(\mathbb{R})$ *to a solution* w *of* (3.2)-(3.3).

Proof. — Boundedness of w_m in $H_{loc}^{1,2}(\mathbb{R})$ follows from Lemma 3.8. In particular, we may assume that $w_m \to w$ weakly locally in $H_{loc}^{1,2}(\mathbb{R})$ and locally uniformly. By Lemma 3.7, clearly, we may likewise suppose that $\kappa_m \to \kappa$ as $m \to \infty$.

Inserting $\eta_m=(w_m-w)\,\psi$, where $\psi\in C_0^\infty(\mathbb{R})$, into (3.10), by uniform local convergence $w_m \to w$ we now obtain that $w_m \to w$ also strongly locally in $H_{loc}^{1,2}(\mathbb{R})$.

Next, choosing a fixed $\eta\in C_0^\infty(\mathbb{R})$ and passing to the limit $m \to \infty$ in (3.10), w is a solution to the equation

$$2\ddot{w} + \frac{4}{\beta-1}e^{-\varkappa}e^{w}=0 \tag{3.11}$$

with $w(t) \leqq w(0)=0$ for all t. I.e. w satisfies (3.3). Moreover, by Lemma 3.8 and (3.11) $\dot{w}(t) \to 1$ monotonically and hence $w(t) \to -\infty$ as $t \to -\infty$.

Denote for convenience

$$C_0 = \frac{4}{\beta - 1} e^{-\varkappa}. \tag{3.12}$$

Multiply (3.11) by $\dot{w}$ and integrate from $-\infty$ to 0. This gives the identity

$$1 = |\dot{w}(-\infty)|^2 = -\int_{-\infty}^{0} \frac{d}{dt}(|\dot{w}|^2)\,dt = C_0 \int_{-\infty}^{0} \frac{d}{dt} e^{w}\,dt = C_0, \tag{3.13}$$

whence w is a solution to (3.2), as claimed.

Q.E.D.

Step 4. – Estimating β.

Note that by (3.12)-(3.13) necessarily $\beta = 4e^{-\varkappa}+1$. The required estimate hence is a consequence of

LEMMA 3.10. – $\kappa \geqq \ln 4 - 1$.

Proof. – Note that by unique solvability of (3.2)-(3.3) $w(t)=w(-t)$. Hence by weak lower semi-continuity from Lemma 3.8 we obtain the estimate

$$\begin{aligned}\kappa \geqq \frac{1}{4} \lim_{\Lambda \to \infty} \int_{-\Lambda}^{\Lambda} |\dot{w} + \operatorname{sgn}(t)|^2\,dt \\ = \frac{1}{2} \lim_{\Lambda \to \infty} \int_{0}^{\Lambda} (|\dot{w}|^2 + 2\dot{w} + 1)\,dt \\ = \frac{1}{2} \lim_{\Lambda \to \infty} \left(\int_{0}^{\Lambda} (|\dot{w}|^2 - 1)\,dt + 2(w(\Lambda) + \Lambda) \right).\end{aligned}$$

However, testing (3.2) with $\dot{w}$ and integrating between 0 and t we deduce

$$|\dot{w}(t)|^2 = 1 - e^{w(t)} = 1 + 2\ddot{w}(t), \tag{3.14}$$

whence

$$\kappa \geqq \lim_{\Lambda \to \infty} (w(\Lambda) + \Lambda + \dot{w}(\Lambda)) = \lim_{\Lambda \to \infty} (w(\Lambda) + \Lambda) - 1.$$

Now, since $\dot{w}(0)=0$, by (3.2) w is strictly decreasing on $[0, \infty[$. Hence the map $t \to s=e^{w(t)}$ is invertible and by (3.14) the differential is given by

$$\frac{dt}{ds}=\frac{1}{e^{w(t)}\dot{w}(t)}=\frac{-1}{s\sqrt{1-s}}.$$

Integrating we obtain that

$$\lim_{t \to \infty}(w(t)+t)=\lim_{w \to -\infty}(w+t(e^w))=\lim_{w \to -\infty}\left(w+\int_{e^w}^{1}\frac{ds}{s\sqrt{1-s}}\right). \quad (3.15)$$

Substituting

$$r=\sqrt{1-s},$$

we may write

$$\begin{aligned}\int_{e^w}^{1}\frac{ds}{s\sqrt{1-s}}&=\int_0^{\sqrt{1-e^w}}\frac{2\,dr}{1-r^2}\\&=\int_0^{\sqrt{1-e^w}}\frac{dr}{1+r}+\int_0^{\sqrt{1-e^w}}\frac{dr}{1-r}\\&=\ln(1+\sqrt{1-e^w})-\ln(1-\sqrt{1-e^w})\\&=\ln 2-\ln\left(\frac{1}{2}e^w\right)+o(1)\\&=\ln 4-w+o(1),\end{aligned} \quad (3.16)$$

where $o(1) \to 0$ as $w \to -\infty$. The lemma follows.

Q.E.D.

The proof of Theorem 3.1 is complete.

Proof of Theorem 1.4. — Let $\bar{w}_m$ be defined by (1.6) with $R_m=\exp(-\tau_m/2)$, $\bar{\kappa}=\kappa$. Let

$$w_m^*(t)=\bar{w}_m(e^{-t/2}).$$

By definition of τ_m and κ_m, and since $\kappa_m \to \kappa$, Lemma 3.9 yields that

$$w_m^*(t) \equiv v_m^2(\tau_m+t)-\tau_m+\kappa \to w(t)+t=:w^*(t)$$

in $H^{1,2}_{loc}(\mathbb{R})$ and locally uniformly. From (3.2), (3.15)-(3.16) we deduce that

$$2\ddot{w}^* + e^{w^*-t} = 0 \quad \text{in }]-\infty, \infty[\tag{3.17}$$

with asymptotic behavior

$$\begin{aligned} w^*(t) &\to \ln 4 \qquad (t \to \infty), \\ w^*(t) - 2t &\to \ln 4 \qquad (t \to -\infty). \end{aligned} \tag{3.18}$$

Hence

$$\bar{w}_m \to 2\bar{w} \in H^{1,2}_{loc}(\mathbb{R}^2 \setminus \{0\})$$

uniformly locally, where

$$\bar{w}(e^{-t/2}) = \frac{1}{2} w^*(t). \tag{3.19}$$

(1.8) is immediate from (3.18). To see (1.7) observe that (3.17), (1.7) resp. are the Euler-Lagrange equations of two functionals

$$\int (|\dot{w}^*|^2 - e^{w^*-t})\, dt, \qquad \frac{1}{2}\int (|\nabla \bar{w}|^2 - e^{2\bar{w}})\, dx,$$

resp. that are transformed into one another under (3.19).

Finally, the characterization of $\bar{w}$ given in (1.9) can be obtained as gollows: Let Φ be the stereographic projection $\Phi: \mathbb{R}^2 \to S^2 \subset \mathbb{R}^3$, and let $W = \bar{w} \circ \Phi^{-1} + \frac{1}{2} \log \det (d\Phi^{-1})$. By (1.8)-(1.9) W is a bounded function on S^2. Moreover, by conformal invariance of (1.7), W (weakly) solves the equation

$$-\Delta W = e^{2W} - 1 \quad \text{on } S^2. \tag{3.20}$$

Hence W is smooth, in fact analytic. But all smooth solutions to (3.20) are of the form $W = \frac{1}{2} \log \det (d\psi)$, ψ a conformal map of S^2. Thus $\bar{w}$ is of the form $w = \frac{1}{2} \log \det (d\psi)$ for some conformal map $\psi: \mathbb{R}^2 \to S^2$. Since $\bar{w}$ is radically symmetric with respect to the origin, the characterization (1.9) follows.

4. PROOF OF THEOREM 1.5

For a non-negative function $u \in C_0^\infty(\mathbb{R}^2)$ denote u^* its radial symmetrization (cp. [8], Chap. VII)

$$u^*(r) = c : \text{meas}\{x \mid u(x) \geqq c\} = \text{meas}(B_r(0)).$$

For a bounded domain $\Omega \subset \mathbb{R}^2$, considering $C_0^\infty(\Omega) \subset C_0^\infty(\mathbb{R}^2)$, the symmetrization-map $*$ extends to a map

$$*: H_0^{1,2}(\Omega) \cap \{u \geqq 0\} \to H_{0,\text{rad}}^{1,2}(B_R(0)) \quad (^3),$$

where meas $(\Omega) = $ meas $(B_R(0))$, with the properties:

$$\int_{B_R(0)} |\nabla u^*|^2 \, dx \leqq \int_\Omega |\nabla u|^2 \, dx, \tag{4.1}$$

$$E_\alpha^*(u^*) := ⨍_{B_R(0)} \exp(\alpha |u^*|^2) \, dx = ⨍_\Omega \exp(\alpha |u|^2) \, dx = E_\alpha(u), \tag{4.2}$$

for all $u \in H_0^{1,2}(\Omega)$, $u \geqq 0$, and all $\alpha > 0$.

Now suppose $\{u_m\} \subset H_0^{1,2}(\Omega)$ is a maximizing sequence for $E_{4\pi}$ in $\Sigma(\Omega)$; however, assume that the supremum of $E_{4\pi}$ on Σ is not attained. By Theorem 1.3 there exists a point $x_0 \in \bar{\Omega}$ such that $|\nabla u_m|^2 \, dx \to \delta_{x_0}$ weakly in measure.

Moreover, we may assume that $u_m \geqq 0$, otherwise we consider the new maximizing sequence $\tilde{u}_m = |u_m|$.

Let $\{u_m^*\} \subset H_{0,\text{rad}}^{1,2}(B_R(0))$ be the symmetrized sequence.

LEMMA 4.1. — *$|\nabla u_m^*|^2 \, dx \to \delta_0$ weakly in measure.*

Proof. — $\|u_m^*\|_{1,2} \leqq 1$, hence we may assume that $u_m^* \to u^*$ weakly in $H^{1,2}$, strongly in L^2. But $*$ preserves the L^2-norm. Thus, since $u_m \to 0$ weakly in $H^{1,2}$ and strongly in L^2, necessarily $u_m^* = 0$. Negating the asser-

(3) We denote $H_{0,\text{rad}}^{1,2}(B_R(0)) = H_0^{1,2}(B_R(0)) \cap \{u(x) = u(|x|)\}$. the space of radial $H_0^{1,2}(\Omega)$- functions.

tion of the lemma, by Theorem 1.3 (ii)

$$E_{4\pi}(u_m) = E^*_{4\pi}(u^*_m) \to E^*_{4\pi}(0) = 1 < \sup_{\Sigma(\Omega)} E_{4\pi},$$

and we have a contradiction.

Q.E.D.

In view of Lemma 4.1 and Theorem 3.1 the proof will be complete once we establish a uniform bound for $\| d(E^*_{4\pi} \circ \pi)(u^*_m) \|$. To achieve this requires the construction of a pseudo-gradient flow for $(E^*_{4\pi} \circ \pi) =: E$ in a neighborhood of $\Sigma = \{ u \in H^{1,2}_{0, \mathrm{rad}}(B_R(0)) \mid \| u \| = 1 \}$. We briefly recall the following concepts:

DEFINITION 4.2. — Let U be a norm-neighborhood of Σ, $0 \notin U$ and let $\tilde{U} = \{ u \in U \mid dE(u) \neq 0 \}$ be the set of regular points of E in U. A locally Lipschitz continuous vector field $e : \tilde{U} \to H^{1,2}_{0, \mathrm{rad}}(B_R(0))$ is a pseudo-gradient vector field for E iff e satisfies:

(i) $\| e(u) \| < 2 \min \{ \| dE(u) \|, 1 \}$,

(ii) $\langle dE(u), e(u) \rangle > \min \{ \| dE(u) \|^2, \| dE(u) \| \}$.

A C^1-functional E on an open subset of a Banach space always admits a pseudo-gradient vector field, cp. [7], p. 204 ff.

The vector field e defines a pseudo-gradient flow

$$\Phi : D(\Phi) \subset \tilde{U} \times \mathbb{R} \to H^{1,2}_{0, \mathrm{rad}}(B_R(0))$$

via the initial-value problem

$$\begin{aligned} \frac{d}{dt} \Phi(u, t) &= e(\Phi(u, t), \\ \Phi(u, 0) &= u. \end{aligned} \tag{4.3}$$

Note that E is non-decreasing along flow-lines, in fact by Definition 4.2 (ii)

$$\left. \frac{d}{dt} E(\Phi(u, t)) \right|_{t=0} > \min \{ \| dE(u) \|^2, \| dE(u) \| \}. \tag{4.4}$$

Finally, note that if $u \in \Sigma$ and $\| dE(\Phi(u, t)) \| \geqq c > 0$ for all t such that $(u, t) \in D(\Phi)$, then by boundedness of e, Definition 4.2 (i), the solution $\Phi(u, t)$ through $\Phi(u, 0) = u$ is defined in a time interval $0 \leqq t \leqq T$ of length $T > 0$ independent of u.

Now consider $v_m = u_m^*/\| u_m^* \| \in \Sigma$, and let

$$t_m(C) := \inf(\{ t \geqq 0 \mid \| dE(\Phi(v_m, t) \| \leqq C \} \cup \{ T \})$$

LEMMA 4.3. — $t_m(C) \to 0$ *as* $C \to \infty$, *uniformly in* m.

Proof. — Otherwise there exists a sequence $C_k \to \infty$ and a number $T_1 > 0$ such that for any k and some $m = m(k)$ the flow $\Phi(v_m, t)$ is defined for all $t \in [0, T_1]$ and

$$\| dE(\Phi(v_m, t)) \| > C_k$$

uniformly in $t \in [0, T_1]$. By (4.4)

$$E(\Phi(v_m, T_1)) = E(v_m) + \int_0^{T_1} \frac{d}{dt} E(\Phi(v_m, t))\, dt \geqq 1 + T_1 C_k$$

becomes arbitrarily large as $k \to \infty$, contradicting Theorem 1.1.

Q.E.D.

Proof of Theorem 1.5. — Choose a sequence $C_k \to \infty$ and let $v_{m,k} = \Phi(v_m, t_m(C_k))$, $\bar{v}_{m,k} = v_{m,k}/\| v_{m,k} \|$. For large k $\{ \bar{v}_{m,k} \}$ satisfies:

$$\left.\begin{array}{c} E_{4\pi}^*(\bar{v}_{m,k}) \geqq E_{4\pi}^*(v_m) \geqq E_{4\pi}(u_m) \\ \| d(E_{4\pi}^* \circ \pi)(\bar{v}_{m,k}) \| \leqq C_k \end{array}\right\} \tag{4.5}$$

uniformly in m, while as $k \to \infty$

$$\| \bar{v}_{m,k} - v_m \| \leqq \| v_{m,k} - v_m \| + 1 - \| v_{m,k} \| \leqq 4\, t_m(C_k) \to 0 \tag{4.6}$$

uniformly in m.

Suppose that for some $k \in \mathbb{N} : |\nabla \bar{v}_{m,k}|^2 dx \to \delta_0$. Then from Theorem 1.4 we conclude that

$$\limsup_{m \to \infty} E_{4\pi}^*(\bar{v}_{m,k}) \leqq e + 1,$$

and the proof of Theorem 1.5 is complete by (4.5).

Otherwise, by Theorem 1.3 (ii) for all $k \in \mathbb{N} : \bar{v}_{m,k} \to \bar{v}_k$ weakly as $m \to \infty$, and

$$E_{4\pi}^*(\bar{v}_k) = \limsup_{m \to \infty} E_{4\pi}^*(\bar{v}_{m,k}) \geqq \beta. \tag{4.7}$$

But by (4.6) and Lemma 4.1:

$$\|\bar{v}_k\|_{1,2}^2 = \lim_{m\to\infty}\int_{B_R(0)} \nabla \bar{v}_{m,k}\nabla \bar{v}_k\,dx = \lim_{m\to\infty}\int_{B_R(0)} \nabla(\bar{v}_{m,k}-v_m)\nabla \bar{v}_k\,dx \to 0$$

as $k\to\infty$, and therefore $E^*_{4\pi}(\bar{v}_k)\to 1<\beta$.

The proof is complete.

Q.E.D.

Proof of Corollary 1.6. — We may assume that $B_R(0)\subset\Omega$.

For any function $u\in H_0^{1,2}(B_R(0))\subset H_0^{1,2}(\Omega)$ with norm $\|u\|_{1,2}=1$

$$\begin{aligned}\operatorname{meas}(\Omega)\cdot E_{4\pi}^{(\Omega)}(u) &= \int_\Omega \exp(4\pi u^2)\,dx\\ &= \int_{B_R(0)} \exp(4\pi u^2)\,dx + \operatorname{meas}(\Omega\setminus B_R(0))\\ &= \operatorname{meas}(B_R(0))\cdot E_{4\pi}^{(B_R(0))}(u)+\operatorname{meas}(\Omega\setminus B_R(0)).\end{aligned}$$

In particular if we let $u=u_0$: the maximizing function constructed by Carleson and Chang, we find that

$$E_{4\pi}^{(\Omega)}(u_0)=\frac{\pi R^2}{\operatorname{meas}(\Omega)}(\beta_0-1)+1$$

which is $>e+1$ iff

$$\pi R^2(\beta_0-1)>e\cdot\operatorname{meas}(\Omega).$$

Hence in this case

$$\sup_{\Sigma(\Omega)} E_{4\pi}>e+1$$

and by Theorem 1.5 a maximizing sequence $\{u_m\}\subset\Sigma(\Omega)$ converges.

Q.E.D.

5. PROOF OF THEOREM 1.7

By Theorem 1.5, under the hypothesis of Theorem 1.7

$$\beta_{4\pi}^* = \sup_{u \in \Sigma} E_{4\pi}(u) > e + 1$$

is achieved in $\Sigma = \Sigma(\Omega)$. Moreover, we have

LEMMA 5.1. – *The set* $K_{4\pi} = \{ u \in \Sigma \mid E_{4\pi}(u) = \beta_{4\pi}^* \}$ *is compact.*

Proof. – Since $\beta_{4\pi}^* > e + 1$, for any $\{u_m\} \subset K_{4\pi}$ such that $u_m \to u$ weakly in $H_0^{1,2}(\Omega)$ by Theorem 1.5 also

$$E_{4\pi}(u_m) \to E_{4\pi}(u) = \beta_{4\pi}^*.$$

Moreover, $\| u \| \leqq \| u_m \| = 1$, whence

$$E_{4\pi}(u) \leqq E_{4\pi}(u/\| u \|) \leqq \beta_{4\pi}^*$$

and, in fact, equality holds. But then $\| u \| = 1$, and $u_m \to u \in K_{4\pi}$ strongly.

Q.E.D.

By compactness of $K_{4\pi}$ the family of norm-neighborhoods

$$N_\varepsilon = \{ u \in \Sigma \mid \exists v \in K_{4\pi} : \| u - v \| < \varepsilon \}$$

constitutes a neighborhood basis for $K_{4\pi}$ in Σ.

LEMMA 5.2. – *For sufficiently small* $\varepsilon > 0$ *we have*

$$\sup_{N_{2\varepsilon} \setminus N_\varepsilon} E_{4\pi} < \beta_{4\pi}^* = \sup_{N_\varepsilon} E_{4\pi}. \tag{5.1}$$

Proof. – By contradiction, assume there exists a sequence $u_m \in N_{2\varepsilon} \setminus N_\varepsilon$ such that $E_{4\pi}(u_m) \to \beta_{4\pi}^*$. We may assume that $u_m \to u$ weakly in $H_0^{1,2}(\Omega)$.

Let $v_m \in K_{4\pi}$ satisfy: $\|u_m - v_m\| < 2\varepsilon$. By compactness, $v_m \to v$ strongly, where $v \in K_{4\pi}$. In particular, v solves (2.6):

$$-\Delta v = \lambda^{-1} v \exp(\alpha |v|^2) \in L^p, \qquad \forall p < \infty.$$

Hence $v \in L^\infty(\Omega)$.

Also note that by lower semi-continuity $\|v-u\| \leqq 2\varepsilon$, whence

$$\|v - (u/\|u\|)\| \leqq \|v-u\| + 1 - \|u\| \leqq 4\varepsilon;$$

I. e. $u/\|u\| \in \bar{N}_{4\varepsilon}$, and hence $E_{4\pi}(u) \leqq E_{4\pi}(u/\|u\|) \leqq \beta^*_{4\pi}$.

Moreover, equality $E_{4\pi}(u) = \beta^*_{4\pi}$ as in the proof of Lemma 5.1 entrains that $\|u\| = 1$, hence $u_m \to u$.

But then $u \notin N_\varepsilon$, in particular $u \notin K_{4\pi}$ and u cannot be relatively maximal.

Hence we obtain that $E_{4\pi}(u) < \beta^*_{4\pi}$.

Now let $w_m = u_m - v_m + v$. By Theorem 1.1

$$\exp(4\pi |w_m|^2) \leqq \exp(8\pi \|v\|^2_{L^\infty}) \exp(8\pi |u_m - v_m|^2)$$

is uniformly bounded in $L^2(\Omega)$, if $16\varepsilon^2 \leqq 1$. Since $w_m \to u$ weakly, therefore $E_{4\pi}(u) = \lim_{m\to\infty} E_{4\pi}(w_m)$.

Finally, $w_m - u_m \to 0$ strongly in $H_0^{1,2}(\Omega)$. By uniform local continuity of $E_{4\pi}$, Lemma 2.1, and compactness of $K_{4\pi}$ it hence follows that for sufficiently small $\varepsilon > 0$: $E_{4\pi}(w_m) - E_{4\pi}(u_m) \to 0$, and $E_{4\pi}(u) = \beta^*_{4\pi}$. The contradiction proves the lemma.

Q.E.D.

Actually, our proof is more involved than needed for the case $\alpha = 4\pi$. However, the proof immediately extends to the more general situation considered in Lemma 5.4 below.

LEMMA 5.3. — *There exists $\alpha^* > 4\pi$, $\varepsilon > 0$ such that for all $\alpha \in [4\pi, \alpha^*[$ there holds*

$$\sup_{N_{2\varepsilon} \setminus N_\varepsilon} E_\alpha < \sup_{N_\varepsilon} E_\alpha =: \beta^*_\alpha. \tag{5.2}$$

Moreover, β^*_α is achieved in N_ε for all such α, and

$$K_\alpha = \{ u \in N_\varepsilon \mid E_\alpha(u) = \beta^*_\alpha \}$$

is compact.

Proof. — By compactness of $K_{4\pi}$ and uniform local continuity of E_α (cp. Lemma 2.1) there exists a neighborhood N of $K_{4\pi}$ such that the map

$\alpha \to E_\alpha(u)$ is continuous, uniformly with respect to $u \in N$. Choose $\varepsilon > 0$ such that (5.1) holds and $N_{2\varepsilon} \subset N$; then (5.2) will be valid for all α in a small neighborhood of 4π.

For such α now let $u_m \in N_\varepsilon$ be a maximizing sequence, $E_\alpha(u_m) \to \beta_\alpha^*$, and let $v_m \in K_{4\pi}$ satisfy $\|u_m - v_m\| \leqq \varepsilon$. We may assume that $v_m \to v$, strongly in $H_0^{1,2}(\Omega)$, where $v \in L^\infty$, and $u_m \to u$ weakly; also let $w_m = u_m - v_m + v \to u$ weakly. Then as in the proof of the preceeding lemma we conclude that for sufficiently small $\varepsilon > 0$, α sufficiently close to 4π we have that

$$E_\alpha(w_m) \to E_\alpha(u), \qquad E_\alpha(u_m) - E_\alpha(w_m) \to 0 \qquad (m \to \infty).$$

Hence $E_\alpha(u) = \beta_\alpha^*$. Moreover,

$$\|v - u\| \leqq \varepsilon, \qquad \|v - (u/\|u\|)\| \leqq 2\varepsilon.$$

I. e.: $u/\|u\| \in \bar{N}_{2\varepsilon}$ and $E_\alpha(u/\|u\|) \leqq \beta_\alpha^*$.

Thus also $E_\alpha(u/\|u\|) \leqq E_\alpha(u)$, and $\|u\| = 1$. It follows that $u \in \Sigma$, i. e. $u \in N_\varepsilon$ and β_α^* is attained. Moreover, $u_m \to u$ strongly.

In particular, if $u_m \in K_\alpha$, this also shows that K_α is compact.

Q.E.D.

By Theorem 1.1 sup E_α is achieved in Σ for $\alpha < 4\pi$. Thus, the proof of Theorem 1.7 is complete. Note that even though our proof does not reveal the existence of a "branch" of relative maximizers for $\alpha > 4\pi$, we can find relative maximizers of E_α for α sufficiently close to 4π arbitrarily close to $K_{4\pi}$.

We would like to examine the set of relative maximizers more closely. Denote

$$C = \{ (\alpha, u) \mid \alpha > 0, \exists N \subset \Sigma : N \text{ open}, \quad u \in K_\alpha = \{ u \in N \mid E_\alpha(u) = \sup_{v \in N} E_\alpha(v) \} \subset\subset N \} \qquad (5.3)$$

the set of pairs (α, u) where $u \in \Sigma$ is a relative maximizer of E_α on Σ belonging to some compact set K_α of relative maximizers which possesses an isolating neighborhood N.

Let

$$I = \{ \alpha \mid \exists (\alpha, u) \in C \}$$

denote the projection of C onto its first component. Note that $I \supset [0, \alpha^*[$. Moreover, LEMMA 5.4. — *Suppose* $K_{\alpha_0} = \{ u \mid (\alpha_0, u) \in C \}$ *is compact. Then* I *contains an open neighborhood of* α_0.

Proof. — Replace 4π by α_0 in the definition of N_ε. The proofs of Lemma 5.2 and 5.3 then remain valid.

Q.E.D.

LEMMA 5.5. — *Suppose* $\alpha_m \to \alpha$, *and suppose* $\{ u_m \} \subset \Sigma$ *is a sequence of relative maximizers of* E_{α_m} *on* Σ *such that* $E_{\alpha_m}(u_m) \geqq \beta > 1$ *uniformly while* $\| u_m \|_{L^\infty} \leqq C < \infty$ *uniformly. Then* $\{ u_m \}$ *is relatively compact.*

Proof. — By (2.6), $u_m \in H_0^{1,2}(\Omega)$ satisfies

$$-\Delta u_m = \lambda_m^{-1} u_m \exp(\alpha_m u_m^2) \quad \text{in } \Omega \tag{5.4}$$

with

$$\lambda_m = \int_\Omega u_m^2 \exp(\alpha_m u_m^2)\, dx.$$

Since $\| u_m \|_{L^\infty} \leqq C$ uniformly, if $u_m \to 0$ weakly, by dominated convergence $E_{\alpha_m}(u_m) \to 1$, contrary to hypothesis. Hence $u_m \nrightarrow 0$ weakly, $\lambda_m \geqq \lambda_0 > 0$ uniformly, and (5.4) gives a uniform a-priori bound for u_m in $H^{2,2}(\Omega)$. The lemma follows from compactness of the embedding $H^{2,2}(\Omega) \to H^{1,2}(\Omega)$.

Q.E.D.

The characterization of C as stated after Theorem 1.7 now follows: Rename

$$\alpha^* = \sup \{ \alpha \mid]0, \alpha] \subset I \}.$$

Then any set K^* of relative maximizers of E_{α^*} either is empty or non-compact. In the second case, either K^* contains a critical point which is not relatively maximizing in its closure, i. e. C looses its stability; or by Lemma 5.5 K^* cannot be uniformly bounded, i. e. C becomes unbounded at $\alpha = \alpha^*$.

In the first case, let $\alpha_m \to \alpha^*$, $u_m \in \Sigma$ satisfy $(\alpha_m, u_m) \in C$. If $\| u_m \|_{L^\infty} \to \infty$, again C is unbounded. Otherwise, by Lemma 5.5 either $E_{\alpha_m}(u_m) \to 1$, and for given $\varepsilon > 0$ and sufficiently large m there exists $v_m \in \Sigma$ such that $\| u_m - v_m \| < \varepsilon$ while $E_{\alpha_m}(v_m) > E_{\alpha_m}(u_m)$. Or $\{ u_m \}$ accumulates at a critical point u^* of E_α^* which is not relatively maximal, and C looses its stability as $\alpha \to \alpha^*$.

This characterization of C apparently cannot be reached by the usual methods of bifurcation theory based on degree theory or the implicit function theorem. Since (2.6) is the Euler-Lagrange equation associated with a constrained variational problem, the linearized equation will always have a non-trivial kernel.

6. PROOF OF THEOREM 1.8

Let $K_{4\pi}$ be the set of relative maximizers of $E_{4\pi}$ on Σ, and let $\varepsilon>0$, $\alpha^*>4\pi$ be as determined in Lemma 5.3 such that (5.2) holds.

Note that by Moser's result Theorem 1.1 for $\alpha>4\pi$ the functional E_α is unbounded on Σ.

Thus, for any such α we can find $u_1\in\Sigma$ such that

$$E_\alpha(u_1)>\beta_\alpha^*=\sup_{N_\varepsilon} E_\alpha>\sup_{N_{2\varepsilon}\setminus N_\varepsilon} E_\alpha=:\beta_\alpha. \tag{6.1}$$

Fix $u_0\in N_\varepsilon$ such that

$$E_\alpha(u_0)>\beta_\alpha \tag{6.2}$$

and let

$$P=\{p\in C^0([0,1];\Sigma)\,|\,p(0)=u_0, p(1)=u_1\}$$

be the set of paths connecting u_0 and u_1 in Σ.

Define

$$\gamma_\alpha:=\sup_{p\in P}\inf_{u\in p} E_\alpha(u). \tag{6.3}$$

Note that by (6.1) necessarily $u_1\notin N_{2\varepsilon}$. Hence any $p\in P$ intersects $N_{2\varepsilon}\setminus N_\varepsilon$ and we infer that $\gamma_\alpha\leqq\beta_\alpha$. By continuity of $\alpha\to E_\alpha(u_{0,1})$ and Lemma 5.3 (6.1)-(6.2) remain valid in a neighborhood A of α. u_0, u_1 and the class P being fixed, we extend the definition of γ_α to such a neighborhood A.

LEMMA 6.1. — *For all* $\alpha\in A$ *we have*

$$\gamma_\alpha\leqq\beta_\alpha<\min\{E_\alpha(u_0), E_\alpha(u_1)\}\leqq\beta_\alpha^*$$

and the map $\alpha\to\gamma_\alpha$ *is non-decreasing in* A.

Proof. — The first part is clear from (6.1)-(6.2). Since for fixed $u\in\Sigma$ the map $\alpha\to E_\alpha(u)$ is non-decreasing, also the second part follows.

Q.E.D.

We intend to show that for a. e. $\alpha \in A$ γ_α is a critical value of E_α. However, for $\alpha > 4\pi$ our functional E_α does not satisfy the Palais-Smale condition and the familiar minimax-principle, cp. [7], cannot be applied without additional a-priori estimates. In order to obtain such estimates, we also vary the parameter α and make use of the fact that γ_α is monotone, hence a. e. differentiable in A. This technique was introduced in [13] to deal with a similar lack of compactness in a different setting.

First we state a technical lemma.

LEMMA 6.2. — *For any $\gamma_* > 1$ there exists $\alpha_* > 4\pi$ with the following property: if $\alpha < \alpha_*$ and if $\{u_m\} \subset \Sigma$ satisfies the conditions*

$$E_\alpha(u_m) \to \gamma \geqq \gamma_*, \tag{6.4}$$

$$\lambda_m = \int_\Omega u_m^2 \exp(\alpha u_m^2)\, dx \leqq C < \infty \tag{6.5}$$

uniformly in $m \in \mathbb{N}$, then there exists an exponent $\alpha' > \alpha$ such that the family $\{U_m = \exp(u_m^2)\}$ is uniformly bounded in $L^{\alpha'}(\Omega)$.

Proof. — We may assume that $u_m \to u$ weakly.

For $\Omega' \subset \Omega$, $L > 0$ we may estimate

$$\int_{\Omega'} \exp(\alpha u_m^2)\, dx$$
$$\leqq \int_{\Omega' \cap \{x \mid |u_m(x)| \leqq L\}} \exp(\alpha u_m^2)\, dx + L^{-2} \int_\Omega u_m^2 \exp(\alpha u_m^2)\, dx$$
$$\leqq C(L) . \operatorname{meas}(\Omega') + \lambda_m L^{-2}.$$

Hence for arbitrary $\varepsilon > 0$, choosing $L = \varepsilon^{-1/2}$, $\operatorname{meas}(\Omega') \leqq \varepsilon / C(L)$ we obtain that

$$\int_{\Omega'} \exp(\alpha u_m^2)\, dx \leqq C\varepsilon,$$

uniformly in m. I. e. the sequence $\left\{ \int \exp(\alpha u_m^2)\, dx \right\}$ is uniformly absolutely continuous, and we may pass to the limit $m \to \infty$ for E_α by Vitali's convergence theorem:

$$E_\alpha(u_m) \to E_\alpha(u) = \gamma \geqq \gamma_* > 1.$$

In particular, $u \neq 0$; in fact, given $\gamma_* > 1$ by Theorem 1.1 for any compact interval $I \subset]0, \infty[$ there exists a uniform number $\delta = \delta(I) > 0$ such that $E_\alpha(u) \geqq \gamma_* > 1$, $\alpha \in I$, implies that $\|u\|^2 \geqq \delta$.

Hence

$$\|u_m - u\|^2 = \|u_m\|^2 - \|u\|^2 + o(1) \leqq 1 - \delta + o(1). \tag{6.6}$$

Now for arbitrary $\varepsilon > 0$ there exists $C(\varepsilon)$ such that

$$U_m = \exp(u_m^2) = \exp(|(u_m - u) + u|^2) \leqq \exp(C(\varepsilon)u^2) . \exp((1+\varepsilon)|u_m - u|^2).$$

Thus, for any $\alpha' < 4\pi(1-\delta)^{-1}$, by Theorem 1.1 and (6.6) the sequence $\{U_m\}$ will be uniformly bounded in $L^{\alpha'}(\Omega)$, as claimed.

Note that α_* may be determined as root of the equation

$$\alpha_*(1 - \delta[4\pi, \alpha_*]) = 4\pi, \tag{6.7}$$

with

$$\delta([4\pi, \alpha]) = \inf\{\|u\| \mid E_\alpha(u) \geqq \gamma_*\}.$$

Since $\delta([4\pi, \alpha])$ is non-increasing as a function of α, there is a unique solution $\alpha_* > 4\pi$ to (6.7).

Q.E.D.

Next observe that from the family $w = w(t_1)$ in [6], p. 1080, we may construct comparison paths for any $\alpha > 4\pi$ (with a convenient choice of u_1) that yield a uniform lower bound $\gamma_\alpha \geqq \gamma_* > 1$ for $\alpha \in]4\pi, \alpha^*[$.

Let $\alpha_* > 4\pi$ be fixed corresponding to this number γ_* and Lemma 6.2. We may assume that $\alpha_* \leqq \alpha^*$.

LEMMA 6.3. — *Suppose the map $\alpha \to \gamma_\alpha$ is differentiable at $\alpha \in A$, $\alpha < \alpha_*$. Then there exists a sequence $\{u_m\}$ in Σ such that $E_\alpha(u_m) \to \gamma_\alpha$ while*

$$\|d(E_\alpha \circ \pi)(u_m)\| \to 0$$

and such that

$$\lambda_m = \int_\Omega u_m^2 \exp(\alpha u_m^2)\, dx \leqq C < \infty,$$

uniformly in $m \in \mathbb{N}$.

Proof. — We may assume that α is an interior point of A. Let $\alpha_m \in A$ be a strictly decreasing sequence $\alpha_m > \alpha$, $\alpha_m \to \alpha$ $(m \to \infty)$.

Suppose $u \in \Sigma$ satisfies the inequality

$$\gamma_\alpha - (\alpha_m - \alpha) \leqq E_\alpha(u) \leqq E_{\alpha_m}(u) \leqq \gamma_{\alpha_m} + (\alpha_m - \alpha). \tag{6.8}$$

In particular,

$$\frac{E_{\alpha_m}(u) - E_\alpha(u)}{\alpha_m - \alpha} \leqq \frac{\gamma_{\alpha_m} - \gamma_\alpha}{\alpha_m - \alpha} + 2.$$

Note that since $u^2 \geqq 0$ we have for any m:

$$\frac{E_{\alpha_m}(u) - E_\alpha(u)}{\alpha_m - \alpha} = \int_\Omega \frac{\exp((\alpha_m - \alpha) u^2) - 1}{\alpha_m - \alpha} \exp(\alpha u^2)\, dx$$

$$\geqq \int_\Omega u^2 \exp(\alpha u^2)\, dx = \lambda(u).$$

Since γ_α is differentiable at α we may assert: There is a uniform constant $C > 0$ (depending possibly on α, but not on m) such that

$$\lambda = \lambda(u) = \int_\Omega u^2 \exp(\alpha u^2)\, dx \leqq C \tag{6.9}$$

for all $u \in \Sigma$ satisfying (6.8) for some $m \geqq 1$.

Consider now any sequence of paths $p_m \in P$ such that

$$\inf_{u \in p_m} E_\alpha(u) \geqq \gamma_\alpha - (\alpha_m - \alpha). \tag{6.10}$$

Then, since $E_\alpha(u) \leqq E_{\alpha_m}(u)$ for all u, (6.8) and therefore (6.9) hold for all $u \in p_m$ such that

$$E_{\alpha_m}(u) \leqq \gamma_{\alpha_m} + (\alpha_m - \alpha).$$

Note that by definition of γ_{α_m} the class

$$q_m = \{ u \in p_m \mid E_{\alpha_m}(u) \leqq \gamma_{\alpha_m} + (\alpha_m - \alpha) \} \neq \varnothing$$

for any $m \geqq 1$. Hence also

$$W_{m_0} = \{ u \in \Sigma \mid u \text{ satisfies } (6.8) \text{ for some } m \geqq m_0 \} \neq \emptyset,$$

for all $m_0 \geqq 1$. Note that $W_m \supset W_{m+1}$, etc.

By Lemma 6.2 there exists $\alpha' > \alpha$, $m_0 \geqq 1$ such that the functions $U = \exp(u^2)$, $u \in W_{m_0}$, are uniformly bounded in $L^{\alpha'}(\Omega)$. In particular,

$$d(E_{\alpha_m} \circ \pi)(u) \to d(E_\alpha \circ \pi)(u) \tag{6.11}$$

as $m \to \infty$, uniformly in $u \in W_{m_0}$, as an indirect argument easily shows. Hence to complete the proof it suffices to show that for a sequence $u_m \in W_m$ there holds

$$d(E_{\alpha_m} \circ \pi)(u_m) \to 0$$

while $m \to \infty$.

Suppose by contradiction that there exist $\delta > 0$, $m_0 \geqq 1$ such that for all $m \geqq m_0$ there holds

$$\| d(E_{\alpha_m} \circ \pi)(u) \| \geqq 4\delta,$$

uniformly with respect to $u \in W_m$.

By (6.11) we may choose m_0 such that also

$$\| d(E_\alpha \circ \pi)(u) \| \geqq 3\delta > 0$$

for all $u \in W_m$, $m \geqq m_0$. Relabelling $\{\alpha_m\}$ if necessary, we may assume that $m_0 = 1$. Moreover, we may assume that $u_0, u_1 \notin W := W_1$.

Invoking (6.11) once more we see that

$$\| d(E_\alpha \circ \pi)(u) \| > 2\delta, \qquad \| d(E_{\alpha_m} \circ \pi)(u) - d(E_\alpha \circ \pi)(u) \| < \delta$$

for u in a suitable neighborhood $\tilde{W}$ of W and for all α_m, $m \geqq m_0$. (Again, we may assume that $m_0 = 1$, $u_0, u_1 \notin \tilde{W}$.)

Hence there exists a Lipschitz continuous vector field $\tilde{e}: \tilde{W} \to H_0^{1,2}(\Omega)$ which is simultaneously a pseudo-gradient vector field for *all* functionals $E_{\alpha_m} \circ \pi$ on $\tilde{W}$, satisfying

$$\langle d(E_{\alpha_m} \circ \pi)(u), \tilde{e}(u) \rangle \geqq \delta > 0 \tag{6.12}$$

for all $u \in \tilde{W}$, all α_m. Since $\langle d(E_{\alpha_m} \circ \pi)(u), u \rangle = 0$, we may assume that $\tilde{e}$ is tangent to Σ.

Let η be a Lipschitz function such that $0 \leqq \eta \leqq 1$, $\eta = 1$ on W, $\eta = 0$ outside $\tilde{W}$. Extend $\tilde{e}$ to Σ by letting

$$e(u) = \begin{cases} \eta(u) \cdot \tilde{e}(u), & u \in \tilde{W} \\ 0, & \text{else} \end{cases}.$$

e again is Lipschitz, bounded and satisfies (6.12).

Let $\Phi : \Sigma(\Omega) \times \mathbb{R} \to \Sigma(\Omega)$ be the solution of (4.3).

Since e is bounded and Lipschitz Φ exists globally. Fix a sequence p_m of paths satisfying (6.11) and let

$$p_m^t = \Phi(p_m, t).$$

with associated $q_m^t = \{ u \in p_m^t \mid E_{\alpha_m}(u) \leqq \gamma_{\alpha_m} + (\alpha_m - \alpha) \}$.

Note that for any $u \in p_m$, $t_0 \geqq 0$ by definition of e;

$$\frac{d}{dt}(E_\alpha \circ \pi)(\Phi(u, t))\big|_{t=t_0} \geqq 0,$$

i. e. p_m^t still satisfies (6.10) for all $t \geqq 0$.

Moreover, for any $u \in q_m^t$, by (6.12)

$$\frac{d}{dt}(E_{\alpha_m} \circ \pi)(\Phi(u, t))\bigg|_{t=0} = \langle d(E_{\alpha_m} \circ \pi)(u), e(u) \rangle \geqq \delta.$$

Hence

$$\frac{d}{dt}\big(\inf_{u \in p_m^t} (E_{\alpha_m} \circ \pi)(u)\big) \geqq \delta,$$

and for $t = 1$ and m sufficiently large there holds

$$\inf_{u \in p_m^1} (E_{\alpha_m} \circ \pi)(u) > \gamma_{\alpha_m}. \tag{6.13}$$

However, u_0, $u_1 \notin \tilde{W}$ and are left fixed by $\Phi(., 1)$. Thus the path $p_m^1 \in P$, and we obtain a contradiction to the definition of γ_{α_m} from (6.13).

The proof is complete.

Q.E.D.

Proof of Theorem 1.8. — Suppose $\alpha \to \gamma_\alpha$ is differentiable at $\alpha \in A$, $\alpha < \alpha_*$, and let $\{ u_m \}$ be the sequence constructed in Lemma 6.3. We may assume that $u_m \to u$ weakly.

By Lemma 6.2 there is $\alpha'>\alpha$ such that $\{U_m=\exp(u_m^2)\}$ is uniformly bounded in $L^{\alpha'}(\Omega)$. In particular, by Vitali's convergence theorem

$$E_\alpha(u_m)\to E_\alpha(u)=\gamma_\alpha>1,$$
$$\lambda_m=\lambda(u_m)\to\lambda=\lambda(u)>0.$$

Hence, with a constant $c=-8\pi\,(\text{meas}\,(\Omega))^{-1}\neq 0$ we have

$$o(1)=\langle d(E_\alpha\circ\pi)(u_m),u\rangle$$
$$=c\left(\lambda_m\int_\Omega \nabla u_m\nabla u\,dx-\int_\Omega u_m u\exp(\alpha u_m^2)\,dx\right)$$
$$\to c\left(\lambda\int_\Omega|\nabla u|^2\,dx-\lambda\right)=0,$$

and it follows that $\|u\|=1$. I. e. $u_m\to u$ strongly, and u is a critical point of E_α in Σ. Since $E_\alpha(u)=\gamma_\alpha<\beta_\alpha^*$, moreover, u is distinct from the relative maximizer $u_\alpha\in\Sigma$, constructed in Theorem 1.7.

Finally, we cover the interval $]4\pi,\alpha_*[$ by suitable intervals A to obtain the existence of (at least) two distinct critical points of E_α in Σ for a. e. $\alpha\in[4\pi,\alpha_*[$.

Q.E.D.

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