ANNALES DE L'I. H. P., SECTION C

E. BERKSON

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Annales de l'I. H. P., section C, tome 5, nº 5 (1988), p. 409-424
<http://www.numdam.org/item?id=AIHPC_1988__5_5_409_0>

Ann. Inst. Henri Poincaré,

Vol. 5, n° 5, 1988, p. 409-424. *Analyse non linéaire*

On the decay properties of the Franklin analyzing wavelet

by

E. BERKSON

University of Illinois, Department of Mathematics,
1409 W. Green Street, Urbana, Illinois 61801, U.S.A.

ABSTRACT. – We study the behaviour of the Frankin analysing wavelet ψ, which is piecewise linear with nodes in $\mathbb{Z}/2$, by giving the size and sign of $\psi(n/2)$.

Key words : Wavelet.

RÉSUMÉ. – On étudie le comportement de l'ondelette de Franklin ψ, affine par morceaux et à nœuds dans $\mathbb{Z}/2$, en précisant la taille et le signe de $\psi(n/2)$.

0. NOTATION

As usual, the symbols $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{C}$ will denote, respectively, the real line, the set of integers, the set of positive integers, and the complex plane. The Fourier transform of a function $f \in L^1(\mathbb{R})$ will be written $\hat{f}$. The real and imaginary parts of a complex number z will be denoted by $\mathcal{R}e\, z$ and $\mathcal{I}m\, z$, respectively. Additional notation will be introduced as needed.

1. INTRODUCTION

The Franklin analyzing wavelet $g:\mathbb{R}\to\mathbb{R}$ has a remarkable ability to generate bases for classical function spaces (*see* [2], Chapter I, for an extensive development and treatment of the nature of g). By definition

$$(1.1)\qquad g(\lambda)\equiv\int_{\mathbb{R}} e^{2\pi i\lambda x}e^{-i\pi x}\,\omega(x)\frac{\sin^2(\pi x/2)}{(\pi x/2)^2}\,dx,$$

where

$$(1.2)\qquad \begin{aligned}\omega(x)=&\frac{\sin^2(\pi x/2)}{1-(2/3)\sin^2(\pi x/2)}\\&\times\left[\frac{\sin^4(\pi x/2)}{1-(2/3)\sin^2(\pi x/2)}+\frac{\cos^4(\pi x/2)}{1-(2/3)\cos^2(\pi x/2)}\right]^{-1/2},\end{aligned}$$

for all $x\in\mathbb{R}$.

In particular, $g(2^{-1}+\lambda)=g(2^{-1}-\lambda)$ for all $\lambda\in\mathbb{R}$. The function g is known to have exponential decay at infinity, and to be piecewise linear with nodes at the points $2^{-1}m$, $m\in\mathbb{Z}$.

We shall show below that the function g has the additional properties stated in the following theorem.

(1.3) THEOREM. — *For each real number x, let $\operatorname{sgn}(x)$ denote the sign of x, and let $\langle x\rangle$ denote the largest $m\in\mathbb{Z}$ such that $m\leqq x$. Then:*

(i) $\operatorname{sgn}(g(n/2))=(-1)^{n/2}$, *for all even* $n\in\mathbb{N}$;

(ii) $\operatorname{sgn}(g(n/2))=(-1)^{\langle n/2\rangle}$, *for all sufficiently large* $n\in\mathbb{N}$;

(iii) $\left|g\left(\frac{n}{2}\right)\right|>\left|g\left(\frac{n+1}{2}\right)\right|$, *for all sufficiently large* $n\in\mathbb{N}$;

(iv) $\sigma n^{-1/2}\beta_1^n\leqq|g(n/2)|\leqq\lambda n^{-1/2}\beta_1^n$, *for all sufficiently large* $n\in\mathbb{N}$, *where σ and λ are positive absolute constants, and* $\beta_1=(2-3^{1/2})^{1/2}$.

Remark. — The second inequality in Theorem (1.3) (iv), which states that $|g(n/2)|=O(n^{-1/2}\beta_1^n)$, reproduces the known exponential decay at infinity of $|g|$ ([2], Chapter I).

The first step in obtaining Theorem (1.3) is to recast (1.1) (for the case $\lambda=n/2$) into the following form more convenient for our purposes (*see* § 2, 3).

(1.4) THEOREM. — *For every positive integer n,*

$$(1.5)\quad g\left(\frac{n+1}{2}\right)=-\frac{2}{\pi}\int_0^{2\pi}[\sin^2(nu)]\,\frac{\sin^2 u}{1-(2/3)\sin^2 u}\times\left[\frac{\sin^4 u}{1-(2/3)\sin^2 u}+\frac{\cos^4 u}{1-(2/3)\cos^2 u}\right]^{-1/2}du+g(1/2).$$

Moreover,

$$(1.6)\qquad g(1/2)=\frac{3^{1/2}}{2\pi}\int_0^{2\pi}\frac{1-\cos x}{2+\cos x}\left[\frac{4-\cos^2 x}{1+2\cos^2 x}\right]^{1/2}dx.$$

The difference expressed on the right of (1.5) does not permit ready estimation of the sign or size of $g\left(\frac{n+1}{2}\right)$. However, by contour integration, Theorem (1.4) has the following consequence which alleviates this difficulty.

(1.7) THEOREM. — *Let* $\alpha_1=2-\sqrt{3}$, $\alpha_2=2+\sqrt{3}$, $\beta_1=\sqrt{\alpha_1}$, $\beta_2=\sqrt{\alpha_2}$. *Then:*

$$(1.8)\qquad \text{for } n\in\mathbb{N},\qquad \frac{\sqrt{2}\pi}{\sqrt{3}}\,g\left(\frac{n+1}{2}\right)=-\Lambda_1^{(n)}+\Lambda_2^{(n)}+2\Lambda_3^{(n)},$$

where

$$(1.9)\quad \Lambda_1^{(n)}=(-1)^{n-1}\int_0^{\alpha_1}x^{n-1}\times\frac{(x+1)^2}{(\alpha_1-x)^{1/2}(\alpha_2-x)^{1/2}}\,\frac{(\alpha_1+x)^{1/2}(\alpha_2+x)^{1/2}}{(x^2+\beta_1^2)^{1/2}(x^2+\beta_2^2)^{1/2}}\,dx,$$

$$(1.10)\quad \Lambda_2^{(n)}=\int_0^{\alpha_1}x^{n-1}\frac{(x-1)^2}{(\alpha_1+x)^{1/2}(\alpha_2+x)^{1/2}}\,\frac{(\alpha_1-x)^{1/2}(\alpha_2-x)^{1/2}}{(x^2+\beta_1^2)^{1/2}(x^2+\beta_2^2)^{1/2}}\,dx,$$

$$(1.11)\quad \Lambda_3^{(n)}=\mathscr{I}m\int_0^{\beta_1}\frac{(iy)^{n-1}}{(\beta_1-y)^{1/2}}\times\frac{y^4+6iy^3-10y^2-6iy+1}{(y^2+\alpha_1^2)^{1/2}(y^2+\alpha_2^2)^{1/2}}(\beta_1+y)^{-1/2}(\beta_2^2-y^2)^{-1/2}dy.$$

The strategy we shall pursue in exploiting (1.8) to deduce Theorem (1.3) (*see* § 4, 5 below) can be roughly outlined as follows. Since $0<\alpha_1<\beta_1<1$, application of Scholium (4.1) to (1.9), (1.10), and (1.11) shows that the term $2\Lambda_3^{(n)}$ dominates the behavior of the right hand side of (1.8). This observation allows us to reduce the assertions of Theorem (1.3) to comparatively simple questions about the behavior near β_1 of the function of y in (1.11) expressed by $\mathcal{I}m\,((i)^{n-1}(y^4+6iy^3-10y^2-6iy+1))$.

2. SKETCH OF (1.5)

Define $g_0 : \mathbb{R} \to \mathbb{R}$ as follows:

$$g_0(\lambda)=\frac{1}{2}\,g\left(\frac{\lambda+1}{2}\right), \quad \text{for all } \lambda \in \mathbb{R}.$$

Application of the following proposition to g_0 establishes (1.5).

(2.1) PROPOSITION. – *Suppose that* $F \in L^1(\mathbb{R})$, F *is continuous and bounded on* $\mathbb{R}$, *the function* $x^2\,\hat{F}(x)$ *is periodic of period* 1 *on* $\mathbb{R}$, *and* F *and* $\hat{F}$ *are real-valued. Then for each non-negative integer n, we have*

$$\text{(2.2)} \qquad F(n+1)-F(0)=\frac{-1}{\pi}\int_0^{2\pi} u^2\,\hat{F}\left(\frac{u}{\pi}\right)\frac{\sin^2[(n+1)u]}{\sin^2 u}\,du.$$

Proof. – This proposition can be readily obtained either from the theory of distributions or by direct classical Fourier analysis. We omit the details.

3. THE APPLICATION OF CONTOUR INTEGRATION

In this section, we shall use (1.5) to obtain (1.6) and, with the aid of suitable contour integration, Theorem (1.7). For $n \in N$, put

$$J_n=\int_0^{2\pi}[\sin^2(nt)]\,\frac{\sin^2 t}{1-(2/3)\sin^2 t} \times\left[\frac{\sin^4 t}{1-(2/3)\sin^2 t}+\frac{\cos^4 t}{1-(2/3)\cos^2 t}\right]^{-1/2} dt.$$

Thus, from (1.5), we have

$$(3.1)\qquad g\left(\frac{n+1}{2}\right)=-\frac{2}{\pi}\,J_n+g(1/2).$$

From repeated applications of the double angle formula for the cosine in the integral defining J_n, we find that

$$(3.2)\qquad \frac{4}{3^{1/2}}\,J_n=\int_0^{2\pi}\frac{1-\cos(2t)}{2+\cos(2t)}\left[\frac{1+2\cos^2(2t)}{4-\cos^2(2t)}\right]^{-1/2}dt$$
$$-\int_0^{2\pi}[\cos(2nt)]\,\frac{1-\cos(2t)}{2+\cos(2t)}\left[\frac{1+2\cos^2(2t)}{4-\cos^2(2t)}\right]^{-1/2}dt.$$

The change of variable $\theta=2t$ on the right of (3.2) gives

$$(3.3)\qquad \frac{4}{3^{1/2}}\,J_n=\int_0^{2\pi}\frac{1-\cos\theta}{2+\cos\theta}\left[\frac{4-\cos^2\theta}{1+2\cos^2\theta}\right]^{1/2}d\theta$$
$$-\int_0^{2\pi}[\cos(n\theta)]\,\frac{1-\cos\theta}{2+\cos\theta}\left[\frac{4-\cos^2\theta}{1+2\cos^2\theta}\right]^{1/2}d\theta.$$

By the Riemann-Lebesgue lemma, the second integral on the right-hand side of (3.3) tends to 0 as $n\to+\infty$, and so does $g\left(\frac{n+1}{2}\right)$. Thus, if we use (3.3) to substitute for J_n in (3.1), and let $n\to+\infty$, we establish (1.6) and:

$$(3.4)\qquad g\left(\frac{n+1}{2}\right)=\frac{\sqrt{3}}{\pi}\,\mathcal{R}e\int_0^{\pi}e^{in\theta}\,\frac{1-\cos\theta}{2+\cos\theta}\left[\frac{4-\cos^2\theta}{1+2\cos^2\theta}\right]^{1/2}d\theta,$$

for $n\in\mathbb{N}$.

We replace the integral on the right of (3.4) with a contour integral by the change of variable $z=e^{i\theta}$. This gives us for all $n\in\mathbb{N}$:

$$(3.5)\qquad g\left(\frac{n+1}{2}\right)$$
$$=-\frac{3^{1/2}}{2^{1/2}\pi}\,\mathcal{I}m\int_\Gamma z^{n-1}\,\frac{(z-1)^2}{(z+\alpha_1)(z+\alpha_2)}\left[-\frac{z^4-14z^2+1}{z^4+4z^2+1}\right]^{1/2}dz,$$

where Γ is the upper semicircle $z=e^{i\theta}$, $0\leqq\theta\leqq\pi$, oriented in the sense of increasing θ. Here, and in what follows, $w^{1/2}\equiv e^{2^{-1}\operatorname{Log} w}$, where Log denotes the principal branch of the logarithm, and α_1, α_2, β_1, and β_2 are as in the statement of Theorem (1.7). It is easy to see that:

$$(3.6)\qquad 0<\alpha_1<\beta_1<1<\beta_2<\alpha_2.$$

For all sufficiently small $\varepsilon>0$, let C_ε be the simple, closed, counterclockwise curve comprised of: Γ; the upper semicircles of radius ε centered at α_1 and $(-\alpha_1)$; the arc $\mathscr{A}_\varepsilon$ of the circle $\mathscr{K}_\varepsilon\equiv\{z:|z-i\beta_1|=\varepsilon\}$ obtained by adjoining to the upper semicircle of $\mathscr{K}_\varepsilon$ the part of the lower semicircle of $\mathscr{K}_\varepsilon$ lying outside the strip $-\frac{\varepsilon}{4}<\mathscr{R}e\,z<\frac{\varepsilon}{4}$; the vertical line segments joining the end-points of $\mathscr{A}_\varepsilon$ to the real axis; and the closed intervals on the real axis $[-1,\,-\alpha_1-\varepsilon]$, $\left[-\alpha_1+\in,\,-\frac{\varepsilon}{4}\right]$ together with the negatives of these last two intervals. Let z_0 be one of the complex numbers α_1, $(-\alpha_1)$, α_2, $(-\alpha_2)$, $i\beta_1$, $(-i\beta_1)$, $(-i\beta_2)$. We introduce an analytic logarithm for $(z-z_0)$ by deleting from $\mathbb{C}$ the vertical downwards-directed ray issuing from z_0, and taking $\left(-\frac{\pi}{2}\right)<\arg(z-z_0)<\frac{3\pi}{2}$. Further, we consider also the case $z_0=i\beta_2$. In this case, we introduce an analytic logarithm for $(z-z_0)$ by deleting from $\mathbb{C}$ the ray $\{x+i\beta_2:x\leqq 0\}$, and then taking $(-\pi)<\arg(z-z_0)<\pi$.

It is clear that C_ε is contained in the simply connected domain D_ε, which is defined to be the intersection of the domains of definition for the analytic logarithms of $(z-z_0)$ mentioned above. For each choice of z_0 listed above, we use the corresponding analytic logarithm defined above to write $\sqrt{z-z_0}\equiv\exp\{2^{-1}\log(z-z_0)\}$ on D_ε. In terms of this notation, we put

$$(3.7)\qquad F(z)=(-i)\frac{\sqrt{z-\alpha_1}\sqrt{z+\alpha_1}\sqrt{z-\alpha_2}\sqrt{z+\alpha_2}}{\sqrt{z-i\beta_1}\sqrt{z+i\beta_1}\sqrt{z-i\beta_2}\sqrt{z+i\beta_2}},$$

$$\text{for}\quad z\in D_\varepsilon.$$

It is not difficult to see that the branches indicated in (3.7) allow us to rewrite (3.5) in the form:

$$(3.8)\qquad g\left(\frac{n+1}{2}\right)=-\frac{3^{1/2}}{2^{1/2}\pi}\,\mathscr{I}m\int_{\Gamma} z^{n-1}\frac{(z-1)^2}{(z+\alpha_1)(z+\alpha_2)}\,F(z)\,dz,$$

$$\text{for}\quad n\in\mathbb{N}.$$

Taking note of the fact that

$$\int_{C_\varepsilon} z^{n-1}\frac{(z-1)^2}{(z+\alpha_1)(z+\alpha_2)}\,F(z)\,dz=0,$$

we decompose this contour integral into the corresponding sum of integrals along the constituent curves of C_ε, then take imaginary parts and let $\varepsilon\to 0^+$. With due attention to the branches indicates in (3.7), we thereby obtain Theorem (1.7) from (3.8).

4. PROOF OF THEOREM (1.3) (i), (ii)

In essence, our method of proof for Theorem (1.3) will proceed by showing that the term $2\Lambda_3^{(n)}$ dominates the right-hand side of (1.8), and then deducing the requisite properties of $\Lambda_3^{(n)}$. The following well-known scholium (easily established by induction) will be instrumental for these purposes.

(4.1) SCHOLIUM. — *For $a>0$, and k a non-negative integer, we have:*

$$(4.2)\qquad \int_0^a \frac{s^k}{(a-s)^{1/2}}\,ds=C_k\,a^{k+(1/2)},$$

and

$$(4.3)\qquad \int_0^a (a-s)^{1/2}\,s^k\,ds=\frac{1}{2(k+1)}\,C_{k+1}\,a^{k+(3/2)},$$

where, for $k\in\mathbb{Z}$, $k\geqq 0$,

$$(4.4)\qquad C_k=\frac{2^{2k+1}(k!)^2}{(2k+1)!}.$$

From Scholium (4.1), (1.9) and (1.10), we obtain the following lemma.

(4.5) LEMMA. — *For* $n \in \mathbb{N}$:

$$\operatorname{sgn}(\Lambda_1^{(n)}) = (-1)^{n-1}, \quad \text{and} \quad \Lambda_2^{(n)} < |\Lambda_1^{(n)}|, \tag{4.6}$$

$$K_1 C_{n-1} \alpha_1^{n-(1/2)} \leqq |\Lambda_1^{(n)}| \leqq K_2 C_{n-1} \alpha_1^{n-(1/2)}, \tag{4.7}$$

$$K_3 \frac{1}{2n} C_n \alpha_1^{n+(1/2)} \leqq \Lambda_2^{(n)} \leqq K_4 \frac{1}{2n} C_n \alpha_1^{n+(1/2)}, \tag{4.8}$$

where K_1, K_2, K_3, K_4 *are positive constants.*

Moreover, using Stirling's formula [1], p. 203, we easily see the following.

(4.9) LEMMA. — *For* $n \in \mathbb{N}$,

$$C_n = \frac{\pi^{1/2} \omega_n}{n^{1/2}},$$

where $\omega_n > 0$, *and* $\omega_n \to 1$ *as* $n \to +\infty$.

Next we observe that by virtue of (1.8), (4.7), (4.8), and (4.9), we have the following lemma.

(4.10) LEMMA. — *For* $n \in \mathbb{N}$:

$$|\Lambda_1^{(n)}| = O(n^{-1/2} \alpha_1^n); \tag{4.11}$$

$$0 < \Lambda_2^{(n)} = O(n^{-3/2} \alpha_1^n); \tag{4.12}$$

$$\frac{\pi}{\sqrt{6}} g\left(\frac{n+1}{2}\right) = \Lambda_3^{(n)} + O(n^{-1/2} \alpha_1^n). \tag{4.13}$$

As an immediate consequence of (4.13) we have the following.

(4.14) COROLLARY. — *For* $n \in \mathbb{N}$,

$$\frac{\pi}{\sqrt{6}} \left| g\left(\frac{n+1}{2}\right) \right| = |\Lambda_3^{(n)}| + O(n^{-1/2} \alpha_1^n).$$

Easy estimates with (1.9) show that

$$|\Lambda_1^{(n)}| \leqq 2(\alpha_1 + 1)^2 3^{-1/4} \alpha_1^{1/2} \int_0^{\alpha_1} \frac{u^{n-1}}{(\alpha_1 - u)^{1/2}} du, \quad \text{for } n \in \mathbb{N}. \tag{4.15}$$

We now take up the proof of Theorem (1.3) (i). From (1.11) we have for $m \in \mathbb{N}$, m odd:

$$(4.16)\quad \Lambda_3^{(m)} = 6(-1)^{(m+1)/2} \int_0^{\beta_1} \frac{y^{m-1}}{(\beta_1 - y)^{1/2}} \times \frac{y - y^3}{(y^2+\alpha_1^2)^{1/2}(y^2+\alpha_2^2)^{1/2}} (\beta_1 + y)^{-1/2} (\beta_2^2 - y^2)^{-1/2} dy,$$

and

$$(4.17)\quad \operatorname{sgn}(\Lambda_3^{(m)}) = (-1)^{(m+1)/2}.$$

We temporarily fix an arbitrary odd $m \in \mathbb{N}$. From (4.6) and (4.12), we have:

$$(4.18)\quad |-\Lambda_1^{(m)} + \Lambda_2^{(m)}| = \Lambda_1^{(m)} - \Lambda_2^{(m)} < \Lambda_1^{(m)}.$$

Easy estimates with (4.16), together with the relations $(\alpha_1 + \alpha_2^2)^{1/2} < 4$, $\beta_1 = \alpha_1^{1/2}$, and $\beta_1 \beta_2 = 1$, give:

$$(4.19)\quad |\Lambda_3^{(m)}| \geqq \frac{3(1-\alpha_1)}{4\beta_1^{1/2}} \int_0^{\beta_1} \frac{y^m}{(\beta_1 - y)^{1/2}} dy.$$

From (4.15), (4.18), and (4.19), we get

$$(4.20)\quad |2\Lambda_3^{(m)}| - |-\Lambda_1^{(m)} + \Lambda_2^{(m)}| \geqq \frac{3(1-\alpha_1)}{2\beta_1^{1/2}} \int_0^{\beta_1} \frac{y^m}{(\beta_1 - y)^{1/2}} dy - 2(\alpha_1+1)^2 3^{-1/4} \alpha_1^{1/2} \int_0^{\alpha_1} \frac{u^{m-1}}{(\alpha_1 - u)^{1/2}} du.$$

By virtue of (4.2), and the fact that $(\alpha_1 + 1)^2 = 6\alpha_1$, we can rewrite (4.20) in the following form.

$$(4.21)\quad |2\Lambda_3^{(m)}| - |-\Lambda_1^{(m)} + \Lambda_2^{(m)}| \geqq 3(1-\alpha_1) \frac{2^{2m}(m!)^2}{(2m+1)!} \beta_1^m - 6\alpha_1 3^{-1/4} \frac{2^{2m}[(m-1)!]^2}{(2m-1)!} \alpha_1^m.$$

Thus, for $m \in \mathbb{N}$, m odd, in order to show that

$$(4.22)\quad |2\Lambda_3^{(m)}| - |-\Lambda_1^{(m)} + \Lambda_2^{(m)}| > 0,$$

it suffices to establish

$$(1-\alpha_1)\frac{m^2}{2m(2m+1)}>2\alpha_1 3^{-1/4}\alpha_1^{m/2}.$$

Thus, since $\alpha_1<3^{-1}$, it suffices for (4.22), in the case of $m\in\mathbb{N}$ with m odd, to have

$$\frac{m^2}{2m(2m+1)}>3^{-(1/4)-(m/2)}, \tag{4.23}$$

which can be shown without difficulty for m odd, $m\geqq 3$. Hence by (4.22) and (1.8), together with (4.17), we see that

$$\operatorname{sgn}\left\{g\left(\frac{m+1}{2}\right)\right\}=(-1)^{(m+1)/2}, \quad \text{for} \quad m \text{ odd}, \quad m\geqq 3. \tag{4.24}$$

Moreover, $\Lambda_3^{(1)}<0$ by (4.17), while by virtue of (4.6) $\Lambda_1^{(1)}>\Lambda_2^{(1)}$. Hence by (1.8), $g(1)<0$. This shows that (4.24) holds for $m=1$. Thus (4.24) holds for all odd $m\in\mathbb{N}$, which is precisely (1.3) (i).

Next we proceed to the demonstration of (1.3) (ii). Suppose that $\nu\in\mathbb{N}$ and ν is even. From (1.11) we see that

$$\begin{aligned}\Lambda_3^{(\nu)}=(-1)^{(\nu+2)/2} & \\ \times\int_0^{\beta_1}\frac{y^{\nu-1}}{(\beta_1-y)^{1/2}}\,&\frac{y^4-10y^2+1}{(y^2+\alpha_1^2)^{1/2}(y^2+\alpha_2^2)^{1/2}} \\ &\times(\beta_1+y)^{-1/2}(\beta_2^2-y^2)^{-1/2}\,dy.\end{aligned} \tag{4.25}$$

Let $\delta_1=5-2\sqrt{6}>0$, $\delta_2=5+2\sqrt{6}$. Thus,

$$y^4-10y^2+1\equiv(\sqrt{\delta_1}-y)(\sqrt{\delta_1}+y)(\delta_2-y^2),$$

and $0<\sqrt{\delta_1}<\beta_1$. We can rewrite (4.25) more concisely as follows:

$$\Lambda_3^{(\nu)}=(-1)^{(\nu+2)/2}\int_0^{\beta_1}\frac{y^{\nu-1}}{(\beta_1-y)^{1/2}}(\sqrt{\delta_1}-y)\,\mathrm{W}(y)\,dy, \tag{4.26}$$

for all even $\nu\in\mathbb{N}$,

where W is a fixed continuous function on $[0, \beta_1]$ such that $W(y)>0$ for all $y\in[0, \beta_1]$. Put $\tau=2^{-1}(\sqrt{\delta_1}+\beta_1)$. We see from (4.26) that

$$(4.27)\quad (-1)^{\nu/2}\Lambda_3^{(\nu)}=\int_\tau^{\beta_1}\frac{y^{\nu-1}}{(\beta_1-y)^{1/2}}\times(y-\sqrt{\delta_1})\,W(y)\,dy+O(\nu^{-1}\tau^\nu),\quad \text{for all even } \nu\in\mathbb{N}.$$

Hence there is a positive constant A such that

$$(4.28)\quad (-1)^{\nu/2}\Lambda_3^{(\nu)}\geqq A\left\{\int_0^{\beta_1}\frac{y^{\nu-1}}{(\beta_1-y)^{1/2}}\,dy-\int_0^{\tau}\frac{y^{\nu-1}}{(\beta_1-y)^{1/2}}\,dy\right\}+O(\nu^{-1}\tau^\nu).$$

Applying (4.2) and (4.9) to the first integral in (4.28), we get a positive constant B such that:

$$(-1)^{\nu/2}\Lambda_3^{(\nu)}\geqq B\,\nu^{-1/2}\beta_1^\nu+O(\nu^{-1}\tau^\nu).$$

Hence

$$(4.29)\quad (-1)^{\nu/2}\Lambda_3^{(\nu)}\geqq\frac{B}{2}\nu^{-1/2}\beta_1^\nu,$$

for all sufficiently large even $\nu\in\mathbb{N}$.

In particular, (4.29) shows that

$$(4.30)\quad \operatorname{sgn}(\Lambda_3^{(\nu)})=(-1)^{\nu/2},$$

for all sufficiently large even $\nu\in\mathbb{N}$.

It follows from (4.7), (4.8), (4.9), and (4.29) that:

$$(4.31)\quad |2\Lambda_3^{(\nu)}|-|-\Lambda_1^{(\nu)}+\Lambda_2^{(\nu)}|\geqq B\,\nu^{-1/2}\beta_1^\nu+O(\nu^{-1/2}\alpha_1^\nu),$$

for all sufficiently large even $\nu\in\mathbb{N}$.

Hence the positive constant B satisfies:

$$(4.32)\quad |2\Lambda_3^{(\nu)}|-|-\Lambda_1^{(\nu)}+\Lambda_2^{(\nu)}|\geqq\frac{B}{2}\nu^{-1/2}\beta_1^\nu,$$

for all sufficiently large even $\nu \in \mathbb{N}$.

Using (4.32) together with (1.8) and (4.30), we easily see that for all sufficiently large even $\nu \in \mathbb{N}$:

$$\operatorname{sgn}\left[g\left(\frac{\nu+1}{2}\right)\right]=(-1)^{\nu/2}, \tag{4.33}$$

and

$$\left|g\left(\frac{\nu+1}{2}\right)\right| \geqq B_1\, \nu^{-1/2}\, \beta_1^{\nu}, \tag{4.34}$$

where B_1 is a positive constant.

Combining (4.33) with (1.3) (i) completes the demonstration of (1.3) (ii).

5. PROOF OF THEOREM (1.3) (iii), (iv)

From (4.25) and (4.30) we see that

$$\begin{aligned} |\Lambda_3^{(2k)}| = \int_0^{\beta_1} & \frac{y^{2k-1}}{(\beta_1-y)^{1/2}} \\ & \times \frac{-y^4+10y^2-1}{(y^2+\alpha_1^2)^{1/2}(y^2+\alpha_2^2)^{1/2}} (\beta_1+y)^{-1/2}(\beta_2^2-y^2)^{-1/2}\,dy, \end{aligned} \tag{5.1}$$

for all sufficiently large $k \in \mathbb{N}$.

From (4.16) we have

$$\begin{aligned} |\Lambda_3^{(2k+1)}| = \int_0^{\beta_1} & \frac{y^{2k-1}}{(\beta_1-y)^{1/2}} \\ & \times \frac{6y^2-6y^4}{(y^2+\alpha_1^2)^{1/2}(y^2+\alpha_2^2)^{1/2}} (\beta_1+y)^{-1/2}(\beta_2^2-y^2)^{-1/2}\,dy, \end{aligned} \tag{5.2}$$

for $k \in \mathbb{N}$.

Using (5.1), (5.2) and Corollary (4.14), we find that:

$$\frac{\pi}{\sqrt{6}}\left\{\left|g\left(\frac{2k+1}{2}\right)\right|-\left|g\left(\frac{2k+2}{2}\right)\right|\right\}$$

(5.3)
$$=\int_0^{\beta_1}\frac{y^{2k-1}}{(\beta_1-y)^{1/2}}\,\Phi(y)\,G(y)\,dy+O\left(k^{-1/2}\alpha_1^{2k}\right),$$

for all sufficiently large $k\in\mathbb{N}$,

where

$$G(y)\equiv(y^2+\alpha_1^2)^{-1/2}(y^2+\alpha_2^2)^{-1/2}(\beta_1+y)^{-1/2}(\beta_2^2-y^2)^{-1/2}>0$$

is continuous on $[0,\beta_1]$, and

$$\Phi(y)\equiv 5y^4+4y^2-1\equiv 5(y^2+1)(y+5^{-1/2})(y-5^{-1/2}).$$

Since $0<5^{-1/2}<\beta_1$, we have:

(5.4) $\quad \Phi<0$ on $[0, 5^{-1/2})$, and $\Phi>0$ on $(5^{-1/2}, \beta_1]$.

By virtue of (3.6) we can choose and fix a number η such that

(5.5)
$$\max\{\alpha_1, 5^{-1/2}\}<\eta<\beta_1.$$

From (5.3) we get

$$\frac{\pi}{\sqrt{6}}\left\{\left|g\left(\frac{2k+1}{2}\right)\right|-\left|g\left(\frac{2k+2}{2}\right)\right|\right\}$$

$$=\int_0^{\eta}\frac{y^{2k-1}}{(\beta_1-y)^{1/2}}\,\Phi(y)\,G(y)\,dy$$

(5.6)
$$+\int_{\eta}^{\beta_1}\frac{y^{2k-1}}{(\beta_1-y)^{1/2}}\,\Phi(y)\,G(y)\,dy+O(k^{-1/2}\alpha_1^{2k})$$

$$=\int_{\eta}^{\beta_1}\frac{y^{2k-1}}{(\beta_1-y)^{1/2}}\,\Phi(y)\,G(y)\,dy+O(k^{-1}\eta^{2k})+O(k^{-1/2}\alpha_1^{2k}),$$

for all sufficiently large $k\in\mathbb{N}$.

Since Φ and G are positive and continuous on $[\eta, \beta_1]$, we can utilize (5.6), (4.2), and Lemma (4.9) to obtain a positive constant r such that:

$$\text{(5.7)}\quad \frac{\pi}{\sqrt{6}}\left\{\left|g\left(\frac{2k+1}{2}\right)\right|-\left|g\left(\frac{2k+2}{2}\right)\right|\right\} \geqq r k^{-1/2}\beta_1^{2k}+O(k^{-1}\eta^{2k})+O(k^{-1/2}\alpha_1^{2k}),$$

for all sufficiently large $k\in\mathbb{N}$.

In view of (5.5), it follows from (5.7) that

$$\text{(5.8)}\quad \left|g\left(\frac{2k+1}{2}\right)\right|>\left|g\left(\frac{2k+2}{2}\right)\right|,$$

for all sufficiently large $k\in\mathbb{N}$.

Next we replace k by $(k+1)$ in (5.1). Using the resulting equation together with (5.2) and Corollary (4.14), we obtain the following relation.

$$\text{(5.9)}\quad \frac{\pi}{\sqrt{6}}\left\{\left|g\left(\frac{2k+2}{2}\right)\right|-\left|g\left(\frac{2k+3}{2}\right)\right|\right\} =\int_0^{\beta_1}\frac{y^{2k+1}}{(\beta_1-y)^{1/2}}\frac{y^4-16y^2+7}{(y^2+\alpha_1^2)^{1/2}(y^2+\alpha_2^2)^{1/2}} \times(\beta_1+y)^{-1/2}(\beta_2^2-y^2)^{-1/2}\,dy+O(k^{-1/2}\alpha_1^{2k}),$$

for all sufficiently large $k\in\mathbb{N}$.

Let $q(y)\equiv y^4-16y^2+7$. It is easy to see that $q(y)>0$ for all $y\in[0, \beta_1]$. Applying this last fact to (5.9), we see that there is a positive constant μ such that

$$\text{(5.10)}\quad \frac{\pi}{\sqrt{6}}\left\{\left|g\left(\frac{2k+2}{2}\right)\right|-\left|g\left(\frac{2k+3}{2}\right)\right|\right\} \geqq\mu\int_0^{\beta_1}\frac{y^{2k+1}}{(\beta_1-y)^{1/2}}\,dy+O(k^{-1/2}\alpha_1^{2k}),$$

for all sufficiently large $k\in\mathbb{N}$.

As previously, this gives

$$(5.11) \qquad \left|g\left(\frac{2k+2}{2}\right)\right| > \left|g\left(\frac{2k+3}{2}\right)\right|$$

for all sufficiently large $k \in \mathbb{N}$.

Combining (5.8) with (5.11) establishes (1.3) (iii).

To complete the proof of Theorem (1.3), we next show (1.3) (iv). From (4.19) together with (4.2) and Corollary (4.14) we see that there is a positive constant R such that:

$$\left|g\left(\frac{m+1}{2}\right)\right| \geqq \mathrm{R}\, m^{-1/2}\, \beta_1^m + O\left(m^{-1/2}\, \alpha_1^m\right),$$

for all odd $m \in \mathbb{N}$.

Hence for all sufficiently large odd $m \in \mathbb{N}$,

$$(5.12) \qquad \left|g\left(\frac{m+1}{2}\right)\right| \geqq \frac{\mathrm{R}}{2}\, m^{-1/2}\, \beta_1^m.$$

Combining (4.34) with (5.12), we obtain (for all sufficiently large $n \in \mathbb{N}$) the lower estimate for $|g(n/2)|$ in (1.3) (iv). From the definition of $\Lambda_3^{(n)}$ in (1.11) together with (4.2) and Lemma (4.9), we see that for $n \in \mathbb{N}$,

$$(5.13) \qquad \left|\Lambda_3^{(n)}\right| = O\left(n^{-1/2}\, \beta_1^n\right).$$

Use of (5.13) in Corollary (4.14) completes the proof of (1.3) (iv).

ACKNOWLEDGEMENT

The work of the author was supported by a grant from the National Science Foundation (U.S.A.).

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(*Manuscrit reçu le* 7 *juillet* 1987.)