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A continuous version of Liapunov's convexity theorem

by

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ABSTRACT. — Given a continuous map $s \mapsto \mu_s$, from a compact metric space into the space of nonatomic measures on T , we show the existence of a family $(A_\alpha^s)_{\alpha \in [0, 1]}$, increasing in α and continuous in s , such that

$$\mu_s(A_\alpha^s) = \alpha \mu_s(T) \quad (\alpha \in [0, 1]).$$

Key words : Liapunov's convexity theorem - Measure theory - Selections.

RÉSUMÉ. — Étant donnée une application continue $s \mapsto \mu_s$, d'un espace métrique compact dans l'espace des mesures nonatomicques sur T , nous montrons l'existence d'une famille $(A_\alpha^s)_{\alpha \in [0, 1]}$, croissante avec α et continue en s , telle que

$$\mu_s(A_\alpha^s) = \alpha \mu_s(T) \quad (\alpha \in [0, 1]).$$

1. INTRODUCTION

Let μ be a non-atomic finite measure on a measurable space T . A result of measure theory states the existence of a family $(A_\alpha)_\alpha$ of subsets of T , increasing with α in $[0, 1]$ and such that

$$\mu(A_\alpha) = \alpha \mu(T).$$

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According to Liapunov's Convexity Theorem on the range of vector measures (see Halmos [2], [3] and Liapunov [4]) the above result holds for a finite family of nonatomic measures μ_i , $i=1, \dots, n$: there exists an increasing family $(A_\alpha)_\alpha$ such that

$$\mu_i(A_\alpha) = \alpha \mu_i(T), \quad i=1, \dots, n.$$

In general, the above is not true for an infinite family $(\mu_s)_s$ of measures (see Liapunov [5]). In this paper we consider a map $s \rightarrow \mu_s$, continuous for s in a compact metric space S . Denoting by $\mathcal{A}(\mu_s)$ the set of increasing families $(A_\alpha^s)_\alpha$ satisfying

$$\mu_s(A_\alpha^s) = \alpha \mu_s(T),$$

we show the existence of a selection $(\tilde{A}_\alpha^s)_\alpha$ of the multivalued map $\mathcal{A}(\mu_s)$ continuously depending on s in the sense of Definition 2 of the following section.

2. NOTATIONS AND PRELIMINARY RESULTS

We consider a measure space (T, \mathcal{F}, μ_0) where μ_0 is a non-atomic positive measure on a σ -algebra \mathcal{F} and $\mu_0(T)=1$. Denote by \mathcal{M} the set of positive finite measures μ on T which are absolutely continuous with respect to μ_0 , hence non-atomic. The metric in \mathcal{M} is induced by the norm $\|\mu\|$ given by the variation of μ .

DEFINITION 1. — A family $(A_\alpha)_{\alpha \in [0, 1]}$, $A_\alpha \in \mathcal{F}$, is called *increasing* if

$$A_\alpha \subseteq A_\beta \quad \text{when } \alpha \leq \beta.$$

An increasing family is called *refining* $A \in \mathcal{F}$ with respect to the measure $\mu = (\mu_1, \dots, \mu_n) \in \mathcal{M}^n$ if $A_0 = \emptyset$, $A_1 = A$ and

$$\mu(A_\alpha) = \alpha \mu(A) \quad (\alpha \in [0, 1]).$$

The set of the families refining T with respect to μ is denoted by $\mathcal{A}(\mu)$.

The proofs of Lemmas 1 and 2 are based on Liapunov's theorem (see Fryszkowski [1]).

LEMMA 1. — Consider a vector measure $\mu \in \mathcal{M}^n$. For each $A \in \mathcal{F}$ there exists a family $(A_\alpha)_{\alpha \in [0, 1]}$ refining A with respect to μ . In particular, the set $\mathcal{A}(\mu)$ is nonempty.

In what follows, S is a compact metric space with distance d .

LEMMA 2. — Let $s \rightarrow \mu_s$ be a continuous map from S into \mathcal{M}^n . Then for every $\varepsilon > 0$ there exists an increasing family $(A_\alpha)_\alpha$ satisfying

- (i) $\mu_0(A_\alpha) = \alpha$ ($\alpha \in [0, 1]$);
- (ii) $|\mu_s(A_\alpha) - \alpha \mu_s(T)| < \varepsilon$ ($\alpha \in [0, 1], s \in S$).

DEFINITION 2. — A map $s \rightarrow (A_\alpha^s)_\alpha$ is called *continuous* on S if for every $s^0 \in S$ and $\varepsilon > 0$ there exists a $\delta > 0$ such that: s, s' and s'' in $B(s^0, \delta)$ implies

$$\sup_{\alpha \in [0, 1]} \mu_s(A_\alpha^{s'} \Delta A_\alpha^{s''}) < \varepsilon.$$

Analogously we set

DEFINITION 3. — The set valued map $s \rightarrow \mathcal{A}(\mu_s)$ is called *continuous* if for every $s^0 \in S$ and $\varepsilon > 0$ there exists a $\delta > 0$ such that: s, s' and s'' in $B(s^0, \delta)$ implies $\forall (A'_\alpha) \in \mathcal{A}(\mu_{s'}), \exists (A''_\alpha) \in \mathcal{A}(\mu_{s''})$ such that

$$\sup_{\alpha \in [0, 1]} \mu_s(A'_\alpha \Delta A''_\alpha) < \varepsilon.$$

We will use the symbol \cup to denote the union of disjoint sets. Finally, we recall that $\rho(\cdot, \cdot)$ defined as $\rho(A, B) = \mu(A \Delta B)$ ($\mu \in \mathcal{M}$) is a pseudometric on \mathcal{F} .

Remarks. — (a) In [5], Liapunov considers a sequence μ_n of measures on $[0, 2\pi]$ defined by a family of densities f_n converging strongly in L^1 to zero. He shows that there cannot exist any Borel subset A of $[0, 2\pi]$ such that for every $n, \mu_n(A) = \frac{1}{2} \mu_n([0, 2\pi])$. By associating μ_n to the point $1/n$ and $\mu_\infty = 0$ to the point 0, we have a map $s \mapsto \mu_s$ from the compact metric space $S = \{1/n : n \in \mathbb{N}\} \cup \{0\}$ into the space of nonatomic measures. The continuity at 0 follows from the strong convergence of (f_n) . This example shows that the assumptions of Theorem 1 below do not guarantee the existence of a constant selection.

(b) A further example is taken from Valadier [7]. Let S and T be the real interval $[0, 1]$, and set $\mu_s(A) = \int_A e^{-st} dt$. Assume there exists a set $\bar{A} \subseteq T$ such that

$$\forall s, \mu_s(\bar{A}) = \frac{1}{2} \mu_s(T).$$

Then

$$\int_{-\infty}^{+\infty} \chi_{\bar{A}}(t) e^{-st} dt = \int_{-\infty}^{+\infty} \frac{1}{2} \chi_T(t) e^{-st} dt.$$

Since the Laplace transformations of χ_A and $\frac{1}{2}\chi_T$, both of compact support, are analytic and coincide on $[0,1]$, they are identical. By the injectivity of the Lapalce Transformation, we have

$$\chi_A = \frac{1}{2}\chi_T,$$

a contradiction. Hence again we have an example where there exist no constant selections.

(c) It seems more natural to express the continuity in terms of the pseudometric $\rho(A, B) = \mu_0(A, B)$. However, Definition 2 is not necessarily equivalent to the continuity with respect to this pseudometric when μ_0 is not absolutely continuous with respect to μ_s^0 .

3. MAIN RESULTS

In order to prove our main theorem we need three additional Lemmas.

LEMMA 3. — Consider a 1-dimensional measure $\mu \in \mathcal{M}$ and an increasing family $(A_\alpha^1)_\alpha$ such that for some $\varepsilon > 0$,

$$|\mu(A_\alpha^1) - \alpha\mu(T)| < \varepsilon \quad (\alpha \in [0, 1]).$$

There exists an increasing family $(A_\alpha^2)_\alpha$ such that

$$(i) \quad \mu(A_\alpha^2) = \alpha\mu(T) \quad (\alpha \in [0, 1])$$

$$(ii) \quad \mu(A_\alpha^1 \triangle A_\alpha^2) < 6\varepsilon \quad (\alpha \in [0, 1]).$$

Proof. — Fix M so that $\frac{1}{M} \geq \frac{\varepsilon}{\mu(T)} \geq \frac{1}{M+1}$. We begin by defining recursively an increasing family $(A_\alpha^2)_\alpha$ for $\alpha = i/M$, $i=0, \dots, M$, such that (i) holds and $A_{i/M}^2 \subseteq A_{(i+1)/M}^1$. Set $A_0^2 = \emptyset$ and assume $A_{i/M}^2$ has been defined for $i=0, \dots, n < M$.

Case 1. — When $\mu(A_{(n+1)/M}^1) \geq \frac{n+1}{M}\mu(T)$, define $A_{(n+1)/M}^2$ by Lemma 1, as a set such that $A_{n/M}^2 \subseteq A_{(n+1)/M}^2 \subseteq A_{(n+1)/M}^1$ and

$$\mu(A_{(n+1)/M}^2) = \frac{n+1}{M}\mu(T).$$

Case 2. — When $\mu(A_{(n+1)/M}^1) < \frac{n+1}{M}\mu(T)$, we first notice that by the

choice of M we have that $\mu(A_{(n+2)/M}^1) \geq \frac{n+1}{M} \mu(T)$; hence we can define

$A_{(n+1)/M}^2$ as a set such that $A_{(n+1)/M}^1 \subseteq A_{(n+1)/M}^2 \subseteq A_{(n+2)/M}^1$ and

$$\mu(A_{(n+1)/M}^2) = \frac{n+1}{M} \mu(T).$$

Notice that $A_{(n+1)/M}^2 \supseteq A_{n/M}^2$, since $A_{(n+1)/M}^1 \supseteq A_{n/M}^2$ by the inductive hypothesis.

In either case, we have

$$\begin{aligned} \mu(A_{(n+1)/M}^2 \Delta A_{(n+1)/M}^1) &= \left| \mu(A_{(n+1)/M}^2) - \mu(A_{(n+1)/M}^1) \right| \\ &\leq \left| \mu(A_{(n+1)/M}^2) - \frac{n+1}{M} \mu(T) \right| \\ &\quad + \left| \mu(A_{(n+1)/M}^1) - \frac{n+1}{M} \mu(T) \right| \\ &< \varepsilon. \end{aligned}$$

By Lemma 1 it is now easy to define a family $(A_\alpha^2)_{\alpha \in [0, 1]}$ such that

- (a) $A_{i/M}^2 \subseteq A_\alpha^2 \subseteq A_\beta^2 \subseteq A_{(i+1)/M}^2$ for $\frac{i}{M} \leq \alpha \leq \beta \leq \frac{i+1}{M}$;
- (b) $\mu(A_\alpha^2) = \alpha \mu(T)$.

Now we check that (ii) holds for $\frac{i}{M} \leq \alpha \leq \frac{i+1}{M}$. We can as well assume that $\mu(T) \geq 6\varepsilon$ otherwise (ii) trivially holds.

$$\begin{aligned} \mu(A_\alpha^1 \Delta A_\alpha^2) &= \mu(A_\alpha^1 \setminus A_\alpha^2) + \mu(A_\alpha^2 \setminus A_\alpha^1) \\ &\leq \mu(A_{(i+1)/M}^1 \setminus A_{i/M}^1) + \mu(A_{i/M}^1 \setminus A_{i/M}^2) \\ &\quad + \mu(A_{(i+1)/M}^2 \setminus A_{i/M}^2) + \mu(A_{i/M}^2 \setminus A_{i/M}^1) \\ &\leq \frac{1}{M} \mu(T) + 2\varepsilon + \frac{1}{M} \mu(T) + \varepsilon \\ &\leq 2 \frac{\varepsilon \mu(T)}{\mu(T) - \varepsilon} + 3\varepsilon = \frac{2\varepsilon}{1 - (\varepsilon/\mu(T))} + 3\varepsilon \\ &\leq \left(\frac{12}{5} + 3 \right) \varepsilon < 6\varepsilon. \quad \blacksquare \end{aligned}$$

COROLLARY. — *The set-valued map $s \rightarrow \mathcal{A}(\mu_s)$ is continuous.*

Proof. — Choose s^0 and $\varepsilon > 0$. Let $\delta > 0$ be such that $d(s, s^0) < \delta$ implies $\|\mu_s - \mu_{s^0}\| < \varepsilon/26$. Fix s, s' and s'' in $B(s^0, \delta)$ and $A'_\alpha \in \mathcal{A}(\mu_{s'})$. Since

$$\begin{aligned} |\mu_{s^0}(A'_\alpha) - \alpha\mu_{s^0}(T)| &= |\mu_{s^0}(A'_\alpha) - \mu_{s'}(A'_\alpha) + \mu_{s'}(A'_\alpha) \\ &\quad - \alpha\mu_{s'}(T) + \alpha\mu_{s'}(T) - \alpha\mu_{s^0}(T)| \\ &\leq 2\|\mu_{s'} - \mu_{s^0}\| < \varepsilon/13, \end{aligned}$$

by Lemma 3 there exists $A''_\alpha \in \mathcal{A}(\mu_{s^0})$ such that $\mu_{s^0}(A'_\alpha \triangle A''_\alpha) \leq 6\varepsilon/13$. Analogously, given A''_α , there exists $A'''_\alpha \in \mathcal{A}(\mu_{s''})$ such that $\mu_{s''}(A''_\alpha \triangle A'''_\alpha) \leq 6\varepsilon/13$.

Hence

$$\begin{aligned} \mu_s(A'_\alpha \triangle A'''_\alpha) &\leq |\mu_s(A'_\alpha \triangle A'''_\alpha) - \mu_{s^0}(A'_\alpha \triangle A'''_\alpha)| + \mu_{s^0}(A'_\alpha \triangle A'''_\alpha) \\ &\leq \|\mu_s - \mu_{s^0}\| + \mu_{s^0}(A'_\alpha \triangle A''_\alpha) + \mu_{s^0}(A''_\alpha \triangle A'''_\alpha) \\ &\leq \varepsilon/26 + 6\varepsilon/13 + \|\mu_{s^0} - \mu_{s''}\| + \mu_{s''}(A''_\alpha \triangle A'''_\alpha) \\ &\leq \varepsilon. \quad \blacksquare \end{aligned}$$

In the following Lemmas, the symbol $\sup_{\lambda_j(s) > 0}$ is a shorthand notation

for $\sup_{\{j \in \mathbb{N} : \lambda_j(s) > 0\}}$.

LEMMA 4. — Let $s \rightarrow \mu_s$ be a continuous map from a metric space S into the space \mathcal{M} and let $(B(s_j, \eta_j))_{j=1, \dots, N}$ be a finite open covering of S . Let $(\lambda_j(\cdot))_{j=1, \dots, N}$ be a continuous partition of unity subordinate to it such that $\lambda_j(s_j) = 1$.

For any center $s_j, j=1, \dots, N$, let be defined a finite increasing family $(\bar{A}_{i/M}^{s_j})_{i=0, \dots, M}$ such that

$$\mu_{s_j}(\bar{A}_{i/M}^{s_j}) = \frac{i}{M} \mu_{s_j}(T) \quad (i \in \{0, \dots, M\}).$$

Then for each $s \in S$ there exists an increasing family $(A_\alpha^s)_\alpha$ that extends the family $(\bar{A}_{i/M}^{s_j})_i$ in the sense that $A_{i/M}^{s_j} = \bar{A}_{i/M}^{s_j}$ for every i and j , and such that the following properties hold:

$$(i) \quad |\mu_s(A_\alpha^s) - \alpha\mu_s(T)| \leq 6 \sup_{\lambda_j(s) > 0} \|\mu_s - \mu_{s_j}\| \quad (\alpha \in [0, 1]);$$

$$(ii) \quad \text{for } \alpha \in \left[\frac{i}{M}, \frac{i+1}{M} \right] \text{ and any center } s_j,$$

$$\mu_{s_j}(A_\alpha^s \triangle A_\alpha^{s_j}) \leq \sup_{\lambda_k(s) > 0} \mu_{s_j}(\bar{A}_{(i+1)/M}^{s_k} \triangle \bar{A}_{(i+1)/M}^{s_j})$$

$$+ \sup_{\lambda_k(s) > 0} \|\mu_{s_j} - \mu_{s_k}\| + \frac{1}{M} (\sup_{\lambda_k(s) > 0} \mu_{s_k}(\mathbb{T}) + \mu_{s_j}(\mathbb{T}));$$

(iii) $\lim_{s \rightarrow s^*} \sup_{\alpha \in [0,1]} \mu_0(A_\alpha^s \triangle A_\alpha^{s^*}) = 0.$

Proof. – For each $s \in S$, first we will define the sets $(A_{i/M}^s)_i$ by interpolating among the given families $(\bar{A}_{i/M}^{s_j})_i$, taking from each set a subset having measure proportional to the corresponding $\lambda_i(s)$. Then we extend the construction for $\alpha \in]i/M, (i+1)/M[$. Finally we check that (i)-(iii) hold.

I. For any set $A \subseteq \mathbb{T}$, we define $A^1 = A$ and $A^0 = \mathbb{T} \setminus A$. We denote by \mathcal{X} the set of all $N \times (M-1)$ matrices $\Gamma = (\gamma_{ij})$ whose elements are in $\{0, 1\}$.

Now we define

$$\begin{aligned} A(\Gamma) = & (\bar{A}_{1/M}^{s_1})^{\gamma_{11}} \cap \dots \cap (\bar{A}_{1/M}^{s_N})^{\gamma_{1N}} \\ & \cap (\bar{A}_{2/M}^{s_1})^{\gamma_{21}} \cap \dots \cap (\bar{A}_{2/M}^{s_N})^{\gamma_{2N}} \\ & \dots \dots \dots \\ & \cap (\bar{A}_{(M-1)/M}^{s_1})^{\gamma_{M-1,1}} \cap \dots \cap (\bar{A}_{(M-1)/M}^{s_N})^{\gamma_{M-1,N}}. \end{aligned}$$

Note that:

- (a) since the family $(\bar{A}_{i/M}^{s_j})_i$ is increasing in i , $A(\Gamma) = \emptyset$ if $\exists i, j: \gamma_{ij} = 1, \gamma_{i+1,j} = 0$; moreover, if $\Gamma_1 \neq \Gamma_2$, then $A(\Gamma_1) \cap A(\Gamma_2) = \emptyset$;
- (b) for any i, j

$$\bar{A}_{i/M}^{s_j} = \bigcup_{\substack{\Gamma \in \mathcal{X} \\ \gamma_{ij} = 1}} A(\Gamma),$$

i. e. the family at the r. h. s. is a partition of $\bar{A}_{i/M}^{s_j}$;

$$(c) \quad \bigcup_{\substack{\Gamma \in \mathcal{X} \\ \gamma_{ij} = 0, \gamma_{ik} = 1}} A(\Gamma) = A_{i/M}^{s_k} \setminus A_{i/M}^{s_j}, \quad \bigcup_{\substack{\Gamma \in \mathcal{X} \\ \gamma_{ij} = 1, \gamma_{ik} = 1}} A(\Gamma) = A_{i/M}^{s_j} \cap A_{i/M}^{s_k}.$$

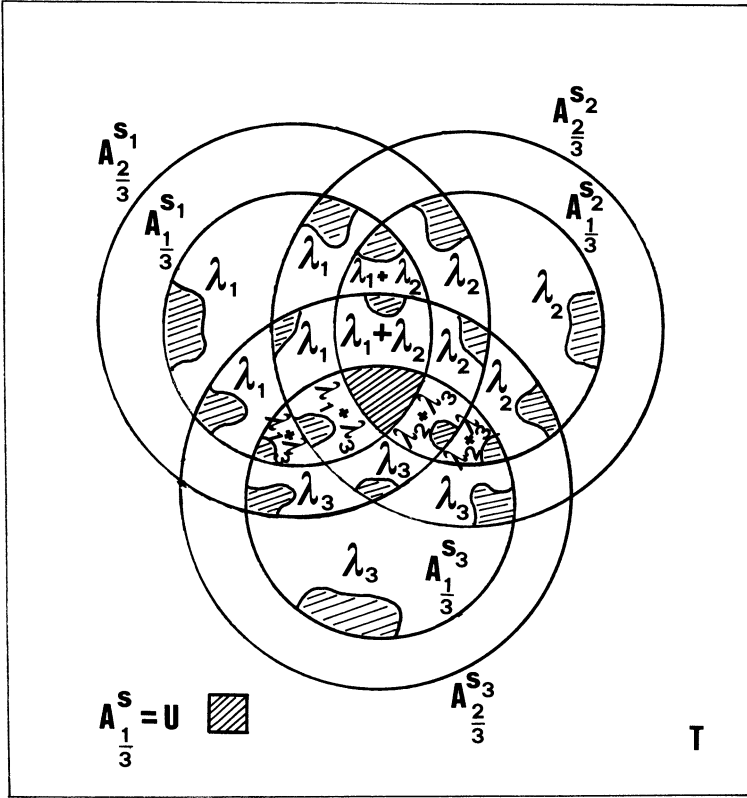
By lemma 1, for each $\Gamma \in \mathcal{X}$ there exists a family $(A(\Gamma)_\alpha)_{\alpha \in [0,1]}$ refining $A(\Gamma)$ with respect to the measure $(\mu_0, \mu_{s_1}, \dots, \mu_{s_N})$. Define

$$\beta_\Gamma^i(s) = \sum_{k=1}^N \gamma_{ik} \lambda_k(s)$$

and

$$A_{i/M}^s = \bigcup_{\Gamma \in \mathcal{X}} A(\Gamma)_{\beta_\Gamma^i(s)} \tag{1}$$

(see Fig., where the case $N = M = 3$ is described).



The family $(A_{i/M}^s)_i$ coincides with $(\bar{A}_{i/M}^{s_j})_i$ for $s=s_j$; in fact we have $\beta_\Gamma^i(s_j) = \gamma_{ij}$ so that, by (b),

$$A_{i/M}^{s_j} = \bigcup_{\Gamma \in \mathcal{X}} A(\Gamma)_{\gamma_{ij}} = \bigcup_{\substack{\Gamma \in \mathcal{X} \\ \gamma_{ij}=1}} A(\Gamma) = \bar{A}_{i/M}^{s_j}.$$

Next we have:

$$\begin{aligned} \mu_{s_j}(A_{i/M}^s) &= \sum_{\Gamma \in \mathcal{X}} \mu_{s_j}(A(\Gamma)_{\beta_\Gamma^i(s)}) = \sum_{\Gamma \in \mathcal{X}} \beta_\Gamma^i(s) \mu_{s_j}(A(\Gamma)) \\ &= \sum_{\Gamma \in \mathcal{X}} \left(\sum_{k=1}^N \gamma_{ik} \lambda_k(s) \right) \mu_{s_j}(A(\Gamma)) \\ &= \sum_{k=1}^N \lambda_k(s) \sum_{\Gamma \in \mathcal{X}} \gamma_{ik} \mu_{s_j}(A(\Gamma)) \end{aligned} \tag{2}$$

$$\begin{aligned}
 &= \sum_{k=1}^N \lambda_k(s) \mu_{s_j} \left(\bigcup_{\substack{\Gamma \in \mathcal{X} \\ \gamma_{ik}=1}} A(\Gamma) \right) \\
 &= \sum_{k=1}^N \lambda_k(s) \mu_{s_j}(A_{i/M}^{s_k}).
 \end{aligned}$$

II. Set, for $\alpha = (1-t)i/M + t(i+1)/M$ ($t \in [0, 1]$) and $s \in S$,

$$A_\alpha^s = \bigcup_{\Gamma \in \mathcal{X}} A(\Gamma)_{(1-t)\beta_\Gamma^i(s) + t\beta_\Gamma^{i+1}(s)}.$$

Remark that by the above definition and (1), it follows that

$$\mu_{s_j}(A_\alpha^s) = (1-t) \mu_{s_j}(A_{i/M}^s) + t \mu_{s_j}(A_{(i+1)/M}^s).$$

We claim that

$$\mu_{s_j}(A_\alpha^s) = \sum_{k=1}^N \lambda_k(s) \mu_{s_j}(A_\alpha^{s_k}) \quad (j=1, \dots, N; \alpha \in [0, 1]; s \in S).$$

In fact, for α as above, we have:

$$\begin{aligned}
 \mu_{s_j}(A_\alpha^s) &= \sum_{\Gamma \in \mathcal{X}} [(1-t)\beta_\Gamma^i(s) + t\beta_\Gamma^{i+1}(s)] \mu_{s_j}(A(\Gamma)) \\
 &= (1-t) \sum_{k=1}^N \lambda_k(s) \sum_{\Gamma \in \mathcal{X}} \gamma_{ik} \mu_{s_j}(A(\Gamma)) \\
 &\quad + t \sum_{k=1}^N \lambda_k(s) \sum_{\Gamma \in \mathcal{X}} \gamma_{i+1,k} \mu_{s_j}(A(\Gamma)) \\
 &= (1-t) \sum_{k=1}^N \lambda_k(s) \mu_{s_j}(A_{i/M}^{s_k}) + t \sum_{k=1}^N \lambda_k(s) \mu_{s_j}(A_{(i+1)/M}^{s_k}) \\
 &= \sum_{k=1}^N \lambda_k(s) [(1-t) \mu_{s_j}(A_{i/M}^{s_k}) + t \mu_{s_j}(A_{(i+1)/M}^{s_k})] \\
 &= \sum_{k=1}^N \lambda_k(s) \mu_{s_j}(A_\alpha^{s_k}).
 \end{aligned}$$

III. We are now in the position of proving (i). Fix $s \in S$ and $\alpha \in [0, 1]$ and set $\omega_s = \sup \{ \|\mu_s - \mu_{s_j}\| : \lambda_j(s) > 0 \}$. We have:

$$\begin{aligned}
 |\mu_s(A_\alpha^s) - \alpha \mu_s(T)| &\leq |\mu_s(A_\alpha^s) - \mu_{s_j}(A_\alpha^s)| \\
 &\quad + |\mu_{s_j}(A_\alpha^s) - \alpha \mu_{s_j}(T)| + \alpha |\mu_{s_j}(T) - \mu_s(T)|
 \end{aligned}$$

$$\begin{aligned}
&\leq 2\omega_s + \left| \sum_{k=1}^N \lambda_k(s) \mu_{s_j}(A_\alpha^{s_k}) - \alpha \mu_{s_j}(T) \right| \\
&\leq 2\omega_s + \sum_{k=1}^N \lambda_k(s) [|\mu_{s_j}(A_\alpha^{s_k}) - \mu_{s_k}(A_\alpha^{s_k})| \\
&\quad + \alpha |\mu_{s_k}(T) - \mu_{s_j}(T)|] \\
&\leq 6\omega_s.
\end{aligned}$$

In order to prove (ii), note first that

$$\begin{aligned}
A_{i/M}^s \triangle A_{i/M}^{s_j} &= \left(\bigcup_{\Gamma \in \mathcal{X}} A(\Gamma)_{\beta_\Gamma^i(s)} \right) \triangle \left(\bigcup_{\Gamma \in \mathcal{X}} A(\Gamma)_{\beta_\Gamma^i(s_j)} \right) \\
&= \bigcup_{\Gamma \in \mathcal{X}} (A(\Gamma)_{\beta_\Gamma^i(s)} \triangle A(\Gamma)_{\beta_\Gamma^i(s_j)})
\end{aligned} \tag{3}$$

and that, by a calculation similar to (2) and by (c),

$$\mu_{s_j} \left(\bigcup_{\substack{\Gamma \in \mathcal{X} \\ \gamma_{ij}=0}} A(\Gamma)_{\beta_\Gamma^i(s)} \right) = \sum_{k=1}^N \lambda_k(s) \mu_{s_j}(A_{i/M}^{s_k} \setminus A_{i/M}^{s_j}), \tag{4}$$

$$\mu_{s_j} \left(\bigcup_{\substack{\Gamma \in \mathcal{X} \\ \gamma_{ij}=1}} (A(\Gamma) \setminus A(\Gamma)_{\beta_\Gamma^i(s)}) \right) = \sum_{k=1}^N \lambda_k(s) \mu_{s_j}(A_{i/M}^{s_j} \setminus A_{i/M}^{s_k}). \tag{5}$$

Therefore, for any i, j , from (3) and recalling that $\beta_\Gamma^i(s_j) = \gamma_{ij}$, we have

$$\mu_{s_j}(A_{i/M}^s \triangle A_{i/M}^{s_j}) = \mu_{s_j} \left(\bigcup_{\substack{\Gamma \in \mathcal{X} \\ \gamma_{ij}=0}} A(\Gamma)_{\beta_\Gamma^i(s)} \right) + \mu_{s_j} \left(\bigcup_{\substack{\Gamma \in \mathcal{X} \\ \gamma_{ij}=1}} (A(\Gamma) \setminus A(\Gamma)_{\beta_\Gamma^i(s)}) \right)$$

and from (4), (5) this last expression is

$$\begin{aligned}
&\sum_{k=1}^N \lambda_k(s) \mu_{s_j}(A_{i/M}^{s_k} \setminus A_{i/M}^{s_j}) + \sum_{k=1}^N \lambda_k(s) \mu_{s_j}(A_{i/M}^{s_j} \setminus A_{i/M}^{s_k}) \\
&= \sum_{k=1}^N \lambda_k(s) \mu_{s_j}(A_{i/M}^{s_k} \triangle A_{i/M}^{s_j}) \\
&\leq \sup \{ \mu_{s_j}(A_{i/M}^{s_k} \triangle A_{i/M}^{s_j}) : \lambda_k(s) > 0 \}.
\end{aligned}$$

Hence (ii) holds for $\alpha = i/M$.

In order to prove (ii) for α in $]i/M, (i+1)/M[$, let us note that

$$\begin{aligned}
A_\alpha^s \setminus A_\alpha^{s'} &\subseteq [(A_{(i+1)/M}^s \setminus A_{(i+1)/M}^{s_j}) \setminus A_\alpha^{s'}] \cup [A_{(i+1)/M}^{s_j} \setminus A_\alpha^{s'}] \\
&\subseteq (A_{(i+1)/M}^s \setminus A_{(i+1)/M}^{s_j}) \cup (A_{(i+1)/M}^{s_j} \setminus A_{i/M}^{s'}),
\end{aligned}$$

so that

$$\mu_{s_j}(A_\alpha^s \setminus A_\alpha^{s_j}) \leq \mu_{s_j}(A_{(i+1)/M}^s \setminus A_{(i+1)/M}^{s_j}) + \mu_{s_j}(A_{(i+1)/M}^{s_j} \setminus A_{i/M}^{s_j})$$

and

$$\mu_{s_j}(A_\alpha^{s_j} \setminus A_\alpha^s) \leq \mu_{s_j}(A_{(i+1)/M}^{s_j} \setminus A_{(i+1)/M}^s) + \mu_{s_j}(A_{(i+1)/M}^s \setminus A_{i/M}^s).$$

Hence

$$\begin{aligned} \mu_{s_j}(A_\alpha^s \triangle A_\alpha^{s_j}) &\leq \mu_{s_j}(A_{(i+1)/M}^s \triangle A_{(i+1)/M}^{s_j}) \\ &\quad + \mu_{s_j}(A_{(i+1)/M}^{s_j} \setminus A_{i/M}^{s_j}) + \mu_{s_j}(A_{(i+1)/M}^s \setminus A_{i/M}^s) \\ &\leq \sup \{ \mu_{s_j}(A_{(i+1)/M}^{s_k} \triangle A_{(i+1)/M}^{s_j}) : \lambda_k(s) > 0 \} \\ &\quad + (1/M) \mu_{s_j}(T) + \sum_{k=1}^N \lambda_k(s) \mu_{s_j}(A_{(i+1)/M}^{s_k} \setminus A_{i/M}^{s_k}) \\ &\leq \sup_{\lambda_k(s) > 0} \mu_{s_j}(A_{(i+1)/M}^{s_k} \triangle A_{(i+1)/M}^{s_j}) + (1/M) \mu_{s_j}(T) \\ &\quad + \sum_{k=1}^N \lambda_k(s) | \mu_{s_j}(A_{(i+1)/M}^{s_k} \setminus A_{i/M}^{s_k}) - \mu_{s_k}(A_{(i+1)/M}^{s_k} \setminus A_{i/M}^{s_k}) | \\ &\quad + \sum_{k=1}^N \lambda_k(s) \mu_{s_k}(A_{(i+1)/M}^{s_k} \setminus A_{i/M}^{s_k}) \\ &\leq \sup_{\lambda_k(s) > 0} \mu_{s_j}(\bar{A}_{(i+1)/M}^{s_k} \triangle A_{(i+1)/M}^{s_j}) \\ &\quad + \sup_{\lambda_k(s) > 0} \| \mu_{s_j} - \mu_{s_k} \| \\ &\quad + (1/M) \sup_{k=1, \dots, N} \mu_{s_k}(T). \end{aligned}$$

This proves (ii).

Finally we prove (iii); for $\alpha = (1-t) i/M + t (i+1)/M$ we have

$$\begin{aligned} \mu_0(A_\alpha^s \triangle A_\alpha^{s^*}) &= \sum_{\Gamma \in \mathcal{X}} \mu_0(A(\Gamma)_{(1-t)\beta_\Gamma^i(s) + t\beta_\Gamma^{i+1}(s)} \\ &\quad \triangle A(\Gamma)_{(1-t)\beta_\Gamma^i(s^*) + t\beta_\Gamma^{i+1}(s^*)}) \\ &= \sum_{\Gamma \in \mathcal{X}} \{ | [(1-t)\beta_\Gamma^i(s) + t\beta_\Gamma^{i+1}(s)] \\ &\quad - [(1-t)\beta_\Gamma^i(s^*) + t\beta_\Gamma^{i+1}(s^*)] | \mu_0(A(\Gamma)) \} \\ &\leq (1-t) \sum_{\Gamma \in \mathcal{X}} | \beta_\Gamma^i(s) - \beta_\Gamma^i(s^*) | \mu_0(A(\Gamma)) \\ &\quad + t \sum_{\Gamma \in \mathcal{X}} | \beta_\Gamma^{i+1}(s) - \beta_\Gamma^{i+1}(s^*) | \mu_0(A(\Gamma)). \end{aligned}$$

By taking the limit as s tends to s^* we conclude the proof. ■

LEMMA 5. — *Let $s \rightarrow \mu_s$ be a continuous map from a compact metric space S into the space \mathcal{M} and, for each $s \in S$, let $(\bar{A}_\alpha^s)_\alpha$ be an increasing family, continuous with respect to s and such that, for some $\varepsilon > 0$,*

$$|\mu_s(\bar{A}_\alpha^s) - \alpha \mu_s(T)| < \varepsilon \quad (\alpha \in [0, 1], s \in S).$$

For every $s \in S$ there exists an increasing family $(A_\alpha^s)_\alpha$ continuous with respect to s and such that

- (i) $|\mu_s(A_\alpha^s) - \alpha \mu_s(T)| < \varepsilon/10 \quad (\alpha \in [0, 1]);$
- (ii) $\sup_{\alpha \in [0, 1]} \mu_s(\bar{A}_\alpha^s \triangle A_\alpha^s) < 10\varepsilon.$

Proof. — By continuity, for each $s \in S$ there is a $\eta_s > 0$ such that $d(s, s') < 2\eta_s$ implies $\|\mu_s - \mu_{s'}\| < \varepsilon/60$ and $\mu_{s'}(\bar{A}_\alpha^s \triangle \bar{A}_\alpha^{s'}) < \varepsilon$. The open balls $B(s, \eta_s)$ cover S . Let $\{B(s_j, \eta_j) : j = 1, \dots, N\}$ be a finite sub-covering and $\{\lambda_j : j = 1, \dots, N\}$ be a continuous partition of unity subordinate to it and such that $\lambda_j(s_j) = 1, j = 1, \dots, N$.

Let $(A_\alpha^{s_j})_\alpha$ be the families defined by Lemma 3 by taking $\mu = \mu_{s_j}$.

Fix j such that $\mu_{s_j}(T) = \max\{\mu_{s_k}(T) : k = 1, \dots, N\}$ and choose $M \geq 2\mu_{s_j}(T)/\varepsilon$. By Lemma 4, extend the collection $(A_{i/M}^{s_k})_{i=0, \dots, M}$ ($k = 1, \dots, N$) to the family $(A_\alpha^s)_{\alpha \in [0, 1]} (s \in S)$.

The continuity of $s \rightarrow (A_\alpha^s)_{\alpha \in [0, 1]}$ follows from (iii) of Lemma 4, recalling that $\mu_s \ll \mu_0$ for each $s \in S$.

The choice of η_s and (i) of Lemma 4 imply that (i) holds. Moreover

$$\mu_{s_j}(\bar{A}_\alpha^s \triangle A_\alpha^s) \leq \mu_{s_j}(\bar{A}_\alpha^s \triangle \bar{A}_\alpha^{s_j}) + \mu_{s_j}(\bar{A}_\alpha^{s_j} \triangle A_\alpha^{s_j}) + \mu_{s_j}(A_\alpha^{s_j} \triangle A_\alpha^s).$$

By the choice of η_s and (ii) of Lemma 3, the r. h. s. is bounded by

$$\varepsilon + 6\varepsilon + \mu_{s_j}(A_\alpha^{s_j} \triangle A_\alpha^s),$$

which, by (ii) of Lemma 4 and the choice of M , yields

$$\mu_{s_j}(\bar{A}_\alpha^s \triangle A_\alpha^s) \leq \left(9 + \frac{1}{60}\right)\varepsilon.$$

Since $\|\mu_{s_j} - \mu_s\| < \varepsilon/60$, (ii) follows. ■

The following theorem shows the existence of a selection (\tilde{A}_α^s) from $\mathcal{A}(\mu_s)$, continuously depending on s .

THEOREM 1. — *Let $s \rightarrow \mu_s$ be a continuous map from a compact metric space S into the space \mathcal{M} . For every $s \in S$ there an increasing family $(\tilde{A}_\alpha^s)_\alpha$ of measurable subsets of T satisfying*

$$\mu_s(\tilde{A}_\alpha^s) = \alpha \mu_s(T) \quad (\alpha \in [0, 1]) \quad (6)$$

and such that the map $s \rightarrow (\tilde{A}_\alpha^s)_\alpha$ is continuous.

Proof. — We assume that we have defined for s in S an increasing family $(A_\alpha^{s,n})_\alpha$ which is continuous with respect to s and satisfies

$$|\mu_s(A_\alpha^{s,n}) - \alpha\mu_s(T)| < 10^{-n}.$$

By Lemma 2, the above is true for $n=1$ taking a family $(A_\alpha^{s,1})_\alpha$ constant with respect to s .

We obtain the existence of an increasing family $(A_\alpha^{s,n+1})_\alpha$ continuous with respect to s and such that

$$|\mu_s(A_\alpha^{s,n+1}) - \alpha\mu_s(T)| < 10^{-(n+1)} \tag{7}$$

and

$$\mu_s(A_\alpha^{s,n+1} \triangle A_\alpha^{s,n}) < 10^{-(n-1)}. \tag{8}$$

In fact, set in Lemma 5 \bar{A}_α^s to be $A_\alpha^{s,n}$ and ε to be 10^{-n} to infer the existence of a family, denoted by $(A_\alpha^{s,n+1})_\alpha$, satisfying (7) and (8).

Consider now the sequence $((A_\alpha^{s,n})_\alpha)_{n \in \mathbb{N}}$ defined by the above recursive procedure: we wish to show that it converges to a family $(\tilde{A}_\alpha^s)_\alpha$ which is continuous with respect to s and satisfies (6).

Property (8) implies that the sequence $(A_\alpha^{s,n})_n$ (s and α fixed) is a Cauchy sequence in \mathcal{F} supplied with the pseudometric $\rho_s(A, B) = \mu_s(A \triangle B)$. The procedure in Oxtoby [6], Chap. 10, defines a limit family $(\tilde{A}_\alpha^s)_\alpha$, which is increasing: $\tilde{A}_\alpha^s = \bigcup_{n \in \mathbb{N}} \bigcap_{m \geq n} A_\alpha^{s,m}$.

By the inequality

$$|\mu_s(A) - \mu_s(B)| \leq \mu_s(A \triangle B)$$

and (7) we have

$$\mu_s(\tilde{A}_\alpha^s) = \lim_{n \rightarrow \infty} \mu_s(A_\alpha^{s,n}) = \alpha\mu_s(T).$$

In order to check the continuity of the map $s \rightarrow (\tilde{A}_\alpha^s)_\alpha$, fix $\varepsilon > 0$ and $s^0 \in S$. Since the inequality (8) is uniform with respect to s and α , there exists an \bar{n} such that $\mu_s(A_\alpha^{s,\bar{n}} \triangle \tilde{A}_\alpha^s) < \varepsilon/5$ for every s in S and α in $[0, 1]$. Let $\delta > 0$ be such that

$$\|\mu_s - \mu_{s^0}\| < \varepsilon/10 \quad [s \text{ in } B(s^0, \delta)]$$

and

$$\sup_{\alpha \in [0, 1]} \mu_s(A_\alpha^{s',\bar{n}} \triangle A_\alpha^{s'',\bar{n}}) < \varepsilon/5 \quad [s, s' \text{ and } s'' \text{ in } B(s^0, \delta)].$$

Then for every $\alpha \in [0, 1]$, s, s' and s'' in $B(s^0, \delta)$, we have:

$$\begin{aligned} \mu_s(\tilde{A}_\alpha^{s'} \triangle \tilde{A}_\alpha^{s''}) &\leq \mu_s(\tilde{A}_\alpha^{s'} \triangle A_\alpha^{s',\bar{n}}) + \mu_s(A_\alpha^{s',\bar{n}} \triangle \tilde{A}_\alpha^{s''}) \\ &\leq \mu_s(\tilde{A}_\alpha^{s'} \triangle A_\alpha^{s',\bar{n}}) + \mu_s(A_\alpha^{s',\bar{n}} \triangle A_\alpha^{s'',\bar{n}}) + \mu_s(A_\alpha^{s'',\bar{n}} \triangle \tilde{A}_\alpha^{s''}) \end{aligned}$$

$$\begin{aligned} &\leq \|\mu_s - \mu_{s'}\| + \mu_{s'}(\tilde{A}_\alpha^{s'} \triangle A_\alpha^{s'}, \bar{\eta}) + \mu_s(A_\alpha^{s'}, \bar{\eta} \triangle A_\alpha^{s'}, \bar{\eta}) \\ &\quad + \|\mu_s - \mu_{s''}\| + \mu_{s''}(A_\alpha^{s'}, \bar{\eta} \triangle \tilde{A}_\alpha^{s''}) \\ &< \varepsilon. \quad \blacksquare \end{aligned}$$

COROLLARY. — Under the same assumptions, for every $\eta > 0$ and for every increasing family $(A_\alpha)_\alpha$ satisfying

$$|\mu_s(A) - \alpha \cdot \mu_s(T)| < \eta \quad (\alpha \in [0, 1], s \in S),$$

the family $(\tilde{A}_\alpha^s)_\alpha$ of Theorem 1 can be chosen as to satisfy, in addition,

$$\mu_s(\tilde{A}_\alpha^s \triangle A_\alpha) < \eta \quad (\alpha \in [0, 1], s \in S).$$

Proof. — Set $A_\alpha^{s,1}$ to be A_α in the proof of Theorem 1. \blacksquare

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