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A description of self-similar Blow-up for dimensions $n \geq 3$

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ABSTRACT. — A precise description of the asymptotic behavior near the blowup singularity for solutions of $u_t - \Delta u = f(u)$ which blowups in finite time $T$ is given.

Key words : Blowup, self similar, nonlinear parabolic equation, thermal runaway.

RÉSUMÉ. — On établie une description précise de la conduite asymptotique autour de la singularité de l’explosion totale pour la solution de l’équation $u_t - \Delta u = f(u)$.

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0. INTRODUCTION

The purpose of this paper is to give a precise description of the asymptotic behavior for solutions \( u(z, t) \) of

\[
\frac{\partial u}{\partial t} = \Delta u + f(u) \tag{0.1}
\]

which blow-up in finite positive time \( T \). We assume \( f(u) = u^p \) \((p > 1)\) or \( f(u) = e^u \), and \( z \in B_R = \{z \in \mathbb{R}^n : |z| < R \} \) where \( R \) is sufficiently large to guarantee blow-up.

Giga and Kohn ([8], [11]) recently characterized the asymptotic behavior of solutions \( u(z, t) \) of (0.1) with \( f(u) = u^p \) near a blow-up singularity assuming a suitable upper bound on the rate of blow-up and provided \( n = 1, 2, \) or \( n \geq 3 \) and \( p \leq \frac{n+2}{n-2} \). For \( B_R \subseteq \mathbb{R}^n \) using recent a priori bounds established by Friedman-McLeod [7], this implies that solutions \( u(z, t) \) of (0.1) with suitable initial-boundary conditions satisfy

\[
(T-t)^\beta u(z, t) \to \beta^\beta \quad \text{as} \quad t \to T^- \tag{0.2}
\]

provided \( |z| \leq C(T-t)^{1/2} \) for arbitrary \( C \geq 0 \) and where \( \beta = \frac{1}{p-1} \).

For \( f(u) = e^u \) and \( n = 1 \) or 2, Bebernes, Bressan, and Eberly [1] proved that solutions \( u(z, t) \) of (0.1) satisfy

\[
u(z, t) + \ln(T-t) \to 0 \quad \text{as} \quad t \to T^- \tag{0.3}
\]

provided \( |z| \leq C(T-t)^{1/2} \) for arbitrary \( C \geq 0 \).

The real remaining difficulty in understanding how the single point blow-up occurs for (0.1) rests on determining the nonincreasing globally Lipschitz continuous solutions of an associated steady-state equation

\[
y'' + \left(\frac{n-1}{x} - \frac{x}{2}\right)y' + F(y) = 0, \quad 0 < x < \infty \tag{0.4}
\]

where \( F(y) = y^p - \beta y \) or \( e^y - 1 \) for \( f(y) = y^p \) or \( e^y \) respectively and where \( y(0) > 0 \) and \( y'(0) = 0 \).

For \( F(y) = y^p - \beta y \) in the cases \( n = 1, 2, \) or \( n \geq 3 \) and \( p \leq \frac{n}{n-2} \), we give a new proof of a special case of a known result ([8], Theorem 1) that the only such positive solution of (0.4) is \( y(x) \equiv \beta^\beta \). For \( F(y) = e^y - 1 \) and \( n = 1, \) Bebernes and Troy [3] proved that the only such solution is \( y(x) \equiv 0 \).
Eberly [5] gave a much simpler proof showing $y(x) = 0$ is the only solution for the same nonlinearity valid for $n = 1$ and 2.

For $3 \leq n \leq 9$, Troy and Eberly [6] proved that (0.4) has infinitely many nonincreasing globally Lipschitz continuous solutions on $[0, \infty)$ for $F(y) = e^y - 1$. Troy [10] proved a similar multiplicity result for (0.4) with $F(y) = y^p - \beta y$ for $3 \leq n \leq 9$ and $p > \frac{n+2}{n-2}$.

This multiple existence of solutions complicates the stability analysis required to precisely describe the evolution of the time-dependent solutions $u(z, t)$ of (0.1) near the blow-up singularity.

In this paper we extend the results of Giga-Kohn [8] and Bebernes-Bressan-Eberly [1] to the dimensions $n \geq 3$ by proving that, in spite of the multiple existence of solutions of (0.4), the asymptotic formulas (0.2) and (0.3) remain the same as in dimensions 1 and 2. The key to unraveling these problems is a precise understanding of the behavior of the nonconstant solutions relative to a singular solution of (0.4) given by

$$S_e(x) = \ln \frac{2(n-2)}{x^2} \quad (0.5)$$

for $f(u) = e^u$ and $n \geq 3$, and

$$S_p(x) = \left\{-4 \beta \left[ \frac{1}{2} (2-n) \right] / x^2 \right\}^\beta \quad (0.6)$$

for $f(u) = u^p$ and $\beta + \frac{1}{2} (2-n) < 0$, $n \geq 3$. This will be accomplished by counting how many times the graphs of a nonconstant self-similar solution crosses that of the singular solution.

1. STATEMENT OF THE RESULTS

We consider the initial value problem

$$\begin{align*}
  u_t - \Delta u &= f(u), & (z, t) &\in \Omega \times (0, T) \\
  u(z, 0) &= \varphi(z), & z &\in \Omega \\
  u(z, t) &= 0, & (z, t) &\in \partial \Omega \times (0, T)
\end{align*} \quad (1.1)$$

where \( \Omega = B_{R} = \{ z \in \mathbb{R}^{n} : |z| < R \} \), \( \varphi \) is nonnegative, radially symmetric, nonincreasing (\( \varphi (z) \geq \varphi (x) \) for \( |z| \leq |x| \leq R \)), and \( \Delta \varphi + f (\varphi) \geq 0 \) on \( \Omega \). The two nonlinearities considered are

\[
f (u) = e^{u} \tag{1.2}
\]
or

\[
f (u) = u^{p}, \quad u \geq 0, \quad p > 1. \tag{1.3}
\]

We assume \( R > 0 \) and \( \varphi (z) \geq 0 \) are such that the radially symmetric solution \( u (z, t) \) blows-up in finite positive time \( T \). By the maximum principle, \( u (., t) \) is radially decreasing for each \( t \in [0, T) \) and \( u_{t} (z, t) > 0 \) for \( (z, t) \in \Omega \times (0, T) \).

Friedman and McLeod [7] proved that blow-up occurs only at \( z = 0 \). The following arguments are essentially those used in [7] to obtain the needed a priori bounds.

Let \( U (t) = u (0, t) \). Since \( \Delta u (0, t) \leq 0 \) because \( u \) is radially symmetric and decreasing, from (1.1) it follows that \( U' (t) \leq f (U (t)) \). Integrating, we have

\[
-\ln (T - t) \leq u (0, t), \quad t \in [0, T) \tag{1.4}
\]
for \( f (u) = e^{u} \), and

\[
\beta^{\varphi} (T - t)^{-\varphi} \leq u (0, t), \quad t \in [0, T) \tag{1.5}
\]
for \( F (u) = u^{p} \).

Define the radially symmetric function \( J (z, t) = u_{t} - \delta f (u) \) where \( \delta > 0 \) is to be determined. Then \( J_{t} - \Delta J - f' (u) J \geq 0 \). For \( 0 < \eta < \min (R, T) \), let \( \Omega_{\eta} = B_{R - \eta} \) be the ball of radius \( R - \eta \) centered at \( 0 \in \mathbb{R}^{n} \). Let \( \Pi_{\eta} = \Omega_{\eta} \times (\eta, T) \). Since blow-up occurs only at \( z = 0 \), \( u (z, t) \) is bounded on the parabolic boundary of \( \Pi_{\eta} \) and \( f (u) \leq C_{0} < \infty \) there. Since \( u_{t} > 0 \) on \( \Omega \times (0, T) \), we have \( u_{t} \geq C > 0 \) on the parabolic boundary of \( \Pi_{\eta} \). Hence, for \( \delta > 0 \) sufficiently small, \( J_{t} \geq C - \delta C_{0} > 0 \) there. By the maximum principle, \( J > 0 \) on \( \Pi_{\eta} \). An integration yields the following upper bound on \( u (0, t) \):

\[
u (0, t) \leq -\ln [\delta (T - t)], \quad t \in [\eta, T] \tag{1.6}
\]
for \( f (u) = e^{u} \), and

\[
u (0, t) \leq \left( \frac{\beta}{\delta} \right)^{\varphi} (T - t)^{-\varphi}, \quad t \in [\eta, T] \tag{1.7}
\]
for \( f (u) = u^{p} \). In fact, since \( u_{t} (., t) \geq 0 \) for \( t \in [0, T) \), these bounds are true for all \( t \in [0, T) \).

As in [7], we also have the existence of \( \tilde{t} < T \) such that

\[
|\nabla u (z, t)| \leq [2 e^{u (0, \tilde{t})}]^{1/2}, \quad (z, t) \in \bar{\Omega} \times [\tilde{t}, T] \tag{1.8}
\]
for \( f(u) = e^u \), and
\[
|\nabla u(z, t)| \leq \left[ \frac{2}{p+1} |u(0, t)|^{p+1} \right]^{1/2}, \quad (z, t) \in \Omega \times [\tau, T) 
\] (1.9)
for \( f(u) = u^p \).

In this paper we prove the following two theorems which describe the asymptotic self-similar blow-up of \( u(z, t) \).

**Theorem 1.** — (a) For \( n \geq 3 \), the solution \( u(z, t) \) of (1.1)-(1.2) satisfies
\( u(z, t) + \ln(T-t) \to 0 \) uniformly on \( \{(z, t) : |z| \leq C(T-t)^{1/2}\} \) for arbitrary \( C \geq 0 \) as \( t \to T^- \).

(b) For \( n \geq 3 \) and \( p > \frac{n}{n-2} \), the solution \( u(z, t) \) of (1.1)-(1.3) satisfies
\( (T-t)^p u(z, t) \to \beta^p \) uniformly on \( \{(z, t) : |z| \leq C(T-t)^{1/2}\} \) for arbitrary \( C \geq 0 \) as \( t \to T^- \).

**Theorem 2.** — Let \( r = |z| \) and \( v(r, t) = u(z, t) \). There is a value \( r_1 \in (0, R) \) such that the following properties hold.
(a) \( v(r_1, 0) = S_*(r_1) \) where \( S_* \) is the singular solution given in (0.5) or (0.6).
(b) \( v(r, 0) < S_*(r) \) for \( 0 < r < r_1 \).
(c) For each \( r \in (0, r_1) \) there is a \( t = t(r) \in (0, T) \) such that \( v(r, t) > S_*(r) \) for \( t \in (t, T) \).

### 2. THE SELF-SIMILAR PROBLEM

Since the solution \( u(z, t) \) of (1.1) is radially symmetric, the initial-boundary value problem can be reduced to a problem in one spatial dimension.

Let \( \Pi' = \{ (r, t) : 0 < r < R, 0 < t < T \} \). If \( r = |z| \), then \( v(r, t) = u(z, t) \) is well-defined on \( \Pi' \) and satisfies
\[
v_t = v_{rr} + \frac{n-1}{r} v_r + f(v), \quad (r, t) \in \Pi'
\]
(2.1)
\[
v(r, 0) = \varphi(r), \quad r \in (0, R)
\]
\[
v_r(0, t) = 0, \quad v(R, t) = 0, \quad t \in (0, T)
\]
(2.2)
To analyze the behavior of \( v \) as \( t \to T^- \), we make the following change of variables:

\[
\sigma = \ln \left[ T/(T-t) \right], \quad x = r(T-t)^{-1/2}
\]

Then \( \Pi' \) transforms into \( \Pi \) where

\[
\Pi = \{(x, \sigma) : \sigma > 0, \ 0 < x < RT^{-1/2} e^{1/2 \sigma} \}.
\]

If \( f(u) = e^u \), set

\[
w(x, \sigma) = v(r, t) + \ln (T-t).
\]

If \( f(u) = u^p \), set

\[
w(x, \sigma) = (T-t)^{\beta} v(r, t).
\]

Then \( w(x, \sigma) \) solves

\[
w_{\sigma} = w_{xx} + c(x) w_x + F(w), \quad (x, \sigma) \in \Pi
\]

\[
w_x(0, \sigma) = 0, \quad \sigma \in (0, \infty)
\]

where \( c(x) = (n-1)/x - x/2; \) if \( f(u) = e^u \), then

\[
F(w) = e^w - 1
\]

\[
w(RT^{-1/2} e^{1/2 \sigma}, \sigma) = -\sigma + \ln T, \quad \sigma \in (0, \infty)
\]

and if \( f(u) = u^p \), then

\[
F(w) = w^p - \beta w
\]

\[
w(0, \sigma) = 0, \quad \sigma \in (0, \infty)
\]

\[
w(x, 0) = \varphi(x T^{1/2}) + \ln T, \quad x \in (0, RT^{-1/2})
\]

Using the \textit{a priori} bounds established in section I for \( u(z, t) \) using the ideas of [7], we have the following \textit{a priori} estimates for \( w(x, \sigma) \). For \( F(w) = e^w - 1 \), from (1.4) and (1.6)

\[
0 \leq w(0, \sigma) \leq -\ln \delta, \quad \sigma \geq 0.
\]

For \( F(w) = w^p - \beta w \), from (1.5) and (1.7)

\[
\beta^p \leq w(0, \sigma) \leq (\beta/\delta)^\beta, \quad \sigma \geq 0.
\]

The estimates (1.8) and (1.9) imply that

\[
-\gamma \leq w_x(x, \sigma) \leq 0 \quad \text{on } \tilde{\Pi}
\]

for some positive constant \( \gamma \), and combining this with (2.7) and (2.8) yields

\[
-\gamma x \leq w(x, \sigma) \leq \mu \quad \text{on } \tilde{\Pi}
\]
where $\gamma$ and $\mu$ are positive constants depending on $\delta$. In fact, for $F(w) = w^p - \beta w$, $w(x, \sigma) = (T-t)^\beta v(r, t) \geq 0$ since $v(r, 0) \geq 0$ and $v_t(r, t) \geq 0$.

### 3. BEHAVIOR NEAR SINGULAR SOLUTIONS

The partial differential equation (2.3) has a time-independent solution for certain choices of $n$ and $p$. More precisely, if $n > 2$ and $F(w) = e^w - 1$, then

$$S_e(x) = \ln \left[ \frac{2(n-2)}{x^2} \right] \quad (3.1)$$

is a singular solution of (2.3). If $F(w) = w^p - \beta w$, $n > 2$ and $p > \frac{n}{n-2}$, then

$$S_p(x) = \left\{ -4 \beta \left[ \frac{1}{2} (2-n) \right] / x^2 \right\}^{\beta} \quad (3.2)$$

is a singular solution of (2.3). These solutions are in fact singular solutions of (2.1) because

$$1 + \frac{1}{2} x S_e' = 0, \quad S_e'' + \frac{n-1}{x} S_e' + \exp(S_e) = 0 \quad (3.3)$$

and

$$\beta S_p + \frac{1}{2} x S_p' = 0, \quad S_p' = 0, \quad S_p'' + \frac{n-1}{x} S_p' + (S_p)^p = 0 \quad (3.4)$$

for $0 < x < \infty$.

Consider first the singular solution $S_e(x)$ of (2.3) with $F(w) = e^w - 1$. Then $S_e(0^+) = \infty > w(0, 0)$ and

$$S_e(RT^{-1/2}) = \ln \left[ \frac{2(n-2)}{RT^{-2}} \right] < \ln T = w(RT^{-1/2}, 0)$$

since $2(n-2) < R^2$ for blow-up in finite time (Lacey [9], Bellout [4]). This proves that $w(x, 0)$ intersects $S_e(x)$ at least once for $0 < x < RT^{-1/2}$.

Similarly for $F(w) = w^p - \beta w$ and $S_p(x)$, we can make the following observations: $S_p(0^+) = \infty > w(0, 0)$ and $S_p(RT^{-1/2}) > 0 = w(RT^{-1/2}, 0)$. If $w(x, 0) \leq S_p(x)$ on $[0, RT^{-1/2}]$, we conclude by the maximum principle that $w(x, \sigma) \leq S_p(x)$ on $\bar{\Omega}$. By the result of Troy [10] (see part b of Lemma 4.4), any positive global nonincreasing time-independent solution $y(x)$ associated with (2.3) must interest $S_p(x)$ transversally at least once.

By the argument given in Giga-Kohn [8] (or see our theorem 5.1),

$w(x, \sigma) \to 0$ as $\sigma \to \infty$ for each $x \geq 0$. In particular, $w(0, \sigma) \to 0$, a contradiction to (2.8).

In either case, we can conclude that there exists a first $x_1 \in (0, RT^{-1/2})$ such that $w(x_1, 0) = S_*(x_1)$ and $w(x, 0) < S_*(x)$ on $(0, x_1)$.

**Lemma 3.1.** — There is a continuously differentiable function $x_1(\sigma)$ with domain $[0, \infty)$ such that $x_1(0) = x_1$ and $w(x_1(\sigma), \sigma) = S_*(x_1(\sigma))$ for all $\sigma \geq 0$.

**Proof.** — Define $D(x, \sigma) = w(x, \sigma) - S_*(x)$. We first claim that $\nabla D \neq (0, 0)$ whenever $D = 0$. We had $v_t(r, t) > 0$ on $\Pi'$. For $f(v) = e^v$,

$$v_t = (T-t)^{-1} \left( w_\sigma + \frac{1}{2} x w_x \right),$$

and for $f(v) = v^p$,

$$v_t = (T-t)^{-p-1} \left( w_\sigma + \beta w + \frac{1}{2} x w_x \right).$$

If $\nabla D = (0, 0)$ at a point in $\Pi$ where $D = 0$, then $D_\sigma = 0$ implies that $w_\sigma = 0$. For $f(v) = e^v$, $D_x = 0$ implies that $1 + \frac{1}{2} x w_x = 0$. For $f(v) = v^p$, $D_x = 0$ implies that $\beta w + \frac{1}{2} x w_x = 0$. In either case, $v_t = 0$ is forced at some point in $\Pi'$, a contradiction.

Secondly, we claim that $D_x \neq 0$ at any value $(\tilde{x}, \tilde{\sigma}) \in \Pi$ where $D(\tilde{x}, \tilde{\sigma}) = 0$ and $D(x, \tilde{\sigma}) < 0$ in a left neighborhood of $\tilde{x}$.

If $D(\tilde{x}, \tilde{\sigma}) = 0$ and $D_x(\tilde{x}, \tilde{\sigma}) = 0$, then equations (2.3), (3.3), and (3.4) imply that $D_{xx}(\tilde{x}, \tilde{\sigma}) = D_\sigma(\tilde{x}, \tilde{\sigma})$. In addition, since $v_t > 0$ we have $D_\sigma(\tilde{x}, \tilde{\sigma}) > 0$. Thus $D_{xx}(\tilde{x}, \tilde{\sigma}) > 0$, which implies that $(\tilde{x}, \tilde{\sigma})$ is a local minimum point for $D$, a contradiction to $D < 0$ on a left neighborhood of $\tilde{x}$. Thus, $D_x(\tilde{x}, \tilde{\sigma}) > 0$.

Recall that $v(r, 0) = \phi(r)$ where $\Delta \phi + f(\phi) \geq 0$. This implies

$$D_{xx}(x, 0) + \frac{n-1}{x} D_x(x, 0) + F(w(x, 0)) - F(S_*(x)) \geq 0$$

for $x$ in a left neighborhood of $x_1$. On a left neighborhood of $x_1$, this in turn yields $(x^{n-1} D_x(x, 0))_x \geq 0$. An integration yields $D_x(x_1, 0) > 0$. By the implicit function theorem, there is a continuously differentiable function $x_1(\sigma)$ such that $x_1(0) = x_1$ and $D(x_1(\sigma), \sigma) = 0$ for some maximal interval $[0, \sigma_0)$. If $\sigma_0 < \infty$, then by continuity $D(x_1(\sigma_0), \sigma_0) = 0$. 

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But $D_x(x_1(\sigma_0), \sigma_0) > 0$, so the implicit function theorem allows an extension of the domain past $\sigma_0$, a contradiction to the maximality of $[0, \sigma_0)$. Thus, $\sigma_0 = \infty$. □

For $f(u) = u^p$, since $w(0, 0) \leq S_p(0^+)$, $w(\sqrt{R^{-1/2}}, 0) \leq S_p(\sqrt{R^{-1/2}})$, and $w(x_1, 0) = S_p(x_1)$ transversally, there must be a last point of intersection between $w(x, 0)$ and $S_p(x)$, say $x_L \in (x_1, \sqrt{R^{-1/2}})$. A construction similar to Lemma 3.1 leads to the existence of a continuously differentiable function $x_L(\sigma)$ with domain $[0, \infty)$ such that $x_L(0) = x_L$ and $w(x_L(\sigma), \sigma) = S_p(x_L(\sigma))$ for $\sigma \geq 0$.

Let $\Pi_1 = \{(x, \sigma): \sigma > 0, 0 < x < x_1(\sigma)\}$. We can now prove the following comparison result on this set.

**Lemma 3.2.** $D(x, \sigma) < 0$ for $(x, \sigma) \in \Pi_1$.

*Proof.* By Lemma 3.1, we have shown that $D \leq 0$ on the parabolic boundary of $\Pi_1$. Since $F(w)$ is a local one-sided Lipschitz continuous function, we can apply the Nagumo-Westphal comparison result to obtain $D \leq 0$ on $\Pi_1$.

If $D(x_0, \sigma_0) = 0$ for some $(x_0, \sigma_0) \in \Pi_1$, then $D_x(x_0, \sigma_0) = 0$, $D_{xx}(x_0, \sigma_0) \leq 0$ and $D_\sigma(x_0, \sigma_0) \neq 0$ [since $\forall D \neq (0, 0)$ when $D = 0$]. But $D_\sigma(x_0, \sigma_0) \neq 0$ implies $D(x_0, \sigma)$ is positive for some $\sigma$ near $\sigma_0$. This contradicts $D \leq 0$ on $\Pi_1$.

Let $x_2 = \sup\{x \in (x_1, \sqrt{R^{-1/2}}]: D(s, 0) \geq 0 \text{ for } s \in [x_1, 0) = 0$ and $D_x(x_1, 0) > 0$, the supremum exists. For $f(u) = e^u$, $x_2 \leq \sqrt{RT^{-1/2}}$, and for $f(u) = u^p$, $x_2 \leq x_L < \sqrt{RT^{-1/2}}$. Define $x_2(\sigma) = x_2 e^{1/2 \sigma}$ and $\Pi_2 = \{(x, \sigma): \sigma > 0, x_1(\sigma) < x < x_2(\sigma)\}$.

**Lemma 3.3.** $D(x_2(\sigma), \sigma) \geq 0$ for all $\sigma \geq 0$. Moreover, $D(x, \sigma) > 0$ for $(x, \sigma) \in \Pi_2$.

*Proof.* Let $E(\sigma) = D(x_2(\sigma), \sigma)$. By definition of $x_2$, $E(0) = D(x_2, 0) \geq 0$. Also, $E'(\sigma) = D_\sigma(x_2(\sigma), \sigma) + \frac{1}{2} x_2(\sigma) D_x(x_2(\sigma), \sigma)$.

We had earlier that $v_t(r, t) \geq 0$ on $\bar{\Pi}'$. Via the change of variables $(r, t) \to (x, \sigma)$, this implies $E'(\sigma) \geq 0$ in the case $f(v) = e^v$ and $e^{-\beta \sigma} \frac{d}{d\sigma} [e^{\beta \sigma} E(\sigma)] = E'(\sigma) + \beta E(\sigma) \geq 0$ in the case $f(v) = v^p$. An integration yields $E(\sigma) \geq 0$ for $\sigma \geq 0$.

On the parabolic boundary of $\Pi_2$, we now have that $D \geq 0$. By the Nagumo-Westphal comparison theorem, $D \geq 0$ on $\Pi_2$. A similar argument as in Lemma 3.2 shows that $D > 0$ on $\Pi_2$. □
COROLLARY 3.4. — For each $N > 0$ there is a $\sigma_N > 0$ such that for each $\sigma > \sigma_N$, $w(x, \sigma)$ intersects $S_*(x)$ at most once for $x \in [0, N]$.

Proof. — For each $N > 0$ choose $\sigma_N$ such that $N = x_2 \exp \left( \frac{1}{2} \sigma_N \right)$.

Lemma 3.2 guarantees that $D(x, \sigma) < 0$ for $x \in [0, x_1(\sigma))$ and Lemma 3.3 guarantees that $D(x, \sigma) > 0$ for $x \in (x_1(\sigma), x_2(\sigma)]$. For $\sigma > \sigma_N$, $[0, N] \subseteq [0, x_2(\sigma)]$ by definition of $\sigma_N$, so $D = 0$ at most once on this interval.

In section 5 we will see that $x_1(\sigma) \to l$ as $\sigma \to \infty$ where $S_*(l) = 0$ or $S_p(l) = \beta^p$.

4. ANALYSIS OF THE STEADY-STATE PROBLEM

The time-independent solutions of (2.3)-(2.4) satisfy
\begin{align*}
y'' + c(x)y' + F(y) &= 0, \quad 0 < x < \infty \quad (4.1) \\
y(0) &= \alpha, \quad y'(0) = 0 \quad (4.2)
\end{align*}

In this section we will analyze the behavior of a particular class of solutions of (4.1) which are possible members of the $\omega$-limit set for the initial-boundary value problems (2.3)-(2.4)-(2.5) or (2.3)-(2.4)-(2.6).

By the a priori bounds stated in section 2, we have that $w(0, \sigma)$ is bounded for $\sigma \geq 0$. More precisely for $F(w) = e^w - 1$, $w(0, \sigma) \in [0, -\ln \delta]$, and for $F(w) = w^p - \beta w$, $w(0, \sigma) \in [\beta^p, (\beta/\delta)^p)$, for $\sigma \geq 0$. We also had $-\gamma \leq w(x, \sigma) \leq 0$ on $\Pi$ and, for $F(w) = w^p - \beta w$, $w \geq 0$ on $\Pi$.

If $F(w) = e^w - 1$, we need to consider those solutions $y(x)$ of (4.1)-(4.2) which satisfy
\begin{align*}
y(0) &= \alpha \geq 0, \quad y'(x) \leq 0 \quad \text{for} \quad x \geq 0, \quad y'(x) \text{bounded below}. \quad (4.3)
\end{align*}

For $n = 1$ or $2$, (4.1)-(4.2)-(4.3) has only the solution $y(x) \equiv 0$ ([3], [5]). For $3 \leq n \leq 9$, (4.1)-(4.2)-(4.3) has infinitely many nonconstant solutions [6]. In this section we prove that all nonconstant solutions of (4.1)-(4.2)-(4.3) must intersect the singular solution $S_*(x)$ at least twice. Hence, the only solution intersecting $S_*(x)$ exactly once is $y(x) \equiv 0$.

For $F(w) = w^p - \beta w$, we consider those solutions $y(x)$ of (4.1)-(4.2) which satisfy
\begin{align*}
y(0) &= \alpha \geq \beta^p, \quad y'(x) \leq 0 \quad \text{and} \quad y(x) > 0 \quad \text{for} \quad x \geq 0. \quad (4.4)
\end{align*}
For \( n = 1, 2, \text{ or } n \geq 3 \) with \( p = \frac{n}{n-2} \), we prove a special case of the known result [8] that the only solution to (4.1)-(4.2)-(4.4) is \( y(x) = \beta^p \). Troy [10] showed that, for \( n \geq 3 \) and \( p > \frac{n+2}{n-2} \), (4.1)-(4.2)-(4.4) has infinitely many nonconstant solutions. In this section we show that any nonconstant solution \( y(x) \) of (4.1)-(4.2)-(4.4) must intersect \( S_p(x) \) at least twice. Hence, the only solution intersecting \( S_p(x) \) exactly once is \( y(x) = \beta^p \).

**Lemma 4.1.** Consider initial value problem (4.1)-(4.2).

(a) Any solution to (4.1)-(4.2)-(4.3) must satisfy \( y(\sqrt{2n}) \leq 0 \).

(b) Any solution to (4.1)-(4.2)-(4.4) must satisfy \( y(\sqrt{2n}) \leq \beta^p \).

**Proof.** (a) In this case, \( F(y) = e^y - 1 \geq y \), so equation (4.1) implies that \( y'' + c(x)y' + y \leq 0 \). Let \( u(x) = \alpha(1-x^2/2n) \). Then \( u'' + c(x)u' + u = 0 \), \( u(0) = y(0) \), and \( u'(0) = y'(0) \). Define \( W(x) = u(x)y'(x) - u'(x)y(x) \). While \( u(x) > 0 \), \( W' + c(x)W \leq 0 \) and \( W(0) = 0 \), so an integration yields that \( W(x) \leq 0 \). But \( (y/u)'(x) = W(x)/[u(x)]^2 \leq 0 \), so integrating from 0 to \( \sqrt{2n} \) yields \( y(\sqrt{2n}) \leq u(\sqrt{2n}) = 0 \).

Note that for \( \alpha > 0 \), if \( y(z) = 0 \), then \( y'(z) < 0 \) by uniqueness to initial value problems, so \( y(x) < 0 \) for \( x > z \).

(b) The function \( F(y) = y^p - \beta y \) in convex, so \( F(y) \geq y - \beta^p \) and equation (4.1) implies that \( v'' + c(x)v' + v \leq 0 \) where \( v(x) = y(x) - \beta^p \). A similar argument as in part (a) shows that \( v(\sqrt{2n}) \leq 0 \), thus, \( y(\sqrt{2n}) \leq \beta^p \).

Note that for \( \alpha > \beta^p \), if \( y(z) = \beta^p \), then \( y'(z) < 0 \) by uniqueness to initial value problems, so \( y(x) < \beta^p \) for \( x > z \). \( \square \)

Define \( h(x) = y'' + \frac{n-1}{x} y' \). For \( F(y) = e^y - 1 \), define \( g(x) = 1 + \frac{1}{2} xy' \) and for \( F(y) = y^p - \beta y \), define \( g(x) = \beta y + \frac{1}{2} xy' \). It can be shown that \( h \) and \( g \) satisfy the following equations:

\[
\begin{align*}
g'' + c(x)g' + [F'(y) - 1]g &= 0, \quad g(0) > 0, \quad g'(0) = 0. \quad (4.5) \\
h'' + c(x)h' + [F'(y) - 1]h &= -F''(y)(y'), \quad h(0) \leq 0, \quad h'(0) = 0. \quad (4.6)
\end{align*}
\]

For \( F(y) = e^y - 1 \),

\[
g' - \frac{1}{2} xg = -\frac{1}{2} xe^y + \frac{1}{2} (2-n) y'. \quad (4.7)
\]

For \( F(y) = y^p - \beta y \),

\[
g' - \frac{1}{2} xg = -\frac{1}{2} xy^p + \left[ \beta + \frac{1}{2} (2-n) \right] y'. \quad (4.8)
\]
Also define $W(x) = g(x)h'(x) - g'(x)h(x)$. Then
\[ W' + c(x)W = -F''(y)(y')^2 g, \quad W(0) = 0, \]
and
\[ W(x) = -x^{1-n} e^{(1/4)x^2} \int_0^x s^{n-1} e^{-(1/4)s^2} F''[y(s)][y'(s)]^2 g(s) ds \quad (10) \]
where $I(x) \geq 0$, while $g > 0$ on $(0, x)$. Note that
\[ \left( \frac{h}{g} \right)'(x) = W(x)/[g(x)]^2, \]
so while $g > 0$ on $(0, x)$, we have
\[ h(x) = \frac{h(0)}{g(0)} - g(x) \int_0^x t^{1-n} e^{(1/4)t^2} I(t)[g(t)]^{-2} dt \quad (4.9) \]

**Lemma 4.2.** Consider initial value problem (4.1)-(4.2).

(a) If $y(x)$ is a solution to (4.1)-(4.2)-(4.3) with $\alpha > 0$, then $g(x)$ must have a zero.

(b) If $y(x)$ is a solution to (4.1)-(4.2)-(4.4) with $\alpha > \beta$, then $g(x)$ must have a zero.

**Proof.** Suppose that $g(x) \geq \varepsilon > 0$ for all $x \geq 0$. Note that $h(0) < 0$ because $\alpha > 0$ [part (a)] or $\alpha > \beta$ [part (b)]. Then (4.9) implies that $h(x) \leq [h(0)/g(0)]g(x) \leq -\delta < 0$ since $h(0)/g(0) < 0$ and since $I(x) \geq 0$. Multiplying by $x^{n-1}$ and integrating yields $y'(x) \leq -\frac{\delta}{n}$. This contradicts the boundedness of $y'$ in equation (4.3) and forces $y$ to be negative eventually, contradicting equation (4.4). Thus, $g(x)$ cannot be bounded away from zero.

Suppose that $g(x) > 0$ for $x \geq 0$ and that $g$ is not bounded away from zero. Suppose there is an increasing unbounded sequence $\{x_k\}^\infty_1$ such that $g'(x_k) = 0$. Equation (4.5) implies that $g''(x_k) = 1 - F'(y(x_k))g(x_k)$. However, Lemma 4.1 implies that $1 - F'(y(x_k)) > 0$ for $k$ sufficiently large. This forces $g''(x_k) > 0$ for $k$ sufficiently large, a contradiction, since this would imply that $g$ has two local minimums without a local maximum between. It must be the case that $g'(x) < 0$ for $x$ sufficiently large and $g(x) \to 0$ as $x \to \infty$.

Suppose there is an increasing unbounded sequence $\{x_k\}^\infty_1$ such that $g''(x_k) = 0$ and $g'(x_k) \leq -L < 0$. Then equation (4.5) implies that $0 = c(x_k)g'(x_k) + [F'(y(x_k)) - 1]g(x_k) \quad (4.7) \quad (c(x_k) \to -\infty, \quad g'(x_k) \leq -L, \quad F'(y(x_k)) - 1$ is bounded, and $g(x_k) \to 0$. But then the right-hand side of

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the last equality must become infinite, a contradiction. Thus, \( g'(x) < 0 \) for
\( x \) large and \( g''(x) \to 0 \).

In equation (4.9), take the limit as \( x \to \infty \) to obtain
\[
\lim_{x \to \infty} h(x) = - \lim_{x \to \infty} g(x) \int_0^x t^{1-n} e^{(1/4)t^2} I(t) [g(t)]^{-2} \, dt
\]
\[
= \lim_{x \to \infty} x^{1-n} e^{(1/4)x^2} I(x) [g'(x)]^{-1} = -\infty
\]
where we have used L'Hôpital's rule. This implies that \( h(x) \leq -\delta < 0 \) for
\( x \) sufficiently large. Multiplying by \( x^{n-1} \) and integrating yields
\( y'(x) \leq K - \frac{\delta}{n} x \) for \( x \) sufficiently large. As before, this contradicts the
boundedness of \( y' \) in equation (4.3) and forces \( y \) to be negative eventually,
contradicting equation (4.4).

In all of the above cases, we arrived at contradictions, so there must be
a value \( x_0 \) such that \( g(x_0) = 0 \), \( g'(x_0) < 0 \), and \( g(x) > 0 \) on \([0, x_0)\). □

**Lemma 4.3.** — Consider problem (4.1)-(4.2)-(4.3).

(a) If \( 1 \leq n \leq 2 \), then the only solution is \( y(x) \equiv 0 \).

(b) If \( n > 2 \), then the only solution which intersects \( S_e(x) \) exactly once is
\( y(x) \equiv 0 \).

**Proof.** — (a) Let \( 1 \leq n \leq 2 \), then \( \frac{1}{2} (2-n) > 0 \). Let \( x_0 \) be the first zero for
\( g(x) \). Suppose there is an \( x_1 > x_0 \) such that \( g'(x_1) = 0 \) and \( g(x) < 0 \) on
\( (x_0, x_1) \). Equation (4.7) implies that
\[
0 \leq -\frac{1}{2} x_1 g(x_1) = g'(x_1) - \frac{1}{2} x_1 g(x_1) = - \frac{1}{2} x_1 e^{y(x_1)} + \frac{1}{2} (2-n) y'(x_1) < 0
\]
which is a contradiction. Thus, \( g'(x) < 0 \) for \( x \geq x_0 \) and so \( g(x) \leq -\varepsilon < 0 \)
for \( x \geq x > x_0 \). But \( h(x) = g(x) - e^y(x) \leq g(x) \leq -\varepsilon \). Multiplying by \( x^{n-1} \) and
integrating yields \( y'(x) \leq K - \frac{\varepsilon}{n} x \), contradicting equation (4.3). As a result,
the only solution of (4.1)-(4.2)-(4.3) for these values of \( n \) is \( y(x) \equiv 0 \).

(b) Let \( n > 2 \). Define \( D(x) = y(x) - S_e(x) \) where \( S_e \) is the singular solution
discussed in section 3. Then
\[
D'' + c(x) D' + \frac{2(n-2)}{x^2} (e^D - 1) = 0, \quad 0 < x < \infty,
\]
\[
D(0^+) = -\infty, \quad D'(0^+) = \infty.
\]
\[
(4.10)
\]
Note that $D' > 0$ while $D < 0$ on $(0, x)$. Suppose that $D(x) < 0$ for all $x \geq 0$. Then $e^D - 1 < 0$ and $D'' + c(x)D' \geq 0$. Integrating this last equation yields
\[ x^{n-1} e^{-(1/4)x^2} D'(x) \geq x^{n-1} e^{-(1/4)x^2} D'(x) =: p > 0. \]
Consequently,
\[ D(x) \geq D(\tilde{x}) + \int_{\tilde{x}}^{x} pt^{-n} e^{(1/4)t^2} dt. \]
But the right-hand side of this inequality must be positive for $x$ sufficiently large, contradicting our assumption. Thus, $D(x)$ must have a first zero $x_1$ and $D'(x) > 0$ on $(0, x_1]$.

By Lemma 4.2, $g(x)$ must have a zero $x_0$. But then $D'(x_0) = \frac{2}{x_0} g(x_0) = 0$ and $x_0 > x_1$. If $D(x_0) < 0$, then there must have been a second zero $x_2$ for $D$. Otherwise, $D(x) > 0$ on $(x_1, x_0]$. Suppose that $D > 0$ for all $x \geq x_0$. Then there is an $\tilde{x}$ sufficiently large such that $D(\tilde{x}) > 0$, $D'(\tilde{x}) < 0$, $D''(\tilde{x}) > 0$, and $c(\tilde{x}) < 0$. Evaluating equation (4.10) at $\tilde{x}$ yields $0 < (D'' + cD + e^D - 1)(\tilde{x}) = 0$, a contradiction. Thus, $D$ must have a second zero $x_2$.

We have shown that there are at least two points of intersection between the graphs of $y(x)$ and $S_e(x)$ for $\alpha > 0$. Thus, the only solution to (4.1)-(4.2)-(4.3) which intersects $S_e(x)$ exactly once is $y(x) \equiv 0$. 

**Lemma 4.4.** — Consider initial value problem (4.1)-(4.2)-(4.4).

(a) If $1 \leq n \leq 2$, or if $n > 2$ and $\beta + \frac{1}{2}(2-n) \geq 0$, then the only solutions is $y(x) \equiv \beta^\gamma$.

(b) If $n > 2$ and $\beta + \frac{1}{2}(2-n) < 0$, then the only solution which intersects $S_e(x)$ exactly once is $y(x) \equiv \beta^\gamma$.

**Proof.** — (a) In this case, $\beta + \frac{1}{2}(2-n) \geq 0$. Let $x_0$ be the first zero for $g(x)$. Suppose there is an $x_1 > x_0$ such that $g'(x_1) = 0$ and $g(x) < 0$ on $(x_0, x_1]$. Equation (4.8) implies that
\[ 0 < -\frac{1}{2} x_1 g(x_1) = g'(x_1) - \frac{1}{2} x_1 g(x_1) \]
\[ = -\frac{1}{2} x_1 [y(x_1)]^\gamma + \left[ \beta + \frac{1}{2}(2-n) \right] y'(x_1) \leq 0 \]
which is a contradiction. Thus \( g'(x_0) < 0 \) for \( x \geq x_0 \) and so \( g(x) \leq -\varepsilon < 0 \) for \( x \geq x > x_0 \). But \( h(x) = g(x) - [y(x)]^p \leq g(x) \leq -\varepsilon \). Multiplying by \( x^{n-1} \) and integrating yields \( y'(x) \leq K - \frac{\varepsilon}{n} x \), which forces \( y(x) \) to have a zero.

This contradicts equation (4.4). As a result, the only solution for these cases is \( y(x) \equiv \beta^p \).

(b) Let \( n > 2 \) and \( f + \frac{1}{2}(2-n) < 0 \left(p > \frac{n}{n-2}\right) \). The result for the cases \( p > \frac{n+2}{n-2} \) is proved by Troy [10]. For the larger range \( p > \frac{n}{n-2} \) we have the following proof. Define \( W(x) = y(x)S_p'(x) - y'(x)S_p(x) \) and \( Q(u) = F(u)/u \). Then \( W' + c(x)W = yS_p[Q(y) - Q(S_p)] \). Note that \( Q(u) \) is an increasing function. Also note that \( W(x) = -2Kx^{-2-\beta}g(x) \) where \( S_p(x) = Kx^{-\beta} \). Thus, \( x^{n-1}W(x) = -2Kx^{n-2-\beta}g(x) \) where \( n-2-\beta > 0 \). As a result, \( x^{n-1}W(x) \rightarrow 0 \) as \( x \rightarrow 0^+ \). Integrating the equation for \( W(x) \), we obtain

\[
x^{n-1}e^{-\frac{1}{4}x^2}W(x) = \int_0^x t^{n-1}e^{-\frac{1}{4}t^2}y(t)S_p(t)[Q(y(t)) - Q(S_p(t))] dt.
\]

If \( 0 < y < S_p \) for all \( x \geq 0 \), then since \( Q(u) \) is increasing, \( W(x) < 0 \) for all \( x \). But then \( g(x) > 0 \) for all \( x \) is forced, a contradiction to Lemma 4.2. Consequently, there must be a value \( z \) such that \( y(z) = S_p(z) \).

Also, \( W(x) < 0 \) for \( x \in [0, x_0) \). At \( x_0, 0 < W'(x_0) \) which implies that \( y'(x_0) > S_p'(x_0) \). [Note that \( W'(x_0) = 0 \) and \( y(x_0) = S_p(x_0) \) imply that \( y'(x_0) = S'(x_0) \) which in turn would imply, by uniqueness to initial value problems, that \( y(x) \equiv S_p(x) \), a contradiction.] So \( z < x_0 \) is necessary.

Let \( x_1 > x_0 \) be small enough so that \( W(x_1) > 0 \). Suppose that \( y > S_p \) for all \( x > z \). Then integrating the equation for \( W(x) \), we have \( W' + c(x)W \geq 0 \) and

\[
x^{n-1}e^{-\frac{1}{4}x^2}W(x) \geq x_1^{n-1}e^{-\frac{1}{4}x_1^2}W(x_1) =: p > 0.
\]

But \( (S_p/y)'(x) = W(x)[y(x)]^2 \), so

\[
(S_p/y)(x) \geq (S_p/y)(x_1) + p \int_{x_1}^x t^{1-n}e^{\frac{1}{4}t^2}[y(t)]^{-2} dt.
\]

For \( x \) sufficiently large, the right-hand side must become larger than 1, in which case \( (S/y)(x) \geq 1 \). That is, there is another value \( q \) where \( y(q) = S_p(q) \).
We have shown that there are at least two points of intersection between
the graphs of \( y(x) \) and \( S_p(x) \) for \( \alpha > \beta^p \). Thus, the only solution to (4.1)-(4.2)-(4.4) which intersects \( S_p(x) \) exactly once is \( y(x) = \beta^p \). \( \square \)

5. THE CONVERGENCE RESULTS

We are now able to precisely describe how the blowup asymptotically
evolves in dimensions \( n \geq 3 \). Let \( w(x, \sigma) \) be the solution of (2.3)-(2.4)-(2.5) or (2.3)-(2.4)-(2.6) depending on the nonlinearity being considered.

By Corollary 3.4 we know that for each \( N > 0 \) there is a \( \sigma_N > 0 \) such that \( w(x, \sigma) \) intersects \( S_*(x) \) at most once on \([0, N]\) for each \( \sigma > \sigma_N \). By

Lemmas 4.3 and 4.4, the only possible steady-state solution of (2.3) with
\[ F(w) = w^p - \beta w, \]
the only possible steady-state solution of (2.3) intersecting
\( \tilde{S}_e(x) \) at most once is \( y(x) = \beta^p \).

Because of these observations we are now able to prove a convergence
or stability result similar to those given in [8] and [1] which prove that
the \( \omega \)-limit set for (2.3)-(2.4)-(2.5) consists of the singleton critical point
\( y(x) = 0 \), and for (2.3)-(2.4)-(2.6), \( y(x) = \beta^p \).

For the sake of completeness, we include the proof of the following
theorem which is influenced by the ones given in [1] and [8].

**Theorem 5.1.** — Let \( n \geq 3 \).

(a) As \( \sigma \to \infty \), the solution \( w(x, \sigma) \) of (2.3)-(2.4)-(2.5) converges to
\( y(x) = 0 \) uniformly in \( x \) on compact subsets of \([0, \infty)\).

(b) As \( \sigma \to \infty \), the solution \( w(x, \sigma) \) of (2.3)-(2.4)-(2.6) converges to
\( y(x) = \beta^p \) uniformly in \( x \) on compact subsets of \([0, \infty)\).

**Proof.** — Define \( w^\tau(x, \sigma) = w(x, \sigma + \tau) \) as the function obtained by
shifting \( w \) in time by the amount \( \tau \). We will show that as \( \tau \to \infty \), \( w^\tau(x, \sigma) \)
converges to the solution \( y(x) \) uniformly on compact subsets of \( \mathbb{R}^+ \times \mathbb{R} \).

Provided that the limiting function is unique, it is equivalent to prove that
given any unbounded increasing sequence \( \{n_j\} \), there exists a subsequence
\( \{n_j\} \) such that \( w^{n_j} \) converges to \( y(x) \) uniformly on compact subsets of
\( \mathbb{R}^+ \times \mathbb{R} \).

Let \( N \in \mathbb{Z}^+ \). For \( i \) sufficiently large, the rectangle given by \( Q_{2N} = \{ (x, \sigma): \)
\( 0 \leq x \leq 2N, |\sigma| \leq 2N \} \) lies in the domain of \( w^{n_i} \). The radially symmetric
function \( \tilde{w}(\zeta, \sigma) = w^n_i(\|\zeta\|, \sigma) \) solves the parabolic equation

\[
\tilde{w}_\sigma = \Delta \tilde{w} - \frac{1}{2} \langle \zeta, \nabla \tilde{w} \rangle + F(\tilde{w})
\]

on the cylinder given by \( \Gamma_{2N} = \{ (\zeta, \sigma) \in \mathbb{R}^n \times \mathbb{R} : |\zeta| \leq 2N, |\sigma| \leq 2N \} \) with \(-2N \leq \gamma \leq 2N, \sigma \leq \mu \) using (2.10).

By Schauder's interior estimates, all partial derivatives of \( \tilde{w} \) can be uniformly bounded on the subcylinder \( \Gamma_N \subseteq \Gamma_{2N} \). Consequently, \( w^n_i, w^o_i, \) and \( w^n_{xx} \) are uniformly Lipschitz continuous on \( Q_N \subseteq Q_{2N} \). Their Lipschitz constants depend on \( N \) but not on \( i \). By the Arzela-Ascoli theorem, there is a subsequence \( \{ \eta_j \}_1 \) and a function \( \tilde{w} \) such that \( w^n_i, w^o_i, w^n_{xx} \) converge to \( \tilde{w}, \tilde{w}_\sigma, \) and \( \tilde{w}_{xx} \), respectively, uniformly on \( Q_N \).

Repeating the construction for all \( N \) and taking a diagonal subsequence, we can conclude that \( w^n_i \rightarrow \tilde{w}, w^o_i \rightarrow \tilde{w}, \) and \( w^n_{xx} \rightarrow \tilde{w}_{xx} \) uniformly on every compact subset in \( \mathbb{R}^+ \times \mathbb{R} \). Clearly \( \tilde{w} \) satisfies (2.3)-(2.4) with \(-\gamma \leq \tilde{w}_x \leq 0 \). For \( n \geq 3 \) and \( F(w) = e^w - 1 \), the limiting function \( \tilde{w} \) intersects \( S_p(x) \) at most once since, by Corollary 3.4, \( w^n(x, \sigma) \) intersects \( S_p(x) \) at most once on \([0, N]\) for each \( \sigma > \sigma_n \), and \( 0 \leq \bar{w}(0, \sigma) \leq -\ln \delta \) for \( \sigma \geq 0 \). For \( n \geq 3 \),

\[
\beta + \frac{1}{2} (2 - n) < 0, \quad \text{and}
\]

\[
F(w) = w^p - \beta w,
\]

Corollary 3.4 guarantees that \( \tilde{w} \) intersects \( S_p(x) \) at most once. By (2.8) we have \( \beta^\sigma \leq \tilde{w}(0, \sigma) \leq (\beta/\delta)^\beta \) for \( \sigma \geq 0 \).

We now prove that \( \tilde{w} \) is independent of \( \sigma \). For the solution \( w(x, \sigma) \) of (2.3)-(2.4)-(2.5) or (2.6), define the energy functional

\[
E(\sigma) = \int_0^\nu \left[ \frac{1}{2} w_x^2 - G(w) \right] dx,
\]

\[
u = RT^{-1/2} e^{1/2 \sigma}, \quad \rho(x) = x^{n-1} e^{-(1/4) x^2}
\]

where \( G(w) = e^w - w \) if \( F(w) = e^w - 1 \), and \( G(w) = w^{p+1}/(p + 1) - 1/2 \beta w^2 \) if \( F(w) = w^p - \beta w \).

Multiplying equation (2.3) by $\rho w_\sigma$ and integrating from 0 to $v$ yields the equation

$$
\int_0^v \rho w_\sigma^2 dx = \int_0^v w_\sigma (\rho w_\sigma)_x dx + \int_0^v \frac{\partial}{\partial \sigma} [\rho G(w)] dx
$$

Moreover,

$$
E'(\sigma) = \int_0^v \frac{\partial}{\partial \sigma} \left[ \frac{1}{2} \rho w_\sigma^2 - \rho G(w) \right] dx
+ \frac{1}{2} \{ \rho (v) \left[ \frac{1}{2} w_x^2 (v, \sigma) - G(w(v, \sigma)) \right]\} \quad (5.2)
$$

Therefore, for all $a, b$ with $0 \leq a < b$, integrating (5.2) with respect to $\sigma$ from $a$ to $b$, and using (5.3), we have

$$
\int_a^b \int_0^v \rho w_x dx d\sigma = - \int_a^b E'(\sigma) d\sigma + \int_a^b \rho (v) w_\sigma (v, \sigma) w_x (v, \sigma) d\sigma
$$

$$
+ \frac{1}{2} \int_a^b \rho (v) \left[ \frac{1}{2} w_x^2 (v, \sigma) - G(w(v, \sigma)) \right] d\sigma =: E(a) - E(b) + \psi(a, b) \quad (5.4)
$$

Recalling that $|w_x| \leq \gamma$ and observing that

$$
w_\sigma (v, \sigma) = -1 - R u_{r}(R, T(1-e^{-\sigma}))
$$

for $f(u) = e^u$, or $w_\sigma (v, \sigma) = -R u_{r}(R, T(1-e^{-\sigma}))$ for $f(u) = u^p$, we see that in either case the quantity is uniformly bounded as $\sigma \to \infty$. We conclude that

$$
\lim_{a \to \infty} \{ \sup_{b > a} \psi(a, b) \} = 0 \quad (5.5)
$$

For any fixed $N$, we shall prove that

$$
\int_{\mathbb{R}^N} \int \rho \overline{w_\sigma^2} dx d\sigma = \lim_{n_j \to \infty} \int_{\mathbb{R}^N} \rho (w_{\sigma}^n)^2 dx d\sigma = 0.
$$
Note that it is not a restriction to assume that \( \lim (n_{j+1} - n_j) = \infty \). For all \( j \) large enough, \( N \leq R T^{-1/2} \exp \left[ \frac{1}{2} (n_j - N) \right] \) and \( n_{j+1} - n_j \geq 2 N \). Hence,

\[
\int_{-N}^{N} \int_{0}^{N} \rho (w_j^\sigma) dx d\sigma \leq \int_{-N}^{N} \int_{0}^{\frac{RT^{-1/2} \exp \left(1/2 n_j\right)}} \rho (w_j^\sigma) dx d\sigma \\
= E(n_j - N) - E(n_{j+1} - N) + \psi (n_j - N, n_{j+1} - N)
\]

by (5.4). As a consequence of (5.5), we have

\[
\int_{Q_N} \int_{-N}^{N} \rho w_j^2 dx d\sigma \leq \limsup_{j \to \infty} [E(n_j - N) - E(n_{j+1} - N)].
\] (5.6)

Fix any \( K \) arbitrarily large. For \( j \) sufficiently large, we have

\[
E(n_j - N) - E(n_{j+1} - N) = \int_{0}^{K} \rho \left\{ \left[ w_j^\sigma(x, -N) \right]^2 - \left[ w_j^{\sigma+1}(x, -N) \right]^2 \right\} dx \\
- \int_{0}^{K} \rho \left\{ G(w_j^\sigma(x, -N) - G(w_j^{\sigma+1}(x, -N)) \right\} dx \\
+ \int_{K}^{RT^{-1/2} \exp \left(1/2 (n_j - N)\right)} \rho \left\{ \frac{1}{2} \left[ w_j^\sigma(x, -N) \right]^2 - G(w_j^\sigma(x, -N)) \right\} dx \\
\int_{K}^{RT^{-1/2} \exp \left(1/2 (n_j - N)\right)} \rho \left\{ \frac{1}{2} \left[ w_j^{\sigma+1}(x, -N) \right]^2 - G(w_j^{\sigma+1}(x, -N)) \right\} dx
\] (5.7)

In (5.7), the first two integrals on the right-hand side converge to zero as \( j \to \infty \). Recalling that \( \left| w_j^\sigma(x, -N) \right| \leq \gamma \) and \( -\gamma x \leq w_j^\sigma(x, -N) \leq \mu \), we see that the sum of the absolute values of the last two integrals is bounded by \( M \int_{K}^{\infty} x^{n-1} e^{-\nu x^2} dx \) where \( M \) is a positive constant. This integral can be made arbitrarily small by choosing \( K \) large enough.

This proves that \( \int_{-N}^{N} \rho w_j^2 dx d\sigma = 0 \) and hence \( w_\sigma = 0 \). Thus, \( \bar{w}(x, \sigma) = \bar{w}(x, 0) = y(x) \) where \( y(x) \) is a nonincreasing globally Lipschitz continuous solution of (4.1)-(4.2) which intersects \( S_e(x) \) at most once. If \( f(u) = e^u \), then \( y(0) \in [0, -\ln \delta] \) and so \( y(x) \equiv 0 \) is the only solution which intersects \( S_e(x) \) exactly (and thus at most) once on \([0, \infty)\). Similarly for \( f(u) = u^p \), \( y(0) \in [\beta^0, (\beta/\delta)^0] \) and the only possible solution is \( y(x) \equiv \beta^0 \).
Since the limiting solution $y(x)$ is unique in either case, $o^{+}(x, \sigma) \to y(x)$ as $\tau \to \infty$ and we have the result asserted. \qed

Proof of Theorem 1. — The last theorem shows that $w(x, \sigma) \to y(x)$ uniformly in $x$ on compact subsets of $[0, \infty)$ as $\sigma \to \infty$.

(a) In the case $f(u) = e^u$, changing back to the variables $(r, t)$, we have that $v(r, t) + \ln(T-t) \to 0$ as $t \to T^-$ provided $r \leq C(T-t)^{1/2}$ for arbitrary $C \geq 0$.

In particular, $v(0, t) + \ln(T-t) \to 0$ as $t \to T^-$. (b) In the case $f(u) = u^p$ we obtain $(T-t)^p v(r, t) \to \beta^p$ as $t \to T^-$ provided $r \leq C(T-t)^{1/2}$ for arbitrary $C \geq 0$. In particular, $(T-t)^p v(0, t) \to \beta^p$ as $t \to T^-$. 

Proof of Theorem 2. — Theorem 5.1 guarantees that the first branch of zeros $x_1(\sigma)$ of $D(x, \sigma) = w(x, \sigma) - S_*(x)$ is bounded and converges to $l$ where $S_*(l) = 0$ or $S_*(l) = \beta^p$.

Define $r_1 = x_1^{1/2}$. Then $D(x_1, 0) = 0$ implies that $v(r_1, 0) = S_*(r_1)$. In addition, $v(r, 0) < S_*(r)$ for $r \in (0, r_1)$.

Since $x_1(\sigma)$ is bounded and since $\frac{d}{d\sigma} D(r T^{-1/2} e^{1/2\sigma}, \sigma) \geq 0$ for each $r \in (0, r_1)$, there is a value $\overline{\sigma} > 0$ such that

$$r T^{-1/2} e^{1/2\overline{\sigma}} = x_1(\overline{\sigma}) D(x_1(\overline{\sigma}), \overline{\sigma}) = 0,$$

and $D(r T^{-1/2} e^{1/2\sigma}, \sigma) > 0$ for $\sigma > \overline{\sigma}$. Changing back to the variables $(r, t)$ with $\overline{\sigma} = \ln[T/(T-t)]$, we obtain $v(r, t) > S_*(r)$ for $t \in (r_1, T)$.

Remark. — After this paper was completed we received the preprint [11] of Giga and Kohn. In the introduction there is a detailed discussion of self-similar solutions and their importance in describing the behavior of solutions near a blow up point. The referee pointed out a number of papers ([12] to [18]) which are related to the ideas used in this paper. Their relevance is discussed in [11]. The referee also pointed out a briefer proof of Lemma 4.1 which we have used.

REFERENCES


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