J. BEBERNES
D. EBERLY

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\[ n \geq 3 \]

by

J. BEBERNES
Department of Mathematics, University of Colorado, 
Boulder, CO 80309 U.S.A.

and

D. EBERLY
Department of Mathematics, University of Texas, 
San Antonio, TX 78285 U.S.A.

ABSTRACT. — A precise description of the asymptotic behavior near the blowup singularity for solutions of \( u_t - \Delta u = f(u) \) which blowups in finite time \( T \) is given.

Key words: Blowup, self similar, nonlinear parabolic equation, thermal runaway.

RÉSUMÉ. — On établie une description précise de la conduite asymptotique autour de la singularité de l’explosion totale pour la solution de l’équation \( u_t - \Delta u = f(u) \).

Classification A.M.S.: 35B05, 35K55, 35K60, 34C15.

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0. INTRODUCTION

The purpose of this paper is to give a precise description of the asymptotic behavior for solutions \( u(z, t) \) of

\[
\frac{\partial u}{\partial t} = \Delta u + f(u)
\]

(0.1)

which blow-up in finite positive time \( T \). We assume \( f(u) = u^p \) (\( p > 1 \)) or \( f(u) = e^u \), and \( z \in B_R = \{ z \in \mathbb{R}^n : |z| < R \} \) where \( R \) is sufficiently large to guarantee blow-up.

Giga and Kohn ([8], [11]) recently characterized the asymptotic behavior of solutions \( u(z, t) \) of (0.1) with \( f(u) = u^p \) near a blow-up singularity assuming a suitable upper bound on the rate of blow-up and provided \( n = 1, 2 \), or \( n \geq 3 \) and \( p \leq \frac{n+2}{n-2} \). For \( B_R \subseteq \mathbb{R}^n \) using recent a priori bounds established by Friedman-McLeod [7], this implies that solutions \( u(z, t) \) of (0.1) with suitable initial-boundary conditions satisfy

\[
(T-t)^\beta u(z, t) \to \beta^\theta \quad \text{as } t \to T^-
\]

(0.2)

provided \( |z| \leq C(T-t)^{1/2} \) for arbitrary \( C \geq 0 \) and where \( \beta = \frac{1}{p-1} \).

For \( f(u) = e^u \) and \( n = 1 \) or 2, Bebernes, Bressan, and Eberly [1] proved that solutions \( u(z, t) \) of (0.1) satisfy

\[
u(z, t) + \ln(T-t) \to 0 \quad \text{as } t \to T^-
\]

(0.3)

provided \( |z| \leq C(T-t)^{1/2} \) for arbitrary \( C \geq 0 \).

The real remaining difficulty in understanding how the single point blow-up occurs for (0.1) rests on determining the nonincreasing globally Lipschitz continuous solutions of an associated steady-state equation

\[
y'' + \left( \frac{n-1}{x} - \frac{x}{2} \right)y' + F(y) = 0, \quad 0 < x < \infty
\]

(0.4)

where \( F(y) = y^p - \beta y \) or \( e^x - 1 \) for \( f(y) = y^p \) or \( e^y \) respectively and where \( y(0) > 0 \) and \( y'(0) = 0 \).

For \( F(y) = y^p - \beta y \) in the cases \( n = 1, 2 \), or \( n \geq 3 \) and \( p \leq \frac{n}{n-2} \), we give a new proof of a special case of a known result ([8], Theorem 1) that the only such positive solution of (0.4) is \( y(x) \equiv \beta^\theta \). For \( F(y) = e^x - 1 \) and \( n = 1 \), Bebernes and Troy [3] proved that the only such solution is \( y(x) \equiv 0 \).
Eberly [5] gave a much simpler proof showing \( y(x) = 0 \) is the only solution for the same nonlinearity valid for \( n = 1 \) and \( 2 \).

For \( 3 \leq n \leq 9 \), Troy and Eberly [6] proved that (0.4) has infinitely many nonincreasing globally Lipschitz continuous solutions on \([0, \infty)\) for \( F(y) = e^y - 1 \). Troy [10] proved a similar multiplicity result for (0.4) with \( F(y) = y^p - \beta y \) for \( 3 \leq n \leq 9 \) and \( p > \frac{n+2}{n-2} \).

This multiple existence of solutions complicates the stability analysis required to precisely describe the evolution of the time-dependent solutions \( u(z, t) \) of (0.1) near the blow-up singularity.

In this paper we extend the results of Giga-Kohn [8] and Bebernes-Bressan-Eberly [1] to the dimensions \( n \geq 3 \) by proving that, in spite of the multiple existence of solutions of (0.4), the asymptotic formulas (0.2) and (0.3) remain the same as in dimensions 1 and 2. The key to unraveling these problems is a precise understanding of the behavior of the nonconstant solutions relative to a singular solution of (0.4) given by

\[
S_e(x) = \ln \frac{2(n-2)}{x^2}
\]

(0.5)

for \( f(u) = e^u \) and \( n \geq 3 \), and

\[
S_p(x) = \left\{-4\beta \left\lfloor \beta + \frac{1}{2}(2-n) \right\rfloor / x^2 \right\}^\beta
\]

(0.6)

for \( f(u) = u^p \) and \( \beta + \frac{1}{2}(2-n) < 0, n \geq 3 \). This will be accomplished by counting how many times the graphs of a nonconstant self-similar solution crosses that of the singular solution.

1. STATEMENT OF THE RESULTS

We consider the initial value problem

\[
\begin{aligned}
&u_t - \Delta u = f(u), \quad (z, t) \in \Omega \times (0, T) \\
u(z, 0) = \varphi(z), \quad z \in \Omega \\
u(z, t) = 0, \quad (z, t) \in \partial \Omega \times (0, T)
\end{aligned}
\]

(1.1)
where $\Omega = B_\mathbb{R} = \{ z \in \mathbb{R}^n : |z| < R \}$, $\phi$ is nonnegative, radially symmetric, nonincreasing ($\phi(z) \geq \phi(x)$ for $|z| \leq |x| \leq R$), and $\Delta \phi + f(\phi) \geq 0$ on $\Omega$. The two nonlinearities considered are

$$f(u) = e^u$$

or

$$f(u) = u^p, \quad u \geq 0, \quad p > 1. \quad (1.3)$$

We assume $R > 0$ and $\phi(z) \geq 0$ are such that the radially symmetric solution $u(z, t)$ blows-up in finite positive time $T$. By the maximum principle, $u(., t)$ is radially decreasing for each $t \in [0, T)$ and $u_t(z, t) > 0$ for $(z, t) \in \Omega \times (0, T)$.

Friedman and McLeod [7] proved that blow-up occurs only at $z = 0$. The following arguments are essentially those used in [7] to obtain the needed a priori bounds.

Let $U(t) = u(0, t)$. Since $\Delta u(0, t) \leq 0$ because $u$ is radially symmetric and decreasing, from (1.1) it follows that $U'(t) \leq f(U(t))$. Integrating, we have

$$-\ln(T-t) \leq u(0, t), \quad t \in [0, T) \quad (1.4)$$

for $f(u) = e^u$, and

$$\beta^\theta (T-t)^{-\theta} \leq u(0, t), \quad t \in [0, T) \quad (1.5)$$

for $F(u) = u^p$.

Define the radially symmetric function $J(z, t) = u_t - \delta f(u)$ where $\delta > 0$ is to be determined. Then $J_t - \Delta J - f'(u)J \leq 0$. For $0 < \eta < \min(R, T)$, let $\Omega_\eta = B_{R-\eta}$ be the ball of radius $R - \eta$ centered at $0 \in \mathbb{R}^n$. Let $\Pi_\eta = \Omega_\eta \times (\eta, T)$. Since blow-up occurs only at $z = 0$, $u(z, t)$ is bounded on the parabolic boundary of $\Omega_\eta$ and $f(u) \leq C_0 < \infty$ there. Since $u_t > 0$ on $\Omega \times (0, T)$, we have $u_t \geq C > 0$ on the parabolic boundary of $\Pi_\eta$. Hence, for $\delta > 0$ sufficiently small, $\Pi \supset C - \delta C_0 > 0$ there. By the maximum principle, $J > 0$ on $\Pi_\eta$. An integration yields the following upper bound on $u(0, t)$:

$$u(0, t) \leq -\ln \frac{\delta}{\delta (T-t)}, \quad t \in [\eta, T) \quad (1.6)$$

for $f(u) = e^u$, and

$$u(0, t) \leq \left( \frac{\beta}{\delta} \right) (T-t)^{-\theta}, \quad t \in [\eta, T) \quad (1.7)$$

for $f(u) = u^p$. In fact, since $u_t(., t) \geq 0$ for $t \in [0, T)$, these bounds are true for all $t \in [0, T)$.

As in [7], we also have the existence of $\tilde{T} < T$ such that

$$|\nabla u(z, t)| \leq [2 e^{u(0, \eta)}]^{1/2}, \quad (z, t) \in \overline{\Omega} \times [\tilde{T}, T) \quad (1.8)$$
for $f(u) = e^u$, and
\[ |\nabla u(z, t)| \leq \left[ \frac{2}{p + 1} [u(0, t)]^{p+1} \right]^{1/2}, \quad (z, t) \in \bar{\Omega} \times [\bar{t}, T) \] (1.9)

for $f(u) = u^p$.

In this paper we prove the following two theorems which describe the asymptotic self-similar blow-up of $u(z, t)$.

**THEOREM 1.**
(a) For $n \geq 3$, the solution $u(z, t)$ of (1.1)-(1.2) satisfies
\[ u(z, t) + \ln(T-t) \to 0 \text{ uniformly on } \{(z, t): |z| \leq C(T-t)^{1/2}\} \text{ for arbitrary } C \geq 0 \text{ as } t \to T^-.
(b) For $n \geq 3$ and $p > \frac{n}{n-2}$, the solution $u(z, t)$ of (1.1)-(1.3) satisfies
\[ (T-t)^{\beta} u(z, t) \to \beta^{\beta} \text{ uniformly on } \{(z, t): |z| \leq C(T-t)^{1/2}\} \text{ for arbitrary } C \geq 0 \text{ as } t \to T^-.

**THEOREM 2.**
Let $r = |z|$ and $v(r, t) = u(z, t)$. There is a value $r_1 \in (0, R)$ such that the following properties hold.
(a) $v(r_1, 0) = S^*(r_1)$ where $S^*$ is the singular solution given in (0.5) or (0.6).
(b) $v(r, 0) < S^*(r)$ for $0 < r < r_1$.
(c) For each $r \in (0, r_1)$ there is a $t = t(r) \in (0, T)$ such that $v(r, t) > S^*(r)$ for $t \in (t, T)$.

2. THE SELF-SIMILAR PROBLEM

Since the solution $u(z, t)$ of (1.1) is radially symmetric, the initial-boundary value problem can be reduced to a problem in one spatial dimension.

Let $\Pi' = \{(r, t): 0 < r < R, 0 < t < T\}$. If $r = |z|$, then $v(r, t) = u(z, t)$ is well-defined on $\Pi'$ and satisfies
\[ v_t = v_{rr} + \frac{n-1}{r} v_r + f(v), \quad (r, t) \in \Pi' \] (2.1)
\[ v(r, 0) = \varphi(r), \quad r \in (0, R) \] (2.2)
\[ v_r(0, t) = 0, \quad v(R, t) = 0, \quad t \in (0, T) \]
To analyze the behavior of $v$ as $t \to T^-$, we make the following change of variables:

$$
\sigma = \ln \left[ T/(T-t) \right], \quad x = r(T-t)^{-1/2}
$$

Then $\Pi'$ transforms into $\Pi$ where

$$
\Pi = \{(x, \sigma) : \sigma > 0, 0 < x < RT^{-1/2} e^{1/2 \sigma}\}.
$$

If $f(u) = e^u$, set

$$
w(x, \sigma) = v(r, t) + \ln (T-t).
$$

If $f(u) = u^p$, set

$$
w(x, \sigma) = (T-t)^\beta v(r, t).
$$

Then $w(x, \sigma)$ solves

$$
w_\sigma = w_{xx} + c(x) w_x + F(w), \quad (x, \sigma) \in \Pi
$$

$$
w_x(0, \sigma) = 0, \quad \sigma \in (0, \infty)
$$

(2.3)

where $c(x) = (n-1)/x - x/2$; if $f(u) = e^u$, then

$$
F(w) = e^w - 1
$$

$$
w(\text{RT}^{-1/2} e^{1/2 \sigma}, \sigma) = -\sigma + \ln T, \quad \sigma \in (0, \infty)
$$

$$
w(x, 0) = \phi(x T^{1/2}) + \ln T, \quad x \in (0, \text{RT}^{-1/2})
$$

(2.5)

and if $f(u) = u^p$, then

$$
F(w) = w^p - \beta w
$$

$$
w(\text{RT}^{-1/2} e^{1/2 \sigma}, \sigma) = 0, \quad \sigma \in (0, \infty)
$$

$$
w(x, 0) = \text{RT}^\beta \phi(x T^{1/2}), \quad x \in (0, \text{RT}^{-1/2})
$$

(2.6)

Using the \textit{a priori} bounds established in section I for $u(z, t)$ using the ideas of [7], we have the following \textit{a priori} estimates for $w(x, \sigma)$. For $F(w) = e^w - 1$, from (1.4) and (1.6)

$$
0 \leq w(0, \sigma) \leq -\ln \delta, \quad \sigma \geq 0.
$$

(2.7)

For $F(w) = w^p - \beta w$, from (1.5) and (1.7)

$$
\beta^\parallel \leq w(0, \sigma) \leq (\beta/\delta)^\parallel, \quad \sigma \geq 0.
$$

(2.8)

The estimates (1.8) and (1.9) imply that

$$
-\gamma \leq w_x(x, \sigma) \leq 0 \quad \text{on } \Pi
$$

(2.9)

for some positive constant $\gamma$, and combining this with (2.7) and (2.8) yields

$$
-\gamma x \leq w(x, \sigma) \leq \mu \quad \text{on } \Pi
$$

(2.10)
where $\gamma$ and $\mu$ are positive constants depending on $\delta$. In fact, for $F(w) = w^p - \beta w$, $w(x, \sigma) = (T - t)^\beta v(r, t) \geq 0$ since $v(r, 0) \geq 0$ and $v_r(r, t) \geq 0$.

3. BEHAVIOR NEAR SINGULAR SOLUTIONS

The partial differential equation (2.3) has a time-independent solution for certain choices of $n$ and $p$. More precisely, if $n > 2$ and $F(w) = e^w - 1$, then

$$S_\epsilon(x) = \ln \left[ 2(n - 2)/x^2 \right]$$

(3.1)

is a singular solution of (2.3). If $F(w) = w^p - \beta w$, $n > 2$ and $p > \frac{n}{n - 2}$, then

$$S_p(x) = \left\{ -4\beta \left[ \beta + \frac{1}{2}(2 - n) \right]/x^2 \right\}^\beta$$

(3.2)

is a singular solution of (2.3). These solutions are in fact singular solutions of (2.1) because

$$1 + \frac{1}{2} x S_\epsilon' = 0, \quad S_\epsilon'' + \frac{n - 1}{x} S_\epsilon' + \exp(S_\epsilon) = 0$$

(3.3)

and

$$\beta S_p + \frac{1}{2} x S_p' = 0, \quad S_p' = 0, \quad S_p'' + \frac{n - 1}{x} S_p' + (S_p)^p = 0$$

(3.4)

for $0 < x < \infty$.

Consider first the singular solution $S_\epsilon(x)$ of (2.3) with $F(w) = e^w - 1$. Then $S_\epsilon(0^+) = \infty > w(0, 0)$ and

$$S_\epsilon(RT^{-1/2}) = \ln [2(n - 2) T R^{-2}] < \ln T = w(R T^{-1/2}, 0)$$

since $2(n - 2) < R^2$ for blow-up in finite time (Lacey [9], Bellout [4]). This proves that $w(x, 0)$ intersects $S_\epsilon(x)$ at least once for $0 < x < R T^{-1/2}$.

Similarly for $F(w) = w^p - \beta w$ and $S_p(x)$, we can make the following observations: $S_p(0^+) = \infty > w(0, 0)$ and $S_p(R T^{-1/2}) > 0 = w(R T^{-1/2}, 0)$. If $w(x, 0) \leq S_p(x)$ on $[0, R T^{-1/2}]$, we conclude by the maximum principle that $w(x, \sigma) \leq S_p(x)$ on $[0, R T^{-1/2}]$. By the result of Troy [10] (see part b of Lemma 4.4), any positive global nonincreasing time-independent solution $y(x)$ associated with (2.3) must interest $S_p(x)$ transversely at least once. By the argument given in Giga-Kohn [8] (or see our theorem 5.1),

w(x, σ) → 0 as σ → ∞ for each x ≥ 0. In particular, w(0, σ) → 0, a contradiction to (2.8).

In either case, we can conclude that there exists a first x₁ ∈ (0, RT⁻¹/²) such that w(x₁, 0) = S*(x₁) and w(x, 0) < S*(x) on (0, x₁).

**Lemma 3.1.** There is a continuously differentiable function x₁(σ) with domain [0, ∞) such that x₁(0) = x₁ and w(x₁(σ), σ) = S*(x₁(σ)) for all σ ≥ 0.

**Proof.** Define D(x, σ) = w(x, σ) - S*(x). We first claim that D x = 0 whenever D = 0. We had vₙ(r, t) > 0 on Π'. For f(v) = e^v,

\[ v_t = (T-t)^{-1}\left(w_\sigma + \frac{1}{2}xw_x\right), \]

and for f(v) = vp,

\[ v_t = (T-t)^{-\beta - 1}\left(w_\sigma + \beta w + \frac{1}{2}xw_x\right). \]

If D = 0 at a point in Π where D = 0, then D x = 0 implies that w_σ = 0. For f(v) = e^v, D x = 0 implies that \(1 + \frac{1}{2}xw_x = 0\). For f(v) = vp, D x = 0 implies that \(\beta w + \frac{1}{2}xw_x = 0\). In either case, vₙ = 0 is forced at some point in Π', a contradiction.

Secondly, we claim that D x ≠ 0 at any value (\(\bar{x}, \bar{\sigma}\)) ∈ Π where D(\(\bar{x}, \bar{\sigma}\)) = 0 and D x(\(\bar{x}, \bar{\sigma}\)) < 0 in a left neighborhood of \(\bar{x}\).

If D(\(\bar{x}, \bar{\sigma}\)) = 0 and D x(\(\bar{x}, \bar{\sigma}\)) = 0, then equations (2.3), (3.3), and (3.4) imply that D xx(\(\bar{x}, \bar{\sigma}\)) = D x(\(\bar{x}, \bar{\sigma}\)). In addition, since vₙ > 0 we have D x(\(\bar{x}, \bar{\sigma}\)) > 0. Thus D xx(\(\bar{x}, \bar{\sigma}\)) > 0, which implies that (\(\bar{x}, \bar{\sigma}\)) is a local minimum point for D, a contradiction to D < 0 on a left neighborhood of \(\bar{x}\). Thus, D x(\(\bar{x}, \bar{\sigma}\)) > 0.

Recall that v(r, 0) = φ(r) where Δφ + f(φ) ≥ 0. This implies

\[ D_{xx}(x, 0) + \frac{n-1}{x}D_x(x, 0) + F(w(x, 0)) - F(S_*(x)) ≥ 0 \]

for x in a left neighborhood of x₁. On a left neighborhood of x₁, this in turn yields (x^n-1 D x(x, 0)) ≥ 0. An integration yields D x(x₁, 0) > 0. By the implicit function theorem, there is a continuously differentiable function x₁(σ) such that x₁(0) = x₁ and D(x₁(σ), σ) = 0 for some maximal interval [0, σ₀]. If σ₀ < ∞, then by continuity D(x₁(σ₀), σ₀) = 0.
But $D_x(x_1(\sigma_0), \sigma_0) > 0$, so the implicit function theorem allows an extension of the domain past $\sigma_0$, a contradiction to the maximality of $[0, \sigma_0)$. Thus, $\sigma_0 = \infty$. □

For $f(u) = u^p$, since $w(0, 0) < S_p(0^+)$, $w(R\tau^{-1/2}, 0) < S_p(R\tau^{-1/2})$, and $w(x_1, 0) = S_p(x_1)$ transversally, there must be a last point of intersection between $w(x, 0)$ and $S_p(x)$, say $x_L \in (x_1, R\tau^{-1/2})$. A construction similar to Lemma 3.1 leads to the existence of a continuously differentiable function $x_L(\sigma)$ with domain $[0, \infty)$ such that $x_L(0) = x_L$ and $w(x_L(\sigma), \sigma) = S_p(x_L(\sigma))$ for $\sigma \geq 0$.

Let $\Pi_1 = \{(x, \sigma) : \sigma > 0, 0 < x < x_1(\sigma)\}$. We can now prove the following comparison result on this set.

**Lemma 3.2.** $-D(x, \sigma) < 0$ for $(x, \sigma) \in \Pi_1$.

**Proof.** By Lemma 3.1, we have shown that $D \leq 0$ on the parabolic boundary of $\Pi_1$. Since $F(w)$ is a local one-sided Lipschitz continuous function, we can apply the Nagumo-Westphal comparison result to obtain $D \leq 0$ on $\Pi_1$.

If $D(x_0, \sigma_0) = 0$ for some $(x_0, \sigma_0) \in \Pi_1$, then $D_x(x_0, \sigma_0) = 0$, $D_{xx}(x_0, \sigma_0) \leq 0$ and $D_\sigma(x_0, \sigma_0) \neq 0$ [since $\nabla D \neq (0, 0)$ when $D = 0$]. But $D_\sigma(x_0, \sigma_0) \neq 0$ implies $D(x_0, \sigma)$ is positive for some $\sigma$ near $\sigma_0$. This contradicts $D \leq 0$ on $\Pi_1$.

Let $x_2 = \sup\{x \in (x_1, R\tau^{-1/2}) : D(s, 0) \geq 0$ for $s \in [x_1, 0) = 0$ and $D_x(x_1, 0) > 0$, the supremum exists. For $f(u) = e^u$, $x_2 \leq R\tau^{-1/2}$, and for $f(u) = u^p$, $x_2 \leq x_L < R\tau^{-1/2}$. Define $x_2(\sigma) = x_2 e^{1/2\sigma}$ and $\Pi_2 = \{(x, \sigma) : \sigma > 0, x_1(\sigma) < x < x_2(\sigma)\}$.

**Lemma 3.3.** $D(x_2(\sigma), \sigma) \geq 0$ for all $\sigma \geq 0$. Moreover, $D(x, \sigma) > 0$ for $(x, \sigma) \in \Pi_2$.

**Proof.** Let $E(\sigma) = D(x_2(\sigma), \sigma)$. By definition of $x_2$, $E(0) = D(x_2, 0) \geq 0$. Also, $E'(\sigma) = D_\sigma(x_2(\sigma), \sigma) + \frac{1}{2}x_2(\sigma)D_x(x_2(\sigma), \sigma)$.

We had earlier that $v_t(r, t) \geq 0$ on $\Pi'$. Via the change of variables $(r, t) \to (x, \sigma)$, this implies $E'(\sigma) \geq 0$ in the case $f(v) = e^v$ and $\frac{d}{d\sigma}[e^{\beta\sigma}E(\sigma)] = E'(\sigma) + \beta E(\sigma) \geq 0$ in the case $f(v) = v^p$. An integration yields $E(\sigma) \geq 0$ for $\sigma \geq 0$.

On the parabolic boundary of $\Pi_2$, we now have that $D \geq 0$. By the Nagumo-Westphal comparison theorem, $D \geq 0$ on $\Pi_2$. A similar argument as in Lemma 3.2 shows that $D > 0$ on $\Pi_2$. □
COROLLARY 3.4. — For each \( N > 0 \) there is a \( \sigma_N > 0 \) such that for each \( \sigma > \sigma_N \), \( w(x, \sigma) \) intersects \( S_\sigma(x) \) at most once for \( x \in [0, N] \).

Proof. — For each \( N > 0 \) choose \( \sigma_N \) such that \( N = x_2 \exp \left( \frac{1}{2} \sigma_N \right) \).

Lemma 3.2 guarantees that \( D(x, \sigma) < 0 \) for \( x \in [0, x_1(\sigma)) \) and Lemma 3.3 guarantees that \( D(x, \sigma) > 0 \) for \( x \in (x_1(\sigma), x_2(\sigma)] \). For \( \sigma > \sigma_N \), \( [0, N] \subseteq [0, x_2(\sigma)] \) by definition of \( \sigma_N \), so \( D = 0 \) at most once on this interval.

In section 5 we will see that \( x_1(\sigma) \to l \) as \( \sigma \to \infty \) where \( S_\sigma(l) = 0 \) or \( S_\sigma(l) = \beta^p \).

4. ANALYSIS OF THE STEADY-STATE PROBLEM

The time-independent solutions of (2.3)-(2.4) satisfy

\[
y'' + c(x)y' + F(y) = 0, \quad 0 < x < \infty \quad (4.1)
\]
\[
y(0) = \alpha, \quad y'(0) = 0 \quad (4.2)
\]

In this section we will analyze the behavior of a particular class of solutions of (4.1) which are possible members of the \( \omega \)-limit set for the initial-boundary value problems (2.3)-(2.4)-(2.5) or (2.3)-(2.4)-(2.6).

By the \textit{a priori} bounds stated in section 2, we have that \( w(0, \sigma) \) is bounded for \( \sigma \geq 0 \). More precisely for \( F(w) = e^w - 1 \), \( w(0, \sigma) \in [0, -\ln \delta] \), and for \( F(w) = w^p - \beta w \), \( w(0, \sigma) \in [\beta^p, (\beta/\delta)^p] \), for \( \sigma \geq 0 \). We also had \( -\gamma \leq w_x(x, \sigma) \leq 0 \) on \( \Omega \) and, for \( F(w) = w^p - \beta w \), \( w \geq 0 \) on \( \Omega \).

If \( F(w) = e^w - 1 \), we need to consider those solutions \( y(x) \) of (4.1)-(4.2) which satisfy

\[
y(0) = \alpha \geq 0, \quad y'(x) \leq 0 \quad \text{for } x \geq 0, \quad y'(x) \text{ bounded below.} \quad (4.3)
\]

For \( n = 1 \) or 2, (4.1)-(4.2)-(4.3) has only the solution \( y(x) \equiv 0 \) ([3], [5]).

For \( 3 \leq n \leq 9 \), (4.1)-(4.2)-(4.3) has infinitely many nonconstant solutions [6]. In this section we prove that all nonconstant solutions of (4.1)-(4.2)-(4.3) must intersect the singular solution \( S_\sigma(x) \) at least twice. Hence, the only solution intersecting \( S_\sigma(x) \) exactly once is \( y(x) \equiv 0 \).

For \( F(w) = w^p - \beta w \), we consider those solutions \( y(x) \) of (4.1)-(4.2) which satisfy

\[
y(0) = \alpha \geq \beta^p, \quad y'(x) \leq 0 \quad \text{and } y(x) > 0 \quad \text{for } x \geq 0. \quad (4.4)
\]
For \( n = 1, 2, \) or \( n \geq 3 \) with \( p \leq \frac{n}{n-2} \) we prove a special case of the known result [8] that the only solution to (4.1)-(4.2)-(4.4) is \( y(x) = \beta^p \). Troy [10] showed that, for \( n \geq 3 \) and \( p > \frac{n+2}{n-2} \), (4.1)-(4.2)-(4.4) has infinitely many nonconstant solutions. In this section we show that any nonconstant solution \( y(x) \) of (4.1)-(4.2)-(4.4) must intersect \( S_\rho(x) \) at least twice. Hence, the only solution intersecting \( S_\rho(x) \) exactly once is \( y(x) = \beta^p \).

**Lemma 4.1.** Consider initial value problem (4.1)-(4.2).

(a) Any solution to (4.1)-(4.2)-(4.3) must satisfy \( y(\sqrt{2n}) \leq 0 \).

(b) Any solution to (4.1)-(4.2)-(4.4) must satisfy \( y(\sqrt{2n}) \leq \beta^p \).

**Proof.** (a) In this case, \( F(y) = e^y - 1 \geq y \), so equation (4.1) implies that \( y'' + c(x)y' + y \leq 0 \). Let \( u(x) = x(1 - x^2/2n) \). Then \( u'' + c(x)u' + u = 0, u(0) = y(0) \), and \( u'(0) = y'(0) \). Define \( W(x) = u(x)y'(x) - u'(x)y(x) \). While \( u(x) > 0 \), \( W' + c(x)W \leq 0 \) and \( W(0) = 0 \), so an integration yields that \( W(x) \leq 0 \). But \( (y/u)'(x) = W(x)/[u(x)]^2 \leq 0 \), so integrating from 0 to \( \sqrt{2n} \) yields \( y(\sqrt{2n}) \leq u(\sqrt{2n}) = 0 \).

Note that for \( \alpha > 0 \), if \( y(z) = 0 \), then \( y'(z) < 0 \) by uniqueness to initial value problems, so \( y(x) < 0 \) for \( x > z \).

(b) The function \( F(y) = y^p - \beta y \) is convex, so \( F(y) \geq y - \beta y \) and equation (4.1) implies that \( v'' + c(x)v' + v \leq 0 \) where \( v(x) = y(x) - \beta^p \). A similar argument as in part (a) shows that \( v(\sqrt{2n}) \leq 0 \), thus, \( y(\sqrt{2n}) \leq \beta^p \).

Note that for \( \alpha > \beta^p \), if \( y(z) = \beta^p \), then \( y'(z) < 0 \) by uniqueness to initial value problems, so \( y(x) < \beta^p \) for \( x > z \) \( \Box \)

Define \( h(x) = y'' + \frac{n-1}{x}y' \). For \( F(y) = e^y - 1 \), define \( g(x) = 1 + \frac{1}{2}xy' \) and for \( F(y) = y^p - \beta y \), define \( g(x) = \beta y + \frac{1}{2}xy' \). It can be shown that \( h \) and \( g \) satisfy the following equations:

\[
g'' + c(x)g' + [F'(y) - 1]g = 0, \quad g(0) > 0, \quad g'(0) = 0. \tag{4.5}
\]

\[
h'' + c(x)h' + [F'(y) - 1]h = -F''(y)(y')^2, \quad h(0) \leq 0, \quad h'(0) = 0. \tag{4.6}
\]

For \( F(y) = e^y - 1 \),

\[
g' - \frac{1}{2}xg = -\frac{1}{2}xe^y + \frac{1}{2}(2-n)y'. \tag{4.7}
\]

For \( F(y) = y^p - \beta y \),

\[
g' - \frac{1}{2}xg = -\frac{1}{2}xy^p + \left[ \beta + \frac{1}{2}(2-n) \right]y'. \tag{4.8}
\]
Also define \( W(x) = g(x)h'(a)-g'(a)h(x) \). Then
\[
W' + c(x)W = -F''(y)(y')^2g, \quad W(0) = 0,
\]
and
\[
W(x) = -x^{1-n}e^{(1/4)x^2} \int_0^x s^{n-1} e^{-(1/4)s^2} F''[y(s)][y'(s)]^2 g(s) ds
\]
where \( I(x) \geq 0 \), while \( g > 0 \) on \((0, x)\). Note that \( \left( \frac{h}{g} \right)^{(a)} = W(x)/[g(x)]^2 \), so while \( g > 0 \) on \((0, x)\), we have
\[
h(x) = \frac{h(0)}{g(0)} - \int_0^x t^{n-1} e^{(1/4)t^2} I(t) [g(t)]^{-2} dt
\]
\( \text{(4.9)} \)

**Lemma 4.2.** — Consider initial value problem (4.1)-(4.2).

(a) If \( y(x) \) is a solution to (4.1)-(4.2)-(4.3) with \( \alpha > 0 \), then \( g(x) \) must have a zero.

(b) If \( y(x) \) is a solution to (4.1)-(4.2)-(4.4) with \( \alpha > \beta^\delta \), then \( g(x) \) must have a zero.

**Proof.** — Suppose that \( g(x) \geq \varepsilon > 0 \) for all \( x \geq 0 \). Note that \( h(0) < 0 \) because \( \alpha > 0 \) [part (a)] or \( \alpha > \beta^\delta \) [part (b)]. Then (4.9) implies that \( h(x) \leq [h(0)/g(0)]g(x) \leq -\delta < 0 \) since \( h(0)/g(0) < 0 \) and since \( I(x) \geq 0 \). Multiplying by \( x^{n-1} \) and integrating yields \( y'(x) \leq -\delta \). This contradicts the boundedness of \( y' \) in equation (4.3) and forces \( y \) to be negative eventually, contradicting equation (4.4). Thus, \( g(x) \) cannot be bounded away from zero.

Suppose that \( g(x) > 0 \) for \( x \geq 0 \) and that \( g \) is not bounded away from zero. Suppose there is an increasing unbounded sequence \( \{ x_k \}_1^\infty \) such that \( g'(x_k) = 0 \). Equation (4.5) implies that \( g''(x_k) = [1-F'(y(x_k))]g(x_k) \). However, Lemma 4.1 implies that \( 1-F'(y(x_k)) > 0 \) for \( k \) sufficiently large. This forces \( g''(x_k) > 0 \) for \( k \) sufficiently large, a contradiction, since this would imply that \( g \) has two local minimums without a local maximum between. It must be the case that \( g'(x) < 0 \) for \( x \) sufficiently large and \( g(x) \to 0 \) as \( x \to \infty \).

Suppose there is an increasing unbounded sequence \( \{ x_k \}_1^\infty \) such that \( g''(x_k) = 0 \) and \( g'(x_k) \leq -L < 0 \). Then equation (4.5) implies that \( 0 = c(x_k)g'(x_k) + [F'(y(x_k)) - 1]g(x_k) \) where \( c(x_k) \to -\infty \), \( g'(x_k) \leq -L \), \( F'(y(x_k)) - 1 \) is bounded, and \( g(x_k) \to 0 \). But then the right-hand side of...
the last equality must become infinite, a contradiction. Thus, \( g'(x) < 0 \) for \( x \) large and \( g'(x) \to 0 \).

In equation (4.9), take the limit as \( x \to \infty \) to obtain

\[
\lim_{x \to \infty} h(x) = - \lim_{x \to \infty} g(x) \int_{0}^{x} t^{1-n} e^{(1/4)t^2} I(t) |g(t)|^{-2} dt = \lim_{x \to \infty} x^{1-n} e^{(1/4)x^2} I(x) |g'(x)|^{-1} = -\infty
\]

where we have used L'Hôpital's rule. This implies that \( h(x) \leq -\delta < 0 \) for \( x \) sufficiently large. Multiplying by \( x^{n-1} \) and integrating yields \( y'(x) \leq K - \frac{\delta}{n} x \) for \( x \) sufficiently large. As before, this contradicts the boundedness of \( y' \) in equation (4.3) and forces \( y \) to be negative eventually, contradicting equation (4.4).

In all of the above cases, we arrived at contradictions, so there must be a value \( x_0 \) such that \( g(x_0) = 0, g'(x_0) < 0, \) and \( g(x) > 0 \) on \([0, x_0)\). \( \square \)

**Lemma 4.3.** — Consider problem (4.1)-(4.2)-(4.3).

(a) If \( 1 \leq n \leq 2 \), then the only solution is \( y(x) \equiv 0 \).

(b) If \( n > 2 \), then the only solution which intersects \( S_e(x) \) exactly once is \( y(x) \equiv 0 \).

**Proof.** — (a) Let \( 1 \leq n \leq 2 \), then \( \frac{1}{2}(2-n) \geq 0 \). Let \( x_0 \) be the first zero for \( g(x) \). Suppose there is an \( x_1 > x_0 \) such that \( g'(x_1) = 0 \) and \( g(x) < 0 \) on \((x_0, x_1]\). Equation (4.7) implies that

\[
0 < -\frac{1}{2} x_1 g(x_1) = g'(x_1) - \frac{1}{2} x_1 g(x_1) = -\frac{1}{2} x_1 e^{y(x_1)} + \frac{1}{2} (2-n) y'(x_1) < 0
\]

which is a contradiction. Thus, \( g'(x) < 0 \) for \( x \geq x_0 \) and so \( g(x) \leq -\varepsilon < 0 \) for \( x \geq \tilde{x} > x_0 \). But \( h(x) = g(x) - e^{y(x)} \leq g(x) \leq -\varepsilon \). Multiplying by \( x^{n-1} \) and integrating yields \( y'(x) \leq K - \frac{\varepsilon}{n} x \), contradicting equation (4.3). As a result, the only solution of (4.1)-(4.2)-(4.3) for these values of \( n \) is \( y(x) \equiv 0 \).

(b) Let \( n > 2 \). Define \( D(x) = y(x) - S_e(x) \) where \( S_e \) is the singular solution discussed in section 3. Then

\[
D'' + c(x) D' + \frac{2(n-2)}{x^2} (e^D - 1) = 0, \quad 0 < x < \infty, \quad D(0^+) = -\infty, \quad D'(0^+) = \infty.
\]

Note that $D' > 0$ while $D < 0$ on $(0, x_1]$. Suppose that $D(x) < 0$ for all $x \geq 0$. Then $e^D - 1 < 0$ and $D'' + c(x)D' \geq 0$. Integrating this last equation yields

$$x^{n-1} e^{-(1/4)x^2} D'(x) \geq x^{n-1} e^{-(1/4)x^2} D'(x) =: p > 0.$$  

Consequently,

$$D(x) \geq D(\bar{x}) + \int_{\bar{x}}^{x} pt^{1-n} e^{(1/4)t^2} dt.$$  

But the right-hand side of this inequality must be positive for $x$ sufficiently large, contradicting our assumption. Thus, $D(x)$ must have a first zero $x_1$ and $D'(x) > 0$ on $(0, x_1]$.

By Lemma 4.2, $g(x)$ must have a zero $x_0$. But then $D'(x_0) = \frac{2}{x_0} g(x_0) = 0$ and $x_0 > x_1$. If $D(x_0) < 0$, then there must have been a second zero $x_2$ for $D$. Otherwise, $D(x) > 0$ on $(x_1, x_0]$. Suppose that $D > 0$ for all $x \geq x_0$. Then there is an $\bar{x}$ sufficiently large such that $D(\bar{x}) > 0$, $D'(\bar{x}) < 0$, $D''(\bar{x}) > 0$, and $c(\bar{x}) < 0$. Evaluating equation (4.10) at $\bar{x}$ yields $0 < (D'' + cD' + e^D - 1)(\bar{x}) = 0$, a contradiction. Thus, $D$ must have a second zero $x_2$.

We have shown that there are at least two points of intersection between the graphs of $y(x)$ and $S_\alpha(x)$ for $\alpha > 0$. Thus, the only solution to (4.1)-(4.2)-(4.3) which intersects $S_\alpha(x)$ exactly once is $y(x) \equiv 0$. □

**Lemma 4.4.** Consider initial value problem (4.1)-(4.2)-(4.4).

(a) If $1 \leq n \leq n \leq 2$, or if $n > 2$ and $\beta + \frac{1}{2} (2-n) \leq 0$, then the only solutions is $y(x) \equiv \beta^\alpha$.

(b) If $n > 2$ and $\beta + \frac{1}{2} (2-n) < 0$, then the only solution which intersects $S_\alpha(x)$ exactly once is $y(x) \equiv \beta^\alpha$.

**Proof.** (a) In this case, $\beta + \frac{1}{2} (2-n) \leq 0$. Let $x_0$ be the first zero for $g(x)$. Suppose there is an $x_1 > x_0$ such that $g'(x_1) = 0$ and $g(x) < 0$ on $(x_0, x_1]$. Equation (4.8) implies that

$$0 < -\frac{1}{2} x_1 g(x_1) = g'(x_1) - \frac{1}{2} x_1 g(x_1)$$  

$$= -\frac{1}{2} x_1 [y(x_1)]^\alpha + \left[ \beta + \frac{1}{2} (2-n) \right] y'(x_1) \leq 0$$

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which is a contradiction. Thus \( g'(x_0) < 0 \) for \( x \geq x_0 \) and so \( g(x) \leq -\varepsilon < 0 \) for \( x \geq x > x_0 \). But \( h(x) = g(x) - [y(x)]^p \leq g(x) \leq -\varepsilon \). Multiplying by \( x^{n-1} \) and integrating yields \( y'(x) \leq K - \frac{\varepsilon}{n}x \), which forces \( y(x) \) to have a zero.

This contradicts equation (4.4). As a result, the only solution for these cases is \( y(x) \equiv \beta^p \).

(b) Let \( n > 2 \) and \( f + \frac{1}{2}(2-n) < 0 \left( p > \frac{n}{n-2} \right) \). The result for the cases \( p > \frac{n+2}{n-2} \) is proved by Troy [10]. For the larger range \( p > \frac{n}{n-2} \) we have the following proof. Define \( W(x) = y(x)S_p'(x) - y'(x)S_p(x) \) and \( Q(u) = F(u)/u \). Then \( W' + c(x)W = yS_p[Q(y) - Q(S_p)] \). Note that \( Q(u) \) is an increasing function. Also note that \( W(x) = -2Kx^{-2p-1}g(x) \) where \( S_p(x) = Kx^{-2p} \). Thus, \( x^{n-1}W(x) = -2Kx^{n-2p-2}g(x) \) where \( n-2-2p > 0 \). As a result, \( x^{n-1}W(x) \to 0 \) as \( x \to 0^+ \). Integrating the equation for \( W(x) \), we obtain

\[
x^{n-1}e^{-(1/4)x^2}W(x) = \int_0^x t^{n-1}e^{-(1/4)t^2}y(t)S_p(t)\left[Q(y(t)) - Q(S_p(t))\right]dt.
\]

If \( 0 < y < S_p \) for all \( x \geq 0 \), then since \( Q(u) \) is increasing, \( W(x) < 0 \) for all \( x \). But then \( g(x) > 0 \) for all \( x \) is forced, a contradiction to Lemma 4.2. Consequently, there must be a value \( z \) such that \( y(z) = S_p(z) \).

Also, \( W(x) < 0 \) for \( x \in [0, x_0) \). At \( x_0 \), \( 0 < W'(x_0) \) which implies that \( y'(x_0) > S_p(x_0) \). [Note that \( W'(x_0) = 0 \) and \( y(x_0) = S_p(x_0) \) imply that \( y'(x_0) = S'(x_0) \) which in turn would imply, by uniqueness to initial value problems, that \( y(x) \equiv S_p(x) \), a contradiction.] So \( z < x_0 \) is necessary.

Let \( x_1 > x_0 \) be small enough so that \( W(x_1) > 0 \). Suppose that \( y > S_p \) for all \( x > z \). Then integrating the equation for \( W(x) \), we have \( W' + c(x)W \geq 0 \) and

\[
x^{n-1}e^{-(1/4)x^2}W(x) \geq x_1^{n-1}e^{-(1/4)x_1^2}W(x_1) =: p > 0.
\]

But \( (S_p/y)'(x) = W(x)/[y(x)]^2 \), so

\[
(S_p/y)(x) \geq (S_p/y)(x_1) + p \int_{x_1}^x t^{1-n}e^{(1/4)t^2}[y(t)]^{-2}dt.
\]

For \( x \) sufficiently large, the right-hand side must become larger than \( 1 \), in which case \( (S/y)(x) \geq 1 \). That is, there is another value \( q \) where \( y(q) = S_p(q) \).
We have shown that there are at least two points of intersection between the graphs of \( y(x) \) and \( S_p(x) \) for \( \alpha > \beta^p \). Thus, the only solution to (4.1)-(4.2)-(4.4) which intersects \( S_p(x) \) exactly once is \( y(x) = \beta^p \). \( \square \)

5. THE CONVERGENCE RESULTS

We are now able to precisely describe how the blowup asymptotically evolves in dimensions \( n \geq 3 \). Let \( w(x, \sigma) \) be the solution of (2.3)-(2.4)-(2.5) or (2.3)-(2.4)-(2.6) depending on the nonlinearity being considered. By Corollary 3.4 we know that for each \( N > 0 \) there is a \( \sigma_N > 0 \) such that \( w(x, \sigma) \) intersects \( S_p(x) \) at most once on \([0, N]\) for each \( \sigma > \sigma_N \). By Lemmas 4.3 and 4.4, the only possible steady-state solution of (2.3) with \( F(w) = e^w - 1 \) which intersects \( S_p(x) \) at most once is \( y(x) = 0 \), and for \( F(w) = w^p - \beta w \), the only possible steady-state solution of (2.3) intersecting \( S_p(x) \) at most once is \( y(x) = \beta^p \).

Because of these observations we are now able to prove a convergence or stability result similar to those given in [8] and [1] which prove that the \( \omega \)-limit set for (2.3)-(2.4)-(2.5) consists of the singleton critical point \( y(x) = 0 \), and for (2.3)-(2.4)-(2.6), \( y(x) = \beta^p \).

For the sake of completeness, we include the proof of the following theorem which is influenced by the ones given in [1] and [8].

**Theorem 5.1.** Let \( n \geq 3 \).

(a) As \( \sigma \to \infty \), the solution \( w(x, \sigma) \) of (2.3)-(2.4)-(2.5) converges to \( y(x) = 0 \) uniformly in \( x \) on compact subsets of \([0, \infty)\).

(b) As \( \sigma \to \infty \), the solution \( w(x, \sigma) \) of (2.3)-(2.4)-(2.6) converges to \( y(x) = \beta^p \) uniformly in \( x \) on compact subsets of \([0, \infty)\).

**Proof.** Define \( w^\tau(x, \sigma) := w(x, \sigma + \tau) \) as the function obtained by shifting \( w \) in time by the amount \( \tau \). We will show that as \( \tau \to \infty \), \( w^\tau(x, \sigma) \) converges to the solution \( y(x) \) uniformly on compact subsets of \( \mathbb{R}^+ \times \mathbb{R} \).

Provided that the limiting function is unique, it is equivalent to prove that given any unbounded increasing sequence \( \{n_j\} \), there exists a subsequence \( \{n_i\} \) such that \( w^{n_i} \) converges to \( y(x) \) uniformly on compact subsets of \( \mathbb{R}^+ \times \mathbb{R} \).

Let \( N \in \mathbb{Z}^+ \). For \( i \) sufficiently large, the rectangle given by \( Q_{2N} = \{(x, \sigma) : 0 \leq x \leq 2N, |\sigma| \leq 2N\} \) lies in the domain of \( w^{n_i} \). The radially symmetric
function \( \tilde{w}(\zeta, \sigma) = w^n_i(\zeta, \sigma) \) solves the parabolic equation

\[
\tilde{w}_\sigma = \Delta \tilde{w} - \frac{1}{2} \langle \zeta, \nabla \tilde{w} \rangle + F(\tilde{w})
\]

on the cylinder given by \( \Gamma_{2N} = \{ (\zeta, \sigma) \in \mathbb{R}^n \times \mathbb{R} : |\zeta| \leq 2N, |\sigma| \leq 2N \} \) with \( -2N \gamma \leq \tilde{w}(\zeta, \sigma) \leq \mu \) using (2.10).

By Schauder's interior estimates, all partial derivatives of \( \tilde{w} \) can be uniformly bounded on the subcylinder \( \Gamma_N \subseteq \Gamma_{2N} \). Consequently, \( w^n_i, \tilde{w}_\sigma, \) and \( \tilde{w}_x \) are uniformly Lipschitz continuous on \( Q_N \subseteq Q_{2N} \). Their Lipschitz constants depend on \( N \) but not on \( i \). By the Arzela-Ascoli theorem, there is a subsequence \( \{ n_j \} \) and a function \( \tilde{w} \) such that \( w^n_i, \tilde{w}_\sigma, \tilde{w}_x \) converge to \( \tilde{w}, \tilde{w}_\sigma, \) and \( \tilde{w}_x \), respectively, uniformly on \( Q_N \).

Repeating the construction for all \( N \) and taking a diagonal subsequence, we can conclude that \( w^n_i \rightarrow \tilde{w}, \tilde{w}_\sigma \rightarrow \tilde{w}, \) and \( \tilde{w}_x \rightarrow \tilde{w}_x \) uniformly on every compact subset in \( \mathbb{R}^+ \times \mathbb{R} \). Clearly \( \tilde{w} \) satisfies (2.3)-(2.4) with \( -\gamma \leq \tilde{w}_x \leq 0 \).

For \( n \geq 3 \) and \( F(w) = e^w - 1 \), the limiting function \( \tilde{w} \) intersects \( S_\sigma(x) \) at most once since, by Corollary 3.4, \( w^n_i(x, \sigma) \) intersects \( S_\sigma(x) \) at most once on \( [0, N] \) for each \( \sigma > \sigma_{N+1} \), and \( 0 \leq \tilde{w}(0, \sigma) \leq -\ln \delta \) for \( \sigma \geq 0 \). For \( n \geq 3 \), \( \beta + \frac{1}{2}(2-n) < 0 \), and

\[
F(w) = w^p - \beta w,
\]

Corollary 3.4 guarantees that \( \tilde{w} \) intersects \( S_\sigma(x) \) at most once. By (2.8) we have \( \beta^\theta \leq \tilde{w}(0, \sigma) \leq (\beta/\delta)^\theta \) for \( \sigma \geq 0 \).

We now prove that \( \tilde{w} \) is independent of \( \sigma \). For the solution \( w(x, \sigma) \) of (2.3)-(2.4)-(2.5) or (2.6), define the energy functional

\[
E(\sigma) = \int_0^\nu \rho(x) \left[ \frac{1}{2}w_x^2 - G(w) \right] dx,
\]

\[
v = RT^{-1/2} e^{1/2 \sigma}, \quad \rho(x) = x^{n-1} e^{-(1/4)x^2}\]

where \( G(w) = e^w - w \) if \( F(w) = e^w - 1 \), and \( G(w) = w^{p+1}/(p+1) - \frac{1}{2} \beta w^2 \) if \( F(w) = w^p - \beta w \).

Multiplying equation (2.3) by $\rho w_\sigma$ and integrating from 0 to $v$ yields the equation

$$
\int_0^v \rho w_\sigma^2 \, dx = \int_0^v w_\sigma (\rho w_\sigma)_x \, dx + \left[ \int_0^v \frac{\partial}{\partial \sigma} [\rho G(w)] \, dx \right]_{x=0}^{x=v}
$$

Moreover,

$$
E'(\sigma) = \int_0^v \frac{\partial}{\partial \sigma} \left[ \frac{1}{2} \rho w_\sigma^2 - \rho G(w) \right] \, dx
$$

where

$$
E'(\sigma) = \int_0^v \frac{\partial}{\partial \sigma} \left[ \frac{1}{2} \rho w_\sigma^2 (v, \sigma) - G(w(v, \sigma)) \right] \, dx
$$

Therefore, for all $a, b$ with $0 \leq a < b$, integrating (5.2) with respect to $\sigma$ from $a$ to $b$, and using (5.3), we have

$$
\int_a^b \int_0^v \rho w_\sigma \, dx \, d\sigma = - \int_a^b E'(\sigma) \, d\sigma + \int_a^b \rho (v) w_\sigma (v, \sigma) w_x (v, \sigma) \, d\sigma
$$

$$
+ \frac{1}{2} \int_a^b \rho (v) \left[ \frac{1}{2} w_\sigma^2 (v, \sigma) - G(w(v, \sigma)) \right] \, d\sigma
$$

$$
= :E(a) - E(b) + \psi(a, b) \quad (5.4)
$$

Recalling that $|w_x| \leq \gamma$ and observing that

$$
w_\sigma (v, \sigma) = -1 - R u_\sigma (R, T(1-e^{-\sigma})))
$$

for $f(u) = e^u$, or $w_\sigma (v, \sigma) = -R u_\sigma (R, T(1-e^{-\sigma}))$ for $f(u) = u^n$, we see that in either case the quantity is uniformly bounded as $\sigma \to \infty$. We conclude that

$$
\lim_{a \to \infty} \left\{ \sup_{b > a} \psi(a, b) \right\} = 0 \quad (5.5)
$$

For any fixed $N$, we shall prove that

$$
\int_{Q_N} \int \rho \bar{w}_\sigma^2 \, dx \, d\sigma = \lim_{n_j \to \infty} \int_{Q_N} \rho (w_{\sigma_j}^n)^2 \, dx \, d\sigma = 0.
$$
Note that it is not a restriction to assume that $\lim (n_{j+1} - n_j) = \infty$. For all $j$ large enough, $N \leq R T^{-1/2} \exp \left[ \frac{1}{2}(n_j - N) \right]$ and $n_{j+1} - n_j \geq 2N$. Hence, 
\[
\int_{-N}^{N} \int_{0}^{N} \rho (w_{n_j}^2) dx d\sigma \leq \int_{-N}^{N} \int_{0}^{RT^{-1/2} \exp (1/2 n_j)} \rho (w_{n_j}^2) dx d\sigma 
= E(n_j - N) - E(n_{j+1} - N) + \psi(n_j - N, n_{j+1} - N)
\]
by (5.4). As a consequence of (5.5), we have 
\[
\int_{Q_N} \rho \overline{w}_{n_j}^2 dx d\sigma \leq \lim \sup_{j \to \infty} [E(n_j - N) - E(n_{j+1} - N)]. \tag{5.6}
\]
Fix any $K$ arbitrarily large. For $j$ sufficiently large, we have 
\[
E(n_j - N) - E(n_{j+1} - N) 
= \int_{0}^{K} \frac{1}{2} \rho \left\{ [w_{n_j}^2(x, -N)]^2 - [w_{n_{j+1}}^2(x, -N)]^2 \right\} dx 
- \int_{0}^{K} \rho [G(w_{n_j}(x, -N) - G(w_{n_{j+1}}(x, -N))] dx 
+ \int_{K}^{RT^{-1/2} \exp (1/2(n_j - N))} \rho \left\{ \frac{1}{2} [w_{n_j}^2(x, -N)]^2 - G(w_{n_j}(x, -N)) \right\} dx 
+ \int_{K}^{RT^{-1/2} \exp (1/2(n_j - N))} \rho \left\{ \frac{1}{2} [w_{n_{j+1}}^2(x, -N)]^2 - G(w_{n_{j+1}}(x, -N)) \right\} dx \tag{5.7}
\]
In (5.7), the first two integrals on the right-hand side converge to zero as $j \to \infty$. Recalling that $|w_{n_j}^2(x, -N)| \leq \gamma$ and $-\gamma x \leq w_{n_j}(x, -N) \leq \mu$, we see that the sum of the absolute values of the last two integrals is bounded by $M \int_{K}^{\infty} x^{n-1} e^{-(1/4)x^2} dx$ where $M$ is a positive constant. This integral can be made arbitrarily small by choosing $K$ large enough.

This proves that $\int_{-N}^{N} \rho \overline{w}_n^2 dx d\sigma = 0$ and hence $\overline{w}_n = 0$. Thus, $\overline{w}(x, \sigma) = \overline{w}(x, 0) = y(x)$ where $y(x)$ is a nonincreasing globally Lipschitz continuous solution of (4.1)-(4.2) which intersects $S_e(x)$ at most once. If $f(u) = e^u$, then $y(0) \in [0, -\ln \delta]$ and so $y(x) \equiv 0$ is the only solution which intersects $S_e(x)$ exactly (and thus at most) once on $[0, \infty)$. Similarly for $f(u) = u^p$, $y(0) \in [\beta^p, (\beta/\delta)^p]$ and the only possible solution is $y(x) \equiv \beta^p$.

Since the limiting solution \( y(x) \) is unique in either case, \( \omega^\tau(x, \sigma) \to y(x) \) as \( \tau \to \infty \) and we have the result asserted. \( \square \)

**Proof of Theorem 1.** -- The last theorem shows that \( w(x, \sigma) \to y(x) \) uniformly in \( x \) on compact subsets of \([0, \infty)\) as \( \sigma \to \infty \).

(a) In the case \( f(u) = e^u \), changing back to the variables \((r, t)\), we have that \( v(r, t) + \ln(T-t) \to 0 \) as \( t \to T^- \) provided \( r \leq C(T-t)^{1/2} \) for arbitrary \( C \geq 0 \). In particular, \( v(0, t) + \ln(T-t) \to 0 \) as \( t \to T^- \).

(b) In the case \( f(u) = u^p \) we obtain \((T-t)^p v(r, t) \to \beta^p \) as \( t \to T^- \) provided \( r \leq C(T-t)^{1/2} \) for arbitrary \( C \geq 0 \). In particular, \((T-t)^p v(0, t) \to \beta^p \) as \( t \to T^- \).

**Proof of Theorem 2.** -- Theorem 5.1 guarantees that the first branch of zeros \( x_1(\sigma) \) of \( D(x, \sigma) = w(x, \sigma) - S_*^*(x) \) is bounded and converges to \( l \) where \( S_*(l) = 0 \) or \( S_*(l) = \beta^p \).

Define \( r_1 = x_1^{1/2} T \). Then \( D(x_1, 0) = 0 \) implies that \( v(r_1, 0) = S_*(r_1) \). In addition, \( v(r, 0) < S_*(r) \) for \( r \in (0, r_1) \).

Since \( x_1(\sigma) \) is bounded and since \( \frac{d}{d\sigma} D(r T^{-1/2} e^{1/2\sigma}, \sigma) \geq 0 \) for each \( r \in (0, r_1) \), there is a value \( \overline{\sigma} > 0 \) such that

\[
r T^{-1/2} e^{1/2\overline{\sigma}} = x_1(\overline{\sigma}) D(x_1(\overline{\sigma}), \overline{\sigma}) = 0,
\]

and \( D(r T^{-1/2} e^{1/2\sigma}, \sigma) > 0 \) for \( \sigma > \overline{\sigma} \). Changing back to the variables \((r, t)\) with \( \overline{\sigma} = \ln[T/(T-t)] \), we obtain \( v(r, t) > S_*(r) \) for \( t \in (\overline{t}, T) \).

**Remark.** -- After this paper was completed we received the preprint [11] of Giga and Kohn. In the introduction there is a detailed discussion of self-similar solutions and their importance in describing the behavior of solutions near a blow up point. The referee pointed out a number of papers ([12] to [18]) which are related to the ideas used in this paper. Their relevance is discussed in [11]. The referee also pointed out a brief proof of Lemma 4.1 which we have used.

**REFERENCES**


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