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Gradient theory of phase transitions with boundary contact energy


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by

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ABSTRACT. — We study the asymptotic behavior as $\varepsilon \to 0^+$ of solutions of the variational problems for the Van der Waals-Cahn-Hilliard theory of phase transitions in a fluid. We assume that the internal free energy, per unit volume, is given by $\varepsilon^2 |\nabla \rho|^2 + W(\rho)$ and the contact energy with the container walls, per unit surface area, is given by $\varepsilon \sigma(\rho)$, where $\rho$ is the density. The result is that such solutions approximate a two-phases configuration satisfying a variational principle related to the equilibrium configuration of liquid drops.

Key words: Phase transitions, variational thermodynamic principles, variational convergence.

RÉSUMÉ. — Nous étudions ici le comportement asymptotique pour $\varepsilon \to 0^+$ des solutions des problèmes variationnels qui viennent de la théorie de Van der Waals-Cahn-Hilliard sur les transitions de phase des fluides. Nous faisons l'hypothèse que l'énergie libre de Gibbs, pour unité de volume, est donnée par $\varepsilon^2 |\nabla \rho|^2 + W(\rho)$ et que l'énergie de contact avec la surface intérieure du conteneur, pour unité de surface, est donnée par $\varepsilon \sigma(\rho)$, où $\rho$ est la densité. Le résultat est que ces solutions approchent...
We continue in this paper the asymptotic analysis of the Van der Waals-Cahn-Hilliard theory of phase transitions in a fluid, by taking also into account, with respect to our earlier results [10], the contact energy between the fluid and the container walls. Our results give a positive answer to some conjectures by M. E. Gurtin [8].

Let us describe briefly the problem we are concerned with; we refer to [10] for further information and bibliography. Consider a fluid, under isothermal conditions and confined to a bounded container $\Omega \subset \mathbb{R}^n$, and assume that the Gibbs free energy, per unit volume, $W = W(u)$ and the contact energy, per unit surface area, $\sigma = \sigma(u)$ between the fluid and the container walls $\partial \Omega$ are prescribed functions of the density distribution (or composition) $u \geq 0$ of the fluid. According to the Van der Waals-Cahn-Hilliard theory, and in particular to the Cahn's approach [2], the stable configurations of the fluid are determined by solving the variational problem

\[ (*) \quad \min \left\{ \int_{\Omega} \left[ \epsilon^{2} |Du|^2 + W(u) \right] dx + \int_{\partial \Omega} \sigma(u) d\mathcal{H}^{n-1} \right\}, \]

where $\epsilon > 0$ is a small parameter, and the minimum is taken among all functions $u \geq 0$ satisfying the constraint

\[ \int_{\Omega} u \, dx = m, \]

$m$ being the prescribed total mass of the fluid. The function $W(t)$ is supposed to vanish only at two points $t = \alpha$ and $t = \beta$ ($\alpha < \beta$), and to be strictly positive everywhere else. Of course, $\mathcal{H}^{n-1}$ denotes the Hausdorff $(n-1)$-dimensional measure.

Our goal is to study the asymptotic behavior as $\epsilon \to 0^+$ of solutions $u_\epsilon$ of $(*)$ by looking for a variational problem solved by the limit point (or points) of $u_\epsilon$ in $L^1(\Omega)$. As conjectured by Gurtin [8], this limit problem does exist and agrees with the so-called liquid-drop problem.
Namely (cf. Theorem 2.1 for a precise statement), if the function $u_0$ is the limit of $u_\varepsilon$ in $L^1(\Omega)$ as $\varepsilon \to 0^+$, then $u_0$ takes only the values $\alpha$ and $\beta$ (i.e., $u_0$ corresponds to a two-phases configuration of the fluid), and the portion $E_0$ of the container occupied by the phase $u_0 = \alpha$ minimizes the geometric area-like quantity

$$\mathcal{H}_{n-1}(\partial E \cap \Omega) + \gamma \mathcal{H}_{n-1}(\partial E \cap \partial \Omega)$$

among all subsets $E$ of $\Omega$ having the same volume as $E_0$. The number $\gamma$ depends only on $W$ and $\sigma$, and it can be explicitly calculated:

$$\gamma = \frac{\hat{\sigma}(\alpha) - \hat{\sigma}(\beta)}{2c_0},$$

where

$$c_0 = \int_{\alpha}^{\beta} W^{1/2}(s) \, ds,$$

and $\hat{\sigma}$ represents a modified contact energy between the fluid and the container walls, whose definition involves the values of $\sigma(t)$ and $W(t)$ for every $t \geq 0$. One has $|\gamma| \leq 1$ in correspondence with the geometrical meaning of $\gamma$, which is the cosine of the contact angle between the fluid phase $\alpha$ and the walls of the container.

The presence of such $\hat{\sigma}$ instead of $\sigma$ disproves a part of the Gurtin's conjecture but, what is more interesting, it is perfectly in accord with theory and experiments by J. W. Cahn and R. B. Heady ([2], [3]) about critical point wetting. They discovered that, in a range of temperatures below the critical one for a binary system, the phase $\alpha$ does not wet the container (i.e. $\gamma = 1$) and a layer of phase $\beta$, which is, on the contrary, perfectly wetting, appears between the phase $\alpha$ and the container walls. A theoretical explanation of such phenomenon was given by Cahn in the case $\varepsilon > 0$.

We confirm in this paper, under very general assumptions and by a fully mathematical proof, the existence of the critical point wetting phenomenon in the asymptotic case $\varepsilon \to 0$. Indeed, we show that $\gamma = 1$ and $\hat{\sigma}(\alpha) = \hat{\sigma}(\beta) + \sigma_{a\beta}$ ($\sigma_{a\beta}$ denotes the energy, per unit surface area, associated to the interface between the phases $\alpha$ and $\beta$), for $\sigma$ and $W$ having the same global behavior exhibited in the semi-empirical figures of [2]. It now suffices to remark that the balance of energy $\hat{\sigma}(\alpha) = \hat{\sigma}(\beta) + \sigma_{a\beta}$ can be interpreted as the contact energy on $\partial E_0 \cap \partial \Omega$ coming from an infinitely
thin layer of the phase $\beta$ interposed between the phase $\alpha$ and the container walls (cf. Section 3 for details).

We think that other very interesting experimental evidences, discussed by Cahn in [2], would deserve a similar careful mathematical treatment. Finally, we would like to thank Morton Gurtin for stimulating and friendly discussions.

1. SOME PRELIMINARY RESULTS

Throughout this paper $\Omega$ will be an open, bounded subset of $\mathbb{R}^n$ ($n \geq 2$) with smooth boundary $\partial \Omega$; $W$ and $\sigma$ will be two non-negative continuous functions defined on $[0, +\infty[$. The function $W(t)$ is supposed to have exactly two zeros at the points $t = \alpha$ and $t = \beta$, with $0 < \alpha < \beta$.

For every $\varepsilon > 0$ and for every non-negative function $u$ in the Sobolev space $H^1(\Omega)$, we define

$$\mathcal{E}_\varepsilon (u) = \int_{\Omega} \left[ \varepsilon^2 |D u(x)|^2 + W(u(x)) \right] dx + \varepsilon \int_{\partial \Omega} \sigma(\tilde{u}(x)) d\mathcal{H}_{n-1}(x) \quad (1)$$

where $D u$ denotes the gradient of $u$, $\tilde{u}$ denotes the trace of $u$ on $\partial \Omega$, and $\mathcal{H}_{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure.

1.1. PROPOSITION. - For every $\varepsilon > 0$ and for every $m \geq 0$ the minimization problem

$$(P_\varepsilon) \quad \min \left\{ \mathcal{E}_\varepsilon (u) : u \in H^1(\Omega), u \geq 0, \int_{\Omega} u(x) dx = m \right\}$$

admits (at least) one solution.

Proof. — The proof is standard. Let

$$U = \left\{ u \in H^1(\Omega) : u \geq 0, \mathcal{E}_\varepsilon (u) \leq c, \int_{\Omega} u(x) dx = m \right\},$$

with $c \in \mathbb{R}$ large enough so that $U \neq \emptyset$. It suffices to prove that $\mathcal{E}_\varepsilon$ is lower semicontinuous on $U$ and $U$ is compact with respect to the topology of $L^2(\Omega)$. 

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Let \( u_\infty \in U \) and \( (u_h) \) be a sequence in \( U \) converging to \( u_\infty \) in \( L^2(\Omega) \): we have to prove that

\[
\mathcal{E}_\varepsilon(u_\infty) \leq \liminf_{h \to +\infty} \mathcal{E}_\varepsilon(u_h).
\]

Without loss of generality we can assume that there exists the limit of \( \mathcal{E}_\varepsilon(u_h) \) as \( h \to +\infty \) and it is finite. Since \( W \geq 0 \) and \( \sigma \geq 0 \), we have that

\[
\int_\Omega |Du|^2 \, dx \leq c/\varepsilon^2, \quad \forall u \in U;
\]

hence, modulo replacing \( (u_h) \) by a subsequence, \( (u_h) \) and \( (\tilde{u}_h) \) converge pointwise to \( u_\infty \) and \( \tilde{u}_\infty \), respectively almost everywhere on \( \Omega \) and \( H_{n-1}^- \)-almost everywhere on \( \partial \Omega \) [recall that the trace operator is compact between \( H^1(\Omega) \) and \( L^2(\partial \Omega, H_{n-1}^-) \)]. Then (2) follows from lower semicontinuity of the Dirichlet integral and from continuity of \( W \) and \( \sigma \), by applying Fatou’s Lemma.

Lower semicontinuity of \( \mathcal{E}_\varepsilon \) implies now that \( U \) is closed in \( L^2(\Omega) \); on the other hand, by (3) and by Poincaré Inequality, \( U \) is bounded in \( H^1(\Omega) \). Then Rellich’s Theorem gives that \( U \) is compact in \( L^2(\Omega) \) and the proof is complete.

The aim of the present paper is to study the asymptotic behavior as \( \varepsilon \to 0^+ \) of \( (P_\varepsilon) \). We shall prove in Section 2 that such asymptotic behavior is related with the following geometric minimization problem:

\[
(P_\varepsilon) \quad \min \{ P_\Omega(E) + \gamma \mathcal{H}_{n-1}(\partial^* E \cap \partial \Omega): E \subseteq \Omega, |E| = m_1 \}.
\]

Here \( \gamma \in [-1, 1] \), \( m_1 \in [0, |\Omega|] \) are fixed real constants; \( |E| \), \( P_\Omega(E) \), \( \partial^* E \) denote respectively the Lebesgue measure of \( E \), the perimeter of \( E \) in \( \Omega \), and the reduced boundary of \( E \). We refer to the book by E. Giusti [6] for these concepts, which go back to the De Giorgi’s approach to the minimal surfaces theory. Anyhow, for reader’s convenience, we recall that \( P_\Omega(E) = \mathcal{H}_{n-1}(\partial E \cap \Omega) \) and \( \partial^* E = \partial E \), provided that the boundary of \( E \) is locally Lipschitz continuous; hence \( (P_\varepsilon) \) consists in finding a subset \( E \) of \( \Omega \), with prescribed volume \( m_1 \), which minimizes a quantity related with the \((n-1)\)-dimensional measure of its boundary.

The problem \( (P_\varepsilon) \) is known as the liquid-drop problem (cf. E. Giusti [5]). Since \( \Omega \) is bounded and \( |\gamma| \leq 1 \), it always admits (at least) one solution. Such existence result could also be obtained by the following proposition, which we need later.

1.2. Proposition. — Let \( \tau : \partial \Omega \times \mathbb{R} \rightarrow \mathbb{R} \) be a Borel function and define, for \( u \in BV(\Omega) \),

\[
F(u) = \int_{\Omega} |Du| + \int_{\partial \Omega} \tau(x, \tilde{u}(x)) \, d\mathcal{H}^{n-1}_\tau(x)
\]

where \( \tilde{u} \) denotes the trace of \( u \) on \( \partial \Omega \). If

\[\left\{ \begin{array}{l}
\tau(x, s_1) - \tau(x, s_2) \leq |s_1 - s_2|,
\forall x \in \partial \Omega, \\
\forall s_1, s_2 \in \mathbb{R}
\end{array} \right.\] (i)

then the functional \( F \) is lower semicontinuous on \( BV(\Omega) \) with respect to the topology of \( L^1(\Omega) \).

Proof. — Fix \( u_\infty \in BV(\Omega) \) and let \( (u_h) \) be a sequence in \( BV(\Omega) \) converging to \( u_\infty \) in \( L^1(\Omega) \). We want to prove that

\[
\limsup_{h \to +\infty} [F(u_\infty) - F(u_h)] \leq 0.
\]

By (i) we deduce that

\[
F(u_\infty) - F(u_h) \leq \int_{\Omega} |Du_\infty| - \int_{\Omega} |Du_h| + \int_{\partial \Omega} |\tilde{u}_\infty - \tilde{u}_h| \, d\mathcal{H}^{n-1}.
\]

Let \( \delta > 0 \) and define \( v_\delta = (1 - \chi_\delta) (u_\infty - u_h) \), where \( \chi_\delta \) is the usual cut-off function, i.e. \( \chi_\delta \in C^0_0(\Omega) \), \( 0 \leq \chi_\delta \leq 1 \), \( \chi_\delta(x) = 1 \) if \( \text{dist}(x, \partial \Omega) \geq \delta \), \( |D \chi_\delta| \leq 2/\delta \). The trace inequality for BV functions (cf. G. Anzellotti and M. Giaquinta [1]), applied to \( v_\delta \), gives that

\[
\int_{\partial \Omega} |\tilde{u}_\infty - \tilde{u}_h| \, d\mathcal{H}^{n-1} 
\leq c_1 \int_{\Omega_\delta} |D(u_\infty - u_h)| + (2c_1/\delta) \int_{\Omega_\delta} |u_\infty - u_h| \, dx + c_2 \int_{\Omega_\delta} |u_\infty - u_h| \, dx,
\]

(1) For \( u \in BV(\Omega) \) and \( E \) measurable subset of \( \Omega \), we denote by \( \int_E |Du| \) the value of the measure \( |Du| \) at the set \( E \). Of course, if \( Du \) is a Lebesgue integrable vector function, then \( \int_E |Du| \) agrees with the ordinary integral \( \int_E |Du(x)| \, dx \).
where $\Omega_{\delta} = \{ x \in \Omega : \text{dist}(x, \partial \Omega) > \delta \}$ and $\Omega_0^\delta = \Omega \setminus \Omega_{\delta}$. Let us remark that $c_1 = 1$ because $\partial \Omega$ is smooth (see [1]), and that

$$\int_{\Omega_{\delta}} |D(u_{\infty} - u_h)| \leq \int_{\Omega_{\delta}} |D u_{\infty}| + \int_{\Omega_{\delta}} |D u_h| + \int_{\partial \Omega_{\delta}} |D(u_{\infty} - u_h)|.$$

Since $u_{\infty} - u_h \in \text{BV}(\Omega)$, we have that

$$\int_{\partial \Omega_{\delta}} |D(u_{\infty} - u_h)| = 0, \quad \forall h \in \mathbb{N}$$

for a set of $\delta > 0$ of full measure; hence

$$F(u_{\infty}) - F(u_h) \leq \int_{\Omega_{\delta}} |D u_{\infty}| - \int_{\Omega_{\delta}} |D u_h| + \left( \frac{2}{\delta} + c_2 \right) \int_{\partial \Omega_{\delta}} |u_{\infty} - u_h| \, dx$$

and, by lower semicontinuity in $L^1(\Omega_{\delta})$ of the functional

$$u \mapsto \int_{\Omega_{\delta}} |Du|,$$

we conclude that

$$\limsup_{h \to +\infty} [F(u_{\infty}) - F(u_h)] \leq 2 \int_{\Omega_{\delta}} |D u_{\infty}|$$

for almost all $\delta > 0$. By taking $\delta \to 0^+$, the inequality (4) is proved.

1.3. Remark. — The previous proposition fails to be true if $\partial \Omega$ is not smooth, or if the function $\tau$ has in (i) a Lipschitz constant $L > 1$. For example, in the case $\Omega = ]0,1[ \times ]0,1[ \quad \text{and} \quad \tau(x, s) = -\lambda s \quad \text{with} \quad \lambda > \sqrt{2}/2$, the corresponding functional $F$ is not lower semicontinuous at the point $u_{\infty} = 0$; it is enough to check lower semicontinuity on the sequence $(u_h)$ given by $u_h(x, y) = 0$ for $x + y \geq 1/h$, $u_h(x, y) = h$ for $x + y < 1/h$. Analogously, in the case $\Omega = \{ x \in \mathbb{R}^2 : |x| < 1 \}$ and $\tau(x, s) = \lambda |s|$ with $\lambda > 1$, the corresponding functional $F$ is not lower semicontinuous at the point $u_{\infty}(x) = |x|$: one can choose $u_h(x) = \min \{ |x|, (h - 1)(1 - |x|) \}$.

However, it is worth noticing that, in the particular case $\tau(x, s) = |s - \psi(x)|$ with $\psi \in L^1(\partial \Omega, \mathcal{H}^{n-1})$, the functional $F$ defined in Proposition 1.2 is lower semicontinuous on $L^1(\Omega)$ even for Lipschitz
continuous \( \partial \Omega \). Indeed, by choosing an open, bounded set \( \Omega' \supseteq \overline{\Omega} \) and a function \( \hat{\psi} \in BV(\Omega') \) whose trace on \( \partial \Omega \) is \( \psi \), we have that

\[
F(u) = \int_{\Omega} |D u| + \int_{\partial \Omega} |\tilde{u}(x) - \psi(x)| \, d\mathcal{H}^{n-1} = \int_{\Omega'} |D v_u| - \int_{\Omega'} |D \hat{\psi}|,
\]

where the function \( v_u \) is defined by \( v_u(x) = u(x) \) for \( x \in \Omega \), \( v_u(x) = \hat{\psi}(x) \), for \( x \in \Omega' \setminus \Omega \). Since the first addendum of the right-hand side is lower semicontinuous with respect to \( u \) in \( L^1(\Omega) \), \( F \) also is lower semicontinuous in \( L^1(\Omega) \).

From now on, we let, for \( t \geq 0 \),

\[
\phi(t) = \int_{0}^{t} W^{1/2}(s) \, ds, \quad (5)
\]

\[
\sigma(t) = \inf \{ \sigma(s) + 2 |\phi(s) - \phi(t)| : s \geq 0 \}, \quad (6)
\]

and, for \( u \in BV(\Omega) \),

\[
\mathcal{E}_0(u) = 2 \int_{\Omega} |D (\phi \circ u)| + \int_{\partial \Omega} \sigma(\tilde{u}(x)) \, d\mathcal{H}^{n-1}, \quad (7)
\]

where, as above, \( \tilde{u} \) denotes the trace of \( u \) on \( \partial \Omega \).

1.4. PROPOSITION. — Let \( (u_h) \) be a sequence of functions of class \( C^1 \) on \( \Omega \). If \( (u_h) \) converges in \( L^1(\Omega) \) to a function \( u_\infty \) and there exists a real constant \( c \) such that

\[
\int_{\Omega} |D (\phi \circ u_h)| \, dx \leq c
\]

for every \( h \in \mathbb{N} \), then \( \phi \circ u_\infty \in BV(\Omega) \) and

\[
\mathcal{E}_0(u_\infty) \leq \liminf_{h \to +\infty} \mathcal{E}_0(u_h).
\]

Proof. — Let us denote \( v_h(x) = \phi(u_h(x)) \) and fix an open subset \( \Omega' \) of \( \Omega \) such that \( \overline{\Omega'} \subset \Omega \). If we consider the smooth function \( \tilde{v}_h(x) = v_h(x) - \vartheta_h \), where

\[
\vartheta_h = \int_{\Omega'} v_h \, dx,
\]

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Poincaré Inequality gives
\[ \int_{\Omega'} |\tilde{v}_h| \, dx \leq c_1(\Omega) \int_{\Omega'} |D\tilde{v}_h| \, dx \leq c_1(\Omega) c \]
for every \( h \in \mathbb{N} \) and for a real constant \( c_1(\Omega) \) depending on \( \Omega \) but independent of \( \Omega' \subseteq \Omega \). It follows that the sequence \((\tilde{v}_h)\) is bounded in \( BV(\Omega) \); hence, by Rellich’s Theorem, there exists a subsequence \((\tilde{v}_{\sigma(h)})\) which converges in \( L^1(\Omega) \) to a function \( \tilde{v}_\infty \).

Since it is not restrictive to assume that \((\tilde{v}_{\sigma(h)})\) and \((v_{\sigma(h)})\) both converge almost everywhere in \( \Omega \), we infer that \((\tilde{\theta}_{\sigma(h)})\) converges in \( \mathbb{R} \) to \( \tilde{\theta}_\infty \), and finally that \((v_{\sigma(h)})\) converges in \( L^1(\Omega) \) to \( \tilde{v}_\infty + \tilde{\theta}_\infty \). We have of course \( \tilde{v}_\infty + \tilde{\theta}_\infty = \varphi \circ u_\infty \), so we conclude that the whole \((v_h)\) converges in \( L^1(\Omega) \) to \( v_\infty = \varphi \circ u_\infty \) and, by semicontinuity, that
\[ \int_{\Omega} |Dv_\infty| \leq \liminf_{h \to +\infty} \int_{\Omega} |Dv_h| \leq c < +\infty. \]

We now consider the inverse function \( \varphi^{-1} \) of \( \varphi \); note that \( \varphi^{-1} \) exists because \( \varphi'(t) = W(t) > 0 \) except for \( t = \alpha, \beta \). Denoting \( \tau(s) = \hat{\theta}(\varphi^{-1}(s)) \), we have that
\[ |\tau(s_1) - \tau(s_2)| \leq 2|s_1 - s_2| \]
for every \( s_1, s_2 \) in the domain of \( \varphi^{-1} \); then Proposition 1.2 yields that
\[ \mathcal{E}_0(u_\infty) = 2 \int_{\Omega} |Dv_\infty| + \int_{\partial \Omega} \tau(\tilde{v}_\infty) \, d\mathcal{H}^{n-1} \]
\[ \leq \liminf_{h \to +\infty} \left[ 2 \int_{\Omega} |Dv_h| \, dx + \int_{\partial \Omega} \tau(\tilde{v}_h) \, d\mathcal{H}^{n-1} \right] = \liminf_{h \to +\infty} \mathcal{E}_0(u_h) \]
and Proposition 1.4 is proved.

We now turn to the liquid-drop problem \((P_0)\) by proving that the class of competing sets can be restricted to smooth sets.

1.5. Proposition. Suppose \( 0 < m_1 < |\Omega| \) and \( |\gamma| \leq 1 \). If \( \lambda \) is a fixed real number such that
\[ \lambda \leq P_\Omega(A) + \gamma \mathcal{H}^{n-1}(\partial(A \cap \Omega) \cap \partial \Omega) \]

for every open, bounded subset $A$ of $\mathbb{R}^n$ which has smooth boundary and satisfies $\mathcal{H}_{n-1}(\partial A \cap \partial \Omega) = 0$, $|A \cap \Omega| = m_1$, then

$$\lambda \leq \min \left\{ P_\Omega(E) + \gamma \mathcal{H}_{n-1}(\partial^* E \cap \partial \Omega) : E \subseteq \Omega, |E| = m_1 \right\}.$$ 

Proof. — We omit the details because we closely follow the proof of the analogous result proved for the case $\gamma = 0$ in Lemmas 1 and 2 of [10].

Let $E_0$ be the set which realizes the minimum of $(P_\Omega)$. By a theorem of E. Gonzalez, U. Massari and I. Tamanini ([7], Th. 1), which was stated for $\gamma = 0$ but holds also in our situation because of its local character, we have that both $E_0$ and $\Omega \setminus E_0$ contain a non-empty open ball. Then, arguing as in Lemma 1 of [10], one can construct a sequence $(E_h)$ of open, bounded, smooth subsets of $\mathbb{R}^n$ such that $|E_h \cap \Omega| = m_1$, $\mathcal{H}_{n-1}(\partial E_h \cap \partial \Omega) = 0$ for every $h \in \mathbb{N}$, and

$$\lim_{h \to +\infty} |(E_h \cap \Omega) \triangle E_0| = 0,$$

$$\lim_{h \to +\infty} P_\Omega(E_h) = P_\Omega(E_0),$$

$$\lim_{h \to +\infty} \mathcal{H}_{n-1}(\partial (E_h \cap \Omega) \cap \partial \Omega) = \mathcal{H}_{n-1}(\partial^* E_0 \cap \partial \Omega).$$

The last assertion is not actually contained in Lemma 1 of [10] but it easily follows from (8) and from

$$\mathcal{H}_{n-1}(\partial (E_h \cap \Omega) \cap \partial \Omega) = \int_{\partial \Omega} \tilde{\chi}_{E_h \cap \Omega} d \mathcal{H}_{n-1},$$

$$\mathcal{H}_{n-1}(\partial^* E_0 \cap \partial \Omega) = \int_{\partial \Omega} \tilde{\chi}_{E_0} d \mathcal{H}_{n-1},$$

where $\tilde{\chi}_T$ denotes the trace on $\partial \Omega$ of the characteristic function of $T$ for $T = E_h \cap \Omega$ and $T = E_0$.

The proof of the proposition is now a straightforward consequence of (9) and (10). 

The next result, stated here without proof, was proved in [10] (Lemma 4).

1.6. PROPOSITION. — Let $A$ be an open subset of $\mathbb{R}^n$ with smooth, non-empty, compact boundary $\partial A$ such that $\mathcal{H}_{n-1}(\partial A \cap \partial \Omega) = 0$. Define the function $h : \mathbb{R}^n \to \mathbb{R}$ by $h(x) = \text{dist}(x, \partial A)$ for $x \in A$, $h(x) = -\text{dist}(x, \partial A)$ for $x \notin A$. Then $h$ is Lipschitz continuous, $|D h(x)| = 1$ for almost all $x \in \mathbb{R}^n$,
and
\[ \lim_{t \to 0} \mathcal{H}_{n-1} (S_t \cap \Omega) = \mathcal{H}_{n-1} (\partial A \cap \Omega) \]

where \( S_t = \{ x \in \mathbb{R}^n : h(x) = t \} \).

2. THE MAIN RESULT

We recall that \( \Omega \) denotes an open, bounded subset of \( \mathbb{R}^n (n \geq 2) \) with smooth boundary, and \( W, \sigma : [0, +\infty[ \to \mathbb{R} \) denote two non-negative continuous functions. We assume also that \( W(t) = 0 \) only for \( t = \alpha \) or \( t = \beta \) with \( 0 < \alpha < \beta \).

2.1. THEOREM. - Fix \( m \in [\alpha | \Omega|, \beta | \Omega|] \) and, for every \( \varepsilon > 0 \), let \( u_\varepsilon \) be a solution of the minimization problem (\( \mathcal{P}_\varepsilon \)). If each \( u_\varepsilon \) is of class \( C^1 \) and there exists a sequence \( (\varepsilon_h) \) of positive numbers, converging to zero, such that

(i) \( W(u_\varepsilon(x)) = 0 \) \( \text{[i.e. } u_\varepsilon(x) = \alpha \text{ or } u_\varepsilon(x) = \beta \text{]} \) for almost all \( x \in \Omega \);

(ii) the set \( E_0 = \{ x \in \Omega : u_\varepsilon(x) = \alpha \} \) is a solution of the minimization problem (\( \mathcal{P}_0 \)) with

\[ \gamma = \frac{\hat{\sigma}(\alpha) - \hat{\sigma}(\beta)}{2c_0}, \quad m_1 = \frac{\beta | \Omega| - m}{\beta - \alpha}, \]

where \[ \text{see (5) and (6)} \]

\[ \hat{\sigma}(t) = \inf \left\{ \sigma(s) + 2 \left| \int_t^s W^{1/2}(y) dy \right| : s \geq 0 \right\} \]

for \( t = \alpha, \beta, \) and

\[ c_0 = \int_\alpha^\beta W^{1/2}(y) dy; \]

(iii) \[ \lim_{h \to +\infty} \varepsilon_h^{-1} \mathcal{S}_{\varepsilon_h}(u_{\varepsilon_h}) \]

\[ = 2c_0 P_\Omega(E_0) + \hat{\sigma}(\alpha) \mathcal{H}_{n-1}(\partial^* E_0 \cap \partial \Omega) \]

\[ + \hat{\sigma}(\beta) \mathcal{H}_{n-1}(\partial \Omega \setminus \partial^* E_0). \]
For some comments about this statement we refer to Remarks 2.5. The proof of Theorem 2.1 is similar to that one of the result with \( \sigma = 0 \) given in [10]. Nevertheless the extension is not trivial, because in the asymptotic \((\epsilon = 0)\) boundary behavior, given by \( \hat{\sigma} \), both the boundary and the interior behavior for \( \epsilon > 0 \), given by \( W \) and \( \sigma \), are involved.

In the language of \( \Gamma \)-convergence theory, the proof of Theorem 2.1 consists in verifying that \((\epsilon^{-1} \mathcal{E}_\epsilon + I_m)\) converges as \( \epsilon \to 0^+ \), in the sense of \( \Gamma (L^1(\Omega))\)-convergence, to the functional \( \mathcal{E}_0 + I_m \), at the points \( u \in L^1(\Omega) \) such that \( W(u(x)) = 0 \) for almost all \( x \in \Omega \) (cf. Section 3 in [10]). The functional \( \mathcal{E}_0 \) was defined in (7); \( I_m \) denotes here the \( 0/+\infty \) characteristic function of the constraint \( \int_{\Omega} u(x) \, dx = m \).

The main steps in the proof of Theorem 2.1 are the following propositions.

2.2. PROPOSITION. — Suppose that \((v_\epsilon)_\epsilon > 0\) is a family in \( \{u \in C^1(\Omega) : u \geq 0\} \) which converges in \( L^1(\Omega) \) as \( \epsilon \to 0^+ \) to a function \( v_0 \). If
\[
\liminf_{\epsilon \to 0^+} \epsilon^{-1} \mathcal{E}_\epsilon(v_\epsilon) < +\infty,
\]
then \( v_0 \in BV(\Omega) \), \( W(v_0(x)) = 0 \) for almost all \( x \in \Omega \), and
\[
\mathcal{E}_0(v_0) \leq \liminf_{\epsilon \to 0^+} \epsilon^{-1} \mathcal{E}_\epsilon(v_\epsilon). \tag{11}
\]

2.3. PROPOSITION. — Let \( A \) be an open, bounded subset of \( \mathbb{R}^n \) with smooth boundary such that \( \mathcal{H}^{n-1}(\partial A \cap \partial \Omega) = 0 \). Define the function \( v_0 : \Omega \to \mathbb{R} \) by \( v_0(x) = \alpha \) for \( x \in A \cap \Omega \), \( v_0(x) = \beta \) for \( x \in \Omega \setminus A \). For every \( r > 0 \) denote
\[
U_r = \left\{ v \in H^1(\Omega) : v \geq 0, \|v - v_0\|_{L^2(\Omega)} < r, \int_{\Omega} v \, dx = \int_{\Omega} v_0 \, dx \right\}.
\]
Then, for every \( r > 0 \), we have that
\[
\limsup_{\epsilon \to 0^+} \inf_{v \in U_r} \epsilon^{-1} \mathcal{E}_\epsilon(v) \leq \mathcal{E}_0(v_0). \tag{12}
\]

2.4. Remark. — For the connection between (12) and the corresponding inequality in the usual definition of \( \Gamma \)-convergence, see Proposition 1.14 of [4].
Proof of Proposition 2.2. — By the continuity of \(W\) and by Fatou's Lemma we have that

\[
\int_{\Omega} W(v_0) \, dx \leq \liminf_{\varepsilon \to 0^+} \int_{\Omega} W(v_\varepsilon) \, dx \leq \liminf_{\varepsilon \to 0^+} \varepsilon^{-1} \sigma(v_\varepsilon) = 0;
\]

since \(W \geq 0\), we have at once proved that \(W(v_0(x)) = 0\) for almost all \(x \in \Omega\).

Now

\[
\int_{\Omega} |D(\varphi \circ v_\varepsilon)| = \int_{\Omega} |\varphi'(v_\varepsilon(x))| \cdot |Dv_\varepsilon(x)| \, dx
\]

\[
= \int_{\Omega} W(v_\varepsilon(x)) \cdot |Dv_\varepsilon(x)| \, dx
\]

\[
\leq \int_{\Omega} \varepsilon \cdot |Dv_\varepsilon|^2 + \varepsilon^{-1} W(v_\varepsilon) \, dx \leq \varepsilon^{-1} \sigma(v_\varepsilon),
\]

so Proposition 1.4 and \(\tilde{\sigma} \leq \sigma\) apply for obtaining

\[
\sigma_0(v_0) \leq \liminf_{\varepsilon \to 0^+} \sigma_0(v_\varepsilon)
\]

\[
\leq \liminf_{\varepsilon \to 0^+} \left\{ \int_{\Omega} \varepsilon \cdot |Dv_\varepsilon|^2 + \varepsilon^{-1} W(v_\varepsilon) \, dx \right. \\
+ \int_{\partial \Omega} \tilde{\sigma}(v_\varepsilon) \, d\mathcal{H}^{n-1} \right\} \leq \liminf_{\varepsilon \to 0^+} \varepsilon^{-1} \sigma(v_\varepsilon).
\]

It remains to prove that \(v_0 \in \text{BV}(\Omega)\). This is obvious because \(v_0\) takes only the values \(\alpha\) and \(\beta\), and \(\varphi \circ v_0 \in \text{BV}(\Omega)\); hence the proof of Proposition 2.2 is complete. ■

Proof of Proposition 2.3. — Let us fix \(r > 0\) and also, for further convenience, \(L \geq 0, M \geq 0\) and \(\delta > 0\). We shall not often indicate in the following the dependence on \(r, L, M, \delta\) as well as on the other data \(n, \Omega, W, \alpha, \beta, \sigma, A\); in particular we shall denote by \(c_1, c_2, \ldots\) real positive constants depending on all such data.

The following lemma contains a purely technical part of the proof.

2.5. Lemma. — Consider, for every \(\varepsilon > 0\), the first-order ordinary differential equation

\[
|y'| = \varepsilon^{-1} (\delta + W(y))^{1/2}.
\]  

(13)
Then there exist three constants $c_1$, $c_2$, $c_3$, independent of $\varepsilon$, and a Lipschitz continuous function $\chi_\varepsilon(s, t)$, defined on the upper half-plane $\mathbb{R} \times [0, +\infty[$, satisfying the following properties:

\begin{equation}
\begin{aligned}
\chi_\varepsilon(s, t) &= \alpha \quad \text{for} \quad s \geq c_1 \varepsilon, \quad t \geq c_1 \varepsilon, \\
\chi_\varepsilon(s, t) &= \beta \quad \text{for} \quad s \leq 0, \quad t \geq c_1 \varepsilon, \\
\chi_\varepsilon(s, t) &= L \quad \text{for} \quad s \leq 0, \\
\chi_\varepsilon(s, t) &= M \quad \text{for} \quad s \geq c_1 \varepsilon; \\
0 &\leq \chi_\varepsilon \leq c_2, \\
|D\chi_\varepsilon| &\leq c_3/\varepsilon;
\end{aligned}
\end{equation}

on the strip $\{s \leq 0, t \leq c_1 \varepsilon\}$ the function $\chi_\varepsilon(s, t)$ depends only on $t$ and fulfils the equation (13) in the set $\{\chi_\varepsilon(t) \neq \beta\}$; on the strip $\{s \geq c_1 \varepsilon, t \leq c_1 \varepsilon\}$ the function $\chi_\varepsilon(s, t)$ depends only on $t$ and fulfils (13) in the set $\{\chi_\varepsilon(t) \neq \alpha\}$; on the strip $\{0 \leq s \leq c_1 \varepsilon, t \geq c_1 \varepsilon\}$ the function $\chi_\varepsilon(s, t)$ depends only on $s$ and fulfils (13) in the set $\{\chi_\varepsilon(s) \neq \alpha\}$.

Proof. — We have to determine $c_1$, $c_2$, $c_3$ and to complete the definition of $\chi_\varepsilon$ on the strips

\begin{align*}
S_1 &= \{s \leq 0, t \leq c_1 \varepsilon\}, & S_2 &= \{s \geq c_1 \varepsilon, t \leq c_1 \varepsilon\}, \\
S_3 &= \{0 \leq s \leq c_1 \varepsilon, t \geq c_1 \varepsilon\},
\end{align*}

and on the square $Q=\{0, c_1 \varepsilon\} \times \{0, c_1 \varepsilon\}$.

Let us begin by $S_1$, where we have the prescribed boundary values $\chi_\varepsilon(s, c_1 \varepsilon) = \beta$, $\chi_\varepsilon(s, 0) = L$. If $\beta = L$, we define $\chi_\varepsilon(t) = \beta$; if $\beta > L$, we solve the Cauchy problem

\[ y'(t) = \varepsilon^{-1}(\delta + W(y(t)))^{1/2}, \quad y(0) = L, \]

and we define $\chi_\varepsilon(t) = \min\{\beta, y(t)\}$; if $\beta < L$, we solve the same Cauchy problem with $-y'$ instead of $y'$ and we define $\chi_\varepsilon(t) = \max\{\beta, y(t)\}$. Since

\[ |\chi_\varepsilon(t)| = \varepsilon^{-1}(\delta + W(\chi_\varepsilon(t)))^{1/2} \leq \varepsilon^{-1} \delta^{1/2} \]

provided that $\chi_\varepsilon(t) \neq \beta$, we have $\chi_\varepsilon(t) = \beta$ for $t \geq \varepsilon |\beta - L| / \delta$; then, in order that $\chi_\varepsilon$ takes the prescribed boundary values $\chi_\varepsilon(s, c_1 \varepsilon) = \beta$, we need $c_1 \geq |\beta - L| / \delta$. The same holds on $S_2$ and $S_3$, so we are led to define

\[ c_1 = \max \{ |\beta - L| / \delta, |\alpha - \beta| / \delta, |\alpha - M| / \delta \}. \]
Define also $c_2 = \max \{\alpha, \beta, L, M\}$, so that

$$0 \leq \chi_\epsilon \leq c_2$$

and

$$|D\chi_\epsilon| \leq \epsilon^{-1} (\delta + \max \{W(s) : 0 \leq s \leq c_2\})^{1/2}$$

on $(\mathbb{R} \times [0, +\infty[) \setminus Q$. Finally, as we know $\chi_\epsilon$ on three sides of the square $Q$, we can extend $\chi_\epsilon$ on $Q$ in such a way that $\chi_\epsilon$ becomes Lipschitz continuous on the whole upper half-plane and (15) is satisfied with

$$c_3 = 3 c_1 (\delta + \max \{W(s) : 0 \leq s \leq c_2\})^{1/2}.$$

The proof of Lemma 2.5 is now complete. ■

Let us return to the proof of Proposition 2.3. The first part of the proof consists in constructing a family $(v_\epsilon)$ in $U_1$, such that $v_\epsilon$ converges to $v_0$ as $\epsilon \to 0^+$, and

$$\inf_{v \in U_1} \mathcal{E}_\epsilon(v)$$

is approximatively equal to $\mathcal{E}_\epsilon(v_\epsilon)$.

Define

![Diagram](image_url)

**FIG. 1.**
and let \( \chi_\varepsilon \) be the function constructed in Lemma 2.5. Let, for \( x \in \Omega \),

\[
v_\varepsilon'(x) = \chi_\varepsilon (d_A(x), d_\Omega(x)).
\]

Look at Figure 1 for understanding the meaning of our construction.

Denoting

\[
S_s = \{x \in A \cap \Omega : d_A(x) = s\},
\]
\[
\Sigma^s = \{x \in \Omega \cap A : d_\Omega(x) = t\},
\]
\[
\Sigma^p = \{x \in \Omega \setminus A : d_\Omega(x) = t\},
\]

Federer's coarea formula and \( |Dd_\Omega| = |Dd_A| = 1 \) (see Proposition 1.6) yield

\[
\int_{\Omega} |v_\varepsilon' - v_0| \, dx
\leq c_4 \left( \left| \{x \in \Omega : d_\Omega(x) \leq c_1 \varepsilon \} \right| + \left| \{x \in A \cap \Omega : d_A(x) \leq c_1 \varepsilon \} \right| \right)
= c_4 \int_0^{c_1 \varepsilon} \left( \mathcal{H}^{n-1}(\Sigma^s \cup \Sigma^p) + \mathcal{H}^{n-1}(S_t) \right) dt;
\]

hence, as \( \partial A \) and \( \partial \Omega \) are smooth, Proposition 1.6 implies

\[
\int_{\Omega} |v_\varepsilon' - v_0| \, dx \leq c_5 \varepsilon
\]
for \( \varepsilon \) small enough. It follows that \( v_\varepsilon' \) converges to \( v_0 \) in \( L^1(\Omega) \) as \( \varepsilon \to 0^+ \) and, defining

\[
\eta_\varepsilon = \int_{\Omega} v_\varepsilon' dx - \int_{\Omega} v_0 \, dx,
\]
we have that

\[
|\eta_\varepsilon| \leq c_5 \varepsilon
\]
(17)
for \( \varepsilon \) small enough.

Let us choose a point \( x_0 \in \Omega \setminus \partial A \) and, for fixing the ideas, assume that \( x_0 \in \Omega \cap A \). In the case \( \Omega \cap A = \emptyset \) or \( x_0 \in \Omega \setminus A \) the changes in the proof.
are trivial. Note that the closed ball $B_\varepsilon = B(x_0, \varepsilon^{1/n})$ is contained, for $\varepsilon$ small enough, in the set $\{v' = \alpha\}$; then the function $v_\varepsilon$, defined on $\Omega$ by $v_\varepsilon = v'_\varepsilon$ for $x \notin B_\varepsilon$, and by
\[
v_\varepsilon(x) = \alpha + h_\varepsilon \left(1 - \varepsilon^{-1/n} \left| x - x_0 \right| \right),
\]
for $x \in B_\varepsilon$, is Lipschitz continuous whenever $h_\varepsilon \in \mathbb{R}$.

We now choose
\[
h_\varepsilon = -n \omega_{n-1}^{-1} \eta_\varepsilon \varepsilon^{(1-n)/n},
\]
with $\omega_{n-1}$ equal to the volume of the unit ball in $\mathbb{R}^{n-1}$, so that
\[
\int_{B_\varepsilon} (v_\varepsilon - v'_\varepsilon) \, dx = \int_{B_\varepsilon} h_\varepsilon \left(1 - \varepsilon^{-1/n} \left| x - x_0 \right| \right) \, dx = -\eta_\varepsilon,
\]
and, by the definition of $\eta_\varepsilon$ and $v_\varepsilon$,
\[
\int_{B_\varepsilon} v_\varepsilon \, dx = \int_{B_\varepsilon} v_0 \, dx
\]
for $\varepsilon$ small enough. Since, by (17),
\[
|h_\varepsilon| \leq c_\varepsilon \varepsilon^{1/n},
\]
we have, for $\varepsilon$ small enough,
\[
0 \leq v_\varepsilon \leq c_\varepsilon,
\]
and
\[
\lim_{\varepsilon \to 0^+} \int_{\Omega} |v_\varepsilon - v_0|^2 \, dx = 0;
\]

hence
\[
\lim_{\varepsilon \to 0^+} \inf_{v \in \mathcal{U}_\varepsilon} \varepsilon^{-1} \mathcal{E}_\varepsilon(v) \leq \lim_{\varepsilon \to 0^+} \sup_{v \in \mathcal{U}_\varepsilon} \varepsilon^{-1} \mathcal{E}_\varepsilon(v_\varepsilon).
\]

The second part of the proof consists in a sharp estimate of the right-hand side of such inequality. For the sake of simplicity, let
\[
\varepsilon^{-1} \mathcal{E}_\varepsilon(v_\varepsilon) = \mathcal{E}_\varepsilon(v_\varepsilon; \Omega) + \mathcal{E}_\varepsilon''(v_\varepsilon)
\]
with

\[ \mathcal{E}'(v_\varepsilon; C) = \int_C [\varepsilon |Dv_\varepsilon|^2 + \varepsilon^{-1} W(v_\varepsilon)] \, dx \quad (C \subseteq \Omega), \]

and

\[ \mathcal{E}''_\varepsilon(v_\varepsilon) = \int_{\partial\Omega} \sigma(\tilde{v}_\varepsilon) \, d\mathcal{H}^{n-1}. \]

By (20) and (21), and by the continuity of \( \sigma \) and of the trace operator, we at once obtain

\[
\limsup_{\varepsilon \to 0^+} \mathcal{E}''(V_\varepsilon) \leq \int_{\partial\Omega} \sigma(\tilde{v}_0) \, d\mathcal{H}^{n-1}
\]

\[ = \sigma(L) \mathcal{H}^{n-1}(\partial\Omega \setminus A) + \sigma(M) \mathcal{H}^{n-1}(\partial\Omega \cap A). \quad (23) \]

The evaluation of \( \mathcal{E}'_\varepsilon(v_\varepsilon; \Omega) \) is more complicated. Let us divide \( \Omega \) in seven parts, corresponding to the construction of \( \chi_\varepsilon \) in Lemma 2.5 and of \( v_\varepsilon \) (see Fig. 1):

\[
B_\varepsilon = B(x_0, \varepsilon^{1/n}),
\]

\[ \Omega_\varepsilon^a = \{ x \in \Omega : d_A(x) > c_1 \varepsilon, d_\Omega(x) > c_1 \varepsilon, x \notin B_\varepsilon \}, \]

\[ \Omega_\varepsilon^p = \{ x \in \Omega : d_A(x) \leq 0, d_\Omega(x) > c_1 \varepsilon \}, \]

\[ \Omega_\varepsilon^{ab} = \{ x \in \Omega : 0 < d_A(x) \leq c_1 \varepsilon, d_\Omega(x) > c_1 \varepsilon \}, \]

\[ \Omega_\varepsilon^{ab} = \{ x \in \Omega : d_A(x) \leq c_1 \varepsilon, d_\Omega(x) \leq c_1 \varepsilon \}, \]

\[ \Omega_\varepsilon^m = \{ x \in \Omega : d_A(x) > c_1 \varepsilon, d_\Omega(x) \leq c_1 \varepsilon \}, \]

\[ \Omega_\varepsilon^0 = \{ x \in \Omega : 0 < d_A(x) \leq c_1 \varepsilon, d_\Omega(x) \leq c_1 \varepsilon \}. \]

On \( B_\varepsilon \) we have, by (19),

\[
\mathcal{E}'_\varepsilon(v_\varepsilon; B_\varepsilon) \]

\[ \leq \varepsilon \| h_\varepsilon \|^2 \varepsilon^{-2/n} |B_\varepsilon| + \varepsilon^{-1} \int_{B_\varepsilon} W(\alpha + h_\varepsilon(1 - \varepsilon^{-1/n}|x - x_0|)) \, dx \]

\[ \leq c_7 \left[ \varepsilon^2 + \int_0^1 W(\alpha + h_\varepsilon(1 - r)) r^{n-1} \, dr \right]. \]

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hence

\[
\limsup_{\varepsilon \to 0^+} \mathcal{E}'(v_\varepsilon : B_\varepsilon) = 0. \tag{24}
\]

On \( \Omega^\varepsilon_a \) and \( \Omega^\varepsilon_\beta \) the function \( v_\varepsilon \) equals respectively \( \alpha \) and \( \beta \), so that

\[
\mathcal{E}'(v_\varepsilon : \Omega^\varepsilon_a) + \mathcal{E}'(v_\varepsilon : \Omega^\varepsilon_\beta) = 0. \tag{25}
\]

On \( \Omega^\varepsilon_{\alpha\beta} \) we have \( v_\varepsilon(x) = \chi_\varepsilon(d_\lambda(x), d_\Omega(x)) \); moreover, by (16), \( \chi_\varepsilon(s, t) = \chi_\varepsilon(s) \) depends only on the first variable and satisfies the equation

\[
-\chi_\varepsilon'(s) = \varepsilon^{-1} (\delta + W(\chi_\varepsilon(s)))^{1/2}
\]
on an interval \( ]0, \tau_\varepsilon[ \), with \( 0 < \tau_\varepsilon < c_1 \varepsilon \), while \( \chi_\varepsilon(s) = \alpha \) for \( s \geq \tau_\varepsilon \). Then, applying Federer’s coarea formula and \( \chi_\varepsilon(0) = \beta \), we obtain that

\[
\mathcal{E}'(v_\varepsilon : \Omega^\varepsilon_{\alpha\beta}) = \int_0^{\tau_\varepsilon} [\varepsilon \chi_\varepsilon(\varepsilon) + \varepsilon^{-1} W(\chi_\varepsilon(0))] \mathcal{H}_{n-1}(S_\varepsilon) \, ds
\]

\[
\leq \left( \sup_{0 \leq s \leq \tau_\varepsilon} \mathcal{H}_{n-1}(S_\varepsilon) \right) \int_0^{\tau_\varepsilon} 2 (\delta + W(\chi_\varepsilon(t)))^{1/2} \, ds
\]

\[
= \left( \sup_{0 \leq s \leq \tau_\varepsilon} \mathcal{H}_{n-1}(S_\varepsilon) \right) \left( 2 \int_0^\beta (\delta + W(t))^{1/2} \, dt \right),
\]

and therefore, by Proposition 1.6,

\[
\limsup_{\varepsilon \to 0^+} \mathcal{E}'(v_\varepsilon : \Omega^\varepsilon_{\alpha\beta}) \leq 2 \mathcal{H}_{n-1}(\partial \Lambda \cap \Omega) \left[ \int_0^\beta (\delta + W(t))^{1/2} \, dt \right] . \tag{26}
\]

The same argument leads to

\[
\limsup_{\varepsilon \to 0^+} \mathcal{E}'(v_\varepsilon : \Omega^\varepsilon_{\beta}) \leq 2 \mathcal{H}_{n-1}(\partial \Omega \cap \Lambda) \left| \int_0^{\lambda} (\delta + W(t))^{1/2} \, dt \right| , \tag{27}
\]

and to

\[
\limsup_{\varepsilon \to 0^+} \mathcal{E}'(v_\varepsilon : \Omega^\varepsilon_{\alpha\beta}) \leq 2 \mathcal{H}_{n-1}(\partial \Omega \cap \Lambda) \left| \int_0^{\lambda} (\delta + W(t))^{1/2} \, dt \right| . \tag{28}
\]

Finally, on \( \Omega^\varepsilon_a \) we have, by (15),

\[
\mathcal{E}'(v_\varepsilon : \Omega^\varepsilon_a) \leq c_8 \varepsilon^{-1} | \Omega^\varepsilon_a |.
\]
Note that, again by coarea formula,

$$|\Omega_0^\varepsilon| = \int_0^{c_1 \varepsilon} \mathcal{H}^{n-1} \left( \{ x \in \Omega : d_A(x) = s, \ d_\Omega(x) \leq c_1 \varepsilon \} \right) ds \leq c_1 \left( \sup_{0 \leq s \leq c_1 \varepsilon} \mathcal{H}^{n-1}(S_s \setminus \Omega_0^\varepsilon) \right),$$

where $\Omega_\rho$ denotes here the set $\{ x \in \Omega : d_\Omega(x) > \rho \}$. Since we have $\mathcal{H}^{n-1}(\partial A \cap \partial \Omega_\rho) = 0$ for almost all $\rho > 0$, Proposition 1.6 gives

$$\limsup_{\varepsilon \to 0^+} \left( \sup_{0 \leq s \leq c_1 \varepsilon} \mathcal{H}^{n-1}(S_s \setminus \Omega_0^\varepsilon) \right) \leq \limsup_{\varepsilon \to 0^+} \left( \sup_{0 \leq s \leq c_1 \varepsilon} \mathcal{H}^{n-1}(S_s \setminus \Omega_0^\varepsilon) \right) = \mathcal{H}^{n-1}(\partial A \cap \partial (\Omega \setminus \Omega_\rho))$$

for almost all $\rho > 0$; by taking the infimum for $\rho > 0$, we conclude that

$$\limsup_{\varepsilon \to 0^+} \mathcal{E}_\varepsilon(v, \Omega_0^\varepsilon) = 0. \quad (29)$$

Now, by collecting (22) to (29), we have that

$$\limsup_{\varepsilon \to 0^+} \inf_{v \in U_\varepsilon} \varepsilon^{-1} \mathcal{E}_\varepsilon(v) \leq 2 \mathcal{H}^{n-1}(\partial A \cap \Omega) \int_0^\delta (\delta + W(t))^{1/2} dt \left( \mathcal{H}^{n-1}(\partial \Omega \cap A) \left( 2 \left| \int_\Omega (\delta + W(t))^{1/2} dt \right| + \sigma(M) \right) + \mathcal{H}^{n-1}(\partial \Omega \cap A) \left( 2 \left| \int_\Omega (\delta + W(t))^{1/2} dt \right| + \sigma(L) \right) \right),$$

The left-hand side does not depend on $\delta$, $L$, and $M$, so, by taking first the infimum for $\delta > 0$, and then the infima for $M \geq 0$ and for $L \geq 0$ of the right-hand side, we obtain, by the definition of $\widehat{\sigma}$ and $c_0$, that

$$\limsup_{\varepsilon \to 0^+} \inf_{v \in U_\varepsilon} \varepsilon^{-1} \mathcal{E}_\varepsilon(v) \leq 2 c_0 \mathcal{H}^{n-1}(\partial A \cap \Omega) + \widehat{\sigma}(\alpha) \mathcal{H}^{n-1}(\partial \Omega \cap A) + \widehat{\sigma}(\beta) \mathcal{H}^{n-1}(\partial \Omega \cap A) \mathcal{H}^{n-1}(\partial \Omega \cap A) + \int_{\partial \Omega} \widehat{\sigma}(\tilde{v}_0) \sigma \mathcal{H}^{n-1}. \quad (30)$$
Remarking that the Fleming-Rishel formula yields

\[ 2 \int_{\Omega} |D(\phi \circ v_0)| = 2 \int_{\mathbb{R}} P_{\Omega}(\{x \in \Omega: \phi(v_0(x)) > t\}) dt \]

\[ = 2 \int_{\phi(a)}^{\phi(b)} P_{\Omega}(A \cap \Omega) dt = 2 c_0 H_{n-1}(\partial A \cap \Omega), \quad (31) \]

the right-hand side of (30) agrees with \( \mathcal{E}_0(v_0) \) and the proof of Proposition 2.3 is complete. □

Now, we can prove Theorem 2.1.

**Proof of Theorem 2.1.** — Assume for simplicity that all \((u_\varepsilon)\) converges, as \(\varepsilon \to 0^+\), to \(u_0\). By constructing, as in the proof of Theorem I of [10], a suitable family of comparison piecewise affine functions, we first obtain that

\[ \liminf_{\varepsilon \to 0^+} \mathcal{E}_\varepsilon(u_\varepsilon) < +\infty; \]

hence Proposition 2.2 gives \( W(u_0(x)) = 0 \) and

\[ \mathcal{E}_0(u_0) \leq \liminf_{\varepsilon \to 0^+} \mathcal{E}_\varepsilon(u_\varepsilon). \]

Now, let \( \mathcal{A} \) be the class of all open, bounded subsets \( A \) of \( \mathbb{R}^n \), with smooth boundary, such that \( H_{n-1}(\partial A \cap \Omega) = 0 \) and \( |A \cap \Omega| = |E_0| = m_1 \). For every \( A \in \mathcal{A} \), we define \( v_0^A(x) = \alpha \) for \( x \in A \cap \Omega \), \( v_0^A(x) = \beta \) for \( x \in \Omega \setminus A \); applying Proposition 2.3 with \( r = 1 \), we infer that

\[ \limsup_{\varepsilon \to 0^+} \inf_{v \in U} \mathcal{E}_\varepsilon(v) \leq \mathcal{E}_0(v_0^A), \]

where

\[ U = \left\{ v \in H^1(\Omega): v \geq 0, \int_{\Omega} |v - v_0^A|^2 dx < 1, \int_{\Omega} v dx = \int_{\Omega} v_0^A dx \right\} \]

Since

\[ \int_{\Omega} v_0^A dx = m, \]
we have, by the minimality of \( u_\varepsilon \), that

\[
\mathcal{E}_\varepsilon (u_\varepsilon) \leq \mathcal{E}_\varepsilon (v), \quad \forall v \in U,
\]

and we conclude that

\[
\mathcal{E}_0 (u_0) \leq \liminf_{\varepsilon \to 0^+} \varepsilon^{-1} \mathcal{E}_\varepsilon (u_\varepsilon) \leq \limsup_{\varepsilon \to 0^+} \varepsilon^{-1} \mathcal{E}_\varepsilon (u_\varepsilon) \leq \mathcal{E}_0 (v_0^\varepsilon) \quad (33).
\]

for every \( A \in \mathcal{A} \). Arguing as for (30) and (31), we obtain

\[
\mathcal{E}_0 (u_0) = 2c_0 P_\Omega (E_0) + \hat{\sigma} (\alpha) \mathcal{H}_{n-1} (\partial^* E_0 \cap \partial \Omega) + \hat{\sigma} (\beta) \mathcal{H}_{n-1} (\partial \Omega \setminus \partial^* E_0) \quad (34)
\]

and

\[
\mathcal{E}_0 (v_0^\varepsilon) = 2c_0 P_\Omega (A) + \hat{\sigma} (\alpha) \mathcal{H}_{n-1} (\partial \Omega \cap A) + \hat{\sigma} (\beta) \mathcal{H}_{n-1} (\partial \Omega \setminus A),
\]

so that

\[
P_\Omega (E_0) + \gamma \mathcal{H}_{n-1} (\partial^* E_0 \cap \partial \Omega) \leq P_\Omega (A) + \gamma \mathcal{H}_{n-1} (\partial (A \cap \Omega) \cap \partial \Omega)
\]

for every \( A \in \mathcal{A} \). Then the required minimality property (ii) of \( E_0 \) follows from Proposition 1.5. Finally, by employing again (33) and Proposition 1.5, with

\[
\lambda = \limsup_{\varepsilon \to 0^+} \varepsilon^{-1} \mathcal{E}_\varepsilon (u_\varepsilon),
\]

we have that

\[
\mathcal{E}_0 (u_0) = \lim_{\varepsilon \to 0^+} \varepsilon^{-1} \mathcal{E}_\varepsilon (u_\varepsilon);
\]

hence the result (iii) follows from (34) and this concludes the proof of Theorem 2.1. ■

2.5. Remarks. — (a) The assumption that \( \partial \Omega \) is smooth in Theorem 2.1 cannot be easily replaced by \( \partial \Omega \) Lipschitz continuous, except for \( \sigma = 0 \) (cf. [10]). In fact, as we already observed in Remark 1.3, the liquid-drop problem (P0) in bounded domains with angles requires a particular treatment.

(b) Well-known growth conditions at infinity on \( W \) guarantee that the minimizers \( u_\varepsilon \) are of class \( C^1 \). Of course, if \( u_\varepsilon \in L^\infty (\Omega) \), then \( u_\varepsilon \) is smooth.

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(c) The (relative) compactness of \((u_e)\) in \(L^1(\Omega)\) may be studied as in Proposition 4 of [10]. It is ensured either by equiboundedness of \((u_e)\) (cf. [9]), or again by a growth condition at infinity on \(W\).

### 3. A DISCUSSION

**ABOUT CRITICAL POINT WETTING**

We make here more precise some statements of Introduction, about the connection between Theorem 2.1 and the critical point wetting theory by J. W. Cahn [2].

According to this author, and looking in particular at page 3668 and Figure 4 of [2], we assume that the contact energy \(\sigma\) is a non-negative, convex, decreasing function of class \(C^1\). Moreover we denote by \(W_T\) the Gibbs free energy at the temperature \(T\) (recall that we are concerned with isothermal phenomena), by \(\alpha_T\) and \(\beta_T\) the corresponding zeros, by \(M_T\) the maximum height of the hump between \(\alpha_T\) and \(\beta_T\). We assume that \(W_T(t)\) increases for \(t \geq \beta_T\). By thermodynamic and experimental reasons (cf. [2], page 3669), we assume also that \(\beta_T\) and \(M_T\) are decreasing in \(T\), \(\alpha_T\) is increasing in \(T\) and \((\beta_T - \alpha_T) \to 0\), \(M_T \to 0\) when \(T\) increases towards a critical temperature \(T_0\) (critical point of a binary system). The \(\varphi\) and \(\hat{\sigma}\) corresponding to \(\sigma\) and \(W_T\) will be denoted by \(\varphi_T\) and \(\hat{\sigma}_T\).

Let us compute now \(\hat{\sigma}_T(t)\) for \(t \geq \alpha_T\). Since \(\sigma\) is decreasing and

\[
\lim_{t \to +\infty} \varphi_T(t) = +\infty,
\]

we obtain that the minimum of \(s \mapsto \sigma(s) + 2|\varphi_T(t) - \varphi_T(s)|\) is attained at a point \(s = \lambda_{t, T} \geq t\). Moreover, either \(\lambda_{t, T} = t\), or

\[
-\sigma'(\lambda_{t, T}) = 2\varphi'(\lambda_{t, T}) = 2W^{1/2}(\lambda_{t, T}).
\]

For \(T_0 - T\) small enough, that is for a temperature \(T\) below and close to the critical one, the hump in the graph of \(2W_T^{1/2}\) between \(\alpha_T\) and \(\beta_T\) does not intersect the graph of \(-\sigma'\) in the same interval; on the other hand, since \(\sigma\) is convex, the decreasing function \(-\sigma'\) does intersect the increasing function \(2W_T^{1/2}\) at a single point \(\lambda_T \geq \beta_T\) (see Fig. 2).
It is easy to check that $\lambda_T$ (independent of $t$) is actually the minimum point of $s \mapsto \sigma(s) + 2|\varphi_T(t) - \varphi_T(s)|$; hence we conclude that

$$\hat{\sigma}_T(t) = \sigma(\lambda_T) + 2(\varphi_T(\lambda_T) - \varphi_T(t)), \quad \forall t \geq \alpha_T,$$

hence

$$\gamma_T = \frac{\hat{\sigma}_T(\alpha_T) - \hat{\sigma}_T(\beta_T)}{2(\varphi_T(\beta_T) - \varphi_T(\alpha_T))} = 1$$

in correspondence with the phenomenon of the perfectly wetting phase $\beta$ quoted in Introduction. If one prefers not to consider the modified energy $\hat{\sigma}_T$, it could be alternatively thought that a very thin layer of a third phase of the fluid, with density $\lambda_T > \beta_T$, appears on the whole boundary of the container.

When the temperature $T$ is much more below $T_0$, a possible relative behavior of $-\sigma'$ and $2W^{1/2}$ is shown in Figure 3, with both $\mu_T$ and $\lambda_T$ relative minima of

$$s \mapsto \sigma(s) + 2|\varphi_T(t) - \varphi_T(s)|$$

for every $t \geq \alpha_T$. 
Note that

\[ \hat{\sigma}_T(\beta_T) = \sigma(\lambda_T) + 2(\varphi_T(\lambda_T) - \varphi_T(\beta_T)), \]

while the value of \( \sigma_T(\alpha_T) \) depends on the areas A and B. Indeed, if \( A \leq B \), then

\[ \hat{\sigma}_T(\alpha_T) = \sigma(\lambda_T) + 2(\varphi_T(\lambda_T) - \varphi_T(\alpha_T)) \]

and \( \gamma_T = 1 \) as above. On the contrary, if \( A > B \), then

\[ \hat{\sigma}_T(\alpha_T) = \sigma(\mu_T) + 2(\varphi_T(\mu_T) - \varphi_T(\alpha_T)) < \sigma(\lambda_T) + 2(\varphi_T(\lambda_T) - \varphi_T(\alpha_T)) \]

and \( \gamma_T < 1 \); since we have analogously \( \gamma_T > -1 \), this means that both the fluid phases wet the container walls. Or, alternatively, two thin layers of fluid, with densities \( \mu_T \) and \( \lambda_T \), are interposed between the phases \( \alpha_T \) and \( \beta_T \) and the container.

Finally, we want to remark that the equation \( \hat{\sigma} = \sigma \) is equivalent to the inequality

\[ |\sigma(s_1) - \sigma(s_2)| \leq 2|\varphi(s_1) - \varphi(s_2)|, \quad \forall 0 \leq s_1 \leq s_2, \quad (35) \]

which gives in particular

\[ \sigma'(\alpha) \geq \varphi'(\alpha) = W^{1/2}(\alpha) = 0 \]
and analogously $\sigma'(\beta) \geq 0$; hence (35) cannot be satisfied in the case $\sigma' < 0$. It would be interesting to know whether the inequality (35), and then the equality $\sigma = \hat{\sigma}$, are verified in some other thermodynamic situation, different from the phenomenon studied in [2] by Cahn.

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