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An isoperimetric inequality in $\mathbb{R}^3$

by

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ABSTRACT. — In this paper we give an inequality of isoperimetric type in $\mathbb{R}^3$, which is a natural extension of the better known isoperimetric inequalities in $\mathbb{R}^2$. It shows the relationship between the "degree" of a function from $\mathbb{R}^3$ to S^3 and a certain functional which is relevant in the study of mesons in Nuclear Physics.

Key words : Isoperimetric inequality, lebesgue measure, minimization problems, Schwarz inequality.

RÉSUMÉ. — Dans cet article on démontre une inégalité isopérimétrique dans $\mathbb{R}^3$, qui généralise de façon naturelle une inégalité isopérimétrique classique dans $\mathbb{R}^2$. Cette inégalité met en évidence la relation existant entre le « degré » des fonctions définies dans $\mathbb{R}^3$ et à valeurs dans S^3 et une certaine fonctionnelle liée à l'étude des mésons en Physique Nucléaire.

I. INTRODUCTION

Let S be any closed hypersurface in $\mathbb{R}^4$ and T_s, the open domain of $\mathbb{R}^4$ whose boundary is S. Then there exists an isoperimetric inequality connecting the 3-dimensional Hausdorff area of S, area (S), with the 4-dimensional

Lebesgue measure of T_s, vol (T_s). Namely:

$$(\mathrm{vol}(T_s))^{3/4} \leqq C_1 \text{ area } (S), \tag{1}$$

where $C_1 = 2^{-7/4}\pi^{-1/2}$. Moreover the equality holds in (1) if and only if S is a sphere.

One says that a surface S is parametric if there exists a function $\Phi: \mathbb{R}^3 \to \mathbb{R}^4$ with $\Phi(\mathbb{R}^3) = S$. If such a function Φ is injective and of class C^1, inequality (1) can be written as:

$$\left| \int_{\mathbb{R}^3} \det(\Phi, \nabla \Phi)\, dx \right|^{3/4} \leqq C_2 \int_{\mathbb{R}^3} \left| \frac{\partial \Phi}{\partial x_1} \wedge \frac{\partial \Phi}{\partial x_2} \wedge \frac{\partial \Phi}{\partial x_3} \right| dx \tag{2}$$

where $C_2 = 2^{-1/4}\pi^{-1/2}$.

In this paper we prove that (2) still holds for a much wider class of functions Φ: let us define the sets

$$Y = \{ \Phi: \mathbb{R}^3 \to \mathbb{R}^4 / \nabla \Phi,\ A(\Phi) \in L^2(\mathbb{R}^3, \mathbb{R}^4) \},$$

where

$$|A(\Phi)|^2 = \sum_{\alpha, \beta = 1}^{3} \left| \frac{\partial \Phi}{\partial x_\alpha} \wedge \frac{\partial \Phi}{\partial x_\beta} \right|^2 = \sum_{\substack{\alpha, \beta = 1, \ldots, 3 \\ i, j = 1, \ldots, 4}} \left(\frac{\partial \Phi^i}{\partial x_\alpha} \frac{\partial \Phi^j}{\partial x_\beta} - \frac{\partial \Phi^i}{\partial x_\beta} \frac{\partial \Phi^j}{\partial x_\alpha} \right)^2,$$

and

$$E = \{ \Phi \in Y / \Phi \text{ satisfies the approximation-property (P)} \},$$

with

$$\text{(P)} \quad \begin{cases} \exists \{\Phi_n\} \subset C^1(\mathbb{R}^3, \mathbb{R}^4) \cap Y \text{ s.t. } \nabla \Phi_n \underset{n \to +\infty}{\longrightarrow} \nabla \Phi, \\ \qquad A(\Phi_n) \underset{n \to +\infty}{\longrightarrow} A(\Phi) \quad \text{in} \quad L^2(\mathbb{R}^3, \mathbb{R}^4). \end{cases}$$

Then, (2) holds for all Φ in E.

Furthermore we will deduce from it the existence of a positive constant C such that for all Φ in E we have:

$$\left| \int_{\mathbb{R}^3} \det(\Phi, \nabla \Phi)\, dx \right|^{3/4} \leqq C \int_{\mathbb{R}^3} (|\nabla \Phi|^2 + |A(\Phi)|^2)\, dx. \tag{4}$$

The above inequality presents an increasing relevancy as a fundamental tool to treat a family of minimization problems to be studied in [E1] and [E2]. In these forthcoming papers we study the existence of minima for the following energy functional:

$$\mathscr{E}(\Phi)=\int_{\mathbb{R}^3}(|\nabla\Phi|^2+|A(\Phi)|^2\,dx \tag{5}$$

in the classes of functions Φ which have degree $k\in\mathbb{Z}$. This minimization problems are connected with the study of stable configurations of meson fields. (For the physical discussion of the problem, *see* [Sk].)

Furthermore, inequalities (2) and (4) may be interesting for other problems, and so we will not only prove them, but also make some complementary remarks and give an estimate for the value of the best constant C in (4). In [E2] this best constant is shown to be achieved in E, but we do not know its exact value. We will then give upper and lower bounds for it.

Let us finally remark that inequalities (2) and (4) are of isoperimetric type. They are the counterpart in $\mathbb{R}^3$ of the classical isoperimetric inequalities in $\mathbb{R}$ and $\mathbb{R}^2$ (*see* [W]). This kind of inequality has very important applications in the calculus of variations. Lately it has been crucial in the study of the well-known Plateau's problem and also in the proof of the existence of harmonic maps with given boundary value in open domains of $\mathbb{R}^2$ (*see* [H], [ST], [W], [BC1] and [BC2]).

Notation and some remarks

For any three vectors of $\mathbb{R}^4$, a, b, c, we denote by $a\wedge b$ (resp. $a\wedge b\wedge c$) the alternating exterior product of a, b (resp. a, b, c) which is an element of $\Lambda^2(\mathbb{R}^4)$ [resp. $\Lambda^3(\mathbb{R}^4)\simeq\mathbb{R}^4$].

Moreover, the following notations will be used:

$$S_R=\{x\in\mathbb{R}^3\,|\,|x|=R\}$$
$$B_R=\{x\in\mathbb{R}^3\,|\,|x|<R\}$$
$$S^N=\{x\in\mathbb{R}^{N+1}\,|\,|x|=1\}$$
$$B^N=\{x\in\mathbb{R}^{N+1}\,|\,|x|<1\}.$$

Finally, for any $A\subset\mathbb{R}^N$, we denote by $|A|$ the N-dimensional Lebesgue measure of A.

II. MAIN RESULTS

PROPOSITION 1. — *Inequality* (2) *holds for any function* Φ *in* E.

COROLLARY 2. — *There exists a positive constant* C *such that for any* Φ *in* E, *we have:*

$$\left|\int_{\mathbb{R}^3} \det(\Phi, \nabla \Phi)\, dx\right|^{3/4} \leqq C\,\mathscr{E}(\Phi). \tag{4}$$

Moreover, C *lies in the interval* $[3^{-1}\,2^{-7/4}\,\pi^{-1/2},\ 3^{-1}\,2^{-5/4}\,\pi^{-1/2})$.

Remark 3. — The estimate given in Corollary 2 for C is not very sharp. Moreover it is good enough for the study of the minimization problems treated in [E1] and [E2]. In lemma 8 below we will see (without proof) more explicitely what can be C.

DEFINITION 4. — Given Φ in E and $S_\Phi = \Phi(\mathbb{R}^3)$, we define the generalized area of S_Φ as the following:

$$A(S_\Phi) = \int_{\mathbb{R}^3} \left| \frac{\partial \Phi}{\partial x_1} \wedge \frac{\partial \Phi}{\partial x_2} \wedge \frac{\partial \Phi}{\partial x_3} \right| dx. \tag{8}$$

We note that $A(S_\Phi)$ actually is the Hausdorff measure of Φ and that it coincides with area (S_Φ) when Φ is a bijection from $\mathbb{R}^3$ into S_Φ.

Let us now consider Φ in $E \cap C^0(\mathbb{R}^3\, \mathbb{R}^4)$, S_Φ and T_Φ (the open set of $\mathbb{R}^4$ whose boundary is S_Φ). This is possible since, as we prove in the next lemma, all the functions Φ in E define a closed surface in $\mathbb{R}^4$, i. e., S_Φ is closed, and so T_Φ is well defined.

LEMMA 5. — *Let* Φ *be a function in* E. *Then there exist* $e \in \mathbb{R}^4$ *such that* $\int_{\mathbb{R}^3} |\Phi(x) - e|\, dx < +\infty$. *That is,* Φ *is equal to e at infinity, in a weak sense.*

Proof. — For all $r \in \mathbb{R}^+$ we define $\bar{\Phi}(r) = \dfrac{1}{|S_r|} \displaystyle\int_{S_r} \Phi(x)\, ds$, which is called the spherical mean of Φ. We consider now $0 < r < r'$ and see that:

$$|\bar{\Phi}(r') - \bar{\Phi}(r)| \leqq \int_r^{r'} |\bar{\Phi}'(s)|\, ds \leqq \left(\int_r^{r'} (\bar{\Phi}'(s))^2 s^2\, ds \right)^{1/2} \left(\int_r^{r'} \frac{ds}{s^2} \right)^{1/2}$$

$$\leqq C \left(\int_{B_{r'} - B_r} |\nabla \Phi|^2\, dx \right)^{1/2} \left(\frac{1}{r} - \frac{1}{r'} \right)^{1/2}.$$

Hence, since $\nabla\Phi$ is in $L^2(\mathbb{R}^3)$, we infer from the above inequalities that $\bar{\Phi}$ is continuous in $(0, +\infty)$ and moreover, there exists $\lim_{r \to +\infty} \bar{\Phi}(r) = e$.

Now, since we know that $\|\Phi - \bar{\Phi}\|_{L^6(\mathbb{R}^3)} \leqq C \|\nabla\Phi\|_{L^2(\mathbb{R}^3)} < +\infty$, it follows that:

$$\|\Phi - e\|_{L^6(\mathbb{R}^3)} < +\infty,$$

and it completes the proof. □

DEFINITION 6. — For any Φ in E, we define two generalized volumes of T_Φ as follows:

$$V_1(\Phi) = \frac{1}{4}\int_{\mathbb{R}^3} \det(\Phi, \nabla\Phi)\, dx \tag{9}$$

$$V_2(\Phi) = \frac{1}{4}\int_{\mathbb{R}^3} |\det(\Phi, \nabla\Phi)|\, dx. \tag{10}$$

These two "volumes" of T_Φ may coincide with $\mathrm{vol}(T_\Phi)$, but give much more information about the way $\Phi(\mathbb{R}^3)$ covers S_Φ than the latter does.

We prove next the main inequality obtained for the elements of E, namely proposition 1.

Proof of Proposition 1. — First we observe that it is enough to consider Φ in $E \cap C^1(\mathbb{R}^3, \mathbb{R}^4)$. Then we consider the two following cases:

(i) Φ is constant near infinity. In this case (1) is simply the standard isoperimetric inequality (*see* [F]) since S_Φ is then 3-rectifiable.

We note next that the définition of $V_1(\Phi)$ takes into account not only the points of $\mathbb{R}^4$ that are in $\Phi(\mathbb{R}^3)$, but the orientation of Φ at those points as well. Then, it is clear that for any Φ in E, there exists $k \in \mathbb{Z}$ such that:

$$V_1(\Phi) = k\, \mathrm{vol}(T_\Phi),$$

and $k = 0$ if and only if Φ is homotopic to a constant function.

Obviously we also have that:

$$A(\Phi) \geqq k\, \mathrm{aire}(S_\Phi)$$

and thus, since $k \in \mathbb{Z}$ we have:

$$|V_1(\Phi)|^{3/4} = |k|^{3/4} (\mathrm{vol}(T_\Phi))^{3/4} \leqq k\, (\mathrm{vol}(T_\Phi))^{3/4}$$
$$\leqq k\, C_1\, \mathrm{area}(S_\Phi) \leqq C_1\, A(\Phi).$$

(ii) Φ is not constant in a neighborhood of infinity. Then we consider a function m in $C^\infty(\mathbb{R}^3, \mathbb{R})$ satisfying:

$$0 \leqq m(x) \leqq 1 \quad \text{for all } x, \qquad m \equiv 1 \quad \text{in } B_1, \qquad m \equiv 0 \quad \text{in } B_2^c,$$

and we define Φ_n as Φm_n where $m_n(.) = m\left(\frac{\cdot}{n}\right)$.

We can apply (i) to Φ_n for all n and pass to the limit to obtain that (1) is satisfied for all Φ in $E \cap C^1(\mathbb{R}^3, \mathbb{R}^4)$, since the $L^3(\mathbb{R}^3, \mathbb{R}^4)$-norm of ∇m_n is independent of n and finite and $E \subset L^6(\mathbb{R}^3, \mathbb{R}^4)$. $\square$

Proof of corollary 2. — Let (i, j, k) any permutation of (1,2,3). Then we have:

$$A(\Phi) \leqq \int_{\mathbb{R}^3} \left|\frac{\partial \Phi}{\partial x_i}\right| \cdot \left|\frac{\partial \Phi}{\partial x_j} \wedge \frac{\partial \Phi}{\partial x_k}\right| dx$$

$$\leqq \left(\int_{\mathbb{R}^3} \left|\frac{\partial \Phi}{\partial x_i}\right|^2 dx\right)^{1/2} \left(\int_{\mathbb{R}^3} \left|\frac{\partial \Phi}{\partial x_j} \wedge \frac{\partial \Phi}{\partial x_k}\right|^2 dx\right)^{1/2},$$

where we have used Schwarz's inequality. Hence, we obtain:

$$3A(\Phi) \leqq \|\nabla \Phi\|_{L^2(\mathbb{R}^3)} \|A(\Phi)\|_{L^2(\mathbb{R}^3)} \tag{11}$$

and this together with inequality (2) proves that (4) holds with $C \leqq 3^{-1} 2^{-5/4} \pi^{-1/2}$. Moreover this inequality is strict, because for any Φ in E we cannot have the equality when applying Schwarz's inequality to $A(\Phi)$.

Finally we prove a lower bound for the best constant C.

Let f be the inverse of the stereographic projection which takes the north pole of S^3 to $+\infty$, and the south pole to 0. f is the following function:

$$f : (x_1, x_2, x_3) \in \mathbb{R}^3 \rightsquigarrow \left(\frac{2x_1}{r^2+1}, \frac{2x_2}{r^2+1}, \frac{2x_3}{r^2+1}, 1 - \frac{2}{r^2+1}\right), \tag{12}$$

where $r^2 = \sum_{i=1}^{3} x_i^2$.

For this particular element of E we find:

$$I = \int_{\mathbb{R}^3} |\nabla f|^2 dx = 12\pi^2,$$

$$J = \int_{\mathbb{R}^3} |A(f)|^2 dx = 6\pi^2 \qquad \text{and} \qquad V_1(f) = 1.$$

Then

$$\mathscr{E}\left(f\left(\frac{\cdot}{\sigma_0}\right)\right)=\frac{1}{2}\inf_{\sigma>0}\mathscr{E}\left(f\left(\frac{\cdot}{\sigma}\right)\right)=(\mathrm{IJ})^{1/2}=6\sqrt{2}\,\pi^2,$$

with $\sigma_0=\left(\frac{\mathrm{I}}{\mathrm{J}}\right)^{1/2}$.

This proves the lower bound for C we sought. □

Remark 7. — Let h be the stereographic projection used above, i. e., $h=f^{-1}$. Then for any Φ in E with $\Phi(\mathbb{R}^3)\subset \mathrm{S}^3$ we may consider the degree of $\Phi\circ h$, which we denote by $d(\Phi)$. We have then:

$$d(\Phi)=\frac{4\,\mathrm{V}_1(\Phi)}{\operatorname{area}(\mathrm{S}^3)}=\frac{1}{2\pi^2}\int_{\mathbb{R}^3}\det(\Phi,\nabla\Phi)\,dx. \tag{13}$$

From the above results we deduce that:

$$\mathscr{E}(\Phi)\geqq 12\,\pi^2\,|d(\Phi)|^{3/4} \tag{14}$$

and for $f=h^{-1}$ we have:

$$d\left(f\left(\frac{\cdot}{\sigma_0}\right)\right)=1 \qquad \text{and} \qquad \mathscr{E}\left(f\left(\frac{\cdot}{\sigma_0}\right)\right)=12\sqrt{2}\,\pi^2.$$

As we said before, these estimates on C are not too precise, but they are good enough for the study of the minimization problems treated in [E2]. In any case in [E2] we prove that C is achieved. More explicitely, in [E2] we prove the following:

LEMMA 8. — *Let* I_k *be equal to the* $\inf\{\mathscr{E}(\Phi)\,|\,d(\Phi)=k\}$. *Then* C *is equal to* I_1, $\mathrm{I}_2\,2^{-3/4}$ *or* $\mathrm{I}_3\,3^{-3/4}$, *and is achieved by a fonction* Φ *of degree* k *if* $\mathrm{C}=\mathrm{I}_k\,k^{-3/4}$. *Moreover, for all* k *in* $\mathbb{Z}$ *we have:*

$$6\,|k|\,.\,|\mathrm{S}^3|<\mathrm{I}_k\leqq 6\sqrt{2}\,|k|\,.\,|\mathrm{S}^3|.$$

Let us now prove a result which concerns the "positive" volume of $\Phi\in\mathrm{E}$ we defined in (10). It will be very useful to us in [E2]:

LEMMA 9. — *Let be* Φ *any element of* E *such that* $\Phi(\mathbb{R}^3)\subset\mathrm{S}^3$ *and* $d(\Phi)\neq 0$. *Then we have:*

$$\left(\int_{\mathbb{R}^3}|\det(\Phi,\nabla\Phi)|\,dx\right)^{3/4}\leqq \mathrm{C}_2\,\mathrm{A}(\Phi), \tag{15}$$

where as in (2), $C_2 = 2^{-1/4}\pi^{-1/2}$.

Proof. — We observe that if $\Phi \in E$ and $\Phi(\mathbb{R}^3) \subset S^3$, then we have:

$$\frac{A(S_\Phi)}{\operatorname{area}(S_\Phi)} = \frac{V_2(\Phi)}{\operatorname{vol}(T_\Phi)} = h \in \mathbb{R}^+.$$

Furthermore, $d(\Phi) \neq 0$ implies that $h \geqq 1$, and then we find:

$$V_2(\Phi)^{3/4} = h^{3/4} \operatorname{vol}(T_\Phi) \leqq h\, C_2 \operatorname{area}(S_\Phi) = C_2\, A(\Phi)$$

and (15) follows immediatly. □

Finally we want to finish this paper with some remarks about isoperimetric inequalities in any dimension. All the results we proved above are only concerned with the dimension $N=3$, because we need them for the applications. Nevertheless, such results could be obtained for any dimension N.

The equivalent of (2) in dimension N would be:

$$\left| \int_{\mathbb{R}^N} \det(\Phi, \nabla\Phi)\, dx \right|^{N/N+1} \leqq C_{N-1} \int_{\mathbb{R}^N} \left| \frac{\partial\Phi}{\partial x_1} \wedge \ldots \wedge \frac{\partial\Phi}{\partial x_N} \right| dx, \qquad (16)$$

where $C_{N-1} = (N+1)^{3/4} |B^{N+1}|^{3/4} |S^N|^{-1}$. In every case the space where Φ must be in order that the volume and area-functionals in (16) are well defined must be carefully chosen.

Moreover, a large class of inequalifities equivalent to (4) would be obtained by applying Schwarz's inequality to the integral $\int_{\mathbb{R}^N} \left| \frac{\partial\Phi}{\partial x_1} \wedge \ldots \wedge \frac{\partial\Phi}{\partial x_N} \right| dx$ in all the possible different ways.

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