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Analyse non linéaire

Hopf bifurcation for fully nonlinear equations in Banach space

by

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ABSTRACT. — We study Hopf bifurcation for some fully nonlinear evolution equations in Banach spaces.

Key-words : Nonlinear evolution equations, Hopf bifurcation.

RÉSUMÉ. — Nous étudions la bifurcation de Hopf pour des équations totalement non linéaires dans les espaces de Banach.

INTRODUCTION

Hopf bifurcation has been widely studied in the last years; see for instance the monographs [*1*] [*5*] [*6*] [*11*] [*13*]. The infinite dimensional case has

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been studied in [1] [2] [4] [5] [7] [8] [9] [12] for semilinear equations. In this paper we study periodic solutions of the problem

$$u'(t) = f(\lambda, u(t)) \tag{0.1}$$

where $f :]-1, 1[\times D \to X$, $(\lambda, x) \to f(\lambda, x)$ is a smooth function, D and X are Banach spaces with $D \subset X$, the operator $A = f_x(0, 0)$ generates an analytic semigroup in X and satisfies the usual spectral properties (see (1.7), (1.8) later). Then, as an application, we may study quasilinear or fully nonlinear equations, such as

$$u_t(t, x) = \varphi(\lambda, u(t, x), u_x(t, x), u_{xx}(t, x)) \tag{0.2}$$

Our main tool is a maximal regularity property for the linear problem

$$\begin{cases} v'(t) = Av(t) + f(t); & 0 \leqslant t \leqslant 2\pi \\ v(0) = v(2\pi) \end{cases} \tag{0.3}$$

We prove that if f is γ-Hölder continuous and periodic and v is any solution of (0.3), then v' and Av are γ-Hölder continuous (see section 1).

Using this result we may treat problem (0.1) by means of classical arguments (see section 2). Our proof follows closely the one of Crandall and Rabinowitz [2], the main difference being the use of maximal regularity which enables us to get strict instead of mild solutions.

1. THE LINEAR CASE

Let X, D be two Banach spaces with D continuously embedded in X. We denote by $\tilde{X}$ (resp. $\tilde{D}$) the complexification of X (resp. D): $\tilde{X} = \{ x + iy ; x, y \in X \}$; $\tilde{D} = \{ x + iy ; x, y \in D \}$.

If $A \in L(D, X)$, set $\tilde{A} : \tilde{D} \to \tilde{X}$, $\tilde{A}(x + iy) = Ax + iAy$.

We assume that:

(1.1) $\begin{cases} \text{the resolvent set } \rho(\tilde{A}) \text{ of } \tilde{A} \text{ contains a sector} \\ S = \{ \xi \in \mathbb{C} ; \xi \neq \omega, |\arg(\xi - \omega)| < \theta \} \text{ with} \\ \omega \in \mathbb{R} \text{ and } \theta \in]\pi/2, \pi] ; \text{ there exists } M > 0 \\ \text{such that} \\ \| (\xi - A)^{-1} \|_{L(X)} \leqslant M / | \xi - \omega | \quad \text{for} \quad \xi \in S \end{cases}$

From (1.1) it follows that $\tilde{A}$ generates an analytic semigroup $e^{t\tilde{A}}$ (not necessarily strongly continuous at 0), defined by means of the usual Dunford integral

$$e^{t\tilde{A}} = \frac{1}{2\pi i} \int_C e^{\xi t} (\xi - \tilde{A})^{-1} d\xi, \quad t > 0$$

where C is a suitable path joining $\infty e^{-i\theta}$ and $\infty e^{i\theta}$. It may be shown that $e^{t\tilde{A}}(X) \subset X$, so that the restriction of $e^{t\tilde{A}}$ to X is an analitic semigroup e^{tA} in X. Moreover, denoting by $\overline{D}$ the closure of D in X, e^{tA} is strongly continuous in $\overline{D}$ and $e^{tA}(X) \subset D$ (see [*14*]).

For $0 < \gamma < 1$, the interpolation space $D_A(\gamma, \infty)$ is given by

$$D_A(\gamma, \infty) = \{ x \in X ; [x]_\gamma = \sup_{0<t\leqslant 1} t^{-\gamma} \| e^{tA}x - x \| < +\infty \}$$

and it is endowed with the norm

$$\| x \|_\gamma = \| x \| + [x]_\gamma$$

It is easy to see that, in the case $\omega = 0$, our definition of $D_A(\gamma, \infty)$ is equivalent to the one given in [*10*] and [*14*].

We shall consider the linear problem

$$(1.2) \qquad \begin{cases} u'(t) = Au(t) + f(t) ; & 0 \leqslant t \leqslant 2\pi \\ u(0) = u(2\pi) \end{cases}$$

with $f \in C^\gamma_\#(X)$ $(0 < \gamma < 1)$, where $C^\gamma_\#(X)$ denotes the space of all γ-Hölder continuous 2π-periodic functions $\varphi : \mathbb{R} \to X$, endowed with the usual norm

$$\| \varphi \|_{C^\gamma_\#(X)} = \sup_{0\leqslant t\leqslant 2\pi} \| \varphi(t) \| + \sup_{0\leqslant s<t\leqslant 2\pi} \frac{\| \varphi(t) - \varphi(s) \|}{(t-s)^\gamma}$$

We shall look for solutions of (1.2) belonging to $C^{1,\gamma}_\#(X) \cap C^\gamma_\#(D)$; $C^{1,\gamma}_\#(X)$ is the space of the differentiable functions $\varphi : \mathbb{R} \to X$ such that φ and φ' belong to $C^\gamma_\#(X)$. $C^{1,\gamma}_\#(X)$ is endowed with the norm

$$\| \varphi \|_{C^{1,\gamma}(X)} = \sup_{0\leqslant t\leqslant 2\pi} \| \varphi(t) \| + \| \varphi' \|_{C^\gamma_\#(X)}$$

We shall use the following inclusion property, whose proof will be given in the appendix.

LEMMA 1.1. — *Let* $0<\gamma<1$, $a<b$ *and let* $v \in C^\gamma([a,b]; D) \cap C^{1,\gamma}([a,b]; X)$. *Then* $v'(t) \in D_A(\gamma, \infty)$ *for each* $t \in [a,b]$, *and there exists* $K > 0$ *such that*

$$(1.3) \quad \| v'(t) \|_\gamma \leqslant K(\| v \|_{C^\gamma([a,b];D)} + \| v \|_{C^{1,\gamma}([a,b];X)}), \quad a \leqslant t \leqslant b \qquad \blacksquare$$

Now we are able to study problem (1.2). First we consider the non resonance case $1 \in \rho(e^{2\pi A})$.

THEOREM 1.2. — *Assume (1.1) and let* 1 *belong to the resolvent set of* $e^{2\pi A}$. *Then for each* $f \in C^\gamma_\#(X)$ *problem (1.2) has a unique solution u given by*

$$(1.4) \quad u(t) = e^{tA}(1 - e^{2\pi A})^{-1} \int_0^{2\pi} e^{(2\pi - s)A} f(s) ds + \int_0^t e^{(t-s)A} f(s) ds, \quad 0 \leqslant t \leqslant 2\pi$$

Moreover, $u \in C^{\gamma}_{\#}(D) \cap C^{1,\gamma}_{\#}(X)$ *and there exists* $H > 0$ *such that*

$$\| u \|_{C^{\gamma}_{\#}(D)} + \| u \|_{C^{1,\gamma}_{\#}(X)} \leqslant H \| f \|_{C^{\gamma}_{\#}(X)} \tag{1.5}$$

In other words, the mapping

$$C^{\gamma}_{\#}(D) \cap C^{1,\gamma}_{\#}(X) \to C^{\gamma}_{\#}(X) ; \quad u \to u' - Au \tag{1.6}$$

is an isomorphism.

Proof. — For $x \in \overline{D}$, consider the initial value problem

$$\begin{cases} u'(t) = Au(t) + f(t) ; & 0 < t \leqslant 2\pi \\ u(0) = x \end{cases}$$

whose unique solution is

$$u(t) = e^{tA}x + \int_0^t e^{(t-s)A} f(s) ds ; \quad 0 \leqslant t \leqslant 2\pi$$

u is 2π-periodic if and only if

$$x = (1 - e^{2\pi A})^{-1} \int_0^{2\pi} e^{(2\pi - s)A} f(s) ds$$

In this case u is given by (1.4), and $u(0) = x \in D$. To prove the regularity properties of u, we set $u = u_1 + u_2$, where u_1 and u_2 are respectively the periodic solutions of the equations

$$u_1'(t) = Au_1(t) + f(0) ; \quad 0 \leqslant t \leqslant 2\pi$$
$$u_2'(t) = Au_2(t) + f(t) - f(0) ; \quad 0 \leqslant t \leqslant 2\pi$$

By (1.4) we get

$$u_1(t) = e^{tA}(1 - e^{2\pi A})^{-1} \int_0^{2\pi} e^{(2\pi - s)A} f(0) ds + \int_0^t e^{(t-s)A} f(0) ds$$

thus

$$Au_1(t) = - f(0) ; \quad 0 \leqslant t \leqslant 2\pi$$

In particular, by the arbitrariness of $f(0)$ in X, we find that $0 \in \rho(A)$ and $u_1(t) = - A^{-1} f(0)$. Since (1.1) implies that the graph norm of A is equivalent to the norm of D, we get $u_1 \in C^{\gamma}_{\#}(D) \cap C^{1,\gamma}_{\#}(X)$.

Using once again (1.4), we obtain

$$u_2(t) = e^{tA}(1 - e^{2\pi A})^{-1} \varphi(2\pi) + \varphi(t)$$

where

$$\varphi(t) = \int_0^t e^{(t-s)A}(f(s) - f(0)) ds ; \quad 0 \leqslant t \leqslant 2\pi$$

is the solution of the problem

$$\begin{cases} \varphi'(t) = A\varphi(t) + f(t) - f(0) ; & 0 \leqslant t \leqslant 2\pi \\ \varphi(0) = 0 \end{cases}$$

and belongs to $C^{\gamma}_{\#}(D) \cap C^{1,\gamma}_{\#}(X)$. Moreover there exists $H_1 > 0$ such that

$$\| \varphi \|_{C^{\gamma}([0,2\pi];D)} + \| \varphi \|_{C^{1,\gamma}([0,2\pi];X)} \leqslant H_1 \| f \|_{C^{\gamma}([0,2\pi];X)}$$

(see [3] [14]).

From lemma 1.1, $\varphi'(2\pi) = A\varphi(2\pi)$ belongs to $D_A(\gamma, \infty)$, so that $A(1 - e^{2\pi A})^{-1}\varphi(2\pi)$ belongs to $D_A(\gamma, \infty)$. This implies

$$u_2 \in C^{\gamma}([0, 2\pi]; D) \cap C^{1,\gamma}([0, 2\pi]; X).$$

(1.5) follows now easily. ■

Now we consider a resonance case. We assume:

$$(1.7) \quad \begin{cases} a) \ i \text{ is a simple isolated eigenvalue of } \tilde{A} \\ b) \ 1 \text{ is a semi-simple isolated eigenvalue of } e^{2\pi\tilde{A}} \text{ with algebraic multiplicity 2.} \end{cases}$$

We remark that conditions (1.7) are satisfied if

$$(1.8) \quad \begin{cases} a) \ (\xi - \tilde{A})^{-1} \text{ is a compact operator for } \xi \in S \\ b) \ i \text{ is a simple eigenvalue of } \tilde{A} \\ c) \ ni \in \rho(\tilde{A}), \quad n = 0, 2, 3, \dots \end{cases}$$

Hypotheses (1.8) coincide with assumptions HL *iii*), *iv*), *v*) in [2]. By (1.7)-(*a*) there exist $x_0, y_0 \in D$ such that

$$(1.9) \qquad \tilde{A}(x_0 \pm iy_0) = \pm i(x_0 \pm iy_0)$$

that is

$$(1.10) \qquad Ax_0 = - y_0, \quad Ay_0 = x_0$$

Moreover

$$(1.11) \qquad e^{t\tilde{A}}(x_0 + iy_0) = e^{it}(x_0 + iy_0)$$

and then

$$(1.12) \qquad e^{tA}x_0 = x_0 \cos t - y_0 \sin t; \quad e^{tA}y_0 = x_0 \sin t + y_0 \cos t$$

Let X_0 be the subspace of X spanned by x_0 and y_0 and let $\tilde{X}_0 = \{ x + iy; x, y \in X_0 \}$. Then a projection on X_0 is given by

$$Q = \frac{1}{2\pi i}\left[\int_{\gamma(i,\varepsilon)} (\xi - A)^{-1}d\xi + \int_{\gamma(-i,\varepsilon)} (\xi - A)^{-1}d\xi\right]$$

where $\gamma(\pm i, \varepsilon)$ is the curve $\{ z \in \mathbb{C}; | z \mp i | = \varepsilon \}$, oriented counterclockwise, and ε is sufficiently small.

We have

$$Q = -2/\pi \int_0^{2\pi} (\cos\theta + \sin\theta \tilde{A})(\varepsilon\cos\theta + i(1+\varepsilon\sin\theta) - \tilde{A})^{-1}(\varepsilon\cos\theta - i(1+\varepsilon\sin\theta) - \tilde{A})^{-1} d\theta$$

so that, as $(\xi - \tilde{A})^{-1}(\bar{\xi} - \tilde{A})^{-1}(X) \subset X$ for $\xi, \bar{\xi} \in \rho(\tilde{A})$, we get $Q(X) \subset X$. Moreover, X_0 is the kernel of $(1 - e^{2\pi A})$, and, setting

$$X_1 = (1 - Q)(X), \quad A_1 x = Ax \quad \text{for} \quad x \in D \cap X_1$$

the restriction of e^{tA} to X_1 is given by

$$e^{tA_1} = \frac{1}{2\pi i}\int_C e^{\xi t}(\xi - \tilde{A}_1)^{-1} d\xi$$

There exist φ_0, η_0 belonging to the dual space X^* of X such that

$$(1.13) \quad \begin{cases} Qx = \langle x, \varphi_0 \rangle x_0 + \langle x, \eta_0 \rangle y_0 & \text{for each} \quad x \in X \\ \langle x_0, \varphi_0 \rangle = \langle y_0, \eta_0 \rangle = 1, & \langle x_0, \eta_0 \rangle = \langle y_0, \varphi_0 \rangle = 0 \end{cases}$$

and then

$$(1.14) \quad \begin{cases} (e^{tA})^*\varphi_0 = \varphi_0 \cos t + \eta_0 \sin t; & (e^{tA})^*\eta_0 = \varphi_0 \sin t - \eta_0 \cos t \\ A^*\varphi_0 = \eta_0, & A^*\eta_0 = -\varphi_0 \end{cases}$$

We are able now to state an existence result for problem (1.2).

THEOREM 1.3. — *Assume (1.1) and (1.7) and let $f \in C^{\gamma}_{\#}(X)$. Then problem (1.2) has a solution if and only if*

$$(1.15) \qquad Q\int_0^{2\pi} e^{(2\pi - s)A} f(s) ds = 0$$

In this case all solutions are given by

$$(1.16) \quad u(t) = c_1 e^{tA} x_0 + c_2 e^{tA} y_0 + e^{tA}(1 - e^{2\pi A_1})^{-1} \int_0^{2\pi} e^{(2\pi - s)A} f(s) ds + \int_0^t e^{(t-s)A} f(s) ds; \quad 0 \leqslant t \leqslant 2\pi$$

with $c_1, c_2 \in \mathbb{R}$. Moreover $u \in C^{\gamma}_{\#}(D) \cap C^{1,\gamma}_{\#}(X)$.

Proof. — Setting $u_0 = Qu$, $u_1 = (1 - Q)u$, $f_0 = Qf$, $f_1 = (1 - Q)f$, $A_0 = QA$, problem (1.2) splits into the following problems:

$$(1.17) \quad \begin{cases} u_0'(t) = A_0 u_0(t) + f_0(t); & 0 \leqslant t \leqslant 2\pi \\ u_0(0) = u_0(2\pi) \end{cases}$$

$$(1.18) \quad \begin{cases} u_1'(t) = A_1 u_1(t) + f_1(t); & 0 \leqslant t \leqslant 2\pi \\ u_1(0) = u_1(2\pi) \end{cases}$$

Due to (1.7), problem (1.18) has a unique solution u_1 given by

$$(1.19) \quad u_1(t) = e^{tA_1}(1 - e^{2\pi A_1})^{-1} \int_0^{2\pi} e^{(2\pi - s)A_1} f_1(s)ds + \int_0^t e^{(t-s)A_1} f_1(s)ds$$

Moreover, if u_0 is a solution of (1.17), then

$$(1.20) \qquad u_0(t) = e^{tA_0}u_0(0) + \int_0^t e^{(t-s)A_0} f_0(s)ds$$

so that, for $t = 2\pi$

$$(1.21) \qquad \int_0^{2\pi} e^{(2\pi - s)A_0} f_0(s)ds = 0$$

which coincides with (1.15). If (1.21) holds, then (1.20) with $u_0(0) = c_1x_0 + c_2y_0$ gives all solutions of (1.17). Now (1.16) follows adding (1.19) and (1.20).

Finally, the regularity of u_1 follows arguing as in the proof of theorem 1.2, and the regularity of u_0 follows easily from (1.12). ■

2. MAIN RESULTS

We are here concerned with periodic solutions of the nonlinear equation

$$(2.1) \qquad u'(\tau) = f(\lambda, u(\tau)); \quad \tau \in \mathbb{R}$$

where

$$(2.2) \qquad \begin{cases} f \in C^\infty(]-1, 1[\times D; X) \\ f(\lambda, 0) = 0, \quad -1 < \lambda < 1 \\ A = f_x(0, 0) \text{ satisfies (1.1) and (1.7)} \end{cases}$$

Since the linear problem $u'(\tau) = Au(\tau)$ has 2π-periodic solutions (see section 1), we look for solutions to (2.1) with period $2\pi\rho$, ρ close to 1. Setting $t = \tau/\rho$ our problem becomes

$$(2.3) \qquad \begin{cases} u'(t) = \rho f(\lambda, u(t)); \quad t \in \mathbb{R} \\ u(0) = u(2\pi) \end{cases}$$

As usual, to solve problem (2.3) we need some transversality condition on the eigenvalues of the operator

$$(2.4) \qquad A(\lambda) = f_x(\lambda, 0)$$

To this aim, we state the following lemma.

LEMMA 2.1. — *If (2.2) holds, there exist* $\tilde{\lambda} \in]0, 1[$ *and* $\alpha, \beta \in C^\infty(]-\tilde{\lambda}, \tilde{\lambda}[; \mathbb{R})$, $x, y \in C^\infty(]-\tilde{\lambda}, \tilde{\lambda}[; D)$ *such that*

$$(2.5) \qquad \begin{cases} A(\lambda)x(\lambda) = \alpha(\lambda)x(\lambda) - \beta(\lambda)y(\lambda) \\ A(\lambda)y(\lambda) = \beta(\lambda)x(\lambda) + \alpha(\lambda)y(\lambda) \end{cases}$$

Proof. — Setting $\tilde{A}(\lambda) : \tilde{D} \to \tilde{X}$, $\tilde{A}(\lambda)(x + iy) = A(\lambda)x + iA(\lambda)y$, the function $\lambda \to \tilde{A}(\lambda)$ belongs to $C^\infty(]-1, 1[; L(\tilde{D}, \tilde{X}))$ and there exists $\lambda_0 \in]0, 1[$ such that for $|\lambda| < \lambda_0$ the operator

$$(2.6) \qquad P(\lambda) = \frac{1}{2\pi i} \int_{\gamma(i,\varepsilon)} (\xi - \tilde{A}(\lambda))^{-1} d\xi$$

is well defined and the function $\lambda \to P(\lambda)$ belongs to $C^\infty(]-\lambda_0, \lambda_0[; L(\tilde{X}))$.

Setting

$$U(\lambda) = P(\lambda)P(0) + (1 - P(\lambda))(1 - P(0))$$

we have $U \in C^\infty(]-\lambda_0, \lambda_0[; L(\tilde{X}))$, $U(\lambda)(\tilde{D}) \subset \tilde{D}$ and $U(\lambda)P(0) = P(\lambda)U(\lambda)$, so that, as $U(0) = 1$, there exists $\lambda_1 \in]0, \lambda_0]$ such that $U(\lambda)$ is invertible for $-\lambda_1 < \lambda < \lambda_1$ and

$$(2.7) \qquad P(\lambda) = U(\lambda)P(0)U(\lambda)^{-1}$$

Recalling that $P(0)(\tilde{X})$ is the subspace of $\tilde{X}$ spanned by w_0 and using (2.7), it is easy to see that $P(\lambda)(\tilde{X})$ is spanned by $w(\lambda) = U(\lambda)w_0$. As $\tilde{A}(\lambda)$ maps $P(\lambda)(\tilde{X})$ into itself, there exists $z(\lambda) \in \mathbb{C}$ such that $\tilde{A}(\lambda)w(\lambda) = z(\lambda)w(\lambda)$, and we have, for λ sufficiently small

$$z(\lambda) = \frac{\langle \tilde{A}(\lambda)w(\lambda), \zeta_0 \rangle}{\langle w(\lambda), \zeta_0 \rangle}$$

so that $z \in C^\infty(]-\tilde{\lambda}, \tilde{\lambda}[; \mathbb{C})$ for some $\tilde{\lambda} \in]0, 1[$.

Now it is sufficient to set, for $|\lambda| < \tilde{\lambda}$

$$w(\lambda) = x(\lambda) + iy(\lambda)$$
$$z(\lambda) = \alpha(\lambda) + i\beta(\lambda)$$

with $x(\lambda)$, $y(\lambda) \in X$ and $\alpha(\lambda)$, $\beta(\lambda) \in \mathbb{R}$. ■

We are able now to state the main result of this paper.

THEOREM 2.2. — *Let (2.2) hold and assume* $\alpha'(0) \neq 0$, *where* α *is given by lemma 2.1. Let* $\gamma \in]0, 1[$ *be fixed.*

Then there exist $\sigma_0 > 0$ *and* C^∞ *functions* $\lambda :]-\sigma_0, \sigma_0[\to \mathbb{R}$; $\sigma \to \lambda(\sigma)$, $\rho :]-\sigma_0, \sigma_0[\to \mathbb{R}$, $\sigma \to \rho(\sigma)$, $u :]-\sigma_0, \sigma_0[\to C^\gamma_\#(D) \cap C^{1,\gamma}_\#(X)$, $\sigma \to u(\sigma)(\cdot)$ *such that*

$$(2.8) \qquad \begin{cases} \lambda(0) = 0, \quad \rho(0) = 1, \quad u(0)(t) = 0 \quad \textit{for} \quad t \in \mathbb{R} \\ u(\sigma)(\cdot) \textit{ is non constant for } \quad \sigma \neq 0 \end{cases}$$

and, setting $u(t,\sigma) = u(\sigma)(t)$, *we have*

$$(2.9) \qquad \begin{cases} u_t(t,\sigma) = \rho(\sigma) f(\lambda(\sigma), u(t,\sigma))\,; & t \in \mathbb{R} \\ u(2\pi,\sigma) = u(0,\sigma) \end{cases}$$

Moreover there exists $\varepsilon_0 > 0$ *such that if* $\bar\lambda \in \mathbb{R}$, $\bar\rho \in \mathbb{R}$ *and* $\bar u \in C^{\gamma}_{\#}(D) \cap C^{1,\gamma}_{\#}(X)$ *verify*

$$(2.10) \qquad \begin{cases} \bar u'(t) = \bar\rho f(\bar\lambda, \bar u(t))\,; \quad t \in \mathbb{R} \\ |\bar\lambda| < \varepsilon_0, \quad |1 - \bar\rho| < \varepsilon_0, \quad \|\bar u\|_{C^{\gamma}_{\#}(D) \cap C^{1,\gamma}_{\#}(X)} < \varepsilon_0 \end{cases}$$

then there exist $\theta \in [0, 2\pi[$, $\sigma \in]-\sigma_0, \sigma_0[$ *such that*

$$(2.11) \qquad \bar\lambda = \lambda(\sigma), \quad \bar\rho = \rho(\sigma), \quad \bar u(t) = u(\sigma)(t + \theta)$$

Proof. — Set

$$\begin{cases} F :]-1, 1[\times]0, 2[\times C^{\gamma}_{\#}(D) \cap C^{1,\gamma}_{\#}(X) \to C^{\gamma}_{\#}(X) \\ F(\lambda, \rho, u) = u' - \rho f(\lambda, u) \end{cases}$$

Then F is of class C^∞ and

$$(2.12) \qquad F_u(\lambda, \rho, u)v = v' - \rho f_x(\lambda, u)v$$

In particular

$$F_u(0, 1, 0)v = v' - Av$$

By theorem 1.3 we have

$$(2.13) \qquad \begin{cases} \operatorname{Ker} F_u(0,1,0) = \{ e^{tA}x\,; x \in X_0 \}\,; \dim \operatorname{Ker} F_u(0,1,0) = 2 \\ \operatorname{Range} F_u(0,1,0) = \left\{ z \in C^{\gamma}_{\#}(X)\,; Q \displaystyle\int_0^{2\pi} e^{[2\pi - s)A} z(s)ds = 0 \right\}; \\ \operatorname{codim} \operatorname{Range} F_u(0,1,0) = 2 \end{cases}$$

Let $V \subset C^{\gamma}_{\#}(D) \cap C^{1,\gamma}_{\#}(X)$ be such that

$$C^{\gamma}_{\#}(D) \cap C^{1,\gamma}_{\#}(X) = \operatorname{Ker} F_u(0,1,0) \oplus V$$

Following [2], we set now

$$G :]-1, 1[\times]-1, 1[\times]0, 2[\times V \to C^{\gamma}_{\#}(X)$$

$$G(\sigma, \lambda, \rho, v) = \begin{cases} 1/\sigma F(\lambda, \rho, \sigma(e^{tA}x_0 + v)), & \sigma \neq 0 \\ F_u(\lambda, \sigma, 0)(e^{tA}x_0 + v), & \sigma = 0 \end{cases}$$

Then G is continuously differentiable and $G(0,0,1,0) = 0$.

In order to find (by the Implicit Function Theorem) $\lambda = \lambda(\sigma)$, $\rho = \rho(\sigma)$, $v = v(\sigma)$ such that $G(\sigma, \lambda(\sigma), \rho(\sigma), v(\sigma)) = 0$, it is sufficient to show that the mapping

$$\begin{cases} \phi : \mathbb{R}^2 \times V \to C^{\gamma}_{\#}(X) \\ \phi(\hat\lambda, \hat\rho, \hat v) = G_\lambda(0,0,1,0)\hat\lambda + G_\rho(0,0,1,0)\hat\rho + G_v(0,0,1,0)\hat v \end{cases}$$

is an isomorphism. By (2.12) and (2.4) we have, for $|\lambda| \leqslant \tilde{\lambda}$

$$\phi(\hat{\lambda}, \hat{\rho}, \hat{v})(t) = -\mathrm{A}'(0)e^{t\mathrm{A}}x_0\hat{\lambda} + e^{t\mathrm{A}}y_0\hat{\rho} + \mathrm{F}_u(0,1,0)\hat{v}$$

Let us show that ϕ is one to one: if $\phi(\hat{\lambda}, \hat{\rho}, \hat{v}) = 0$, then

$$-\int_0^{2\pi} e^{(2\pi-s)\mathrm{A}}\mathrm{A}'(0)e^{s\mathrm{A}}x_0 ds\,\hat{\lambda} + 2\pi y_0\hat{\rho} + \int_0^{2\pi} e^{(2\pi-s)\mathrm{A}}(\mathrm{F}_u(0,1,0)\hat{v})(s)ds = 0$$

Applying φ_0 and recalling (1.13) and (2.13) we get

$$\int_0^{2\pi} \langle e^{(2\pi-s)\mathrm{A}}\mathrm{A}'(0)e^{s\mathrm{A}}x_0, \varphi_0 \rangle\, ds = 0 \tag{2.14}$$

On the other hand, from (1.14) and from the equalities

$$\begin{cases} \mathrm{A}'(0)x_0 = -\mathrm{A}x'(0) + \alpha'(0)x_0 - \beta'(0)y_0 - y'(0) \\ \mathrm{A}'(0)y_0 = -\mathrm{A}y'(0) + \beta'(0)x_0 + \alpha'(0)y_0 + x'(0) \end{cases}$$

(which follow from (2.5)) we get

$$\int_0^{2\pi} \langle e^{(2\pi-s)\mathrm{A}}\mathrm{A}'(0)e^{s\mathrm{A}}x_0, \varphi_0 \rangle\, ds = 2\pi\alpha'(0) \tag{2.15}$$

hence (2.14) implies $\hat{\lambda} = 0$ and then $\hat{\rho} = 0$, $\hat{v} = 0$.

Let us prove now that ϕ is onto: it is sufficient to show that $\mathrm{C}^{\gamma}_{\#}(\mathrm{X})$ is spanned by Range $\mathrm{F}_u(0,1,0)$ and by the functions

$$\psi_1(t) = \mathrm{A}'(0)e^{t\mathrm{A}}x_0; \quad t \in \mathbb{R}$$
$$\psi_2(t) = e^{t\mathrm{A}}y_0; \quad t \in \mathbb{R}$$

ψ_1 and ψ_2 are independent: in fact, if $c_1\psi_1 + c_2\psi_2 = 0$, then

$$c_1\int_0^{2\pi} e^{(2\pi-s)\mathrm{A}}\mathrm{A}'(0)e^{s\mathrm{A}}x_0 ds + 2\pi c_2 y_0 = 0$$

and applying φ_0 we get $2\pi c_1\alpha'(0) = 0$, so that $c_1 = c_2 = 0$ and ψ_1, ψ_2 are independent. Since ψ_1 and ψ_2 do not belong to Range $\mathrm{F}_u(0,1,0)$ and codim Range $\mathrm{F}_u(0,1,0) = 2$ (see (2.13)), ϕ is an isomorphism.

By the Implicit Function Theorem there exist $\sigma_0 \in]0,1[$, $r_0 > 0$, and $\lambda :]-\sigma_0, \sigma_0[\to \mathbb{R}$, $\rho :]-\sigma_0, \sigma_0[\to \mathbb{R}$, $v :]-\sigma_0, \sigma_0[\to \mathrm{V}$ such that

$$\begin{cases} \sigma \in]-\sigma_0, \sigma_0[, \ |\lambda| < r_0, \ |\rho - 1| < r_0, \ \|v\|_{\mathrm{V}} = \|v\|_{\mathrm{C}^{\gamma}_{\#}(\mathrm{D}) \cap \mathrm{C}^{1,\gamma}_{\#}(\mathrm{X})} < r_0 \\ \mathrm{G}(\sigma, \lambda, \rho, v) = 0 \end{cases} \tag{2.16}$$

if and only if $\lambda = \lambda(\sigma)$, $\rho = \rho(\sigma)$, $v = v(\sigma)$.

For each $\sigma \in]-\sigma_0, \sigma_0[$ set

$$u(\sigma)(t) = \sigma(e^{t\mathrm{A}}x_0 + v(\sigma)(t)) \tag{2.17}$$

Then $u(\sigma)$ is a solution of (2.3) for $\lambda = \lambda(\sigma)$, $\rho = \rho(\sigma)$.

It remains to prove uniqueness. Also here we follow [2], choosing $V = P_V(X)$, where P_V is defined by

$$(2.18)\quad (P_V v)(t) = v(t) - \frac{1}{2\pi}\int_0^{2\pi} \langle e^{(2\pi-s)A}v(s), \varphi_0 \rangle \, ds \, e^{tA}x_0 + \\ + \frac{1}{2\pi}\int_0^{2\pi} \langle e^{(2\pi-s)A}v(s), \eta_0 \rangle \, ds \, e^{tA}y_0 \,;\ t \in \mathbb{R}$$

Let $\bar{\lambda} \in \mathbb{R}$, $\bar{\rho} \in \mathbb{R}$, $\bar{u} \in C^{\gamma}_{\#}(D) \cap C^{1,\gamma}_{\#}(X)$ verify (2.10) with ε_0 to choose below. There exist σ_1 and σ_2 such that

$$\bar{u}(t) = \sigma_1 e^{tA}x_0 + \sigma_2 e^{tA}y_0 + \bar{v}(t), \quad \bar{v} = P_V\bar{u}$$

Choose $\theta \in [0, 2\pi[$ such that $\bar{u}(t+\theta) = \sigma e^{tA}x_0 + \bar{v}(t+\theta)$ for some $\sigma \in \mathbb{R}$, and set $\hat{u}(t) = \bar{u}(t+\theta)$, $\hat{v}(t) = \bar{v}(t+\theta)$ so that

$$\hat{u}(t) = \sigma e^{tA}x_0 + \hat{v}(t)\,; \quad t \in \mathbb{R}$$

As easily checked, by (2.18) $\hat{v} \in V$.

We have now to show that if ε_0 is sufficiently small, then

$$(2.19)\qquad |\sigma| < \sigma_0, \quad |\bar{\lambda}| < r_0, \quad \frac{1}{|\sigma|}\|\hat{v}\|_V < r_0, \quad |1-\bar{\rho}| < r_0$$

where r_0 is defined in (2.16). Once (2.19) is proved, it follows $\bar{\lambda} = \lambda(\sigma)$, $\bar{\rho} = \rho(\sigma)$, $\hat{u} = u(\sigma)$.

To find such an ε_0 we first remark that, since

$$\phi(\lambda, \rho, v) = -A'(0)e^{tA}x_0\lambda + e^{tA}y_0\rho + v' - Av$$

is an isomorphism of $\mathbb{R}^2 \times V$ onto $C^{\gamma}_{\#}(X)$, there exists a constant $k > 0$ such that

$$(2.20)\quad |\sigma\bar{\lambda}| + |(1-\bar{\rho})\sigma| + \|\hat{v}\|_V \leqslant \\ \leqslant k\| -A'(0)e^{tA}x_0\bar{\lambda}\sigma + e^{tA}y_0(1-\bar{\rho})\sigma + \hat{v}' - A\hat{v}\|_{C^{\gamma}_{\#}(X)} \leqslant \\ \leqslant k\{\|A'(0)e^{tA}x_0\|_{C^{\gamma}_{\#}(X)}|\sigma||\bar{\lambda}| + \|e^{tA}y_0\|_{C^{\gamma}_{\#}(X)}|\sigma||\bar{\rho}-1| + \\ + \|A\hat{v}\|_{C^{\gamma}_{\#}(X)}|\bar{\rho}-1| + \|y_0\||\sigma||\bar{\rho}-1| + \\ + \|f(\bar{\lambda}, \sigma e^{tA}x_0 + \hat{v}) - A(\sigma e^{tA}x_0 + \hat{v})\|_{C^{\gamma}_{\#}(X)}\}$$

Now, by the equalities

$$f(\bar{\lambda}, \sigma e^{tA}x_0 + \hat{v}(t)) - A(\sigma e^{tA}x_0 + \hat{v}(t)) = \\ = \int_0^1 d\theta \int_0^1 d\eta\, [f_{xx}(\bar{\lambda}, \theta\eta(e^{tA}x_0 + \hat{v}(t))(\sigma e^{tA}x_0 + \hat{v}(t), \sigma e^{tA}x_0 + \hat{v}(t))\eta + \\ + f_{x\lambda}(\bar{\lambda}\theta, 0)(\sigma e^{tA}x_0 + \hat{v}(t))\bar{\lambda}]$$

and

$$f(\bar\lambda, \sigma e^{tA}x_0+\hat v(t))-A(\sigma e^{tA}x_0+\hat v(t))-f(\bar\lambda,\sigma e^{sA}x_0+\hat v(s))+A(\sigma e^{sA}x_0+\hat v(s))=$$
$$=\int_0^1 d\theta\int_0^1 d\eta\int_0^1 d\xi\,[f_{xxx}(\bar\lambda,\theta\eta\xi(\sigma e^{tA}x_0+\hat v(t))+(1-\xi)\theta\eta(\sigma e^{sA}x_0+\hat v(s)))$$
$$(\sigma e^{tA}x_0+\hat v(t),\sigma e^{tA}x_0+\hat v(t),\theta\eta(\sigma e^{tA}x_0+\hat v(t)-\sigma e^{sA}x_0-\hat v(s)))+$$
$$+f_{xx}(\bar\lambda,\theta\eta(\sigma e^{sA}x_0+\hat v(s))((\sigma e^{tA}x_0+\hat v(t),\sigma e^{tA}x_0+\hat v(t))-$$
$$-(\sigma e^{sA}x_0+\hat v(s),\sigma e^{sA}x_0+\hat v(s)))+f_{x\lambda}(\bar\lambda\theta,0)(\sigma e^{tA}x_0+\hat v(t)-\sigma e^{sA}x_0-\hat v(s))]$$

it follows that there exists $k_1>0$ such that

$$\|f(\bar\lambda,\sigma e^{tA}x_0+\hat v(t))-A(\sigma e^{tA}x_0+\hat v(t))\|_{C^{\gamma}_{\#}X)}$$
$$\leqslant k_1(|\bar\lambda|\,\|\hat v\|_V+|\sigma|^3+\|\hat v\|_V^3+|\sigma|^2+\|\hat v\|_V^2+|\bar\lambda\sigma|)$$

and then, taking into account (2.20), there exists $k_2>0$ such that

$$\|\hat v\|_V\leqslant k_2(|\bar\rho-1|\,|\sigma|+|\bar\lambda|\,|\sigma|+|\bar\rho-1|\,\|\hat v\|_V+ \\ +|\bar\lambda|\,\|\hat v\|_V+|\sigma|^2+\|\hat v\|_V^2+|\sigma|^3+\|\hat v\|_V^3) \tag{2.21}$$

Let now $\varepsilon_0<r_0$ be such that

$$\|1-P_V\|_{L(C^{\gamma}_{\#}(D)\cap C^{1,\gamma}_{\#}(X))}\varepsilon_0<\sigma_0\|e^{tA}x_0\|_{C^{\gamma}_{\#}(D)\cap C^{1,\gamma}_{\#}(X)} \tag{2.22}$$

$$k_2(2\varepsilon_0+\|P_V\|_{L(C^{\gamma}_{\#}(D)\cap C^{1,\gamma}_{\#}(X))}\varepsilon_0+\|P_V\|^2_{L(C^{\gamma}_{\#}(D)\cap C^{1,\gamma}_{\#}(X))}\varepsilon_0^2)<1/2 \tag{2.23}$$

$$2k_2(2+\sigma_0+\sigma_0^2)<r_0 \tag{2.24}$$

Then we have $\|\hat v\|_V\leqslant\|P_V\|\,\varepsilon_0$, and, from (2.22), $|\sigma|<\sigma_0$. From (2.23) it follows

$$\|\hat v\|_V<2k_2(|\bar\rho-1|\,|\sigma|+|\bar\lambda|\,|\sigma|+|\sigma|^2+|\sigma|^3)$$

and now (2.24) implies

$$\frac{1}{|\sigma|}\|\hat v\|_V<r_0$$

and the proof is finished. ■

Remark. — The degenerate case $\alpha'(0)=0$ has been treated in [*8*] [*9*] for semilinear equations in Hilbert space.

EXAMPLE 2.3. — Let us consider the equation

$$u_t(t,x)=\varphi(\lambda,u(t,x),u_x(t,x),u_{xx}(t,x)) \tag{2.25}$$

where the function $(\lambda,p)\to\varphi(\lambda,p)$ belongs to $C^\infty(\,]-1,1[\,\times\mathbb{R}^3;\mathbb{R})$. We set

$X=C_\#(\mathbb{R})$

$D=C^2_\#(\mathbb{R})=\{\psi\in C^2([0,2\pi];\mathbb{R});\ \psi(0)=\psi(2\pi),\ \psi'(0)=\psi'(2\pi),\ \psi''(0)=\psi''(2\pi)\}$

Thus the function $f(\lambda,u)=\varphi(\lambda,u,u',u'')$ belongs to $C^\infty(\,]-1,1[\,\times D;X)$.

We assume:

(2.26) $\varphi_{p_3}(0,0) > 0$

(2.27) $\varphi_{p_1}(0,0) = k^2\varphi_{p_3}(0,0),\ k\varphi_{p_2}(0,0) = 1$

(2.28) $\varphi_{p_1\lambda}(0,0) \neq k^2\varphi_{p_3\lambda}(0,0)$

for some $k \in \mathbb{Z}$.

By (2.26) it follows that equation (2.25) is parabolic for λ and u small.

Using the notations of section 2 we have:

$$A(\lambda)v = \varphi_{p_1}(\lambda, 0)v + \varphi_{p_2}(\lambda, 0)v' + \varphi_{p_3}(\lambda, 0)v'', \quad v \in D$$

so that the eigenvalues of $\tilde{A}(\lambda)$ are given by

$$\varphi_{p_1}(\lambda, 0) + ih\varphi_{p_2}(\lambda, 0) - h^2\varphi_{p_3}(\lambda, 0), \quad h \in \mathbb{Z}$$

It can be seen easily that (2.26) implies that $A = f_u(0,0)$ satisfies (1.1). Moreover, by assumption (2.27), A verifies (1.8); finally, the transversality condition $\alpha'(0) \neq 0$ is satisfied thanks to (2.28). Thus we can apply theorem 2.2. ■

APPENDIX

Proof of lemma 1.1. — Let $B : D \to X$, $Bx = Ax - \omega x$, and let $M_0, M_1 > 0$ be such that (see [*14*])

$$\| e^{tB} \|_{L(X)} \leqslant M_0, \quad \| Be^{tB} \|_{L(X)} \leqslant M_1/t, \quad t > 0$$

We have, for any $s > 0$ and $t, t + h \in [a, b]$:

$$\| s^{1-\gamma} Be^{sB} v'(t) \| \leq s^{-\gamma} M_1 \| v' \|_{C([a,b]; X)} \tag{A.1}$$

$$\begin{aligned} \| s^{1-\gamma} Be^{sB} v'(t) \| &\leqslant \\ &\leqslant \| s^{1-\gamma} Be^{sB}(v'(t) - h^{-1}(v(t+h) - v(t))) \| + \| s^{1-\gamma} Be^{sB} h^{-1}(v(t+h) - v(t)) \| = \\ &= \| s^{1-\gamma} Be^{sB} \int_0^1 (v'(t) - v'(t+\sigma h)) d\sigma \| + \| s^{1-\gamma} Be^{sB} h^{-1}(v(t+h) - v(t)) \| \\ &\quad M_1 h^{\gamma} s^{-\gamma} \| v' \|_{C^{\gamma}([a,b]; X)} + M_0 s^{1-\gamma} h^{\gamma-1} \| Bv \|_{C([a,b]; X)} \end{aligned} \tag{A.2}$$

For $s \geq (b - a)/2$, from (A.2) it follows

$$\| s^{1-\gamma} Be^{sB} v'(t) \| \leqslant (b - a)^{-\gamma} 2^{\gamma} M_1 \| v' \|_{C([a,b]; X)} \tag{A.3}$$

and for $s \leqslant (b - a)/2$, setting in (A.2) $| h | = s$, we find

$$\| s^{1-\gamma} Be^{sB} v'(t) \| \leqslant M_0 \| Bv \|_{C^{\gamma}([a,b]; X)} + M_1 \| v' \|_{C^{\gamma}([a,b]; X)} \tag{A.4}$$

For each $r > 0$ we have (see [*14*]):

$$r^{-\gamma} \| e^{rB} v'(t) - v'(t) \| = r^{-\gamma} \left\| \int_0^r Be^{sB} v'(t) ds \right\|$$

and then, from (A.3) and (A.4), it follows that $r^{-\gamma} \| e^{rB} v'(t) - v'(t) \|$ is bounded independently of r and t. Recalling that $e^{rA} = e^{r\omega} e^{rB}$, the conclusion follows easily. ∎

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