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Analyse non linéaire

Reinforcement problems in the calculus of variations (*)

by

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ABSTRACT. — We study the torsion of an elastic bar surrounded by an increasingly thin layer made of increasingly hard material. In the model problem the ellipticity constant tends to zero in the outer layer; the equations considered may be fully nonlinear.

Depending on the link between thickness and hardness we obtain three different expressions of the limit problem.

RÉSUMÉ. — On étudie la torsion d'une barre élastique enveloppée d'une couche très mince d'un matériau très dur. Dans le problème modèle la constante d'ellipticité de la couche extérieure est très petite; les équations considérées peuvent être complètement non-linéaires.

En modifiant la relation entre l'épaisseur et la rigidité, on obtient trois différents problèmes-limites.

Mots-clés : Reinforcement, Γ-convergence, Integral functionals, Non-equicoercive problems.

I. INTRODUCTION

Several recent papers (see [*2*] [*3*] and the references quoted there) deal with the reinforcement problem for an elastic bar, whose mathematical setting may be outlined as follows.

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Let Ω be a bounded open set in $\mathbb{R}^n$, surrounded by a layer Σ_ε whose thickness r_ε goes to zero as $\varepsilon \to 0$; set $\Omega_\varepsilon = \overline{\Omega} \cup \Sigma_\varepsilon$, and denote by u_ε a minimum point in $W_0^{1,p}(\Omega_\varepsilon)$ of the functional

$$(1.1) \qquad G(u) + \varepsilon \int_{\Sigma_\varepsilon} f(x, Du)dx,$$

where G is lower semicontinuous in the L^p topology of $W^{1,p}(\Omega)$, and $f(x, z)$ is non-negative and convex in z.

The reinforcement problem consists in studying the behaviour of u_ε as ε tends to zero.

Under suitable assumptions we characterize the Γ-limit of the functionals (1.1) in the L^p topology: it is known (see [*4*]) that this immediately gives informations on the convergence of u_ε.

We prove in particular that

$$\Gamma^-(L^p) \lim_{\varepsilon \to 0} \left[G(u) + \varepsilon \int_{\Sigma_\varepsilon} f(x, Du)dx \right] = G(u) + L \int_{\partial\Omega} \gamma(\sigma, uv)dH^{n-1}(\sigma),$$

where $L = \lim_{\varepsilon \to 0} \varepsilon / r_\varepsilon^{p-1}$, v is the outward normal vector to Ω and γ depends only on f.

In [*3*] it was proved an analogous result valid in the two-dimensional case, with $G(u) = \int_\Omega |Du|^2 dx$ and $f(x, z) = |z|^2$; the functional (1.1) is then related to the torsion of an elastic bar with cross section Ω enclosed in an increasingly thin shell made of an increasingly hard material.

Using again some techniques of elliptic equations, the results of [*3*] were generalized in [*2*] to the many-dimensional case, with

$$G(u) = \int_\Omega a_{ij}(x) D_i u D_j u dx \qquad \text{and} \qquad f(x, z) = b_{ij}(x) z_i z_j.$$

Other similar results may be found in [*1*], sections 1, 3, 5, and in [*5*], chapter 13.

In this paper we use the direct methods of Calculus of Variations and Γ-convergence, which allow us to give some answers even in the fully nonlinear case.

II. NOTATIONS AND STATEMENT OF THE RESULT

In what follows we denote by Ω a bounded open subset of $\mathbb{R}^n$ with $C^{1,1}$ boundary, and by v its outward normal vector. Let $d: \partial\Omega \to \mathbb{R}$ be a Lipschitz function satisfying

$$0 < \alpha \leqslant d(\sigma) \leqslant \beta \qquad \text{for every} \quad \sigma \in \partial\Omega.$$

For all $\varepsilon > 0$ fix $r_\varepsilon > 0$ so that $\lim_{\varepsilon \to 0} r_\varepsilon = 0$, and set

$$\Sigma_\varepsilon = \{ \sigma + t\nu(\sigma) : \sigma \in \partial\Omega, 0 < t < r_\varepsilon d(\sigma) \}$$
$$\Omega_\varepsilon = \overline{\Omega} \cup \Sigma_\varepsilon .$$

Our assumptions on $\partial\Omega$ ensure that the mapping $(\sigma, t) \mapsto \sigma + t\nu(\sigma)$ is invertible on Σ_ε if ε is sufficiently small, hence for every $x \in \Sigma_\varepsilon$ there exist $\sigma(x) \in \partial\Omega$ and $t(x) \in]0, r_\varepsilon d(\sigma(x))[$ such that

$$x = \sigma(x) + t(x)\nu(\sigma(x)) .$$

If $x \in \Sigma_\varepsilon$ we will briefly write $d(x)$ in place of $d(\sigma(x))$. Take $p > 1$, and let $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfy

(2.1) for all $x \in \mathbb{R}^n$ the function $f(x, .)$ is convex;

(2.2) there exists a function ω: $[0, +\infty[\to [0, +\infty[$ which is continuous, increasing, vanishing at the origin and such that

$$| f(x_1, z) - f(x_2, z) | \leqslant \omega(| x_1 - x_2 |)(1 + | z |^p);$$

(2.3) for all $x \in \mathbb{R}^n$, $z \in \mathbb{R}^n$

$$\lambda | z |^p \leqslant f(x, z) \leqslant \Lambda(1 + | z |^p) \qquad (0 < \lambda \leqslant \Lambda);$$

(2.4) there exists a non-negative continuous function $\gamma(x, z)$ which is convex and p-homogeneous as a function of z and satisfies

$$\sup \{ | f(x, z) - \gamma(x, z) | : x \in \mathbb{R}^n \} \leqslant \rho(| z |)(1 + | z |^p),$$

where $\rho : [0, +\infty[\to [0, +\infty[$ is continuous, decreasing and vanishes at infinity.

Finally consider a functional $\mathrm{G} : \mathrm{W}^{1,p}(\Omega) \to [0, +\infty]$ such that

(2.5) G is lower semicontinuous in the topology $\mathrm{L}^p(\Omega)$;

(2.6) $\mathrm{G}(u) \geqslant \int_\Omega | \mathrm{D}u |^p dx$ for every $u \in \mathrm{W}^{1,p}(\Omega)$.

When $u \in \mathrm{L}^p(\mathbb{R}^n)$ is such that $u |_\Omega \in \mathrm{W}^{1,p}(\Omega)$, we will simply write $\mathrm{G}(u)$ instead of $\mathrm{G}(u |_\Omega)$.

For every $u \in \mathrm{L}^p(\mathbb{R}^n)$ and $\varepsilon > 0$ set

$$\mathrm{F}_\varepsilon(u) = \begin{cases} \mathrm{G}(u) + \varepsilon \int_{\Sigma_\varepsilon} f(x, \mathrm{D}u)dx & \text{if } u \in \mathrm{W}_0^{1,p}(\Omega_\varepsilon) \\ +\infty & \text{otherwise}. \end{cases}$$

We want to characterize the Γ-limit of F_ε in the topology $\mathrm{L}^p(\Omega)$, depending on the behaviour of r_ε. Indeed, it is well known that the Γ-convergence of a sequence of functionals is strictly related to the convergence of their minimum points and minimum values: more precisely, let X be a metric space, $(F_\varepsilon)_{\varepsilon>0}$ mappings from X into $\overline{\mathbb{R}}$, and $x \in X$. We set

$$\Gamma^-(X) \liminf_{\varepsilon\to 0} F_\varepsilon(x) = \inf \{ \liminf_{\varepsilon\to 0} F_\varepsilon(x_\varepsilon) : x_\varepsilon \to x \text{ in } X \}$$

$$\Gamma^-(X) \limsup_{\varepsilon\to 0} F_\varepsilon(x) = \inf \{ \limsup_{\varepsilon\to 0} F_\varepsilon(x_\varepsilon) : x_\varepsilon \to x \text{ in } X \} .$$

If these two Γ-limits agree, their common value will be denoted by

$$\Gamma^-(X) \lim_{\varepsilon\to 0} F_\varepsilon(x) .$$

We have:

THEOREM [II.1] (see [4], Theorem 2.6). — *Let* X *be a metric space and* F, $(F_\varepsilon)_{\varepsilon>0}$ *mappings from* X *into* $\overline{\mathbb{R}}$ *such that*

i) the family (F_ε) *is equicoercive, i. e. for every* $\lambda > 0$ *there exists a compact subset* K_λ *of* X *such that*

$$\{ x \in X : F_\varepsilon(x) \leqslant \lambda \} \subseteq K_\lambda \quad \textit{for every } \varepsilon > 0 ;$$

ii) $\Gamma^-(X) \lim_{\varepsilon\to 0} F_\varepsilon = F$.

Then F *has a minimum on* X *and* $\min_X F = \lim_{\varepsilon\to 0} (\inf_X F_\varepsilon)$ *; moreover if* $\lim_{\varepsilon\to 0} F_\varepsilon(x_\varepsilon) = \lim_{\varepsilon\to 0} (\inf_X F_\varepsilon)$ *and* $x_\varepsilon \to \hat{x}$ *in* X, *then* $\hat{x}$ *is a minimum point for* F.

Define for all $u \in W^{1,p}(\Omega)$

$$G_\infty(u) = \begin{cases} G(u) & \text{if } u \in W_0^{1,p}(\Omega) \\ +\infty & \text{otherwise} \end{cases}$$

and for every $L \geqslant 0$

$$G_L(u) = G(u) + L \int_{\partial\Omega} d^{1-p}(\sigma)\gamma(\sigma, uv) dH^{n-1}(\sigma) .$$

We will prove in section III the following result:

THEOREM [II.2]. — *Assume that* $\lim_{\varepsilon\to 0} \varepsilon/r_\varepsilon^{p-1} = L \in [0, +\infty]$*; then for every* $u \in W^{1,p}(\Omega)$ *we have*

$$\Gamma^-(L^p(\mathbb{R}^n)) \lim_{\varepsilon\to 0} F_\varepsilon(u) = G_L(u) .$$

REMARK [II.3]. — The result above yields immediately that if $\lim_{\varepsilon\to 0} \varepsilon/r_\varepsilon^{p-1}$ does not exist then the functionals F_ε do not Γ-converge in the topology $L^p(\mathbb{R}^n)$.

III. PROOF OF THE RESULT

We will later need the following results:

LEMMA [III.1]. — *Let* $a: \mathbb{R}^n \to \mathbb{R}$ *be a continuous function and* $u \in W^{1,p}(\mathbb{R}^n)$ *; then*

$$\lim_{\varepsilon \to 0} \frac{1}{r_\varepsilon} \int_{\Sigma_\varepsilon} a(x) \, | \, u(x) \, |^p dx = \int_{\partial\Omega} d(\sigma) a(\sigma) \, | \, u(\sigma) \, |^p d\mathrm{H}^{n-1}(\sigma) \, .$$

Proof. — We denote by the letter c any positive constant. Set for every $\varepsilon > 0$

$$\omega_\varepsilon = \sup \{ \, | \, a(x + y) - a(x) \, | : x \in \partial\Omega, \, | \, y \, | \leqslant \beta r_\varepsilon \} \, .$$

Then $\lim_{\varepsilon \to 0} \omega_\varepsilon = 0$ and

$$\begin{aligned}
&\lim_{\varepsilon \to 0} \left| \frac{1}{r_\varepsilon} \int_{\Sigma_\varepsilon} a(x) \, | \, u(x) \, |^p dx - \int_{\partial\Omega} d(\sigma) a(\sigma) \, | \, u(\sigma) \, |^p d\mathrm{H}^{n-1}(\sigma) \right| \\
&\leqslant \lim_{\varepsilon \to 0} \frac{1}{r_\varepsilon} \int_{\partial\Omega} d\mathrm{H}^{n-1}(\sigma) \int_0^{r_\varepsilon d(\sigma)} \left| a(\sigma + t\nu(\sigma)) \, | \, u(\sigma + t\nu(\sigma)) \, |^p - a(\sigma) \, | \, u(\sigma) \, |^p \right| dt \\
&\leqslant \lim_{\varepsilon \to 0} \frac{1}{r_\varepsilon} \int_{\partial\Omega} d\mathrm{H}^{n-1}(\sigma) \int_0^{r_\varepsilon d(\sigma)} [c \left| \, | \, u(\sigma + t\nu(\sigma)) \, |^p - | \, u(\sigma) \, |^p \right| + \omega_\varepsilon \, | \, u(\sigma) \, |^p] dt \\
&\leqslant c \lim_{\varepsilon \to 0} \Bigg[\omega_\varepsilon \int_{\partial\Omega} | \, u(\sigma) \, |^p d\mathrm{H}^{n-1}(\sigma) \\
&\qquad + \frac{1}{r_\varepsilon} \int_{\partial\Omega} d\mathrm{H}^{n-1}(\sigma) \int_0^{r_\varepsilon d(\sigma)} | u(\sigma + t\nu(\sigma)) - u(\sigma) | (| u(\sigma) |^{p-1} + | u(\sigma + t\nu(\sigma)) |^{p-1}) dt \Bigg] \\
&\leqslant c \lim_{\varepsilon \to 0} \frac{1}{r_\varepsilon} \int_{\partial\Omega} d\mathrm{H}^{n-1}(\sigma) \Bigg[\int_0^{r_\varepsilon d(\sigma)} | \, \mathrm{D}u(\sigma + t\nu(\sigma)) \, | \, dt \\
&\qquad \cdot \int_0^{r_\varepsilon d(\sigma)} (\, | \, u(\sigma) \, |^{p-1} + | \, u(\sigma + t\nu(\sigma)) \, |^{p-1}) dt \Bigg] \\
&\leqslant c \lim_{\varepsilon \to 0} \frac{1}{r_\varepsilon} \left[r_\varepsilon^{p-1} \int_{\partial\Omega} d\mathrm{H}^{n-1}(\sigma) \int_0^{r_\varepsilon d(\sigma)} | \, \mathrm{D}u(\sigma + t\nu(\sigma)) \, |^p dt \right]^{\frac{1}{p}} \\
&\qquad \cdot \left[r_\varepsilon^{\frac{1}{p-1}} \int_{\partial\Omega} d\mathrm{H}^{n-1}(\sigma) \int_0^{r_\varepsilon d(\sigma)} (\, | \, u(\sigma) \, |^p + | \, u(\sigma + t\nu(\sigma)) \, |^p) dt \right]^{\frac{p-1}{p}} \\
&\leqslant c \lim_{\varepsilon \to 0} \left[\int_{\Sigma_\varepsilon} | \mathrm{D}u(x) |^p dx \right]^{\frac{1}{p}} \left[r_\varepsilon \int_{\partial\Omega} | u(\sigma) |^p d\mathrm{H}^{n-1}(\sigma) + \int_{\Sigma_\varepsilon} | u(x) |^p dx \right]^{\frac{p-1}{p}} = 0.
\end{aligned}$$

■

LEMMA [III.2]. — *Let* $g:\mathbb{R}^n \to \mathbb{R}$ *be* C^1, *convex, p-homogeneous and such that for all* $z \in \mathbb{R}^n$

$$\lambda\,|z|^p \leqslant g(z) \leqslant \Lambda\,|z|^p \qquad (0 < \lambda \leqslant \Lambda).$$

Then for every $x, y \in \mathbb{R}^n$

$$|\langle y, Dg(y)\rangle|^p\, g(x) \geqslant |\langle x, Dg(y)\rangle|^p\, g(y).$$

Proof. — By the homogeneity of g we may assume $|x| = |y| = 1$, hence if we set $\psi_y(x) = \dfrac{g(x)}{|\langle x, Dg(y)\rangle|^p}$ for $\langle x, Dg(y)\rangle \neq 0$, we must prove that

$$\min\{\psi_y(x) : |x| = 1\} = \psi_y(y).$$

Let $\hat{x}$ be a minimum point of ψ_y on $\{|x| = 1\}$: then

$$Dg(\hat{x})\,|\langle \hat{x}, Dg(y)\rangle|^p - p g(\hat{x})\,|\langle \hat{x}, Dg(y)\rangle|^{p-2}\langle \hat{x}, Dg(y)\rangle\, Dg(y) = \mu \hat{x}\,|\langle \hat{x}, Dg(y)\rangle|^{2p}.$$

Multiplying by $\hat{x}$ and using Euler's theorem on p-homogeneous functions we obtain $\mu = 0$, that is

$$Dg(\hat{x})\,|\langle \hat{x}, Dg(y)\rangle|^2 = p g(\hat{x})\langle \hat{x}, Dg(y)\rangle\, Dg(y). \tag{3.1}$$

Now take $\bar{x} = t\hat{x}$ such that $g(\bar{x}) = g(y)$; by eventually taking $-\bar{x}$ instead of $\bar{x}$, we obtain from (3.1) that for a suitable $\eta > 0$

$$Dg(\bar{x}) = \eta Dg(y).$$

By the convexity of g we have

$$g(y) \geqslant g(\bar{x}) + \langle y - \bar{x}, Dg(\bar{x})\rangle,$$

whence $\langle y - \bar{x}, Dg(y)\rangle \leqslant 0$; analogously we have

$$g(\bar{x}) \geqslant g(y) + \langle \bar{x} - y, Dg(y)\rangle,$$

so that $\langle y - \bar{x}, Dg(y)\rangle \geqslant 0$ and finally

$$\langle y, Dg(y)\rangle = \langle \bar{x}, Dg(y)\rangle.$$

But then

$$\psi_y(\hat{x}) = \psi_y(\bar{x}) = \frac{g(\bar{x})}{|\langle \bar{x}, Dg(y)\rangle|^p} = \frac{g(y)}{|\langle y, Dg(y)\rangle|^p} = \psi_y(y). \qquad \blacksquare$$

Proof of Theorem [II.2] *in the case* $L < +\infty$.

We begin with the inequality $F^+ \leqslant G_L$.

For every $\varepsilon > 0$ set

$$\varphi_\varepsilon(x) = \max\left\{0, 1 - \frac{\operatorname{dist}(x, \Omega)}{r_\varepsilon d(x)}\right\};$$

then $0 \leqslant \varphi_\varepsilon \leqslant 1$ on Σ_ε and $\varphi_\varepsilon = 0$ outside Ω_ε; moreover $\|D\varphi_\varepsilon\|_{L^\infty(\Sigma_\varepsilon)} \leqslant c/r_\varepsilon$. Let $u \in W^{1,p}(\Omega)$: by the regularity of $\partial\Omega$ we may assume that $u \in W^{1,p}(\mathbb{R}^n)$.

Setting $u_\varepsilon = u\varphi_\varepsilon$ we have $u_\varepsilon \in W_0^{1,p}(\Omega_\varepsilon)$ and $u_\varepsilon \to u 1_\Omega$ in $L^p(\mathbb{R}^n)$; in addition for every $t \in]0, 1[$

$$(3.2)\quad F_\varepsilon(u_\varepsilon) = G(u) + \varepsilon \int_{\Sigma_\varepsilon} f(x, uD\varphi_\varepsilon + \varphi_\varepsilon Du)dx$$
$$\leqslant G(u) + \varepsilon \int_{\Sigma_\varepsilon} \left[tf\left(x, \frac{uD\varphi_\varepsilon}{t}\right) + (1-t)f\left(x, \frac{\varphi_\varepsilon Du}{1-t}\right)\right]dx$$
$$\leqslant G(u) + \varepsilon \int_{\Sigma_\varepsilon} f\left(x, \frac{uD\varphi_\varepsilon}{t}\right)dx + \varepsilon\Lambda(1-t)\int_{\Sigma_\varepsilon}\left[1 + \frac{|Du|^p}{(1-t)^p}\right]dx.$$

Fix $M > 0$ and set $A_{M,\varepsilon} = \{ x \in \mathbb{R}^n : | u(x)D\varphi_\varepsilon(x) | < Mt \}$: then

$$(3.3)\quad \varepsilon \int_{\Sigma_\varepsilon} f\left(x, \frac{uD\varphi_\varepsilon}{t}\right)dx \leqslant \varepsilon \int_{\Sigma_\varepsilon} \gamma\left(x, \frac{uD\varphi_\varepsilon}{t}\right)dx$$
$$+\varepsilon\int_{\Sigma_\varepsilon \cap A_{M,\varepsilon}} \rho(0)(1+M^p)dx + \varepsilon\int_{\Sigma_\varepsilon - A_{M,\varepsilon}} \rho(M)\left(1 + \left|\frac{cu}{tr_\varepsilon}\right|^p\right)dx$$
$$\leqslant \varepsilon\int_{\Sigma_\varepsilon} \gamma\left(x, \frac{uD\varphi_\varepsilon}{t}\right)dx + \varepsilon \text{ meas } (\Sigma_\varepsilon)\rho(0)(2+M^p) + \varepsilon\rho(M)\left(\frac{c}{tr_\varepsilon}\right)^p \int_{\Sigma_\varepsilon} | u |^p dx.$$

Let $\delta(x) = \text{dist}(x, \Omega)$: we have

$$(3.4)\quad \varepsilon\int_{\Sigma_\varepsilon} \gamma\left(x, \frac{uD\varphi_\varepsilon}{t}\right)dx = \frac{\varepsilon}{t^p}\int_{\Sigma_\varepsilon}\gamma\left(x, \delta\frac{uDd}{d^2 r_\varepsilon} - (1-\delta)\frac{udD\delta}{(1-\delta)d^2 r_\varepsilon}\right)dx$$
$$\leqslant \frac{\varepsilon}{(tr_\varepsilon)^p}\int_{\Sigma_\varepsilon}\left[\delta(x)\gamma\left(x, \frac{uDd}{d^2}\right) + (1-\delta(x))\gamma\left(x, \frac{-uD\delta}{(1-\delta)d}\right)\right]dx$$
$$\leqslant \frac{\varepsilon}{r_\varepsilon^{p-1}}\frac{\Lambda\beta}{t^p}\int_{\Sigma_\varepsilon}\left|\frac{uDd}{\alpha^2}\right|^p dx + \frac{\varepsilon}{(tr_\varepsilon)^p}\frac{1}{(1-\beta r_\varepsilon)^{p-1}}\int_{\Sigma_\varepsilon}\gamma\left(x, \frac{-uD\delta}{d}\right)dx.$$

By (3.2) (3.3) (3.4) and by Lemma [III.1] we obtain

$$F^+(u) \leqslant \limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon)$$
$$\leqslant G(u) + L\rho(M)\left(\frac{c}{t}\right)^p \limsup_{\varepsilon\to 0}\frac{1}{r_\varepsilon}\int_{\Sigma_\varepsilon}| u |^p dx + \frac{L}{t^p}\limsup_{\varepsilon\to 0}\frac{1}{r_\varepsilon}\int_{\Sigma_\varepsilon}| u |^p\gamma\left(x, \frac{v}{d}\right)dx$$
$$= G(u) + \frac{L}{t^p}\int_{\partial\Omega} d(\sigma) | u(\sigma) |^p\left[\rho(M)c^p + \gamma\left(\sigma, \frac{v}{d}\right)\right]dH^{n-1}(\sigma).$$

Letting $M \to +\infty$ and $t \to 1$ yields

$$F^+(u) \leqslant G_L(u).$$

We now prove the inequality $G_L \leqslant F^-$.

To this aim, we must show that if $u_\varepsilon \in W_0^{1,p}(\Omega_\varepsilon)$ and $u_\varepsilon \to u$ in $L^p(\mathbb{R}^n)$

$$G_L(u) \leqslant \liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon).$$

Without loss of generality we may assume that for a suitable sequence (ε_h)

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) = \lim_{h\to\infty} F_{\varepsilon_h}(u_{\varepsilon_h}) < +\infty .$$

so that (omitting for simplicity the subscript h)

$$\int_\Omega |Du_\varepsilon|^p dx + \lambda\varepsilon \int_{\Sigma_\varepsilon} |Du_\varepsilon|^p dx \leqslant K, \tag{3.5}$$

hence $u \in W^{1,p}(\Omega)$. By the semicontinuity of G, it will suffice to prove that

$$\liminf_{\varepsilon\to 0} \varepsilon \int_{\Sigma_\varepsilon} f(x, Du_\varepsilon)dx \geqslant L \int_{\partial\Omega} d^{1-p}(\sigma)\gamma(\sigma, \nu(\sigma))\,|u(\sigma)|^p d H^{n-1}(\sigma).$$

Fix $M > 0$ and set $B_{M,\varepsilon} = \{ x \in \mathbb{R}^n : |Du_\varepsilon(x)| < M \}$; then by (2.4) (3.5) it follows

$$\begin{aligned}
&\varepsilon \int_{\Sigma_\varepsilon} f(x, Du_\varepsilon)dx \\
&\quad\geqslant \varepsilon \int_{\Sigma_\varepsilon} \gamma(x, Du_\varepsilon)dx - \varepsilon \int_{\Sigma_\varepsilon \cap B_{M,\varepsilon}} \rho(0)(1+M^p)dx - \varepsilon \int_{\Sigma_\varepsilon - B_{M,\varepsilon}} \rho(M)(1+|Du_\varepsilon|^p)dx \\
&\quad\geqslant \varepsilon \int_{\Sigma_\varepsilon} \gamma(x, Du_\varepsilon)dx - \varepsilon \operatorname{meas}(\Sigma_\varepsilon)\rho(0)(2+M^p) - \frac{K}{\lambda}\rho(M).
\end{aligned}$$

Since M is arbitrary, we may only prove that

$$\liminf_{\varepsilon\to 0} \varepsilon \int_{\Sigma_\varepsilon} \gamma(x, Du_\varepsilon)dx \geqslant L \int_{\partial\Omega} d^{1-p}(\sigma)\gamma(\sigma, \nu(\sigma))\,|u(\sigma)|^p d H^{n-1}(\sigma). \tag{3.6}$$

If we set for every $h \in \mathbb{N}$

$$\gamma_h(x, z) = \inf\{ \gamma(x, w) + h\,|z - w|^p : w \in \mathbb{R}^n \},$$

then for a suitable sequence (c_h) vanishing as $h \to +\infty$ we have

(3.7)
- *i)* for every $x \in \mathbb{R}^n$, $\gamma_h(x, .)$ is convex, p-homogeneous and of class C^1;
- *ii)* for every $x_1, x_2, z \in \mathbb{R}^n$
$$|\gamma_h(x_1, z) - \gamma_h(x_2, z)| \leqslant (1 + c_h)\omega(|x_1 - x_2|)\,|z|^p;$$
- *iii)* for every $x, z \in \mathbb{R}^n$ we have
$$(\lambda - c_h)\,|z|^p \leqslant \gamma_h(x, z) \leqslant \Lambda\,|z|^p;$$
- *iv)* for every $x, z \in \mathbb{R}^n$ we have
$$0 \leqslant \gamma(x, z) - \gamma_h(x, z) \leqslant c_h\,|z|^p.$$

By (3.5) and (3.7) *iv*) we have

$$\lim_{h\to\infty}\left\{\liminf_{\varepsilon\to 0}\varepsilon\int_{\Sigma_\varepsilon}\gamma_h(x, Du_\varepsilon)dx - \mathrm{L}\int_{\partial\Omega} d^{1-p}(\sigma)\gamma_h(\sigma, \nu(\sigma))\,|\,u(\sigma)\,|^p d\mathrm{H}^{n-1}(\sigma)\right\}$$
$$= \liminf_{\varepsilon\to 0}\varepsilon\int_{\Sigma_\varepsilon}\gamma(x, Du_\varepsilon)dx - \mathrm{L}\int_{\partial\Omega} d^{1-p}(\sigma)\gamma(\sigma, \nu(\sigma))\,|\,u(\sigma)\,|^p d\mathrm{H}^{n-1}(\sigma),$$

therefore in (3.6) we may assume also that $\gamma(x, .)$ is a C^1 function. For every $\sigma\in\partial\Omega$ we set $v(\sigma) = D_z\gamma(\sigma, \nu(\sigma))$; then by Euler's theorem we have

$$\langle v(\sigma), \nu(\sigma)\rangle = p\gamma(\sigma, \nu(\sigma)) \geqslant p\lambda > 0.$$

By the regularity of $\partial\Omega$, the mapping $(\sigma, t) \mapsto \sigma + tv(\sigma)$ is invertible on Σ_ε if ε is sufficiently small, so that there exists $r_\varepsilon(\sigma) > 0$ such that

$$\Sigma_\varepsilon = \{\,\sigma + tv(\sigma) : \sigma\in\partial\Omega, 0 < t < r_\varepsilon(\sigma)\,\}.$$

By using the regularity assumptions on $\partial\Omega$ and $d(\sigma)$, one may easily verify that

$$(3.8)\qquad \lim_{\varepsilon\to 0}\frac{r_\varepsilon d(\sigma)}{r_\varepsilon(\sigma)} = \langle v(\sigma), \nu(\sigma)\rangle$$

uniformly in $\sigma\in\partial\Omega$. By simple changes of variables we obtain, writing briefly $Du_\varepsilon(x)$ in place of $Du_\varepsilon(\sigma + tv(\sigma))$,

$$\int_{\Sigma_\varepsilon}\gamma(x, Du_\varepsilon)dx$$
$$= \int_{\partial\Omega} d\mathrm{H}^{n-1}(\sigma)\int_0^{r_\varepsilon(\sigma)}\gamma(\sigma + tv(\sigma), Du_\varepsilon(x))\langle v(\sigma), \nu(\sigma)\rangle(1 + tq(t, \sigma))dt$$

where q is a suitable bounded function. Then

$$\liminf_{\varepsilon\to 0}\varepsilon\int_{\Sigma_\varepsilon}\gamma(x, Du_\varepsilon)dx$$
$$= \liminf_{\varepsilon\to 0}\varepsilon\int_{\partial\Omega} d\mathrm{H}^{n-1}(\sigma)\int_0^{r_\varepsilon(\sigma)}\gamma(\sigma + tv(\sigma), Du_\varepsilon(x))\langle v(\sigma), \nu(\sigma)\rangle\, dt.$$

By (3.5) and (3.7) *ii*) we obtain

$$\left|\varepsilon\int_{\partial\Omega} d\mathrm{H}^{n-1}(\sigma)\int_0^{r_\varepsilon(\sigma)}[\gamma(\sigma + tv(\sigma), Du_\varepsilon(x)) - \gamma(\sigma, Du_\varepsilon(x))]\langle v(\sigma), \nu(\sigma)\rangle\, dt\right|$$
$$\leqslant \varepsilon\int_{\partial\Omega} d\mathrm{H}^{n-1}(\sigma)\int_0^{r_\varepsilon(\sigma)}\omega(t\,|\,v(\sigma)\,|)\,|\,Du_\varepsilon(x)\,|^p\langle v(\sigma), \nu(\sigma)\rangle\, dt$$
$$\leqslant \varepsilon\omega_\varepsilon\int_{\Sigma_\varepsilon}|\,Du_\varepsilon\,|^p dx \leqslant \omega_\varepsilon\frac{\mathrm{K}}{\lambda}$$

with ω_ε vanishing as $\varepsilon \to 0$. Therefore

$$(3.9) \quad \liminf_{\varepsilon\to 0} \varepsilon \int_{\Sigma_\varepsilon} \gamma(x, Du_\varepsilon)dx = \liminf_{\varepsilon\to 0} \varepsilon \int_{\partial\Omega} dH^{n-1}(\sigma) \int_0^{r_\varepsilon(\sigma)} \gamma(\sigma, Du_\varepsilon(x)) \langle v(\sigma), \nu(\sigma) \rangle dt .$$

For all $\sigma \in \partial\Omega$ we have

$$(3.10) \quad | u_\varepsilon(\sigma) |^p = \left| \int_0^{r_\varepsilon(\sigma)} \langle Du_\varepsilon(\sigma + tv(\sigma)), v(\sigma) \rangle dt \right|^p \leqslant | r_\varepsilon(\sigma) |^{p-1} \int_0^{r_\varepsilon(\sigma)} | \langle Du_\varepsilon(x), v(\sigma) \rangle |^p dt .$$

Using (3.10) and Lemma [III.2] we finally get

$$\begin{aligned} &\varepsilon \int_{\partial\Omega} dH^{n-1}(\sigma) \int_0^{r_\varepsilon(\sigma)} \gamma(\sigma, Du_\varepsilon(x)) \langle v(\sigma), \nu(\sigma) \rangle dt \\ &\geqslant \varepsilon \int_{\partial\Omega} dH^{n-1}(\sigma) \int_0^{r_\varepsilon(\sigma)} \gamma(\sigma, \nu(\sigma)) | \langle Du_\varepsilon(x), v(\sigma) \rangle |^p \langle v(\sigma), \nu(\sigma) \rangle^{1-p} dt \\ &\geqslant \varepsilon \int_{\partial\Omega} | r_\varepsilon(\sigma) \langle v(\sigma), \nu(\sigma) \rangle |^{1-p} | u_\varepsilon(\sigma) |^p \gamma(\sigma, \nu(\sigma)) dH^{n-1}(\sigma) , \end{aligned}$$

and (3.6) follows from (3.8) (3.9) and from the convergence of u_ε to u in $L^p(\partial\Omega)$. ■

Proof of Theorem [II.2] *in the case* $L = +\infty$.

Since $F_\varepsilon \leqslant G_\infty$ for all ε, the inequality $F^+ \leqslant G_\infty$ is trivial. As for the inequality $F^- \geqslant G_\infty$ we remark that, since $\varepsilon/r_\varepsilon^{p-1} \to +\infty$, given any $L > 0$ we have $\varepsilon \geqslant L r_\varepsilon^{p-1}$ for ε small enough; therefore for every $u \in W_0^{1,p}(\Omega_\varepsilon)$

$$F_\varepsilon(u) \geqslant G(u) + L r_\varepsilon^{p-1} \int_{\Sigma_\varepsilon} f(x, Du)dx .$$

By the first part of the proof (case $L < +\infty$) we obtain

$$F^-(u) \geqslant G(u) + L \int_{\partial\Omega} | u(\sigma) |^p d^{1-p}(\sigma)\gamma(\sigma, \nu(\sigma))dH^{n-1}(\sigma) ,$$

whence

$$F^-(u) \geqslant G_\infty(u)$$

by taking the supremum with respect to L. ■

We now apply the result to the study of the asymptotic behaviour (as $\varepsilon \to 0$) of the minimum values and minimum points of the functionals

$$F_\varepsilon(u) + \int_{\mathbb{R}^n} fudx ,$$

with $f \in L^q(\mathbb{R}^n)$, $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

Since $u \mapsto \int_{\mathbb{R}^n} fudx$ is continuous in the strong topology of $L^p(\mathbb{R}^n)$, we have (see [*4*], Theorem 2.3)

$$\Gamma^-(L^p(\mathbb{R}^n)) \lim_{\varepsilon\to 0}\left(F_\varepsilon(u) + \int_{\mathbb{R}^n} fudx\right) = G_L(u) + \int_{\mathbb{R}^n} fudx .$$

Therefore, in order to apply Theorem [II.1], we must prove the equi-coerciveness of $\left(F_\varepsilon(u) + \int_{\mathbb{R}^n} fudx\right)_{\varepsilon>0}$ in the topology $L^p(\mathbb{R}^n)$. By the inequalities (2.3) (2.6), we only have to show that if $u_\varepsilon \in W_0^{1,p}(\Omega_\varepsilon)$ and

$$\int_\Omega |Du_\varepsilon|^p dx + \varepsilon\lambda\int_{\Sigma_\varepsilon} |Du_\varepsilon|^p dx + \int_{\Omega_\varepsilon} fudx \leqslant c \tag{3.11}$$

then (u_ε) is relatively compact in $L^p(\mathbb{R}^n)$.

THEOREM [III.3]. — *Assume* $0 < L \leqslant +\infty$; *if* (3.11) *holds, then* (u_ε) *is relatively compact in* $L^p(\mathbb{R}^n)$.

Proof. — By (3.11) it immediately follows that for any $\delta > 0$ there exists a constant c_δ such that

$$\int_\Omega |Du_\varepsilon|^p dx+\varepsilon\int_{\Sigma_\varepsilon} |Du_\varepsilon|^p dx \leqslant c_\delta + \delta\int_{\Omega_\varepsilon} |u_\varepsilon|^p dx . \tag{3.12}$$

For all $\sigma \in \partial\Omega$ and $t \in [0, r_\varepsilon d(\sigma)]$

$$\begin{aligned} |u_\varepsilon(\sigma + tv(\sigma))|^p &= \left|\int_t^{r_\varepsilon d(\sigma)} \langle Du_\varepsilon(\sigma + sv(\sigma)), v(\sigma)\rangle ds\right|^p \\ &\leqslant (r_\varepsilon d(\sigma) - t)^{p-1}\int_0^{r_\varepsilon d(\sigma)} |Du_\varepsilon(\sigma + sv(\sigma))|^p ds ; \end{aligned} \tag{3.13}$$

if ε is small enough, for suitable constants c we have

$$\begin{aligned} \int_{\Sigma_\varepsilon} |u_\varepsilon|^p dx &= \int_{\partial\Omega} dH^{n-1}(\sigma)\int_0^{r_\varepsilon d(\sigma)} |u_\varepsilon(\sigma + tv(\sigma))|^p(1 + tq(\sigma, t))dt \\ &\leqslant c\int_{\partial\Omega} dH^{n-1}(\sigma)\left[\int_0^{r_\varepsilon d(\sigma)} (r_\varepsilon d(\sigma)-t)^{p-1}dt\int_0^{r_\varepsilon d(\sigma)} |Du_\varepsilon(\sigma+sv(\sigma))|^p(1+sq(\sigma,s))ds\right] \\ &\leqslant cr_\varepsilon^p\int_{\Sigma_\varepsilon} |Du_\varepsilon|^p dx . \end{aligned}$$

Since $r_\varepsilon^{p-1}/\varepsilon$ is bounded,

$$\int_{\Sigma_\varepsilon} |u_\varepsilon|^p dx \leqslant cr_\varepsilon\varepsilon\int_{\Sigma_\varepsilon} |Du_\varepsilon|^p dx . \tag{3.14}$$

It follows from (3.13), with $t = 0$, that

$$\int_{\partial\Omega} |u_\varepsilon(\sigma)|^p d\mathrm{H}^{n-1}(\sigma) \leqslant c\varepsilon \int_{\Sigma_\varepsilon} |\mathrm{D}u_\varepsilon|^p dx,$$

so that

$$\begin{aligned} \int_\Omega |u_\varepsilon|^p dx &\leqslant c\left[\int_\Omega |\mathrm{D}u_\varepsilon|^p dx + \int_{\partial\Omega} |u_\varepsilon(\sigma)|^p d\mathrm{H}^{n-1}(\sigma)\right] \\ &\leqslant c\left[\int_\Omega |\mathrm{D}u_\varepsilon|^p dx + \varepsilon\int_{\Sigma_\varepsilon} |\mathrm{D}u_\varepsilon|^p dx\right]. \end{aligned} \tag{3.15}$$

If δ is properly chosen, by (3.12) (3.14) and (3.15) we obtain

$$\int_\Omega |\mathrm{D}u_\varepsilon|^p dx + \varepsilon\int_{\Sigma_\varepsilon} |\mathrm{D}u_\varepsilon|^p dx \leqslant c,$$

whence $\int_{\Sigma_\varepsilon} |u_\varepsilon|^p dx \to 0$ and $\|u_\varepsilon\|_{\mathrm{W}^{1,p}(\Omega)} \leqslant c$. ■

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