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JOHN I. E. URBAS

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# The generalized Dirichlet problem for equations of Monge-Ampère type

by

**John I. E. URBAS**
Centre for Mathematical Analysis, Australian National University,
G. P. O. Box 4, Canberra A. C. T. 2601, Australia

ABSTRACT. — We consider a generalization of the Dirichlet problem for equations of Monge-Ampère type under weaker hypotheses than are generally necessary for the solvability of the Dirichlet problem in the classical sense. We prove that there exists a unique smooth convex function which satisfies the equation in the classical sense and the boundary condition in a certain generalized sense.

RÉSUMÉ. — On considère une généralisation du problème de Dirichlet pour les équations du type Monge-Ampère sous des hypothèses plus faibles que celles qui sont en général nécessaires pour assurer l'existence au sens classique du problème de Dirichlet. On démontre qu'il existe une fonction convexe régulière unique qui satisfait l'équation au sens classique et qui vérifie la condition à la frontière en un sens généralisé.

## 1. INTRODUCTION

In recent years the Dirichlet problem for equations of Monge-Ampère type,

$$(1.1) \qquad \det D^2u = f(x, u, Du) \quad \text{in} \quad \Omega, \qquad u = \phi \quad \text{on} \quad \partial\Omega,$$

has received considerable attention. Here $\Omega$ is a uniformly convex domain in $\mathbb{R}^n$, $\phi$ is a function defined on $\partial\Omega$, $f$ is a positive function on $\Omega \times \mathbb{R} \times \mathbb{R}^n$, and $Du$ and $D^2u$ denote the gradient and the Hessian respectively of the function $u$, which is assumed to be convex. The existence of smooth convex solutions of (1.1) has been established under various hypotheses and using a variety of techniques in the work of Pogorelov [*17*] [*18*] [*19*] [*20*], Cheng and Yau [*9*], P. L. Lions [*15*] [*16*], Caffarelli, Nirenberg and Spruck [*8*], Krylov [*14*], Ivochkina [*13*], Gilbarg and Trudinger [*12*] and Trudinger and Urbas [*25*]. The interior second derivative estimate of Trudinger and Urbas [*26*], combined with an approximation argument based on the existence theorems for smooth solutions derived in the above works, reduces the solvability of (1.1) with $C^{1,1}$ uniformly convex $\Omega$, $\phi \in C^{1,1}(\overline{\Omega})$ and $f \in C^{1,1}(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ to the derivation of an *a priori* maximum modulus estimate for convex solutions of (1.1) and the construction of suitable local barriers. In general, each of these procedures requires different structure conditions on the function $f$, and several such conditions are given in the papers mentioned above. A condition ensuring a maximum modulus estimate for convex solutions of (1.1) will be given later. A condition of the second type, which enables us to obtain a gradient estimate with $C^{1,1}$ uniformly convex $\Omega$ and arbitrary $\phi \in C^{1,1}(\overline{\Omega})$, provided we already have a maximum modulus bound for the solution, is the following:

$$(1.2) \quad f(x,z,p) \leq \mu(|z|)\,d(x,\partial\Omega)^{\beta}(1+|p|)^{\alpha} \quad \text{for all} \quad (x,z,p) \in \mathcal{N} \times \mathbb{R} \times \mathbb{R}^n,$$

where $\mu$ is positive and nondecreasing, $\mathcal{N}$ is a neighbourhood of $\partial\Omega$, and $\alpha \geq 0$ and $\beta > -1$ are constants such that $\beta \geq \alpha - n - 1$. This is proved in [*12*] and [*25*] for the case $\beta \geq 0$, and the extension to $\beta \in (-1, 0)$ is given in [*27*]. Conditions which allow us to construct globally Lipschitz or Hölder continuous convex subsolutions of (1.1) are given in [*8*] [*9*] [*15*] [*16*] and [*27*]. Other conditions ensuring the existence of globally smooth convex solutions of (1.1) have been formulated by Ivochkina [*13*]. However, in general these involve a restriction on the size of $|\phi|_{2;\partial\Omega}$, and therefore do not ensure the solvability of (1.1) for arbitrary $\phi \in C^{1,1}(\overline{\Omega})$.

In this paper we are concerned with the case that $f$ does not necessarily satisfy a condition such as (1.2). As is shown in [*25*] and [*29*], in this case we cannot generally solve (1.1) in the classical sense. However, we shall prove that under suitable hypotheses there is a unique convex solution $u \in C^2(\Omega) \cap L^\infty(\Omega)$ of the equation

$$(1.3) \qquad \det D^2u = f(x, u, Du) \quad \text{in} \quad \Omega,$$

which need not satisfy the boundary condition

$$(1.4) \qquad u = \phi \quad \text{on} \quad \partial\Omega$$

in the classical sense, but does satisfy it in a certain optimal or generalized sense. Our result is the following.

THEOREM 1.1. — *Let $\Omega$ be a $C^{1,1}$ uniformly convex domain in $\mathbb{R}^n$, $\phi \in C^{1,1}(\overline{\Omega})$ and $f \in C^{1,1}(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ a positive function such that*

$$(1.5) \qquad f_z \geq 0 \quad in \quad \Omega \times \mathbb{R} \times \mathbb{R}^n,$$

*and*

$$(1.6) \qquad f(x, N, p) \leq g(x)/h(p) \quad for\ all \quad (x, p) \in \Omega \times \mathbb{R}^n,$$

*where N is a constant and g, h are positive functions in $L^1(\Omega)$, $L^1_{loc}(\mathbb{R}^n)$ respectively such that*

$$(1.7) \qquad \int_\Omega g < \int_{\mathbb{R}^n} h.$$

*Suppose furthermore that*

$$(1.8) \quad f(x, z, p) \leq \tilde{g}(x)\tilde{h}(p) \quad for\ all \quad x \in \mathcal{N}, \qquad z \leq \sup_{\partial\Omega} \phi, \quad p \in \mathbb{R}^n,$$

*where $\mathcal{N}$ is a neighbourhood of $\partial\Omega$, and $\tilde{g} \in L^q(\mathcal{N})$, $q > n$, and $\tilde{h} \in L^\infty_{loc}(\mathbb{R}^n)$ are positive functions. Then there is a unique convex function $u \in C^2(\Omega) \cap L^\infty(\Omega)$ satisfying* (1.3) *and*

$$(1.9) \qquad \limsup_{x \to y} u(x) \leq \phi(y) \quad for\ all \quad y \in \partial\Omega,$$

*and such that if $v \in C^2(\Omega)$ is a convex function solving* (1.3) *and* $\limsup_{x \to y} v(x) \leq \phi(y)$ *for all $y \in \partial\Omega$, then $v \leq u$ in $\Omega$.*

We therefore see that $u$ can be characterized as the supremum of all the convex subsolutions of (1.3) which lie below $\phi$ on $\partial\Omega$, and the proof of the theorem shows that $u$ is also the infimum of all the convex supersolutions of (1.3) which lie above $\phi$ on $\partial\Omega$. We refer to $u$ as the solution of the *generalized Dirichlet problem*, although it should be noted that only the boundary condition needs to be interpreted in a generalized sense.

A version of Theorem 1.1 yielding a generalized solution of (1.3) was proved by Bakel'man [5] using polyhedral approximation in the case that $\Omega$ is bounded and convex, $\phi \in C^0(\partial\Omega)$ and $f$ has the form

$$(1.10) \qquad f(x, z, p) = g(x)/h(p) \quad \text{for all} \quad (x, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n,$$

where $g$, $h$ are positive functions in $L^1(\Omega)$, $L^1_{loc}(\mathbb{R}^n)$ respectively satisfying (1.7).

Theorem 1.1 includes a case of geometric interest, the equation of prescribed Gauss curvature

$$(1.11) \qquad \det D^2u = K(x)(1 + |Du|^2)^{(n+2)/2}.$$

We state this special case separately.

THEOREM 1.2. — *Let $\Omega$ be a $C^{1,1}$ uniformly convex domain in $\mathbb{R}^n$ and $\phi \in C^{1,1}(\overline{\Omega})$. Let $K \in C^{1,1}(\Omega) \cap L^q(\Omega)$, $q > n$, be a positive function satisfying*

$$(1.12) \qquad \int_\Omega K < \omega_n.$$

*Then there is a unique convex function* $u \in C^2(\Omega) \cap L^\infty(\Omega)$ *satisfying* (1.11) *and* (1.9), *and such that if* $v \in C^2(\Omega)$ *is a convex solution of* (1.11) *and* $\limsup_{x \to y} v(x) \leq \phi(y)$ *for all* $y \in \partial\Omega$, *then* $v \leq u$ *in* $\Omega$.

To prove Theorem 1.1 we first solve approximating Dirichlet problems and obtain a convex generalized solution $u$ of (1.3) which clearly satisfies all the conditions except regularity. The main difficulty in proving the regularity of $u$ is the fact that the known interior second derivative estimates for convex solutions of (1.1) require some control of the boundary behaviour of the solutions; in particular, the estimate in [*26*] depends on the boundary values $\phi$ and the modulus of continuity of the solution on $\partial\Omega$ (see [*20*] for a counterexample). This kind of information is precisely what is lacking in our case, so we first need to reduce the problem to a situation in which the interior second derivative estimate can be applied. This requires some measure theory which we shall develop in Section 2. In Section 2 we also give an exposition of the theory of generalized solutions of Monge-Ampère equations, since it follows easily from the measure theoretic results we prove, and because it is needed to prove Theorem 1.1. In the final section we prove Theorem 1.1.

Most of our notation is standard, as for example in [*12*]. Any other notation we use will be explained at the appropriate point.

I wish to thank Professor Neil Trudinger for his advice and interest in this work.

## 2. PRELIMINARY MEASURE THEORY AND GENERALIZED SOLUTIONS

In this section we describe the measure theory we need to prove Theorem 1.1.

Let $\Omega$ be a bounded convex domain in $\mathbb{R}^n$ and $u$ a convex function defined on $\Omega$. It will be useful to have an extension of $u$ to $\overline{\Omega}$, so for $y \in \overline{\Omega}$ we define

$$u(y) = \liminf_{x \to y} u(x) \tag{2.1}$$

whenever the right hand side is finite. Notice that this definition makes $u$ lower semicontinuous wherever $u$ is finite.

We associate some set functions with $u$ in the following way. We let

$$M = \text{boundary of } \{ (x, t) \in \Omega \times \mathbb{R} : u(x) \leq t \}. \tag{2.2}$$

Then M is a convex hypersurface in $\mathbb{R}^{n+1}$. For $y \in \overline{\Omega}$, we define

$$\tilde{\chi}_u(y) = \{ p \in \mathbb{R}^n : \text{there exists a supporting hyperplane of M at } (y, u(y)) \text{ with slope } p \}. \tag{2.3}$$

For any $E \subset \overline{\Omega}$, we define

$$\tilde{\chi}_u(E) = \bigcup_{y \in E} \tilde{\chi}_u(y), \tag{2.4}$$

$$\chi_u(E) = \tilde{\chi}_u(E \cap \Omega) \tag{2.5}$$

and

$$\chi_u^*(E) = \tilde{\chi}_u(E \cap \partial\Omega). \tag{2.6}$$

Notice that we have

$$\tilde{\chi}_u(\overline{\Omega}) = \mathbb{R}^n. \tag{2.7}$$

The set function $\chi_u$ is often called the *normal mapping* of $u$, and is of considerable use in the theory of convex functions. It appears that the mappings $\tilde{\chi}_u$ and $\chi_u^*$ have not previously been used.

A result of Aleksandrov [*1*] implies that the mapping $\tilde{\chi}_u$ is one to one modulo a set of measure zero in the following sense.

LEMMA 2.1. — *Let $\Omega$ and $u$ be as above. Then*

$$|\{ p \in \mathbb{R}^n : p \in \tilde{\chi}_u(x) \cap \tilde{\chi}_u(y) \quad \textit{for distinct } x, y \in \overline{\Omega} \}| = 0. \tag{2.8}$$

We omit the proof of Lemma 2.1. The corresponding result for the normal mapping $\chi_u$ is proved in [*9*] and [*21*].

Using Lemma 2.1 it can be shown without much difficulty that $\mathscr{A} = \{ E \subset \overline{\Omega} : \tilde{\chi}_u(E)$ is Lebesgue measurable $\}$ is a $\sigma$-algebra. Unions and intersections are straightforward; to handle the case of complements we use the identity

$$\tilde{\chi}_u(\overline{\Omega} - E) = [\mathbb{R}^n - \tilde{\chi}_u(E)] \cup [\tilde{\chi}_u(\overline{\Omega} - E) \cap \tilde{\chi}_u(E)].$$

Lemma 2.1 then ensures that the second set on the right hand side has measure zero.

We shall also use the following result which is proved in [*6*] and [*7*]. We state it in a form which is convenient for our purposes.

LEMMA 2.2. — *Let $\Omega$ be a bounded convex domain in $\mathbb{R}^n$ and $\{ u_m \}$ a sequence of convex functions converging in $C^0(\overline{\Omega})$ to a convex function $u$. Let $\{ x_m \} \subset \overline{\Omega}$ be a sequence converging to $x_0 \in \overline{\Omega}$ and $\{ p_m \} \subset \mathbb{R}^n$ a sequence converging to $p_0 \in \mathbb{R}^n$ such that $p_m \in \tilde{\chi}_{u_m}(x_m)$. Then $p_0 \in \tilde{\chi}_u(x_0)$.*

Now let $E \subset \overline{\Omega}$ be a closed set and let $\{ p_i \} \subset \tilde{\chi}_u(E)$ be a sequence converging to $p_0 \in \mathbb{R}^n$. Then we have $p_i \in \tilde{\chi}_u(x_i)$ for suitable $\{ x_i \} \subset E$, and by passing to a subsequence, we can assume that $\{ x_i \}$ converges to a point $x_0 \in E$. But then, by Lemma 2.2, we have $p_0 \in \tilde{\chi}_u(x_0)$. Thus we see that $\tilde{\chi}_u$ maps closed sets to closed sets, and hence the $\sigma$-algebra $\mathscr{A}$ contains all the Borel subsets of $\overline{\Omega}$.

Now let $R \in L^1(\mathbb{R}^n)$ be a positive function, and for each set $E \in \mathscr{A}$, define

$$(2.9) \qquad \tilde{\omega}(u)(E) = \tilde{\omega}(u, R)(E) = \int_{\tilde{\chi}_u(E)} R,$$

$$(2.10) \qquad \omega(u)(E) = \omega(u, R)(E) = \int_{\chi_u(E)} R.$$

and

$$(2.11) \qquad \omega^*(u)(E) = \omega^*(u, R)(E) = \int_{\chi_u^*(E)} R.$$

Generally we will denote these set functions by $\tilde{\omega}(u)$, $\omega(u)$ and $\omega^*(u)$; the R is included only when it is necessary to avoid confusion. From Lemma 2.1 it follows that $\tilde{\omega}(u)$, $\omega(u)$ and $\omega^*(u)$ are countably additive measures on $\overline{\Omega}$. They are finite because $R \in L^1(\mathbb{R}^n)$, and the Borel subsets of $\overline{\Omega}$ are measurable with respect to these measures, so they are Radon measures. In particular, the following regularity properties hold:

$$(2.12) \quad \tilde{\omega}(u)(E) = \inf \{ \tilde{\omega}(u)(U) : U \text{ is relatively open, } E \subset U \subset \overline{\Omega} \}$$

for each Borel set $E \subset \overline{\Omega}$, and

$$(2.13) \qquad \tilde{\omega}(u)(U) = \sup \{ \tilde{\omega}(u)(K) : K \text{ is compact, } K \subset U \}$$

for each relatively open set $U \subset \overline{\Omega}$. Similar statements with $\tilde{\omega}(u)$ replaced by $\omega(u)$ and $\omega^*(u)$ are of course also true. We note also that for each Borel set $E \subset \overline{\Omega}$ we have

$$(2.14) \qquad \tilde{\omega}(u)(E) = \omega(u)(E) + \omega^*(u)(E).$$

The next few results are concerned with the behaviour of the measures we have defined with respect to convergence of convex functions. A sequence $\{ \mu_i \}$ of Radon measures on $\overline{\Omega}$ is said to converge weakly on $\overline{\Omega}$ to a Radon measure $\mu$ if for each $\phi \in C^0(\overline{\Omega})$ we have

$$(2.15) \qquad \int \phi d\mu_i \to \int \phi d\mu.$$

This is equivalent to the following:

$$(2.16) \qquad \limsup \mu_i(K) \le \mu(K)$$

for each compact set $K \subset \overline{\Omega}$, and

$$(2.17) \qquad \liminf \mu_i(U) \ge \mu(U)$$

for each relatively open set $U \subset \overline{\Omega}$. A sequence $\{ \mu_i \}$ of Radon measures on $\Omega$ is said to converge weakly on $\Omega$ to a Radon measure $\mu$ if (2.15) holds for all $\phi \in C_0^0(\Omega)$, or equivalently, if (2.16) and (2.17) hold for each compact set $K \subset \Omega$ and each open set $U \subset \Omega$ respectively.

LEMMA 2.3. — *Let $\Omega$ be a bounded convex domain in $\mathbb{R}^n$, $R \in L^1(\mathbb{R}^n)$ a positive function and $\{u_m\}$ a sequence of convex functions converging in $C^0(\Omega)$ to a convex function $u$. Then $\tilde{\omega}(u_m)$ converges to $\tilde{\omega}(u)$ weakly on $\overline{\Omega}$.*

*Proof.* — Let $K \subset \overline{\Omega}$ be compact and let U be a neighbourhood of $\tilde{\chi}_u(K)$. We assert that for any $N > 0$ there exists $m_0 = m_0(N)$ such that for all $m \geq m_0$ we have

$$\tilde{\chi}_{u_m}(K) \cap \overline{B_N(0)} \subset U.$$

If not, then there exist a subsequence $\{u_{m_k}\}$ of $\{u_m\}$, which we also denote by $\{u_m\}$, a sequence $\{x_m\} \subset K$ and $p_m \in [\tilde{\chi}_{u_m}(x_m) \cap \overline{B_N(0)}] - U$ such that $x_m \to x_0 \in K$ and $p_m \to p_0 \in \overline{B_N(0)} - U$. By Lemma 2.2, we have $p_0 \in \tilde{\chi}_u(x_0)$, which is a contradiction.

Now let $\varepsilon > 0$ and choose $N > 0$ so large that $\displaystyle\int_{\mathbb{R}^n - B_N(0)} R < \varepsilon$. Then for sufficiently large $m$ we have

$$\begin{aligned}\tilde{\omega}(u_m)(K) &= \int_{\chi_{u_m}(K)\cap\overline{B_N(0)}} R + \int_{\chi_{u_m}(K)-\overline{B_N(0)}} R \\ &\leq \int_U R + \varepsilon.\end{aligned}$$

Thus

$$\limsup \tilde{\omega}(u_m)(K) \leq \int_U R + \varepsilon\,.$$

Letting $\varepsilon \to 0$ and using the regularity of the Lebesgue measure we obtain

$$\limsup \tilde{\omega}(u_m)(K) \leq \tilde{\omega}(u)(K)\,. \tag{2.18}$$

Now let $U \subset \overline{\Omega}$ be relatively open and let $K = \overline{\Omega} - U$. Using (2.18) and the fact that $\tilde{\omega}(u_m)(\overline{\Omega}) = \tilde{\omega}(u)(\overline{\Omega}) < \infty$ for each $m$, we obtain

$$\begin{aligned}\tilde{\omega}(u)(U) &= \tilde{\omega}(u)(\overline{\Omega}) - \tilde{\omega}(u)(K) \\ &\leq \tilde{\omega}(u_m)(\overline{\Omega}) - \limsup \tilde{\omega}(u_m)(K) \\ &= \liminf\,[\tilde{\omega}(u_m)(\overline{\Omega}) - \tilde{\omega}(u_m)(K)] \\ &= \limsup \tilde{\omega}(u_m)(U)\,.\end{aligned} \tag{2.19}$$

The lemma is therefore proved.

COROLLARY 2.4. — *Let $\Omega$ be a bounded convex domain in $\mathbb{R}^n$, $R \in L^1(\mathbb{R}^n)$ a positive function and $\{u_m\}$ a sequence of convex functions converging in $C^0(\Omega)$ to a convex function $u$. Then $\omega(u_m)$ converges to $\omega(u)$ weakly on $\Omega$.*

Notice that in general, if $\{u_m\}$ is a sequence of convex functions converging to a convex function $u$ in $C^0(\Omega)$ or even in $C^0(\overline{\Omega})$, we do not have $\omega(u_m) \to \omega(u)$ or $\omega^*(u_m) \to \omega^*(u)$ weakly on $\overline{\Omega}$. However, we do have the following result.

LEMMA 2.5. — *Let $\Omega$ be a bounded convex domain in $\mathbb{R}^n$, $R \in L^1(\mathbb{R}^n)$ a positive function and $\{ u_m \}$ a sequence of convex functions converging in $C^0(\Omega)$ to a convex function u, and suppose that we have*

$$\omega(u_m)(E) \leq \int_E g \tag{2.20}$$

*for each Borel set $E \subset \mathcal{N}$, where $\mathcal{N}$ is a neighbourhood of $\partial\Omega$ and $g \in L^1(\mathcal{N})$ is a non-negative function. Then $\omega^*(u_m)$ converges to $\omega^*(u)$ weakly on $\overline{\Omega}$.*

*Proof.* — Let $U \subset \partial\Omega$ be relatively open and $\varepsilon > 0$. Let $V \subset \mathcal{N}$ be a relatively open set such that $U = \partial\Omega \cap V$ and $\int_V g < \varepsilon$. Then by Lemma 2.3 and (2.14) we have

$$\begin{aligned}\omega(u)(V) + \omega^*(U) &\leq \liminf\, [\omega(u_m)(V) + \omega^*(u_m)(U)] \\ &\leq \limsup \omega(u_m)(V) + \liminf \omega^*(u_m)(U) \\ &\leq \varepsilon + \liminf \omega^*(u_m)(U).\end{aligned}$$

Letting $\varepsilon \to 0$ we obtain

$$\omega^*(u)(U) \leq \liminf \omega^*(u_m)(U). \tag{2.21}$$

By the definition of $\omega^*(u)$, we see that (2.21) holds for all relatively open $U \subset \overline{\Omega}$. Also, from Corollary 2.4, we have

$$\omega(u)(\Omega) \leq \liminf \omega(u_m)(\Omega). \tag{2.22}$$

Using (2.7), (2.14) and (2.22) we obtain

$$\begin{aligned}\omega(u)(\Omega) + \omega^*(u)(\overline{\Omega}) &= \lim\, [\omega(u_m)(\Omega) + \omega^*(u_m)(\overline{\Omega})] \\ &= \liminf \omega(u_m)(\Omega) + \limsup \omega^*(u_m)(\overline{\Omega}) \\ &\geq \omega(u)(\Omega) + \limsup \omega^*(u_m)(\overline{\Omega}).\end{aligned}$$

Thus

$$\omega^*(u)(\overline{\Omega}) \geq \limsup \omega^*(u_m)(\overline{\Omega}). \tag{2.23}$$

Now let $K \subset \overline{\Omega}$ be compact and let $U = \overline{\Omega} - K$. Then

$$\begin{aligned}\omega^*(u)(K) &= \omega^*(u)(\overline{\Omega}) - \omega^*(u)(U) \\ &\geq \limsup \omega^*(u_m)(\overline{\Omega}) - \liminf \omega^*(u_m)(U) \\ &\geq \limsup\, [\omega^*(u_m)(\overline{\Omega}) - \omega^*(u_m)(U)] \\ &= \limsup \omega^*(u_m)(K).\end{aligned} \tag{2.24}$$

The lemma is therefore proved.

*Remarks.* — *i*) The definitions (2.9), (2.10) and (2.11) make sense if we assume that $R \in L^1_{loc}(\mathbb{R}^n)$ rather than $R \in L^1(\mathbb{R}^n)$. Of course, the proofs of Lemmas 2.3 and 2.5 are then no longer valid, although Corollary 2.4

is still true. Versions of Corollary 2.4 with positive $R \in C^0(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ are proved in [*17*]. We shall not need this additional generality.

*ii*) We have defined the set functions $\tilde{\chi}_u$, $\chi_u$ and $\chi_u^*$ only for bounded convex domains $\Omega$ in $\mathbb{R}^n$. To define $\tilde{\chi}_u$ and $\chi_u^*$ we require the convexity of $\Omega$, but $\chi_u$, and also the associated measure $\omega(u)$, can be defined for a convex function $u$ on any domain $\Omega$ in $\mathbb{R}^n$. It is easy to see that Corollary 2.4 holds for arbitrary domains in $\mathbb{R}^n$.

We shall also need the notion of a generalized solution of the equation (1.3). Such concepts were introduced by Aleksandrov [*2*] and Bakel'man [*3*]. Expositions of these ideas, based primarily on the work of Aleksandrov [*2*], have been given by Pogorelov [*17*] and Cheng and Yau [*9*]. A different approach is given by Rauch and Taylor [*21*]. There are several different but equivalent ways of formulating the definition, depending on the structure of $f$.

For our purposes, and also because we have not proved the most general version of Corollary 2.4, it suffices to assume that $f \in C^0(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ and to adopt the following definition. A convex function $u \in C^0(\Omega)$ is said to be a *generalized solution* of the equation (1.3) if for any positive function $R \in L^1(\mathbb{R}^n) \cap C^0(\mathbb{R}^n)$ we have

$$\int_{\chi_u(E)} R = \int_E f(x, u(x), Du(x))R(Du(x))dx \tag{2.25}$$

for each Borel set $E \subset \Omega$. Recall that if $u$ is a convex function, then $Du$ exists almost everywhere.

If $u \in C^2(\Omega)$ is a convex solution of (1.3), then the gradient mapping $Du : \Omega \to \mathbb{R}^n$ is one to one on $\{ x \in \Omega : \det D^2u(x) > 0 \}$ with Jacobian $\det D^2u$ and $| Du(\{ x \in \Omega : \det D^2u(x) = 0 \}) | = 0$, so that a classical solution is a generalized solution. We note also that it is sufficient to make the definition with a fixed function R. However, for different equations it turns out to be convenient to use different functions.

LEMMA 2.6. — *Let $\Omega$ be a domain in $\mathbb{R}^n$ and $\{ f_m \} \subset C^0(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ sequence of non-negative functions converging to $f$ in $C^0(\Omega \times \mathbb{R} \times \mathbb{R}^n)$. Let $u_m \in C^0(\Omega)$ be a generalized solution of*

$$\det D^2u_m = f_m(x, u_m, Du_m) \quad in \quad \Omega, \tag{2.26}$$

*and suppose that $\{ u_m \}$ converges in $C^0(\Omega)$ to a convex function $u$. Then $u$ is a generalized solution of* (1.3).

*Proof.* — Let $R \in L^1(\mathbb{R}^n) \cap C^0(\mathbb{R}^n)$ be a positive function. Since $u_m, u$ are convex and $u_m \to u$ in $C^0(\Omega)$ we have $Du_m \to Du$ almost everywhere, and because $f_m \to f$ in $C^0(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ and $R \in C^0(\mathbb{R}^n)$ we have for almost all $x \in \Omega$,

$$\lim_{m \to \infty} f_m(x, u_m(x), Du_m(x))R(Du_m(x)) = f(x, u(x), Du(x))R(Du(x)) .$$

Now let $E \subset\subset \Omega$ be a Borel set. Then for any subdomain $\Omega'$ such that $E \subset\subset \Omega' \subset\subset \Omega$, we have

$$\sup_{\Omega'} |u_m| \leq C(\Omega'),$$

with C independent of $m$, and hence

$$\sup_{E} |Du_m| \leq 2C(\Omega')d(E, \partial\Omega')^{-1}.$$

Thus

$$\sup_{x\in E} f_m(x, u_m(x), Du_m(x))R(Du_m(x)) \leq C,$$

where C is independent of $m$. By the dominated convergence theorem we have

$$\lim_{m\to\infty} \int_E f_m(x, u_m(x), Du_m(x))R(Du_m(x))dx = \int_E f(x, u(x), Du(x))R(Du(x))dx,$$

and since

$$\omega(u_m)(F) = \int_{\chi_{u_m}(F)} R = \int_F f_m(x, u_m(x), Du_m(x))R(Du_m(x))dx$$

for all Borel sets $F \subset \Omega$, and $\omega(u_m) \to \omega(u)$ weakly on $\Omega$, we obtain

$$\int_{\chi_u(E)} R = \int_E f(x, u(x), Du(x))R(Du(x))dx \tag{2.27}$$

for all Borel sets $E \subset\subset \Omega$.

Now let $E \subset \Omega$ be a Borel set and let $E_m = E \cap \left\{ x \in \Omega : d(x, \partial\Omega) > \frac{1}{m} \right\}$. Then (2.27) holds with E replaced by $E_m$. From this and the monotone convergence theorem we see that (2.27) holds for all Borel sets $E \subset \Omega$, so the lemma is proved. //

## 3. PROOF OF THEOREM 1.1

We have now developed all the measure theory we need to prove Theorem 1.1. We also require some *a priori* estimates and an existence theorem concerning convex solutions of (1.1). The first of these is proved in [*25*] (see also [*15*] [*16*]).

LEMMA 3.1. — *Let $\Omega$ be a $C^{1,1}$ uniformly convex domain in $\mathbb{R}^n$, $\phi \in C^{1,1}(\overline{\Omega})$ and $f \in C^{1,1}(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ a positive function satisfying* (1.5), (1.6) (1.7) *and* (1.2) *with $\beta \geq 0$. Then there is a unique convex solution $u \in C^2(\Omega) \cap C^{0,1}(\overline{\Omega})$ of the classical Dirichlet problem* (1.1).

The next result is an *a priori* maximum modulus estimate for convex solutions of (1.1) and is due to Bakel'man [*4*]. The proof is short, so we include it here.

LEMMA 3.2. — *Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be a convex solution of the Dirichlet problem* (1.1), *where $f$ is positive and satisfies the conditions*

$$f \text{ is nondecreasing with respect to } z \text{ in } \Omega \times \mathbb{R} \times \mathbb{R}^n, \tag{3.1}$$

*and*

$$f(x, N, p) \leq \frac{g(x)}{h(p)} \quad \text{for all } x \in \Omega, \quad p \in \mathbb{R}^n, \tag{3.2}$$

*where N is a constant and $g$, $h$ are positive function in $L^1(\Omega)$, $L^1_{loc}(\mathbb{R}^n)$ respectively such that*

$$\int_\Omega g < \int_{\mathbb{R}^n} h. \tag{3.3}$$

*Then we have the estimate*

$$\min \{ \inf_{\partial\Omega} \phi, N \} - C \operatorname{diam} \Omega \leq u \leq \sup_{\partial\Omega} \phi, \tag{3.4}$$

*where C depends only on $n$, $g$ and $h$.*

*Proof.* — The second inequality holds because $u$ is convex. To prove the first we choose $R_0$ so large that

$$\int_\Omega g = \int_{B_{R_0}(0)} h.$$

Using the fact that the gradient mapping $Du : \Omega \to \mathbb{R}^n$ is one to one with Jacobian $\det D^2u$, we obtain for any $R > R_0$,

$$\int_{Du(\Omega_N)} h \leq \int_{\Omega_N} g < \int_{B_R(0)} h,$$

where $\Omega_N = \{ x \in \Omega : u(x) < N \}$, so there is a $p \in B_R(0) - Du(\Omega_N)$. We then have, by the convexity of $u$,

$$\begin{aligned} \inf_{\Omega_N} u &\geq \inf_{\partial\Omega_N} u - |p| \operatorname{diam} \Omega \\ &\geq \min \{ \inf_{\partial\Omega} \phi, N \} - R \operatorname{diam} \Omega . \end{aligned}$$

We now let $R \to R_0$ and estimate $R_0$ in terms of $n$, $g$ and $h$. //

We also require interior bounds for the second derivatives and their Hölder seminorms. For the first we require only the special case of affine boundary values which is proved in [*12*] [*15*] [*16*] and [*26*]. The second is taken from [*24*], which is based on the work of Evans [*10*].

LEMMA 3.3. — *Let $\Omega$ be a bounded convex domain in $\mathbb{R}^n$ and $f$ a positive function in $C^{1,1}(\Omega \times \mathbb{R} \times \mathbb{R}^n)$. Then if $u \in W^{4,n}_{loc}(\Omega) \cap C^3(\Omega) \cap C^{0,1}(\overline{\Omega})$*

*is a convex solution of* (1.3) *and u is equal to an affine function on* $\partial\Omega$, *we have for any* $\Omega' \subset\subset \Omega$,

$$\sup_{\Omega'} |D^2 u| \leq C, \tag{3.5}$$

*where* C *is a constant depending only on n*, $|u|_{1;\Omega}$, *f*, diam $\Omega$ *and* $d(\Omega', \partial\Omega)$.

LEMMA 3.4. — *Let* $u \in W^{4;n}_{loc}(\Omega) \cap C^{1,1}(\bar{\Omega})$ *be a convex solution of* (1.3) *on a bounded domain* $\Omega$ *in* $\mathbb{R}^n$, *where* $f \in C^{1,1}(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$ *is positive. Then for any* $\Omega' \subset\subset \Omega$ *we have*

$$[D^2 u]_{\alpha;\Omega'} \leq C, \tag{3.6}$$

*where* $\alpha \in (0, 1)$ *depends only on n*, $|u|_{2;\Omega}$ *and f, and* C *depends in addition on* diam $\Omega$ *and* $d(\Omega', \partial\Omega)$.

Notice that by elliptic regularity theory ([*12*], Lemma 17.16), $C^2(\Omega)$ convex solutions of (1.3) with positive $f \in C^{1,1}(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ are in fact in $C^{3,\alpha}(\Omega) \cap W^{4,p}_{loc}(\Omega)$ for all $\alpha \in (0, 1)$ and $p < \infty$.

We now proceed to the proof of Theorem 1.1. Let $\{f_m\}$ be a sequence of bounded positive functions in $C^{1,1}(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ satisfying $f_{m,z} \geq 0$, $f_m \leq f$ and $f_m = f$ for $d(x, \partial\Omega) > \dfrac{1}{m}$, $|z| + |p| < m$. By Lemma 3.1 there is a unique convex solution $u_m \in C^2(\Omega) \cap C^{0,1}(\bar{\Omega})$ of the Dirichlet problem

$$\det D^2 u_m = f_m(x, u_m, Du_m) \quad \text{in} \quad \Omega, \qquad u_m = \phi \quad \text{on} \quad \partial\Omega, \tag{3.7}$$

and by Lemma 3.2 the sequence $\{u_m\}$ is uniformly bounded, so we can choose a subsequence converging in $C^0(\Omega)$ to a convex function $u$ which, by Lemma 2.6, is a generalized solution of the equation (1.3). It is clear that $u$ satisfies the condition (1.9).

Next, let $v \in C^2(\Omega)$ be a convex solution of $\det D^2 v = f(x, v, Dv)$ in $\Omega$ satisfying $\limsup_{x \to y} v(x) \leq \phi(y)$ for all $y \in \partial\Omega$. Then by the comparison principle we have $v \leq u_m$ in $\Omega$ for all $m$, so letting $m \to \infty$ we obtain $v \leq u$ in $\Omega$.

It now remains only to show that $u \in C^2(\Omega)$. This will be carried out in the following lemmas. We define a function $H \in L^1(\mathbb{R}^n)$ by

$$H(p) = \min\left\{\frac{1}{1 + \tilde{h}(p)}, \frac{1}{(1 + |p|^2)^n}\right\}. \tag{3.8}$$

H is positive because $\tilde{h} \in L^1_{loc}(\mathbb{R}^n)$, and since $f_m \leq f$, we obtain from (1.8) that for any Borel set $E \subset \Omega$,

$$\int_{\chi_{u_m}(E)} H \leq \int_E \tilde{g} \tag{3.9}$$

and

$$\int_{\chi_u(\mathrm{E})} \mathrm{H} \leq \int_{\mathrm{E}} \tilde{g}\,. \tag{3.10}$$

LEMMA 3.5. — *For all* $\varepsilon \in (0, 1)$, $p_0 \in \mathbb{R}^n$ *and* $\nu \in \mathrm{S}^{n-1}$ *we have*

$$\int_{\mathrm{A}_{p_0,\nu,\varepsilon}} \mathrm{H} \geq \mathrm{C}\varepsilon^{n-1}\,, \tag{3.11}$$

*where*

$$\mathrm{A}_{p_0,\nu,\varepsilon} = \left\{ p \in \mathbb{R}^n : \frac{p - p_0}{|p - p_0|} \in \mathrm{B}_\varepsilon(\nu) \cap \mathrm{S}^{n-1} \right\} \tag{3.12}$$

*and* C *depends on* $n$, $p_0$ *and* H.

*Proof.* — Let $w$ be defined by

$$w(|p|) = \inf_{|q| \leq |p|} \mathrm{H}(q)\,.$$

Then

$$\begin{aligned}
\int_{\mathrm{A}_{p_0,\nu,\varepsilon}} \mathrm{H}(p)dp &\geq \int_{\mathrm{A}_{p_0,\nu,\varepsilon}} w(|p|)dp \\
&\geq \int_{\mathrm{A}_{p_0,\nu,\varepsilon}} w(|p_0| + |p - p_0|)dp \\
&= \left( \int_{\mathbb{R}^n} w(|p_0| + |p - p_0|)dp \right) \frac{\mathscr{H}^{n-1}(\mathrm{B}_\varepsilon(\nu) \cap \mathrm{S}^{n-1})}{\mathscr{H}^{n-1}(\mathrm{S}^{n-1})}
\end{aligned}$$

which gives the required result. //

Before stating the next lemma we recall the definition of the *generalized Gauss map* of a convex hypersurface. If M is a convex hypersurface in $\mathbb{R}^n$, the generalized Gauss image of a set $\mathrm{E} \subset \mathrm{M}$ is given by

$$\mathrm{G}(\mathrm{E}) = \bigcup_{y \in \mathrm{E}} \{ \eta \in \mathrm{S}^{n-1} : \eta \text{ is the outer unit normal to a supporting hyperplane of M at } y \}. \tag{3.13}$$

Thus G is a set function.

The next lemma is a generalization of a result used in [*28*] to prove the regularity of extremal solutions of the equation of prescribed Gauss curvature.

LEMMA 3.6. — *Assume that all the hypotheses of Theorem 1.1 are satisfied and let* $x_0 \in \partial\Omega$ *be a point such that*

$$u(x_0) = \liminf_{x \to x_0} u(x) < \phi(x_0)\,. \tag{3.14}$$

*Then*

$$\chi_u^*(x_0) = \phi\,.$$

*Proof.* — Let

$$M_m = \text{boundary} \quad \text{of} \quad \{ (x, t) \in \Omega \times \mathbb{R} : u_m(x) \leq t \},$$

where $\{ u_m \}$ is the sequence of $C^2(\Omega) \cap C^{0,1}(\overline{\Omega})$ approximations to $u$. We also define the measures $\tilde{\omega}(u)$, $\omega(u)$ and $\omega^*(u)$ on $\overline{\Omega}$ by (2.9), (2.10) and (2.11) with R replaced by H. Analogous measures are defined for each $u_m$.

Let

$$\gamma = \phi(x_0) - u(x_0) > 0 . \tag{3.15}$$

Then by the continuity of $\phi$, we have for some $\delta \in (0, 1)$ with $\Omega \cap B_\delta(x_0) \subset \mathcal{N}$,

$$\inf_{x \in \partial\Omega \cap B_\delta(x_0)} \phi(x) - u(x_0) \geq \frac{3\gamma}{4} . \tag{3.16}$$

For convenience, we assume that $u(x_0) = 0$. We have $M_m \to M$ in the sense that

$$\lim_{m \to \infty} [\sup_{\xi \in M} d(\xi, M_m) + \sup_{\xi \in M_m} d(\xi, M)] = 0 , \tag{3.17}$$

and M contains the line segment

$$l(x_0) = \{ x_0 \} \times [0, \phi(x_0)] .$$

Therefore, for all sufficiently large $m$, we have

$$(\overline{\Omega} \times \{ \gamma/4 \} ) \cap M_m \neq \phi .$$

Let $(x_m, \gamma/4)$ be the point of $(\overline{\Omega} \times \{ \gamma/4 \} ) \cap M_m$ nearest to $(x_0, \gamma/4)$. We then have $| x_m - x_0 | \to 0$ as $m \to \infty$.

Now let $\varepsilon \in (0, \delta)$ and $x \in \partial\Omega \cap B_\varepsilon(x_0)$. Then

$$\begin{aligned} \frac{u_m(x) - u_m(x_m)}{| x - x_m |} &\geq \frac{u_m(x) - u_m(x_m)}{| x - x_0 | + | x_0 - x_m |} \\ &\geq \gamma/4\varepsilon \end{aligned}$$

for all $m$ sufficiently large, say $m \geq m_0 = m_0(\varepsilon)$. Thus it follows that

$$\chi^*_{um}(\partial\Omega \cap B_\varepsilon(x_0)) \subset \{ p \in \mathbb{R}^n : | p | \geq \gamma/4\varepsilon \} \tag{3.18}$$

for all $m \geq m_0$, and hence

$$\begin{aligned} \omega^*(u_m)(\partial\Omega \cap B_\varepsilon(x_0)) &\leq \int_{\mathbb{R}^n - B_{\gamma/4\varepsilon}(0)} H(p) dp \\ &\leq \int_{\mathbb{R}^n - B_{\gamma/4\varepsilon}(0)} (1 + | p |^2)^{-n} dp \\ &\leq C_1(n, \gamma)\varepsilon^n . \end{aligned} \tag{3.19}$$

By (3.9) and Lemma 2.5 we have $\omega^*(u_m) \to \omega^*(u)$ weakly on $\overline{\Omega}$, and therefore

$$\omega^*(u)(\partial\Omega \cap B_\varepsilon(x_0)) \leq C_1(n, \gamma)\varepsilon^n \tag{3.20}$$

for all $\varepsilon \in (0, \delta)$.

Now assume that there is an affine function $w$ such that graph $w$ is a supporting hyperplane of M at $(x_0, u(x_0))$ and let $\tilde{u} = u - w$, so that $\tilde{u}(x_0) = 0$ and $\tilde{u} \geq 0$ in $\Omega$. For $t > 0$ let

$$\Gamma_t = \{ x \in \overline{\Omega} : (x, t) \in (\overline{\Omega} \times \{ t \}) \cap \tilde{M} \}, \tag{3.21}$$

where $\tilde{M}$ is defined by (2.2) with $u$ replaced by $\tilde{u}$. Then $\Gamma_t$ is a closed convex $n - 1$ dimensional surface in $\overline{\Omega}$, and $x_0 \in \Gamma_t$ for all $t$. Let $B_R(y) \supset B_{R/2}(z)$ be balls such that $\overline{\Omega} \subset \overline{B_{R/2}(z)}$ and $\partial\Omega \cap \partial B_R(y) = \partial\Omega \cap \partial B_{R/2}(z) = \{ x_0 \}$. Let $G_t$ and G denote the generalized Gauss maps of $\Gamma_t$ and $\Gamma = \partial B_R(y)$ respectively. Then for each $t > 0$ we have, by a simple geometric argument,

$$G_t(\Gamma_t - B_\varepsilon) \subset G(\Gamma - B_\varepsilon)$$

for $\varepsilon > 0$, where $B_\varepsilon = B_\varepsilon(x_0)$.

Let $z$ be an affine function such that graph $z$ is a supporting hyperplane of $\tilde{M}$ at $(x, \tilde{u}(x)) \in \tilde{M} \cap ((\overline{\Omega} - B_\varepsilon) \times \mathbb{R})$. Then, provided $Dz \neq 0$, we have

$$\frac{Dz}{|Dz|} \in G_t(\Gamma_t - B_\varepsilon),$$

where $t = \tilde{u}(x)$, and hence

$$\chi_{\tilde{u}}(\overline{\Omega} - B_\varepsilon) \subset \left\{ p \in \mathbb{R}^n : \frac{p}{|p|} \in G(\Gamma - B_\varepsilon) \right\} \cup \{ 0 \}.$$

Since $\chi_{\tilde{u}}(\overline{\Omega}) = \mathbb{R}^n$, we then have

$$\chi_{\tilde{u}}(\overline{\Omega} \cap B_\varepsilon) \supset \left\{ p \in \mathbb{R}^n : \frac{p}{|p|} \in G(\Gamma \cap B_\varepsilon) \right\}, \tag{3.22}$$

and hence

$$\chi_u(\overline{\Omega} \cap B_\varepsilon) \supset Dw + \left\{ p \in \mathbb{R}^n : \frac{p}{|p|} \in G(\Gamma \cap B_\varepsilon) \right\}. \tag{3.23}$$

From (3.23) and Lemma 3.5 we obtain

$$\begin{aligned} \tilde{\omega}(u)(\overline{\Omega} \cap B_\varepsilon) &= \int_{\chi_u(\Omega \cap B_\varepsilon)} H(p)dp \\ &\geq C_2(n, H, R, Dw)\varepsilon^{n-1}. \end{aligned} \tag{3.24}$$

Combining (3.20) and (3.24), and using (3.10), we now obtain for all $\varepsilon \in (0, \delta)$,

$$\begin{aligned} C_2\varepsilon^{n-1} &\leq \int_{\overline{\Omega} \cap B_\varepsilon(x_0)} \tilde{g} + \omega^*(\partial\Omega \cap B_\varepsilon(x_0)) \\ &\leq \| \tilde{g} \|_{L^q(\mathcal{N})} \varepsilon^{n(1-1/q)} + C_1\varepsilon^n. \end{aligned} \tag{3.25}$$

This gives a contradiction for $\varepsilon$ sufficiently small, since $q > n$, so the lemma is proved.

A lemma similar to the following one was proved by Pogorelov [*19*] [*20*]. Here we present a much simpler proof which is taken from Cheng and Yau [*9*] and is based on an idea of Nirenberg.

LEMMA 3.7. — *Assume that all the hypotheses of Theorem 3.1 are satisfied. Then every supporting hyperplane of* graph $u$ *contains at most one point of the boundary of* graph $u$.

*Proof.* — Suppose not. Then there is a supporting hyperplane of graph $u$ which contains a line segment joining two points of the boundary of graph $u$. From Lemma 3.6, we deduce that if $(x_1, u(x_1))$ and $(x_2, u(x_2))$ are these points, then $u(x_1) = \phi(x_1)$ and $u(x_2) = \phi(x_2)$. Let the supporting hyperplane in question be graph $w$. By replacing $u$ by $u - w$, we may assume that $u$ satisfies $\det D^2 u = f(x, u + w, Du + Dw)$ in $\Omega$ in the generalized sense, and also, that the supporting hyperplane is $\{ x \in \mathbb{R}^{n+1} : x_{n+1} = 0 \}$, and that the line segment $[(x_1, u(x_1)), (x_2, u(x_2))]$ is $\{ (t, 0, \ldots, 0) : |t| \leq b \}$.

Since $\partial\Omega$ is $C^{1,1}$ there exists $\varepsilon > 0$ such that there are two $C^{1,1}$ functions $\eta_1$, $\eta_2$ defined on $\left\{ x \in \Omega : x_1 = 0, \sum_{i>1} x_i^2 < \varepsilon \right\}$ such that the two components of $\partial\Omega \cap \left\{ x \in \mathbb{R}^n : \sum_{i>1} x_i^2 < \varepsilon \right\}$ are given by

$$\left\{ (\eta_i(x_2, \ldots, x_n), x_2 \ldots, x_n) : \sum_{i>1} x_i^2 < \varepsilon \right\}, \qquad i = 1, 2 .$$

Let

$$\phi_i(x_2, \ldots, x_n) = \phi(\eta_i(x_2, \ldots, x_n), x_2, \ldots, x_n) .$$

Then $\phi_i \in C^{1,1}$, $\phi_i \geq 0$ and $\phi_i(0) = 0$ for $i = 1, 2$. Hence there exist A, $\varepsilon_1 > 0$ such that

$$\phi_i(x_2, \ldots, x_n) \leq A \sum_{j>1} x_j^2$$

for all $\sum_{j>1} x_j^2 < \varepsilon_1$, with equality holding only when $x_2 = x_3 = \ldots = x_n = 0$.

Let

$$\Omega' = \Omega \cap \left\{ (x_1, \ldots, x_n) : \sum_{j>1} x_j^2 < \varepsilon_1 \right\} .$$

Since $u$ is convex, we have

$$u(x_1, \ldots, x_n) \leq A \sum_{j>1} x_j^2 \quad \text{on} \quad \Omega' ,$$

and equality holds only when $x_2 = x_3 = \ldots = x_n = 0$. For each $\Omega'' \subset\subset \Omega$, we have maximum modulus and gradient estimates for $u$ on $\Omega''$. Thus for some $\varepsilon_2 > 0$ we have

$$\phi \neq \Omega'' = \{ x \in \Omega : d(x, \partial\Omega) > \varepsilon_2 \} \cap \Omega' \subset\subset \Omega,$$

and

$$\det D^2 u \geq \lambda \quad \text{in} \quad \Omega''$$

in the generalized sense for some positive constant $\lambda$.

Let $\tilde{u}$ be defined by

$$\tilde{u}(x_1, \ldots, x_n) = \frac{\lambda}{2^{n+1}A^{n-1}} x_1^2 + A \sum_{j>1} x_j^2 .$$

Then $\tilde{u} > u$ on $\partial\Omega''$, $\tilde{u}(0) = u(0) = 0$, and $\det D^2 \tilde{u} = \lambda/2$. Since $\partial\Omega''$ is compact, there exists $\delta > 0$ such that $\tilde{u} = \delta > 0$ on $\partial\Omega''$. From the comparison principle for generalized solutions of $\det D^2 u = f(x)$ (see [*9*]) we obtain $\tilde{u} - \delta \geq u$ on $\Omega''$. This gives a contradiction because $\tilde{u}(0) = u(0)$. Thus the lemma is proved. //

We are now in a position to apply Lemmas 3.3 and 3.4 to prove the regularity of $u$. Let $x_0 \in \Omega$. We will show that $u$ is of class $C^2$ in a neighbourhood of $x_0$. Since $x_0$ is an arbitrary point of $\Omega$, it follows that $u \in C^2(\Omega)$. Let $w$ be an affine function such that graph $w$ is a supporting hyperplane of graph $w$ at $(x_0, u(x_0))$. By Lemma 3.7, we see that graph $w \cap \partial$ (graph $u$) contains at most one point, say $y$, if there is such a point. By replacing $u$ by $u - w$ as usual, we may assume that $w = 0$, $(x_0, u(x_0)) = (0, 0)$, and $y = (a, 0, \ldots, 0)$ with $a > 0$. Using the fact that $u$ is lower semi-continuous on $\overline{\Omega}$, we see that for $\varepsilon > 0$ sufficiently small, $z(x) = -\varepsilon x_1$ defines an affine function such that graph $z$ is a hyperplane passing through $(x_0, u(x_0))$, and graph $z \cap \partial$ (graph $u$) $= \phi$. Thus for $\delta > 0$ sufficiently small, $U = \{ x \in \Omega : u(x) < z(x) + 4\delta \}$ is an open set containing $x_0$, and $U \subset\subset \Omega$. For all sufficiently large $m$, the sets $\{ x \in \Omega : u_m(x) < z(x) + 3\delta \}$ are contained in a fixed compact subset of U, and $\{ x \in \Omega : u_m(x) < z(x) + \delta \}$ contains a fixed compact neighbourhood of $x_0$. We have uniform estimates for $| u_m |$ and $| Du_m |$ on $\{ x \in \Omega : u_m(x) < z(x) + 3\delta \}$ for sufficiently large $m$. Using Lemmas 3.3 and 3.4 we obtain uniform estimates for $| D^2 u_m |$ on $\{ x \in \Omega : u_m(x) < z(x) + 2\delta \}$, and then for $[D^2 u_m]_\alpha$ on $\{ x \in \Omega : u_m(x) < z(x) + \delta \}$ for sufficiently large $m$. We therefore have uniform estimates for $| u_m |_{2,\alpha}$ on a neighbourhood of $x_0$, from which it follows that $u$ is $C^2$ on this neighbourhood. The proof of Theorem 1.1 is therefore complete. //

*Remarks.* — *i*) The hypotheses of Theorem 1.1 can be weakened slightly. We need only assume (1.8) to hold in a neighbourhood $\mathcal{N}$ of those points $x_0 \in \partial\Omega$ at which a barrier ensuring $u(x_0) = \phi(x_0)$ cannot be constructed, and rather than $\tilde{g} \in L^q(\mathcal{N})$, $q > n$, we may assume that $\tilde{g} \in L^1(\mathcal{N})$ and

$\lim_{\varepsilon \to 0} \varepsilon^{1-n} \int_{\Omega \cap B_\varepsilon(x_0)} \tilde{g} = 0$ for each $x_0 \in \mathcal{N} \cap \partial\Omega$. Notice that we still have $\omega^*(u_m) \to \omega^*(u)$ weakly on $\mathcal{N}$.

*ii*) In the two dimensional case the solution is regular without assuming (1.8). In this case we do not need to assume that $\phi$ and $\partial\Omega$ are of class $C^{1,1}$ either, since the regularity of the solution is automatically ensured by the regularity of $f$, which can in fact be weakened to $f \in C^{0,\alpha}(\Omega \times \mathbb{R} \times \mathbb{R}^2)$ for some $\alpha \in (0, 1)$ (see [*22*] [*23*]).

*iii*) In the proof of Lemma 3.6 we do not use the $C^{1,1}$ regularity of $\phi$ or $\partial\Omega$. In fact, it suffices to assume only that $\phi \in C^0(\overline{\Omega})$ and $\Omega$ is uniformly convex. However, since we cannot generally specify *a priori* on which parts of $\partial\Omega$ the solution $u$ does not attain the boundary values $\phi$, we cannot really weaken the hypotheses on $\Omega$, except in the case $\phi$ is an affine function.

*iv*) Notice that no global continuity of the solution is asserted. If $u$ attains the boundary values $\phi$ in a neighbourhood of a point $x_0 \in \partial\Omega$, and $f$ satisfies appropriate structure conditions, then suitable barrier arguments yield modulus of continuity estimates for $u$ at $x_0$. The results of [*29*] show that the solution $u$ is Hölder continuous in a neighbourhood of a point $x_0 \in \partial\Omega$, provided $f$ satisfies suitable structure conditions, even in the case $u(x_0) \neq \phi(x_0)$.

*v*) Using the results of [*28*] and [*29*] it is not difficult to construct examples showing that the solution obtained in Theorem 1.1 need not attain the prescribed boundary values anywhere on $\partial\Omega$.

*vi*) Theorem 1.2 can be regarded as an analogue of a result proved by Giaquinta [*11*] for the equation of prescribed mean curvature

$$D_i\left(\frac{D_i u}{\sqrt{1 + |Du|^2}}\right) = H(x).$$

He proves that if $\Omega$ is a bounded Lipschitz domain in $\mathbb{R}^n$, $\phi \in C^0(\overline{\Omega})$, $H \in C^{0,1}(\overline{\Omega})$ and there exists a constant $\gamma \in (0, 1)$ such that

$$\left| \int_A H \right| \leq \gamma P(A)$$

for every measurable set $A \subset \Omega$, where $P(A)$ denotes the perimeter of $A$, then there is a function $u \in C^2(\Omega)$ minimizing the functional

$$F(v) = \int_\Omega (1 + |Dv|^2)^{1/2} + \int_\Omega Hv + \int_{\partial\Omega} |v - \phi| \, dH^{n-1}$$

with respect to all $v \in BV(\Omega)$.

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