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Minimax principles for lower semicontinuous functions and applications to nonlinear boundary value problems


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Minimax principles for lower semicontinuous functions and applications to nonlinear boundary value problems

by

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ABSTRACT. — Let X be a real Banach space and I a function on X such that $I = \Phi + \psi$ with $\Phi \in C^1(X, \mathbb{R})$ and $\psi : X \to (-\infty, +\infty]$ convex, proper and lower semicontinuous. A point $u \in X$ is said to be critical if

$$\langle \Phi'(u), v - u \rangle + \psi(v) - \psi(u) \geq 0 \quad \forall v \in X.$$ 

The paper contains a number of existence theorems for critical points of functions of the above mentioned type. Critical levels of saddle type are characterized by minimax principles. The results are applied to variational inequalities and variational equations with single- and multivalued operators, which arise from studying certain elliptic boundary value problems.

RESUMÉ. — Soit $X$ un espace de Banach et $I$ une fonction sur $X$ de la forme $I = \Phi + \psi$, où $\Phi$ est $C^1$ et $\psi$ est convexe s. c. i., pouvant prendre la valeur $+\infty$. On définit une notion naturelle de point critique, et l'on démontre des théorèmes d'existence par des méthodes de minimax, du type Liusternik-Schnivelman. On applique ces résultats à des équations et inéquations variationnelles de type elliptique.


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INTRODUCTION

The purpose of this paper is to generalize some minimax methods in critical point theory to a class of functions which are not necessarily continuous. Let $X$ be a real Banach space. Recall that for a continuously differentiable function $\Phi: X \to \mathbb{R}$, a point $u \in X$ is said to be critical if $\Phi'(u) = 0$. The corresponding number $\Phi(u)$ is called a critical value. It is well known that local maxima and minima are critical points. If $\Phi$ satisfies some appropriate compactness conditions (usually of Palais-Smale type), one may also find other critical points by minimizing $\Phi$ over certain families of subsets of $X$. More precisely, if $\Gamma$ is such a family, one can give sufficient conditions in order that the value

$$c = \inf_{A \in \Gamma} \sup_{u \in A} \Phi(u)$$

be critical. For an account of recent results in critical point theory for $C^1$ functions by minimax methods the reader is referred to [20] [21] [24]. Very recently critical point theory has been generalized by Chang [8] to locally Lipschitz continuous functions and by Struwe [26] [27] to functions which are of class $C^1$ with respect to certain families of subspaces of $X$. In this paper we present another generalization.

Let $X$ be a real Banach space and $\psi: X \to (-\infty, +\infty]$ a convex lower semicontinuous function. The set $D(\psi) = \{ u \in X : \psi(u) < +\infty \}$ is called the effective domain of $\psi$. Denote by $X^*$ the dual of $X$ and by $\langle \cdot, \cdot \rangle$ the duality pairing between $X^*$ and $X$. For $u \in D(\psi)$ the set

$$\partial\psi(u) = \{ u^* \in X^* : \psi(v) - \psi(u) \geq \langle u^*, v - u \rangle \quad \forall v \in X \}$$

is called the subdifferential of $\psi$ at $u$ [3, §II.2]. We shall consider functions $I = \Phi + \psi$ with $\Phi \in C^1(X, \mathbb{R})$ and $\psi$ as above. A point $u \in D(\psi)$ is said to be critical if $-\Phi'(u) \in \partial\psi(u)$, or equivalently, if $u$ satisfies the inequality

$$\langle \Phi'(u), v - u \rangle + \psi(v) - \psi(u) \geq 0 \quad \forall v \in X.$$

Inequalities of this type arise in a number of problems of physics [12].

In [10] [11] Dias and Hernández invoked results from critical point theory for $C^1$ functions in order to study eigenvalue problems $\lambda u \in \partial\psi(u)$ for $X$ a Hilbert space and $\psi$ as above. They used the fact that the operator $id + \partial\psi$ has a single-valued inverse which is of gradient type. Unfortunately, this approach does not seem to give results we want to obtain.
It is easy to see (cf. Proposition 1.1) that local minima are critical points of $I$. In order to be able to obtain other critical points we need a compactness condition (which is introduced in Section 1) and a deformation result. For $C^1$ functions the required deformation is effected by moving along integral lines of a pseudogradient vector field [21, Theorem 1.9; 24, Theorem 1.1]. In the case of functions which are only lower semicontinuous such a construction does not seem to be readily available, mainly because a noncritical value $c$ may be « semicritical » in the sense that there may exist a critical point $\tilde{u}$ with $I(\tilde{u}) < c$ and a sequence $u_n \to \tilde{u}$ with $I(u_n) \to c$. In Proposition 2.3 we obtain a result which in a sense is a weak version of the usual deformation theorem. Our deformation (denoted by $\alpha$) has the inconvenient property that $I(\alpha(u))$ may increase for some $u$. The proofs of existence of nonminimum critical points (which become rather technical because of that) are effected by combining Proposition 2.3 with Ekeland’s variational principle. The idea of using Ekeland’s principle to obtain critical points other than local minima (actually, to prove the Mountain Pass Theorem), may be found in [2] [6].

The paper is organized as follows: Section 1 contains preliminary material. In particular, we introduce a compactness condition and recall Ekeland’s variational principle. In Section 2 we prove a deformation result and in Section 3 we show that the Mountain Pass Theorem of Ambrosetti and Rabinowitz [1] [21] [24] and some related results [22] [23] [24] remain valid for functions satisfying our assumptions. Section 4 is devoted to generalizations of results of Clark [9] [21] [24] and Ambrosetti and Rabinowitz [1] [21] [24] concerning the existence of multiple critical points for even functions. In Section 5 we apply abstract results of Sections 3 and 4 to elliptic boundary value problems. Our examples include variational inequalities and variational equations with single- and multivalued operators.

After completing this paper I have been informed by I. Ekeland that for lower semicontinuous functions $I : X \to (-\infty, +\infty]$ having the property that $I(u) + c \|u\|^2$ is convex for some $c \geq 0$, there is a regularization procedure due to J. M. Lasry [28, Lemma 7], which associates with $I$ a family $(I_\varepsilon)_{0 < \varepsilon < 1}$ of functions such that $I_\varepsilon(u) \to I(u)$ $\forall u \in X$ as $\varepsilon \to 0$ and $I_\varepsilon \in C^1(X, \mathbb{R})$. Furthermore, $I_\varepsilon(u) \leq I(u)$ $\forall u \in X$, $I_\varepsilon$ and $I$ have the same critical points and $I_\varepsilon$ satisfies the Palais-Smale condition whenever $I$ satisfies a condition of similar type (cf. (PS)' below). Note that for such $I$ our Theorem 4.3 is an easy consequence of the above-mentioned result of Clark: $I_\varepsilon(u) \leq I(u)$, $I$ satisfies the hypotheses of Clark’s theorem and has therefore at least $k$ pairs of nontrivial critical points; hence so does $I$. Note also that in some of the applications to boundary value problems in Section 5 (Theorem 5.1 and Theorem 5.8 with subsequent corollaries) $I(u) + c \|u\|^2$ will not be convex for any $c \geq 0$. 

1. PRELIMINARIES

Let $X$ be a real Banach space and $I$ a function on $X$ satisfying the following hypothesis:

(H) $I = \Phi + \psi$, where $\Phi \in C^1(X, \mathbb{R})$ and $\psi : X \to (-\infty, +\infty]$ is convex, proper (i.e., $\psi \not\equiv +\infty$) and lower semicontinuous (l.s.c. in short).

A point $u \in X$ is said to be a critical point of $I$ if $u \in D(\psi)$ and if it satisfies the inequality

$$\langle \Phi'(u), v - u \rangle + \psi(v) - \psi(u) \geq 0 \quad \forall v \in X.$$  

Note that $X$ can be replaced by $D(\psi)$ in (1). A number $c \in \mathbb{R}$ such that $I^{-1}(c)$ contains a critical point will be called a critical value. We shall use the following notation:

$$K = \{ u \in X : u \text{ is a critical point} \},$$

$$I_c = \{ u \in X : I(u) \leq c \}, \quad K_c = \{ u \in K : I(u) = c \}.$$

1.1. PROPOSITION. — If $I$ satisfies (H), each local minimum is necessarily a critical point of $I$.

Proof. — Let $u$ be a local minimum of $I$. Using convexity of $\psi$, it follows that for all small $t > 0$,

$$0 \leq I((1-t)u + tv) - I(u) = \Phi(u + t(v-u)) - \Phi(u) + \psi((1-t)u + tv) - \psi(u)$$

$$\leq \Phi(u + t(v-u)) - \Phi(u) + t(\psi(v) - \psi(u)).$$

Dividing by $t$ and letting $t \to 0$ we obtain (1). \qed

We shall assume that $I$ satisfies the following compactness condition of Palais-Smale type:

(PS) If $(u_n)$ is a sequence such that $I(u_n) \to c \in \mathbb{R}$ and

$$\langle \Phi'(u_n), v - u_n \rangle + \psi(v) - \psi(u_n) \geq - \varepsilon_n \| v - u_n \| \quad \forall v \in X,$$

where $\varepsilon_n \to 0$, then $(u_n)$ possesses a convergent subsequence.

Condition (PS) can also be formulated as follows:

(PS') If $(u_n)$ is a sequence such that $I(u_n) \to c \in \mathbb{R}$ and

$$\langle \Phi'(u_n), v - u_n \rangle + \psi(v) - \psi(u_n) \geq \langle z_n, v - u_n \rangle \quad \forall v \in X,$$

where $z_n \to 0$, then $(u_n)$ possesses a convergent subsequence.

1.2. PROPOSITION. — Conditions (PS) and (PS') are equivalent.

Observe that (3) expresses the fact that $z_n - \Phi'(u_n) \in \partial \psi(u_n)$, or equiva-
lently, that \( z_n \in \Phi'(u_n) + \partial \psi(u_n) \). In this notation the similarity to (a version of) the usual Palais-Smale condition becomes more transparent. Observe also that if \( \psi \equiv 0 \), then \( I \in C^1 \) and our definition of critical point as well as our condition (PS) coincide with the usual ones.

In order to prove Proposition 1.2 we need an additional result. In what follows we shall use the same symbol \( \| \| \) to denote the norms in \( X \) and \( X^* \).

1.3. **Lemma.** — Let \( X \) be a real Banach space and \( \chi : X \to (-\infty, +\infty] \) a l.s.c. convex function with \( \chi(0) = 0 \). If

\[
\chi(x) \geq -\|x\| \quad \forall x \in X,
\]

there exists a \( z \in X^* \) such that \( \|z\| \leq 1 \) and

\[
\chi(x) \geq \langle z, x \rangle \quad \forall x \in X.
\]

It is well known that \( \chi \) is bounded below by an affine function, i.e., \( \chi(x) \geq \langle z, x \rangle - \beta \) for some \( z \in X^* \) and \( \beta \in \mathbb{R} \) [3, Proposition II.2.1]. The lemma asserts that under our assumptions we can choose \( z \) with norm \( \leq 1 \) and \( \beta = 0 \).

**Proof of lemma.** — The proof was suggested to us by P. O. Lindberg. In the space \( X \times \mathbb{R} \) define

\[
A = \{(x, t) : \|x\| < -t\} \quad \text{and} \quad B = \{(x, t) : \chi(x) \leq t\}.
\]

It is easy to verify that \( A \) and \( B \) are convex (in fact, \( B \) is the epigraph of \( \chi \)) and \( A \) is open. Moreover, \( A \cap B = \emptyset \) because \( \chi(x) \geq -\|x\| \). Consequently, there exists a hyperplane separating \( A \) and \( B \), i.e., we can find \( \alpha, \beta \in \mathbb{R} \) and \( w \in X^* \) such that

\[
\langle w, x \rangle - \alpha t - \beta \geq 0 \quad \forall (x, t) \in A,
\]

\[
\langle w, x \rangle - \alpha t - \beta \leq 0 \quad \forall (x, t) \in B.
\]

Since \( (0, 0) \in \overline{A} \cap B, \beta = 0 \). Taking \( t = -\|x\| \) in the first of these inequalities gives

\[
\langle w, x \rangle \geq -\|x\| \quad \forall x \in X.
\]

It follows that \( \alpha \geq 0 \) and \( \|w\| \leq \alpha \). If \( \alpha = 0 \), then \( w = 0 \) and there is no hyperplane. So \( \alpha > 0 \). Set \( z = w/\alpha \) and \( t = \chi(x) \) in the second of the above inequalities. Then \( \|z\| \leq 1 \) and \( \langle z, x \rangle \leq t = \chi(x) \). \( \square \)

**Proof of Proposition 1.2.** — It suffices to prove that (2) and (3) are equivalent and it is clear that (3) implies (2). So suppose that (2) is satisfied. If \( \epsilon_n \leq 0 \), we may take \( z_n = 0 \). If \( \epsilon_n > 0 \), let \( x = v - u_n \) and \( \chi(x) = (\langle \Phi'(u_n), x \rangle + \psi(x + u_n) - \psi(u_n))/\epsilon_n \). Then (2) reads

\[
\chi(x) \geq -\|x\| \quad \forall x \in X.
\]
According to Lemma 1.3 there is a \( \zeta_n \in X^* \) with \( \| \zeta_n \| \leq 1 \) and \( \chi(x) \geq \langle \zeta_n, x \rangle \). Setting \( z_n = \varepsilon_n \zeta_n \) gives
\[
\langle \Phi'(u_n), v - u_n \rangle + \psi(v) - \psi(u_n) / \varepsilon_n \geq \langle z_n / \varepsilon_n, v - u_n \rangle.
\]
Hence (3) is satisfied and \( z_n \to 0 \) because \( \varepsilon_n \to 0 \). \qed

1.4. Proposition. — Suppose that \( I \) satisfies (H) and (PS) and let \( (u_n) \) be a sequence verifying the hypotheses of (PS). If \( u \) is an accumulation point of \( (u_n) \), then \( u \in K_c \). In particular, \( K_c \) is a compact set.

Proof. — We may assume that \( u_n \to u \). Passing to the limit in (2) and using the fact that \( \lim \psi(u_n) \geq \psi(u) \), we obtain (1). Hence \( u \in K \). Moreover, since inequality (1) cannot be strict for \( v = u \), \( \lim \psi(u_n) = \psi(u) \). Consequently, \( I(u_n) \to I(u) = c \) and \( u \in K_c \).

If \( (u_n) \subset K_c \), then \( I(u_n) = c \) and (2) is satisfied with \( \varepsilon_n = 0 \). It follows that a subsequence of \( (u_n) \) converges to some \( u \in X \). By the first part of the proposition, \( u \in K_c \). Hence \( K_c \) is compact. \qed

1.5. Remark. — Suppose that \( \psi \) is the indicator function of a non-empty closed convex set \( K \), i.e., \( \psi(u) = 0 \) if \( u \in K \) and \( \psi(u) = + \infty \) otherwise. Then \( u \) is a critical point of \( I \) if and only if \( u \in \mathbb{K} \) and
\[
\langle \Phi'(u), v - u \rangle \geq 0 \quad \forall v \in \mathbb{K}.
\]
Furthermore, \( I \) satisfies (PS) if and only if each sequence \( (u_n) \subset \mathbb{K} \) such that \( \Phi(u_n) \to c \in \mathbb{R} \) and
\[
\langle \Phi'(u_n), v - u_n \rangle \geq \langle z_n, v - u_n \rangle \quad \forall v \in \mathbb{K}.
\]
where \( z_n \to 0 \), has a convergent subsequence.

We shall make repeated use of the following variational principle of Ekeland.

1.6. Proposition. [13, p. 444 and 456]. — Let \( (Z, d) \) be a complete metric space and \( \Pi : Z \to (- \infty, + \infty] \) a proper l.s.c. function bounded below. Given \( \delta, \lambda > 0 \) and \( x \in Z \) with
\[
\Pi(x) \leq \inf_{z \in Z} \Pi(z) + \delta,
\]
there exists a point \( y \in Z \) such that
\[
\Pi(y) \leq \Pi(x), \quad d(x, y) \leq 1 / \lambda \quad \text{and} \quad \Pi(z) - \Pi(y) \geq - \delta \lambda d(y, z) \quad \forall z \in Z.
\]

An easy consequence of this result is the following

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1.7. THEOREM. — If $I$ is bounded below and satisfies (H) and (PS), then

$$c = \inf_{u \in X} I(u)$$

is a critical value.

Proof. — Let $(w_n)$ be a sequence satisfying $I(w_n) \leq c + 1/n$. By Proposition 1.6 with $\delta = 1/n$ and $\lambda = 1$, we find another sequence, $(u_n)$, such that $I(u_n) \leq c + 1/n$ and

$$I(w) - I(u_n) \geq (-1/n) \| w - u_n \| \quad \forall w \in X.$$

Set $w = (1 - t)u_n + tv, \ t \in (0, 1)$. Since $\psi$ is convex,

$$\Phi(u_n + t(v - u_n)) - \Phi(u_n) + t(\psi(v) - \psi(u_n)) \geq I(w) - I(u_n) \geq (-1/n) \| w - u_n \| = (-1/n)t \| v - u_n \|.$$

Dividing by $t$ and letting $t \to 0$ we obtain

$$\langle \Phi'(u_n), v - u_n \rangle + \psi(v) - \psi(u_n) \geq (-1/n) \| v - u_n \|.$$

So by (PS) and Proposition 1.4, $(u_n)$ possesses a subsequence converging to $u \in K_c$. \qed

2. EXISTENCE OF DEFORMATIONS

2.1. LEMMA. — Suppose that $I$ satisfies (H) and (PS) and let $N$ be a neighbourhood of $K_c$. Then for each $\varepsilon > 0$ there exists an $\delta \in (0, \varepsilon)$ such that if $u_0 \not\in N$ and $c - \varepsilon \leq I(u_0) \leq c + \varepsilon$, then

$$\langle \Phi'(u_0), v_0 - u_0 \rangle + \psi(v_0) - \psi(u_0) < -3\varepsilon \| v_0 - u_0 \|$$

for some $v_0 \in X$.

Proof. — If the conclusion is false, there exists a sequence $(u_n) \subset X - N$ such that $I(u_n) \to c$ and

$$\langle \Phi'(u_n), v - u_n \rangle + \psi(v) - \psi(u_n) \geq (-1/n) \| v - u_n \| \quad \forall v \in X.$$

So by (PS) and Proposition 1.4, a subsequence of $(u_n)$ converges to $u \in K_c$. This, however, is impossible because $u_n \not\in N$ for any $n$ and $N$ is a neighbourhood of $K_c$. \qed

2.2. LEMMA. — Suppose that $I$ satisfies (H) and (PS). Let $N$ be a neighbourhood of $K_c$ and $\varepsilon$ a positive number such that (4) holds. Then for
each \(u_0 \in I_{c+\varepsilon} - N\) there exists a \(v_0 \in X\) and an open neighbourhood \(U_0\) of \(u_0\) with the following properties:

\[
(5) \quad \langle \Phi'(u), v_0 - u \rangle + \psi(v_0) - \psi(u) \leq \|v_0 - u\| \quad \forall u \in U_0,
\]

\[
(6) \quad \langle \Phi'(u), v_0 - u \rangle + \psi(v_0) - \psi(u) \leq -3\varepsilon \|v_0 - u\| \quad \forall u \in U_0
\]

such that \(I(u) \geq c - \varepsilon\).

Furthermore, if \(u_0 \in K\), \(v_0 = u_0\), otherwise \(v_0\), \(U_0\) and a number \(\delta_0 > 0\) can be chosen so that \(v_0 \not\in \overline{U_0}\) and

\[
(5') \quad \langle \Phi'(u), v_0 - u \rangle + \psi(v_0) - \psi(u) \leq -\delta_0 \|v_0 - u\| \quad \forall u \in U_0.
\]

**Proof.** — Assume first \(u_0 \in K\). Then \(u_0\) satisfies (1), i.e.,

\[
\langle \Phi'(u_0), u - u_0 \rangle + \psi(u) - \psi(u_0) \geq 0 \quad \forall u \in X.
\]

It follows that if \(U_0\) is a sufficiently small neighbourhood of \(u_0\),

\[
\langle \Phi'(u), u_0 - u \rangle + \psi(u_0) - \psi(u) \leq \langle \Phi'(u) - \Phi'(u_0), u_0 - u \rangle \leq \|\Phi'(u) - \Phi'(u_0)\| \|u_0 - u\| \leq \|u_0 - u\| \quad \forall u \in U_0.
\]

So (5) with \(v_0 = u_0\) is satisfied. By Lemma 2.1, \(u \in N\) whenever \(u \in K\) and \(c - \varepsilon \leq I(u) \leq c + \varepsilon\). Hence \(I(u_0) < c - \varepsilon\). If \(I(u) < c - \varepsilon\) in a neighbourhood of \(u_0\), \(U_0\) may be chosen in this neighbourhood and condition (6) is empty. If each neighbourhood of \(u_0\) contains points with \(I(u) \geq c - \varepsilon\), it follows from the continuity of \(\Phi\) that \(\psi(u) - \psi(u_0) \geq d\) for some constant \(d > 0\) and all \(u\) sufficiently close to \(u_0\) and such that \(I(u) \geq c - \varepsilon\). So if \(U_0\) is small enough,

\[
\langle \Phi'(u), u_0 - u \rangle + \psi(u_0) - \psi(u) \leq \|\Phi'(u)\| \|u_0 - u\| - d
\]

\[
\leq -3\varepsilon \|u_0 - u\| \quad \forall u \in U_0, \quad I(u) \geq c - \varepsilon.
\]

Suppose now that \(u_0 \not\in K\) and \(I(u_0) < c - \varepsilon\). Since \(u_0\) is not a solution of (1), there exists a \(v_0 \in X\) such that

\[
(7) \quad \langle \Phi'(u_0), v_0 - u_0 \rangle + \psi(v_0) - \psi(v_0) < 0.
\]

Let \(w_0 = tv_0 + (1-t)u_0\), \(0 < t < 1\). By convexity of \(\psi\),

\[
\langle \Phi'(u_0), w_0 - u_0 \rangle + \psi(w_0) - \psi(u_0) \leq t(\langle \Phi'(u_0), v_0 - u_0 \rangle + \psi(v_0) - \psi(u_0)) < 0,
\]

so we may assume that \(v_0\) is arbitrarily close to \(u_0\). As in the preceding part of the proof, \(\psi(u) - \psi(u_0) \geq d > 0\) for all \(u\) close enough to \(u_0\) and such that \(I(u) \geq c - \varepsilon\). Using (7), this implies that if \(U_0\) and \(\|v_0 - u_0\|\) are sufficiently small, then \((\psi(v_0) - \psi(u_0)) + (\psi(u_0) - \psi(u)) \leq \frac{1}{2}d - d = -\frac{1}{2}d\) and

\[
\langle \Phi'(u), v_0 - u \rangle + \psi(v_0) - \psi(u) \leq \|\Phi'(u)\| \|v_0 - u\| - \frac{1}{2}d \leq -3\varepsilon \|v_0 - u\|
\]

\[
\forall u \in U_0, \quad I(u) \geq c - \varepsilon.
\]
Thus (6) is satisfied. Since \( v_0 \neq u_0 \), we may assume that \( v_0 \notin \overline{U}_0 \). In order to verify (5') note that since the left-hand side of (7) is negative,

\[
\langle \Phi'(u_0), v_0 - u_0 \rangle + \psi(v_0) - \psi(u_0) < - \delta_0 \| v_0 - u_0 \|
\]

for some \( \delta_0 > 0 \). Using continuity of \( \Phi' \) and l.s.c. of \( \psi \) it follows that, after shrinking \( U_0 \) if necessary,

\[
\langle \Phi'(u), v_0 - u \rangle + \psi(v_0) - \psi(u) < - \delta_0 \| v_0 - u \| \quad \forall u \in U_0.
\]

The remaining case of \( u_0 \notin K \), \( I(u_0) \geq c - \epsilon \) is easy: by Lemma 2.1 we find \( v_0 \) such that (4) is satisfied. By continuity of \( \Phi' \) and l.s.c. of \( \psi \) there is a neighbourhood \( U_0 \) of \( u_0 \), \( v_0 \notin \overline{U}_0 \), with the property that

\[
\langle \Phi'(u), v_0 - u \rangle + \psi(v_0) - \psi(u) < - 3\epsilon \| v_0 - u \| \quad \forall u \in U_0.
\]

Hence (5') and (6). \( \square \)

A family of mappings \( \varepsilon(\cdot, s) \equiv \varepsilon_s(\cdot) : W \to X, \ 0 \leq s \leq \bar{s}, \ \bar{s} > 0 \), is said to be a deformation if \( \varepsilon \in C(W \times [0, \bar{s}], X) \) and \( \varepsilon_0 = \text{id}_W \) (the identity on \( W \)).

2.3. Proposition. — Suppose that \( I \) satisfies (H) and (PS). Let \( N \) be a neighbourhood of \( K \), \( c \) a positive number. Then there exists an \( \epsilon \in (0, \bar{\epsilon}) \) such that for each compact subset \( A \) of \( X - N \) with

\[
c \leq \sup_{u \in A} I(u) \leq c + \epsilon
\]

we can find a closed set \( W \) containing \( A \) in its interior and a deformation \( \varepsilon_s : W \to X, \ 0 \leq s \leq \bar{s} \), having the following properties:

\[
\begin{align*}
\| u - \varepsilon_s(u) \| & \leq s \quad \forall u \in W, \\
I(\varepsilon_s(u)) - I(u) & \leq 2s \quad \forall u \in W, \\
I(\varepsilon_s(u)) - I(u) & \leq - 2\epsilon s \quad \forall u \in W \text{ with } I(u) \geq c - \epsilon
\end{align*}
\]

and

\[
\sup_{u \in A} I(\varepsilon_s(u)) - \sup_{u \in A} I(u) \leq - 2\epsilon s.
\]

Furthermore, if \( W_0 \) is a closed set and \( W_0 \cap K = \emptyset \), \( W \) and \( \varepsilon_s \) can be constructed so that

\[
I(\varepsilon_s(u)) - I(u) \leq 0 \quad \forall u \in W \cap W_0.
\]

Proof. — Choose \( \epsilon \in (0, \bar{\epsilon}) \) so that Lemma 2.2 can be applied. For each \( u_0 \in A \), let \( U_0 \) be the set constructed in that lemma. If \( \epsilon_0 \in K \), we may assume \( U_0 \) is so small that \( U_0 \cap W_0 = \emptyset \). The sets \( U_0 \) cover \( A \). Let \( \{U_i\}_{i \in J} \) be a finite subcovering. Denote by \( u_i \) and \( v_i \) the points corresponding to \( U_i \) in the same way as \( u_0 \) and \( v_0 \) correspond to \( U_0 \). We may assume that the subcovering has the property that if \( i_0 \in J \) and \( u_{i_0} \in K \), then the distance from \( u_{i_0} \) to each \( U_i \) with \( i \neq i_0 \) is positive. Indeed, if this is not the case.
for some $u_{i_0}$ we choose a closed neighbourhood $D$ of $u_{i_0}$ such that $D \subset U_{i_0}$, $u_i \notin D$ for $i \neq i_0$, and obtain a new covering by deleting $D$ from all $U_i$ with $i \neq i_0$ (strictly speaking, we obtain a refinement of the subcovering $(U_i)_{i \neq 0}$).

Let $\rho_i$ be a continuous function such that $\rho_i(u) > 0 \forall u \in U_i$ and $\rho_i(u) = 0$ otherwise. Let $\sigma_i(u) = \rho_i(u)/\sum_{j \neq i} \rho_j(u) \forall u \in V = \bigcup_{i \neq 0} U_i$. Define the mappings $\alpha_s$ as follows. If $u_{i_0} \in A \cap K$ and $u \in U_{i_0} - \bigcup_{i \neq i_0} U_i$,

$$\alpha_s(u) = \begin{cases} u + s(u_{i_0} - u)/\|u_{i_0} - u\| & \text{for } 0 \leq s < \|u_{i_0} - u\| \\ u_{i_0} & \text{for } s \geq \|u_{i_0} - u\|. \end{cases}$$

(13)

For all other $u \in V$,

$$\alpha_s(u) = u + s \sum_{i \in J} \sigma_i(u)(v_i - u)/\|v_i - u\|. \tag{14}$$

It is immediately seen that $\alpha_0 = id$ and (8) is satisfied. Note that if $u \in U_{i_0} - \bigcup_{i \neq i_0} U_i$ and $u \neq u_{i_0}$, then for all $s \leq \|u - u_{i_0}\|$, $\alpha_s(u)$ will be the same regardless which one of the formulas (13) and (14) we use. Consequently, $\alpha_s$ is well defined and continuous for sufficiently small positive $s$.

Suppose that $\alpha_s(u)$ is given by (14) and set $\alpha_s(u) = u + sw$. By Taylor’s formula,

$$I(\alpha_s(u)) = I(u + sw) = \Phi(u) + s \langle \Phi'(u), w \rangle + r(s) + \psi(u + sw),$$

where

$$|r(s)| \leq s \sup_{0 \leq t \leq s} \|\Phi'(u + tw) - \Phi'(u)\|.$$ 

Let $\delta$ be a given number such that

$$0 < 3\delta \leq \min \{ 1, \varepsilon, \delta_i \}$$

($\delta_i$ correspond to $U_i$ in (5')). Since $A$ is compact, there exists a closed set $W$ containing $A$ in its interior and an $\delta > 0$ such that

$$|r(s)| \leq \delta s \forall s \leq \delta, u \in W \text{ and } w \in X \text{ with } \|w\| \leq 1.$$ 

By (14),

$$\alpha_s(u) = u + sw = \left(1 - s \sum_{i \in J} \sigma_i(u)\|v_i - u\|^{-1}\right)u + s \sum_{i \in J} \sigma_i(u)\|v_i - u\|^{-1}v_i.$$ 

If $s$ is sufficiently small, $0 \leq s \sum_{i \in J} \sigma_i(u)\|v_i - u\|^{-1} \leq 1$ for all $u$ such that

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\( x_s(u) \) is given by (14). Using convexity of \( \psi \) and (15) it follows that
\[
I(x_s(u)) \leq \Phi(u) + s \sum_{i \in J} \sigma_i(u) \| v_i - u \|^{-1} \langle \Phi'(u), v_i - u \rangle + \delta_s
\]
\[
+ \left( 1 - s \sum_{i \in J} \sigma_i(u) \| v_i - u \|^{-1} \right) \psi(u) + s \sum_{i \in J} \sigma_i(u) \| v_i - u \|^{-1} \psi(v_i)
\]
\[
= I(u) + s \sum_{i \in J} \sigma_i(u) \| v_i - u \|^{-1} \langle \Phi'(u), v_i - u \rangle + \psi(v_i) - \psi(u) \rangle + \delta_s.
\]
By (5), each term under the last summation sign is less than or equal to \( \sigma_i(u) \). Hence
\[
I(x_s(u)) \leq I(u) + s + \delta_s.
\]
So (9) is satisfied for all small \( s \). In the same way we see from (6) and (5') that
\[
I(x_s(u)) \leq I(u) - 3es + \delta_s \quad \forall u \in W \text{ with } I(u) \geq c - \epsilon
\]
and
\[
I(x_s(u)) \leq I(u) - 3\delta s + \delta s \quad \forall u \in W \cap W_0
\]
(recall that \( U_i \cap W_0 = \phi \) whenever \( u_i \in K \)). This gives (10) and (12).

Suppose now that \( x_s(u) \) is given by (13). Then
\[
x_s(u) = u + sw = (1 - s \| u_i_0 - u \|^{-1})u + s \| u_i_0 - u \|^{-1}u_i_0
\]
whenever \( s < \| u_i_0 - u \| \equiv s_0 \). So for such \( s \) (9) and (10) follow as in the preceding case. If \( s \geq s_0 \), \( I(x_s(u)) = I(u) = I(u) + s_0 + \delta s_0 \leq I(u) + 2s \) and \( I(x_s(u)) = I(u_i_0) \leq c - \epsilon \). So (9) and (10) are satisfied for small \( s \). (Note that not only (9) and (10) but also (16) and (17) hold; this will be useful in the proof of Corollary 2.4.) To verify (12) recall that if \( x_s(u) \) is given by (13), then \( u \in U_i_0 \) with \( u_i_0 \in K \). Hence \( U_i_0 \cap W_0 = \phi \) and \( u \notin W \cap W_0 \).

It remains to prove (11). If \( \sup_{u \in A} I(x_s(u)) \leq c - \frac{1}{2} \epsilon \), (11) is satisfied for all \( s \leq 1/4 \) because \( \sup_{u \in A} I(u) \geq c \). Suppose that \( \sup_{u \in A} I(x_s(u)) > c - \frac{1}{2} \epsilon \). Set
\[
B = \{ u \in A : I(u) > c - \epsilon \}.
\]
By (9), \( \sup_{u \in A} I(x_s(u)) = \sup_{u \in B} I(x_s(u)) \) whenever \( s \) is small \( (s \leq \epsilon/4) \). Using this and (10) it follows that
\[
\sup_{u \in A} I(x_s(u)) - \sup_{u \in B} I(u) = \sup_{u \in B} I(x_s(u)) - \sup_{u \in B} I(u)
\]
\[
\leq \sup_{u \in B} (I(x_s(u)) - I(u)) \leq -2\epsilon s.
\]
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2.4. Corollary. — Assume that $\Phi$ and $\psi$ are even. If $A$ is symmetric (i.e., $-A = A$), $x_s$ may be chosen to be odd.

Proof. — We may assume that $W$ is symmetric. Let

$$
\beta_s(u) = \frac{1}{2} (x_s(u) - x_s(-u)).
$$

Then $\beta_s$ is odd and satisfies (8). Write $x_s(u) = u + h_s(u)$. By Taylor’s formula,

$$
I(\beta_s(u)) = \Phi(u) + \frac{1}{2} \langle \Phi'(u), h_s(u) - h_s(-u) \rangle + r_s(s) + \psi\left(\frac{1}{2}(u + h_s(u)) + \frac{1}{2}(u - h_s(-u))\right).
$$

Since $\Phi$ is even and $\psi$ even and convex,

$$
I(\beta_s(u)) \leq \frac{1}{2} (\Phi(u) + \langle \Phi'(u), h_s(u) \rangle + \psi(u + h_s(u))) + \frac{1}{2} (\Phi(-u) + \langle \Phi'(-u), h_s(-u) \rangle + \psi(-u + h_s(-u))) + \delta s.
$$

Hence by Taylor’s formula again,

$$
I(\beta_s(u)) \leq \frac{1}{2} I(x_s(u)) + \frac{1}{2} I(x_s(-u)) + 2\delta s.
$$

Using this and (16), $I(\beta_s(u)) \leq I(u) + s + 3\delta_s \leq I(u) + 2s$ for $s$ small. So $\beta_s$ satisfies (9). In the same way one sees that (17) and (18) imply (10) and (12). Finally, (11) follows upon observing that (19) remains true whenever (9) and (10) hold.

3. Mountain Pass Theorem

Let $Z$ be a topological space and $X$ a real Banach space. A mapping $f : Z \to X$ is said to be bounded if the set $f(Z)$ is bounded in $X$. Denote by $C(Z, X)$ the set of all continuous bounded mappings from $Z$ to $X$, metrized by

$$
d(f, g) = \sup_{z \in Z} \| f(z) - g(z) \|.
$$

It is well known that $C(Z, X)$ is a Banach space. Let $I : X \to (-\infty, +\infty]$ be a given function and define a new function $\Pi : C(Z, X) \to (-\infty, +\infty]$ by setting

$$
\Pi(f) = \sup_{z \in Z} I(f(z)).
$$
3.1. Lemma. — Suppose that \( I : X \to (-\infty, +\infty] \) is l.s.c. Then also the function \( \Pi \) is l.s.c.

Proof. — Suppose that \( f_n \to f \). Since \( I \) is l.s.c., \( I(f(z)) \leq \lim \inf I(f_n(z)) \forall z \in Z \). Hence

\[
\Pi(f) = \sup_{z \in Z} I(f(z)) \leq \lim \inf \sup_{z \in Z} I(f_n(z)) = \lim \inf \Pi(f_n).
\]

Denote by \( B_\rho(u) \) the open ball of radius \( \rho \) centered at \( u \), by \( \partial B_\rho(u) \) the boundary of \( B_\rho(u) \) and let \( B_\rho = B_\rho(0), \partial B_\rho = \partial B_\rho(0) \).

3.2. Theorem (Mountain Pass Theorem). — Suppose that \( I : X \to (-\infty, +\infty] \) is a function satisfying (H), (PS) and

i) \( I(0) = 0 \) and there exist \( \alpha, \rho > 0 \) such that \( I_{\mid B_\rho} \geq \alpha \), ii) \( I(e) \neq 0 \) for some \( e \notin B_\rho \).

Then \( I \) has a critical value \( c \geq \alpha \) which may be characterized by

\[
c = \inf_{f \in \Gamma} \sup_{t \in [0,1]} I(f(t)),
\]

where \( \Gamma = \{ f \in C([0,1], X) : f(0) = 0, f(1) = e \} \).

Proof. — Since \( f([0,1]) \cap \partial B_\rho \neq \emptyset \) \( \forall f \in \Gamma \), \( c \geq \alpha \). Suppose that \( c \) is not a critical value of \( I \). Then \( N = \phi \) is a neighbourhood of \( K_\varepsilon \). We may therefore use Proposition 2.3, with \( N = \phi \) and \( \varepsilon = c \), to obtain a number \( \varepsilon \in (0, \delta) \). By the definition of \( c \), \( I_{c-\varepsilon/4} \) is not path connected and 0 and \( e \) lie in different path components, \( W_0 \) and \( W_e \).

We shall need an auxiliary family of mappings from \([0,1]\) to \( X \) (\( \Gamma \) is not suitable for our purposes because \( \alpha \circ f \) may not be in \( \Gamma \) when \( f \) is). Let

\[
\Gamma_1 = \{ f \in C([0,1], X) : f(0) \in W_0 \cap I_{c-\varepsilon/2}, f(1) \in W_e \cap I_{c-\varepsilon/2} \}
\]

and

\[
c_1 = \inf_{f \in \Gamma_1} \sup_{t \in [0,1]} I(f(t)).
\]

Since \( \Gamma_1 \subset \Gamma \), \( c_1 \leq c \). If \( c_1 < c \), there exists an \( f \in \Gamma_1 \) such that \( \sup_{t \in [0,1]} I(f(t)) < c \). Since \( f(0) \) can be joined to 0 and \( f(1) \) to \( e \) by paths lying in \( I_{c-\varepsilon/4} \), there is a \( g \in \Gamma \) such that \( \sup_{t \in [0,1]} I(g(t)) < c \), contradicting the definition of \( c \). Hence \( c = c_1 \).

We claim that \( \Gamma_1 \) is a closed subset of \( C([0,1], X) \) (in particular, \( \Gamma_1 \) is a complete metric space). To prove this, let \( (f_n) \) be a sequence in \( \Gamma_1 \) such that \( f_n \to f \). Denote \( f(0) = u, \ f_n(0) = u_n \). By l.s.c. of \( I \), \( I(u) \leq \lim \inf I(u_n) \leq c - \varepsilon/2 \). Since \( \psi \) is convex and \( \Phi \) continuous, there exist positive numbers \( \delta_n \to 0 \) such that \( \forall t \in [0,1], \)

\[
\psi(tu_n + (1 - t)u) \leq t\psi(u_n) + (1 - t)\psi(u)
\]

and

\[
\Phi(tu_n + (1 - t)u) \leq \Phi(u) + \delta_n \leq t\Phi(u_n) + (1 - t)\Phi(u) + 2\delta_n.
\]

Therefore,
\begin{equation}
(20) \quad I(tu_n + (1-t)u) \leq tI(u_n) + (1-t)I(u) + 2\delta_n \leq c - \varepsilon/2 + 2\delta_n \leq c - \varepsilon/4
\end{equation}
for all large $n$. So for such $n$ the segment joining $u_n$ to $u$ lies in $W_\varepsilon$. In particular, $u \in W_\varepsilon$. Since also $I(u) \leq c - \varepsilon/2$, $u \in W_\varepsilon \cap I_{c - \varepsilon/2}$. Likewise, $f(1) \in W_\varepsilon \cap I_{c - \varepsilon/2}$. Hence $f \in \Gamma_1$.

Since $\Gamma_1$ is a complete metric space and $\Pi(f) = \sup \{ I(f(t)) \}$ is l.s.c. (according to Lemma 3.1), we may use Proposition 1.6 with $Z = \Gamma_1$, $\delta = \varepsilon$ and $\lambda = 1$ in order to obtain an $f \in \Gamma_1$ such that $\Pi(f) \leq c + \varepsilon$ and
\begin{equation}
(21) \quad \Pi(g) - \Pi(f) \geq - \varepsilon d(f, g) \quad \forall g \in \Gamma_1.
\end{equation}

Let $A = f([0, 1])$ and let $\alpha_s$ be the deformation given by Proposition 2.3 (note that $\Pi(f) \geq c_1 = c$). Set $g = \alpha_s \circ f$. For sufficiently small $s$, $\alpha_s \circ f \in \Gamma_1$. Indeed, if $I(f(0)) \in (c - \varepsilon, c - \varepsilon/2]$, then $I(\alpha_s \circ f(0)) \leq I(f(0)) \leq c - \varepsilon/2$ by (10) and if $I(f(0)) \leq c - \varepsilon$, $I(\alpha_s \circ f(0)) \leq I(f(0)) + 2s \leq c - \varepsilon/2$ according to (9). Hence $\alpha_s \circ f(0) \in W_0 \cap I_{c - \varepsilon/2}$. Likewise, $\alpha_s \circ f(1) \in W_\varepsilon \cap I_{c - \varepsilon/2}$. So $\alpha_s \circ f \in \Gamma_1$. Since $d(f, g) \leq s$ according to (8), it follows from (11) and (21) that
\begin{equation}
- 2\varepsilon s \geq \Pi(g) - \Pi(f) \geq - \varepsilon d(f, g) \geq - \varepsilon s.
\end{equation}

This contradiction shows that $K_\varepsilon \not\equiv \phi$. □

### 3.3. Corollary

Suppose that $I : X \to (-\infty, +\infty]$ satisfies (H) and (PS). If $0$ is a local minimum of $I$ and if $I(e) \leq I(0)$ for some $e \neq 0$, then $I$ has a critical point different from $0$ and $e$. In particular, if $I$ has two local minima, then it has at least three critical points.

**Proof.** — We may assume without loss of generality that $I(0) = 0$. If one can find $\alpha, \rho > 0$ such that $\rho < \| e \|$ and $I|_{\partial B_\rho} \geq \alpha$, the existence of a critical point different from $0$ and $e$ follows from Theorem 3.2. Suppose that such $\alpha$, $\rho$ do not exist. Let $r < \| e \|$ be a positive number such that $I|_{B_r} \geq 0$ and let $0 < \rho < r$. We shall use Proposition 1.6 with $Z = B_r$, $\Pi = I|_{B_r}$, $\delta = 1/n^2$ and $\lambda = n$. Since $\inf_{u \in B_\rho} I(u) = 0$, there exist $w_n \in \partial B_\rho$ such that
\begin{equation}
0 \leq I(u_n) \leq I(w_n) \leq 1/n^2, \quad \| u_n - w_n \| \leq 1/n
\end{equation}
and
\begin{equation}
(22) \quad I(z) - I(u_n) \geq (-1/n) \| z - u_n \| \quad \forall z \in B_r.
\end{equation}

Choosing $n$ large enough we may assume that $u_n \in B_r$. Let $v \in X$ and $z = (1 - t)u_n + tv$. If $t$ is small positive, $z \in B_r$. By (22) and the fact that $\psi$ is convex,
\begin{equation}
(23) \quad (-1/n)t \| v - u_n \| \leq I((1 - t)u_n + tv) - I(u_n) \leq \Phi(u_n + t(v - u_n)) - \Phi(u_n) + t(\psi(v) - \psi(u_n)).
\end{equation}
Dividing by $t$ and letting $t \to 0$ we obtain
\[ \langle \Phi'(u_n), v - u_n \rangle + \psi(v) - \psi(u_n) \geq (-1/n) \| v - u_n \|. \]
Since $r$ is arbitrary, it follows from (PS) that after passing to a subsequence, $u_n \to u \in \partial B_r$. So $u$ is a critical point and $0 \neq u \neq e$. A similar argument using Proposition 1.6 can be found in [14, Proposition 5 and 19, Theorem 4].

If $I$ has two local minima, $u_0$ and $u_1$, we may assume without loss of generality that $u_0 = 0$ and $I(u_1) \leq I(u_0) = 0$. By the first part of the corollary, there exists a critical point $u$ different from $u_0$ and $u_1$. Since also $u_0$ and $u_1$ are critical (by Proposition 1.1), the proof is complete.

3.4. THEOREM (Generalized Mountain Pass Theorem). — Suppose that $I : X \to (-\infty, +\infty)$ satisfies (H) and (PS). Let $X = X_1 \oplus X_2$, where $\dim X_1 < \infty$, and suppose that

1) there are constants $\alpha, \beta > 0$ such that $I|_{\partial B_\beta \cap X_1} \geq \alpha$,
2) there is a constant $R > \beta$ and an $e \in X_2$, $\| e \| = 1$, such that the restriction of $I$ to the boundary $\partial Q$ of $Q = (\bar{B}_R \cap X_1) \oplus \{ r e : 0 \leq r \leq R \}$ is nonpositive. Then $I$ has a critical value $c \geq \alpha$ which may be characterized by

\[ c = \inf_{\Gamma} \sup_{x \in Q} I(f(x)), \]

where $\Gamma = \{ f \in C(Q, X) : f|_{\partial Q} = \text{id}|_{\partial Q} \}$.

Proof. — Assume for the moment that $c > \alpha$. Suppose $c$ is not a critical value and apply Proposition 2.3 with $N = \phi$ and $\varepsilon = c$ to obtain an $\varepsilon \in (0, \bar{\varepsilon})$. Denote by $\approx$ the homotopy relation and let

\[ \Gamma_1 = \{ f \in C(Q, X) : f|_{\partial Q} \approx \text{id}|_{\partial Q} \text{ in } I_{c-\varepsilon/4}, \ I : f|_{\partial Q} \leq c - \varepsilon/2 \} \]

and

\[ c_1 = \inf_{\Gamma_1} \sup_{x \in Q} I(f(x)). \]

Since $\Gamma \subset \Gamma_1$, $c_1 \leq c$. If $c_1 < c$, we find an $f \in \Gamma_1$ such that $\sup_{x \in Q} I(f(x)) < c$.

Since $f|_{\partial Q} \approx \text{id}|_{\partial Q}$ in $I_{c-\varepsilon/4}$ and $Q$ is homeomorphic to a closed finite dimensional Euclidean ball, $\text{id}|_{\partial Q}$ can be extended to a mapping $g : Q \to X$ such that $\sup_{x \in Q} I(g(x)) < c$ (the existence of such an extension follows from general results in homotopy theory [16, Proposition 1.9.2]; one can also construct $g$ explicitly — see the proof of Lemma 4.5). This contradicts the definition of $c$ because $g \in \Gamma$. So $c_1 = c$.

We claim that $\Gamma_1$ is a closed subset of $C(Q, X)$. Let $(f_n)$ be a sequence
in $\Gamma_1$ such that $f_n \to f$. Since $I$ is l.s.c., $I \circ f \mid_{\partial Q} \leq c - \varepsilon/2$. Using the argument of the proof of Theorem 3.2 (cf. (20)) and the fact that $f(\partial Q)$ is compact, one shows that for $n$ large,

$$I(t f_n(x) + (1 - t) f(x)) \leq c - \varepsilon/4 \quad \forall x \in \partial Q, \quad t \in [0, 1].$$

Hence $f \mid_{\partial Q} \approx f_n \mid_{\partial Q} \approx \text{id}_{\partial Q}$ in $I_{c-\varepsilon/4}$. So $f \in \Gamma_1$.

The remaining part of the proof follows the last paragraph of the proof of Theorem 3.2 (with some obvious changes: we set $A = f(Q)$ instead of $f([0, 1])$ and observe that $\alpha_s \circ f \in \Gamma_1$ because $\alpha_s \circ f \approx f$ in $I_{c-\varepsilon/2}$).

We still have to show that $c \geq \varepsilon$. The argument can be found in [24, proof of Theorem 4.1] but for the sake of completeness we include it here. Since $I \mid_{\partial B_r \cap X_2} \geq \varepsilon$, it suffices to prove that for each $f \in \Gamma$, $f(Q) \cap \partial B_r \cap X_2 \neq \emptyset$.

In other words, we must find an $x \in Q$ such that $f(x) \in \partial B_r \cap X_2$. Denote by $P_1$ and $P_2$ the projections from $X$ to $X_1$ and $X_2$ associated with the decomposition $X = X_1 \oplus X_2$. For $x \in Y = X_1 \oplus \text{span} \{e\}$, write $x = x_1 + re$ with $x_1 \in X_1$ and $r \in \mathbb{R}$. Define a mapping $h : Y \to Y$ by the formula

$$h(x_1 + re) = P_1 f(x_1 + re) + \|P_2 f(x_1 + re)\| e.$$

Since $h \mid_{\partial Q} = \text{id}_{\partial Q}$, it follows from the properties of Brouwer's degree [25] that

$$\deg (h, \mathring{Q}, \rho e) = \deg (\text{id}, \mathring{Q}, \rho e) = 1$$

($\mathring{Q}$ denotes the interior of $Q$ in $Y$). Consequently, there exists an $x \in Q$ such that $P_1 f(x) = 0$ and $\|P_2 f(x)\| = \rho$. So $f(x) \in \partial B_r \cap X_2$ as required.

3.5. THEOREM (Saddle Point Theorem). — Suppose that

$$I : X \to (-\infty, +\infty]$$

satisfies (H) and (PS). Let $X = X_1 \oplus X_2$, $\dim X_1 < \infty$, and suppose that

i) there exist constants $\rho > 0$ and $\alpha_1$ such that $I \mid_{\partial B_{\rho} \cap X_1} \leq \alpha_1$,

ii) there is a constant $\alpha_2 > \alpha_1$ such that $I \mid_{X_2} \geq \alpha_2$.

Then $I$ has a critical point $c \geq \alpha_2$ which may be characterized by

$$c = \inf_{f \in \Gamma} \sup_{x \in D} I(f(x)),$$

where $D = \overline{B}_r \cap X_1$ and $\Gamma = \{ f \in C(D, X) : f \mid_{\partial D} = \text{id}_{\partial D} \}$.

Proof (outline). — It follows from a degree-theoretic argument [22, proof of Theorem 1.2 ; 24, proof of Theorem 3.1] that $f(D) \cap X_2 \neq \emptyset$. So $c \geq \alpha_2$.

By repeating the reasoning used in the proof of Theorem 3.4 (with $D$ and $\partial D$ replacing $Q$ and $\partial Q$) it can be shown that $K_c \neq \emptyset$. □
4. CRITICAL POINTS OF EVEN FUNCTIONS

Let $X$ be a real Banach space and $\Sigma$ the collection of all symmetric subsets of $X - \{0\}$ which are closed in $X$. A nonempty set $A \in \Sigma$ is said to have \textit{genus} $k$ (denoted $\gamma(A) = k$) if $k$ is the smallest integer with the property that there exists an odd continuous mapping $h : A \to \mathbb{R}^k - \{0\}$. If such an integer does not exist, $\gamma(A) = \infty$. For the empty set $\emptyset$ we define $\gamma(\emptyset) = 0$. Properties of genus are summarized below. Denote by $d(u, A)$ the distance from $u$ to the set $A$ and let

$$N_\delta(A) = \{ u \in X : d(u, A) \leq \delta \}.$$

4.1. Proposition. — Let $A, B \in \Sigma$.

i) If there exists an odd continuous mapping $f : A \to B$, then $\gamma(A) \leq \gamma(B)$.

ii) If $A \subseteq B$, then $\gamma(A) \leq \gamma(B)$.

iii) $\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$.

iv) If $\gamma(B) < \infty$, $\gamma(A - B) \geq \gamma(A) - \gamma(B)$.

v) If $A$ is homeomorphic to $B$ by an odd homeomorphism, $\gamma(A) = k$.

vi) If $A$ is compact, then $\gamma(A) < \infty$ and $\gamma(N_\delta(A)) = \gamma(A)$ for all sufficiently small $\delta > 0$.

vii) If $U \subset \mathbb{R}^k$ is an open, bounded and symmetric neighbourhood of $0$, then $\gamma(\partial U) = k$.

Proofs and a more detailed discussion of the notion of genus can be found e.g. in [21] [24].

Let $\mathcal{S}$ be the collection of all nonempty closed and bounded subsets of $X$. In $\mathcal{S}$ we introduce the Hausdorff metric $\text{dist}$ [18, § 15, VII], given by

$$\text{dist} (A, B) = \max \{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \}.$$

The space $(\mathcal{S}, \text{dist})$ is complete [18, § 29, IV]. Denote by $\Gamma$ the subcollection of $\mathcal{S}$ consisting of all nonempty compact symmetric subsets of $X$ and let

$$\Gamma_j = \text{cl} \{ A \in \Gamma : 0 \notin A, \gamma(A) \geq j \}$$

(cl is the closure in $\Gamma$). It is easy to verify that $\Gamma$ is closed in $\mathcal{S}$, so $(\Gamma, \text{dist})$ and $(\Gamma_\delta, \text{dist})$ are complete metric spaces.

4.2. Lemma. — If $A \in \Gamma_j$ and $0 \notin A$, $\gamma(A) \geq j$.

Proof. — Let $(A_n)$ be a sequence in $\Gamma_j$ such that $A_n \to A$, $0 \notin A_n$ and $\gamma(A_n) \geq j$. By (vi) of Proposition 4.1, there is a $\delta > 0$ such that $\gamma(A) = \gamma(N_\delta(A))$. Since $A_n \to A$, $A_n \subset N_\delta(A)$ for almost all $n$. So $j \leq \gamma(A_n) \leq \gamma(N_\delta(A)) = \gamma(A)$. \[\square\]
Observe that if $I = \Phi + \psi$ satisfies (H) and $\Phi, \psi$ are even, then $\Phi'(0) = 0$ and $\psi(0)$ is a (global) minimum of $\psi$. So $u = 0$ is necessarily a critical point of $I$.

4.3. THEOREM. — Suppose that $I : X \to (-\infty, +\infty]$ satisfies (H) and (PS), $I(0) = 0$ and $\Phi, \psi$ are even. Define

$$c_j = \inf_{A \in F_j} \sup_{u \in A} I(u).$$

If $-\infty < c_j < 0$ for $j = 1, \ldots, k$, then $I$ has at least $k$ distinct pairs of nontrivial critical points.

Proof. — Given $j$, $1 \leq j \leq k$, suppose that $c_j = \ldots = c_{j+p} \equiv c$ for some $p \geq 0$. Note that $0 \notin K_c$ because $c < 0$. We shall show that $\gamma(K_c) \geq p + 1$. Arguing indirectly, assume $\gamma(K_c) \leq p$. Let $\rho > 0$ be such that $\gamma(N_{2\rho}(K_c)) = \gamma(K_c)$. Define

$$\Pi(A) = \sup_{u \in A} I(u).$$

Then $\Pi$ is a function on $\Gamma_j$ and $\Pi$ is l.s.c. by an argument similar to that of Lemma 3.1 (note that $\forall u \in \Lambda$ there is a sequence $u_n \to u$ with $u_n \in A_n$). Let $N = N_\rho(K_c)$ and $\bar{c} = \min\{1, \rho, -c\}$. Apply Proposition 2.3 to obtain an $\varepsilon < \bar{c}$. Choose $A_1 \in \Gamma_{j+p}$ such that $\Pi(A_1) \leq c + \varepsilon^2$. Since $c + \varepsilon^2 \leq c + \varepsilon < 0$, $0 \notin A_1$ and it follows from Lemma 4.2 that $\gamma(A_1) \geq j + p$. Let $A_2 = A_1 - N_{2\rho}(K_c)$. Then $\Pi(A_2) \leq c + \varepsilon^2$ and, according to (iv) of Proposition 4.1, $\gamma(A_2) \geq \gamma(A_1) - \gamma(N_{2\rho}(K_c)) \geq j + p - p = j$. By Proposition 1.6 with $\delta = \varepsilon^2$ and $\lambda = 1/\varepsilon$, there is an $A \in \Gamma_j$ with

$$\Pi(A) \leq c + \varepsilon^2, \quad \text{dist}(A, A_2) \leq \varepsilon$$

and

$$(23) \quad \Pi(B) - \Pi(A) \geq -\varepsilon \text{ dist}(A, B) \quad \forall B \in \Gamma_j.$$ 

Since $\varepsilon < \rho$ and $A \in \Gamma_j$, $A \cap N_\rho(K_c) = \emptyset$ and $\Pi(A) \geq c$. A satisfies therefore the hypotheses of Proposition 2.3 and Corollary 2.4 and we obtain an odd deformation $\alpha_s : A \to X$. Since $c + \varepsilon^2 < 0$, $0 \notin A$ and $\gamma(A) \geq j$. Let $B = \alpha_s(A)$ with $s$ small. Then $\gamma(B) \geq \gamma(A) \geq j$ according to (i) of Proposition 4.1. So $B \in \Gamma_j$. By (8), $\text{dist}(A, B) \leq s$. It follows therefore from (23) and (11) that

$$-2\varepsilon s \geq \Pi(B) - \Pi(A) \geq -\varepsilon \text{ dist}(A, B) \geq -\varepsilon s.$$ 

This is the desired contradiction.

We have shown that $\gamma(K_c) \geq p + 1$. In particular, $\gamma(K_{c_j}) \geq 1$, so each $K_{c_j}$ has at least two points, $u_j$ and $-u_j$. This gives the required number of critical points if all $c_j$ are distinct. If they are not, $p > 0$ for some $j$. Hence $\gamma(K_{c_j}) \geq 2$ and $K_{c_j}$ is an infinite set. □
4.4. THEOREM. — Suppose that \( I : X \to (-\infty, +\infty] \) satisfies (H) and (PS), \( I(0) = 0 \) and \( \Phi, \psi \) are even. Assume also that

i) there exists a subspace \( X_1 \) of \( X \) of finite codimension and numbers \( \alpha, \beta > 0 \) such that \( I|_{\partial B_\beta \cap X_1} \geq \alpha \),

ii) there is a finite dimensional subspace \( X_2 \) of \( X \), \( \dim X_2 > \text{codim} X_1 \), such that \( I(u) \to -\infty \) as \( ||u|| \to \infty, u \in X_2 \).

Then \( I \) has at least \( \dim X_2 - \text{codim} X_1 \) distinct pairs of nontrivial critical points.

**Proof.** — Assume that \( I \) has no critical points in \( -d \) for some \( d > 0 \) (otherwise there are infinitely many critical points and there is nothing to prove). Set \( m = \text{codim} X_1, k = \dim X_2, Q = \overline{B_R} \cap X_2 \), where \( R > \rho \) is chosen so that \( I|_Q \leq -d \). Define for \( 1 \leq j \leq k \)

\[ \mathcal{F} = \{ f \in C(Q, X) : f \text{ is odd and } f|_Q \approx \text{id}_Q \text{ in } -d \text{ by an odd homotopy} \}, \]

\[ \Lambda_j = \{ f(Q - V) : f \in \mathcal{F}, V \text{ is open in } Q \text{ and symmetric}, V \cap \partial Q = \emptyset \}
\]

\[ \text{and for each } Y \subset V \text{ such that } Y \in \Sigma, \gamma(Y) \leq k - j \}, \]

\[ \Lambda_j = \{ A \subset X : A \text{ is compact, symmetric and for each open set } U \supset A \}
\]

\[ \text{there is an } A_0 \in \Lambda_j \text{ such that } A_0 \subset U \}
\]

and

\[ c_j = \inf \sup_{A \in \Lambda_j, U \supset A} I(u). \]

Since \( \text{id}_Q \in \mathcal{F}, Q \in \Lambda_j \) and \( \Lambda_j \neq \emptyset \) for \( j = 1, \ldots, k \). It is easy to see that \( \Lambda_j \) is a closed subset of \( \mathcal{F} \) (and therefore a complete metric space). Indeed, suppose \( A_n \in \Lambda_j \) and \( A_n \to A \). Let \( U \) be an arbitrary open set containing \( A \).

Then \( U \supset A_n \) for almost all \( n \) and, since \( A_n \in \Lambda_j \), there exists an \( A_0 \in \Lambda_j \) such that \( A_0 \subset U \). Hence \( A \in \Lambda_j \).

In order to continue the proof we need two results which we state separately.

4.5. LEMMA. — For \( m + 1 \leq j \leq k, c_j \geq \alpha \).

**Proof.** — Suppose \( c_j < \alpha \). Then \( A \cap X_1 \cap \partial B_\beta = \emptyset \) for some \( A \in \Lambda_j \).

Since \( X - (X_1 \cap \partial B_\beta) \) is an open set containing \( A \), we can first find an \( \Lambda_0 = f(Q - V) \in \Lambda_j \) which does not intersect \( X_1 \cap \partial B_\beta \) and then a \( Y \subset V \) such that \( Y \in \Sigma, \gamma(Y) \leq k - j \) and \( f(Q - Y) \cap X_1 \cap \partial B_\beta = \emptyset \). Let \( F(t, y), y \in \partial Q, 0 \leq t \leq 1 \), be an odd homotopy joining \( f|_Q \) to \( \text{id}_Q \) in \( -d \). Define

\[ \begin{cases} f(y, 2s) & \text{for } 0 \leq s \leq \frac{1}{2}R \\ F(y, 2s/R - 1) & \text{for } \frac{1}{2}R \leq s \leq R, \end{cases} \]

where \((y, s) \in \partial Q \times [0, R]\) are polar coordinates of \(x \in Q\). Since 
\[ f_1(y, s) \in I_{-d} \]
for \(s \geq \frac{1}{2} R\) and \(I_{\|\cdot\|_{B_\rho} \cap X_1} \geq \alpha > 0\), \(f_1(Q - Y_1) \cap X_1 \cap \partial B_\rho = \emptyset\). Now we use a standard argument (see e.g. \[24, Proposition 6.11\]). Let \(W\) be the component of \(f_1^{-1}(B_\rho)\) containing the origin. For \(x \in \partial Q\), \(f_1(x) = x \notin B_\rho\). So \(W \cap \partial Q = \emptyset\) and consequently, \(W\) is open and bounded in \(X_2\). According to \((vii)\) of Proposition 4.1, \(\gamma(\partial W) = k\). Set \(C = f_1^{-1}(\partial B_\rho)\). Then \(C \supset \partial W\) and \(\gamma(C) = k\). Furthermore, by \((iv)\) of Proposition 4.1, 
\[ \gamma(C - Y_1) \geq \gamma(C) - \gamma(Y_1) \geq k - (k - j) = j. \]

Let \(X = X_1 \oplus \tilde{X}_1\) and denote by \(P\) the projection from \(X\) to \(\tilde{X}_1\) along \(X_1\). Since \(f_1(Q - Y_1) \cap X_1 \cap \partial B_\rho = \emptyset\) and \(f_1(C - Y_1) \subset \partial B_\rho\), 
\[ Z = Pf_1(C - Y_1) \subset \tilde{X}_1 - \{0\}. \]

Hence \(Z \in \Sigma\) and by \((i)\) of Proposition 4.1, \(\gamma(Z) \geq \gamma(C - Y_1) \geq j\). On the other hand, \(\dim \tilde{X}_1 = m < j\), so \(\gamma(Z) < j\). \(\square\)

Denote by \(N_\delta(Z)\) the interior of \(N_\delta(Z)\) in \(X\).

4.6. **Lemma.** — The sets \(\Lambda_j\) have the following properties:

\begin{enumerate}
  \item[i)] \(\Lambda_{j+1} \subset \Lambda_j\).
  \item[ii)] If \(A \in \Lambda_j\), \(W\) is a closed and symmetric set containing \(A\) in its interior and \(\alpha : W \rightarrow X\) an odd mapping such that \(\alpha|_{W \cap I_{-d}} \approx id_{W \cap I_{-d}}\) in \(I_{-d}\) by an odd homotopy, then \(\alpha(A) \in \Lambda_j\).
  \item[iii)] If \(Z \in \Sigma\) is compact, \(\gamma(Z) \leq p\) and \(I|_Z > -d\), then there exists a \(\delta > 0\) such that for each \(A \in \Lambda_{j+p}\), \(A - \tilde{N}_\delta(Z) \in \Lambda_j\).
\end{enumerate}

Proof. — i) Let \(A \in \Lambda_{j+1}\) and choose an open set \(U \ni A\). There exists an \(A_0 = f(Q - V) \in \Lambda_{j+1}\) such that \(A_0 \subset U\). For each \(Y \subset V\) such that \(Y \in \Sigma\), \(\gamma(Y) \leq k - (j + 1) < k - j\). So \(A_0 \in \Lambda_j\) and \(A \in \Lambda_j\).

ii) Let \(U \ni \alpha(A)\) be open. Let \(W_1\) be an open set such that \(A \subset W_1 \subset W\) and \(\alpha(W_1) \subset U\). Since \(A \in \Lambda_j\), there is an \(A_0 = f(Q - V) \in \Lambda_j\) such that \(A_0 \subset W_1\). Extend \(\alpha\) to an odd mapping \(\tilde{\alpha} : f(Q) \cup W \rightarrow X\). Since \(f(\partial Q) \subset W \cap I_{-d}\), \(\tilde{\alpha} : f|_{\partial Q} \approx f|_{\partial Q} \approx id_{\partial Q}\) in \(I_{-d}\). Consequently, \(\tilde{\alpha} : f \in I_\Sigma\). Furthermore, \(\tilde{\alpha} : f(Q - V) = \alpha : f(Q - V) = \alpha(A_0) \subset \alpha(W_1) \subset U\), so \(\alpha(A_0) \in \Lambda_j\) and \(\alpha(A) \in \Lambda_j\).

iii) By \((vi)\) of Proposition 4.1, we may choose a \(\delta > 0\) such that \(\gamma(N_\delta(Z)) = \gamma(Z)\). Denote \(Z_0 = N_\delta(Z)\) and \(\tilde{Z}_0 = \tilde{N}_\delta(Z)\). Let \(U \ni A - \tilde{Z}_0\) be open and set \(U_0 = U \cup \tilde{Z}_0\). Then \(A \subset U_0\). Since \(A \in \Lambda_{j+p}\), there is an \(A_0 = f(Q - V) \in \Lambda_{j+p}\). \(A_0 \subset U_0\). Note that

\[
A_0 - \tilde{Z}_0 = f(Q - V) - \tilde{Z}_0 = f((Q - V) \cap (Q - f^{-1}(\tilde{Z}_0))) = f(Q - (V \cup f^{-1}(\tilde{Z}_0))).
\]

Since \(I|_Z > -d\) and \(X - I_{-d}\) is an open set, \(I|_{Z_0} > -d\) provided \(\delta\) is sufficiently small.

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small. It follows that \( f^{-1}(\hat{Z}_0) \cap \partial Q = \emptyset \) because \( I_{f(vQ)} \leq -d \). Consequently, the set \( V \cup f^{-1}(\hat{Z}_0) \) is open in \( Q \), symmetric and does not intersect \( \partial Q \).

Furthermore, if \( Y \subset V \cup f^{-1}(\hat{Z}_0) \) and \( Y \in \Sigma \), there exist \( Y_1, Y_2 \in \Sigma \) such that \( Y = Y_1 \cup Y_2 \) and \( Y_1 \subset V \), \( Y_2 \subset f^{-1}(\hat{Z}_0) \) (e.g., \( Y_1 = \{ x \in Y : d(x, Q - V) \geq d(x, Q - f^{-1}(\hat{Z}_0)) \} \) and \( Y_2 \) is obtained by reversing the inequality). Since \( A_0 \in \Lambda_{j+p} \), \( \gamma(Y_1) \leq k - (j + p) \). By \( i) - iii) \) of Proposition 4.1, \( \gamma(Y_2) \leq \gamma(f^{-1}(\hat{Z}_0)) \leq \gamma(\hat{Z}_0) \leq \rho \) and \( \gamma(Y) \leq \gamma(Y_1) + \gamma(Y_2) \leq k - (j+p) + p = k - j \).

Now it follows from (24) that \( A_0 - \hat{Z}_0 \in \Lambda_j \). Since \( A_0 - \hat{Z}_0 \subset U \) and \( U \) was chosen arbitrarily, \( A - \hat{Z}_0 \in \Lambda_j \). □

**Proof of Theorem 4.4 continued.** — By Lemma 4.5 and i) of Lemma 4.6,

\[ \alpha \leq c_{m+1} \leq \ldots \leq c_k. \]

Suppose that \( c_j = \ldots = c_{j+p} \equiv c \) for some \( j, m+1 \leq j \leq k \), and \( p \geq 0 \). Since \( c > 0 \), \( 0 \not\in K_c \). We complete the proof by demonstrating that \( \gamma(K_c) \geq p + 1 \). To obtain a contradiction, assume \( \gamma(K_c) \leq p \). Choose \( \rho > 0 \) so that \( \gamma(N_{2\rho}(K_c)) = \gamma(K_c) \). Let \( N = N_{\rho}(K_c) \) and \( \bar{\epsilon} = \min \{ 1, \rho \} \).

Let \( \epsilon < \bar{\epsilon} \) be the number given by Proposition 2.3. Recall that \( \Lambda_j \) is a complete metric space and \( \Pi(A) = \sup_{u \in A} I(u) \). There exists an \( A_1 \in \Lambda_{j+p} \) such that \( \Pi(A_1) \leq c + \epsilon^2 \). Let \( A_2 = A_1 - N_{2\rho}(K_c) \). If \( \rho \) is sufficiently small, it follows from \( iii) \) of Lemma 4.6 that \( A_2 \in \Lambda_j \). According to Proposition 1.6 (with \( \delta = \epsilon^2 \) and \( \lambda = 1/\epsilon \)), we can find an \( A \in \Lambda_j \) such that

\[ \Pi(A) \leq c + \epsilon^2, \quad \text{dist} (A, A_2) \leq \epsilon \]

and

\[ \Pi(B) - \Pi(A) \geq - \epsilon \text{ dist} (A, B) \quad \forall B \in \Lambda_j. \]

Since \( \epsilon < \rho \), \( A \cap N_{\rho}(K_c) = \emptyset \). Moreover, \( c \leq \Pi(A) \leq c + \epsilon^2 \leq c + \epsilon \). So according to Proposition 2.3 and Corollary 2.4, there exists an odd deformation \( \chi : W \rightarrow X \) satisfying (8)-(12). Let \( B = \chi_s(A) \) with \( s \) small. It follows from (12) (with \( W_0 = I_{-\delta} \)) that \( \chi_s \) satisfies the hypotheses of \( ii) \) of Lemma 4.6. Hence \( B \in \Lambda_j \). Using (8), (11) and (25) we obtain

\[ -2\epsilon s \geq \Pi(B) - \Pi(A) \geq - \epsilon \text{ dist} (A, B) \geq - \epsilon s, \]

a contradiction. □

**4.7. Remark.** — The minimax characterization of critical values obtained in the proof of Theorem 4.4 is not fully satisfactory because it depends on the a priori assumption that the function \( I \) does not have critical values below a certain level.

4.8. COROLLARY. — Suppose that the hypotheses of Theorem 4.4 are satisfied with ii) replaced by

\textit{ii'}) for any positive integer \( k \) there is a \( k \)-dimensional subspace \( X_2 \) of \( X \) such that \( I(u) \to -\infty \) as \( ||u|| \to \infty \), \( u \in X_2 \).

Then \( I \) has infinitely many distinct pairs of nontrivial critical points.

\textit{Proof.} — Obvious. \( \square \)

5. APPLICATIONS

Throughout this section we assume that \( \Omega \subset \mathbb{R}^N \) is a bounded domain with smooth boundary \( \Gamma \). \( H^m(\Omega) \equiv H^m \) and \( H^m_0(\Omega) \equiv H^m_0 \) are the usual Sobolev spaces of real-valued functions in \( \Omega \). In \( H^m \) we shall use the inner product \( (u, v) = \int_\Omega (\nabla^m u \cdot \nabla^m v + uv)dx \), and in \( H^m_0, (u, v) = \int_\Omega \nabla^m u \cdot \nabla^m vdx \).

The corresponding norm will be denoted by \( || \cdot || \). Standard results in Sobolev spaces, in a form suitable for our purposes, can be found in [21] [24] and in references given there. Let \( p^* \) be the critical exponent for the Sobolev embedding \( H^m \hookrightarrow L^p \). Recall that

\[ p^* = \begin{cases} 
2N/(N - 2m) & \text{if } 2m < N \\
+\infty & \text{otherwise}
\end{cases} \]

and the embedding is compact if \( p < p^* \).

We shall employ some results from the theory of maximal monotone operators [3] [4]. A mapping \( A \) from \( X \) to \( X^* \) is said to be a \textit{multivalued operator} if it maps each \( u \in X \) onto an element \( Au \in 2^X \). The \textit{domain} of \( A \) is defined by \( D(A) = \{ u \in X : Au \neq \emptyset \} \). \( A \) is called \textit{monotone} if

\[ \langle v_1 - v_2, u_1 - u_2 \rangle \geq 0 \quad \forall u_1, u_2 \in D(A), v_1 \in Au_1, v_2 \in Au_2 \]

and \textit{maximal monotone} if it admits no proper monotone extension, i.e., if there is no monotone operator \( B \) such the graph of \( B \) properly contains the graph of \( A \). The subdifferential \( \partial \psi \) of a proper convex l.s.c. function \( \psi : X \to (-\infty, + \infty] \) is maximal monotone [3, Theorem II.2.1].

Below we shall show in a number of examples how the results of Sections 3 and 4 can be applied to boundary value problems for semilinear elliptic operators. For simplicity, we choose the linear part of the operator to be \(-\Delta + \text{const.}\) or \((-\Delta)^m + \text{const.}\). It will, however, be clear from the proofs that other uniformly elliptic operators which induce symmetric bilinear forms and have reasonably smooth coefficients could be chosen as well. We also prefer to work with nonlinearities not explicitly depending on \( x \in \Omega \), although more general results could easily be obtained. In Theorems 5.1, 5.8 and Corollaries 5.9, 5.10 we could also replace \( |u|^{p-2}u \)
by a more general odd function satisfying suitable superlinearity conditions
at 0 and $\infty \{1, (p_3) - (p_3)\}$ on p. 362-363, see also 21, 24.

Our first example is concerned with a variational inequality on a convex
set $K \subset H_0^1$.

5.1. Theorem. — Let $\lambda \in \mathbb{R}$, $g \in L^2$, $g < 0$ a.e. in $\Omega$, $2 < p < p^*$ and

$$K = \{ u \in H_0^1 : u \geq 0 \text{ a.e. in } \Omega \}.$$

Then the variational inequality

$$u \in K : \int_{\Omega} \nabla u \cdot \nabla (v - u) dx - \lambda \int_{\Omega} u(v - u) dx - \int_{\Omega} u^{p-1}(v - u) dx$$

$$\geq \int_{\Omega} g(v - u) dx \quad \forall v \in K$$

has a nontrivial solution (in addition to the trivial one $u = 0$).

Proof. — Let $I = \Phi + \psi$ with

$$\Phi(u) = \frac{1}{2} \| u \|^2 - \frac{1}{2} \lambda \int_{\Omega} u^2 dx - p^{-1} \int_{\Omega} u^{p-1} dx - \int_{\Omega} g u dx$$

and $\psi$ the indicator function of $K$. It is easy to verify that $\Phi \in C^1$ with $\Phi'$

given by

$$(\Phi'(u), v) = (u, v) - \lambda \int_{\Omega} uv dx - \int_{\Omega} u^{p-1} v dx - \int_{\Omega} g v dx.$$

So $I$ satisfies (H) and $u$ is a solution of the variational inequality if and
only if $u$ is a critical point of $I$, i.e., if $u \in K$ and

$$(\Phi'(u), v - u) \geq 0 \quad \forall v \in K.$$

We shall show that $I$ satisfies (PS). Choose a constant $d \in \left( p^{-1}, \frac{1}{2} \right)$. Let

$$(u_n) \text{ be a sequence in } K \text{ such that } \Phi(u_n) \to c \in \mathbb{R}, \varepsilon_n \to 0 \text{ and}$$

$$(26) \quad (\Phi'(u_n), v - u_n) \geq - \varepsilon_n \| v - u_n \| \quad \forall v \in K.$$

Set $v = 2u_n$. Then $(\Phi'(u_n), u_n) \geq - \varepsilon_n \| u_n \|$. Since $\Phi(u_n) \leq c + 1$ for almost
all $n$,

$$c + 1 + \| u_n \| \geq \Phi(u_n) - d(\Phi'(u_n), u_n)$$

$$= \left( \frac{1}{2} - d \right) \| u_n \|^2 + \int_{\Omega} \left\{ (d - p^{-1})u_n^2 + \lambda \left( d - \frac{1}{2} \right) u_n^2 + (d - 1)g u_n \right\} dx.$$

Since $d > p^{-1}$, $p > 2$ and $(d - 1)g u_n \geq 0$ a.e. in $\Omega$, there exists an $R > 0$ such that the integrand is positive for $u_n > R$. Hence

$$c + 1 + \|u_n\| \geq \left(\frac{1}{2} - d\right)\|u_n\|^2 - C,$$

where $C$ is a constant. So the sequence $(u_n)$ is bounded. Denote $\Phi_1(u) = \Phi(u) - \frac{1}{2}\|u\|^2$. Using standard results in Sobolev spaces we see that after passing to a subsequence, $u_n \to \bar{u}$ weakly, $\|u_n\| \to a$ and $\Phi'(u_n) \to \Phi'((\bar{u})$ strongly (because $\Phi'_1$ is compact as $p < p^*$). It follows from (26) with $v = \bar{u}$ that

$$(u_n, \bar{u} - u_n) + (\Phi'(u_n) + (\bar{u} - u_n)) \geq -\epsilon_n\|\bar{u} - u_n\|.$$

Passing to the limit we obtain $\|\bar{u}\|^2 - a^2 \geq 0$, i.e., $\lim \|u_n\| \leq \|\bar{u}\|$. Thus $u_n \to \bar{u}$ strongly.

In order to obtain a nontrivial critical point we shall use Theorem 3.2. Let $u \in H_0^1(\Omega)$ such that $u = 0$ and $\Phi(u) = \frac{1}{2}\|u\|^2/n$. Then

$$\Phi(u_n)\|u_n\|^{-2} = \frac{1}{2} - \frac{1}{2}\lambda \int_{\Omega} z_n^2 dx - p^{-1}\|u_n\|^{p-2} \int_{\Omega} z_n^p dx - \|u_n\|^{-1} \int_{\Omega} g z_n dx \leq 1/n.$$

Assume after passing to a subsequence that $z_n \to \bar{z}$ weakly in $H_0^1$ and strongly in $L^2$. If $\bar{z} = 0$, $\lim \inf \Phi(u_n)\|u_n\|^{-2} \geq \frac{1}{2}$. So $\bar{z} \neq 0$. But then $\int_{\Omega} g \bar{z} dx < 0$ and consequently, $\Phi(u_n)\|u_n\|^{-2} \to +\infty$. This contradiction shows that $i)$ is satisfied for some $\lambda, \rho > 0$. \qed

Consider now a functional $I = \Phi + \psi : H_0^1(\Omega) \to (-\infty, +\infty]$ with

$$\Phi(u) = -\frac{1}{2}\lambda \int_{\Omega} u^2 dx,$$

$$\psi(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx + \int_{\Omega} |\nabla u| dx = \frac{1}{2}\|u\|^2 + \int_{\Omega} |\nabla u| dx.$$

Observe that $D(\psi) = H_0^1$. Denote by $\lambda_j$ the $j$-th eigenvalue of $-\Delta$ in $H_0^1$ (counted according to its multiplicity) and by $e_j$ a corresponding eigenfunction chosen so that $(e_i, e_j) = \delta_{ij}$ (Kronecker's $\delta$).

5.2. Theorem. — Suppose that $\lambda_k < \lambda < \lambda_{k+1}$. Then $I$ has at least $k$ distinct pairs of nontrivial critical points.
Proof. — It is easy to see that $I$ satisfies (H). To verify (PS), let $I(u_n) \to c$, 
$\varepsilon_n \to 0$ and

$$
(27) \quad (I'(u_n), v - u_n) + \psi(v) - \psi(u_n) \geq - \varepsilon_n \| v - u_n \| \quad \forall v \in H_0^1.
$$

First we show that the sequence $(u_n)$ is bounded. Suppose $\| u_n \| \to \infty$, let $z_n = u_n/\| u_n \|$ and assume after passing to a subsequence that $z_n \to \bar{z}$ weakly in $H_0^1$ and strongly in $L^2$. Since

$$
I(u_n) \| u_n \|^{-2} = \frac{1}{2} + \| u_n \|^{-1} \int_{\Omega} |\nabla z_n| dx - \frac{1}{2} \lambda \int_{\Omega} z_n^2 dx \to 0,
$$

$\bar{z} \neq 0$. Take $v = u_n + \varphi$ in (27). Then

$$
\frac{1}{2} \| \varphi \|^2 + (u_m \varphi) + \int_{\Omega} (|\nabla (u_n + \varphi) | - |\nabla u_n |) dx - \lambda \int_{\Omega} u_n \varphi dx \geq - \varepsilon_n \| \varphi \|.
$$

After dividing by $\| u_n \|$ and taking limits we obtain

$$
\int_{\Omega} \nabla \bar{z} \cdot \nabla \varphi dx - \lambda \int_{\Omega} \bar{z} \varphi dx \geq 0.
$$

Since $\varphi$ was chosen arbitrarily, the left-hand side is equal to zero for all $\varphi \in H_0^1$. This contradicts the fact that $\bar{z} \neq 0$ and $\lambda$ is not an eigenvalue. So the sequence $(u_n)$ is bounded. We may therefore assume that $u_n \to \bar{u}$ weakly in $H_0^1$, strongly in $L^2$ and $\| u_n \| \to a$. Set $v = \bar{u}$ in (27). Then

$$
\frac{1}{2} \| \bar{u} \|^2 - \frac{1}{2} \| u_n \|^2 + \int_{\Omega} |\nabla \bar{u}| dx - \int_{\Omega} |\nabla u_n | dx - \lambda \int_{\Omega} u_n (\bar{u} - u_n) dx \geq - \varepsilon \| \bar{u} - u_n \|.
$$

Letting $n \to \infty$ and using the fact that $\lim \inf \int_{\Omega} |\nabla u_n | dx \geq \int_{\Omega} |\nabla \bar{u}| dx$, we obtain $a = \lim \| u_n \| \leq \| \bar{u} \|$. Hence $u_n \to \bar{u}$ strongly in $H_0^1$.

We shall show that i) and ii) of Theorem 4.4 are satisfied. Suppose that no $\alpha, \rho > 0$ such that $I_{|\varepsilon B_\alpha} \leq \alpha$ exist. We may then find a sequence $(u_n)$ such that $u_n \to 0$ and $I(u_n) \leq \| u_n \|^2/n$. Let $z_n = u_n/\| u_n \|$. It follows that

$$
I(u_n) \| u_n \|^{-2} = \frac{1}{2} + \| u_n \|^{-1} \int_{\Omega} |\nabla z_n| dx - \frac{1}{2} \lambda \int_{\Omega} z_n^2 dx \leq 1/n.
$$

After passing to a subsequence, $z_n \to \bar{z}$ weakly in $H_0^1$, strongly in $L^2$. Now it is easy to see that $\lim \inf I(u_n) \| u_n \|^{-2} \geq \frac{1}{2}$ if $\bar{z} = 0$ and $\lim I(u_n) \| u_n \|^{-2} = + \infty$ if $\bar{z} \neq 0$. So i) is satisfied with $X_1 = H_0^1$.
To verify ii), let \( X_2 = \text{span} \{ e_1, \ldots, e_k \} \) and \( u = \alpha_1 e_1 + \ldots + \alpha_k e_k \).

Since \( \int_{\Omega} e_i^2 \, dx = \lambda_i^{-1} \int_{\Omega} |\nabla e_i|^2 \, dx = \lambda_i^{-1} \),

\[
I(u) = \frac{1}{2} \sum_{i=1}^k \left( 1 - \frac{\lambda}{\lambda_i} \right) \alpha_i^2 + \int_{\Omega} |\nabla (\alpha_1 e_1 + \ldots + \alpha_k e_k)| \, dx
\]

\[
\leq \frac{1}{2} (1 - \frac{\lambda}{\lambda_k}) (\alpha_1^2 + \ldots + \alpha_k^2) + C (\alpha_1^2 + \ldots + \alpha_k^2)^\frac{1}{2} = \frac{1}{2} (1 - \frac{\lambda}{\lambda_k}) ||u||^2 + C ||u||,
\]

where \( C \) is a constant. Since \( \lambda > \lambda_k \), \( I(u) \rightarrow -\infty \) as \( ||u|| \rightarrow \infty \), \( u \in X_2 \).

Now it follows from Theorem 4.4 that \( I \) has at least \( \dim X_2 - \text{codim} X_1 = k \) pairs of nontrivial critical points.

5.3. COROLLARY. — If \( \lambda_k < \lambda < \lambda_{k+1} \), there exist at least \( k \) distinct pairs of nontrivial solutions of the inequality

\[
(28) \quad u \in H^2 \cap H_0^1 : \int_{\Omega} (-\Delta u)(v - u) \, dx + \int_{\Omega} |\nabla v| \, dx - \int_{\Omega} |\nabla u| \, dx \geq \lambda \int_{\Omega} u(v - u) \, dx \quad \forall v \in H_0^1.
\]

Proof. — Extend \( \psi \) to a functional \( \tilde{\psi} : L^2 \rightarrow (-\infty, +\infty) \) by setting \( \tilde{\psi}(u) = \psi(u) \) for \( u \in H_0^1 \) and \( \tilde{\psi}(u) = +\infty \) otherwise. Since \( D(\tilde{\psi}) = D(\psi) \), \( u \in H_0^1 \) is a critical point of \( I \) if and only if

\[
\tilde{\psi}(v) - \tilde{\psi}(u) \geq \lambda \int_{\Omega} u(v - u) \, dx \quad \forall v \in L^2,
\]

or equivalently, if \( \lambda u \in \partial \tilde{\psi}(u) \). On the other hand, according to [5, Theorem 15], \( \lambda u \in \partial \tilde{\psi}(u) \) if and only if \( u \) satisfies (28). So the result follows from Theorem 5.2.

Let \( F \) and \( G \) be two functions satisfying the following assumptions.

\( F : \mathbb{R} \rightarrow [0, +\infty] \) is even, l.s.c., convex and \( F(0) = 0 \); \( G : \mathbb{R} \rightarrow \mathbb{R} \) is even, of class \( C^1 \), \( G(0) = 0 \), \( G'(t) = g(t) \) and \( |g(t)| \leq c_1 + c_2 |t|^{p-1} \forall t \in \mathbb{R} \),

where \( 2 \leq p < p^* \) and \( c_1, c_2 \) are positive constants. Let \( I \) be a functional on \( H_0^1 \) such that \( I = \Phi + \psi \) and

\[
\Phi(u) = -\int_{\Omega} G(u) \, dx,
\]

\[
\psi(u) = \begin{cases} 
\frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx + \int_{\Omega} F(u) \, dx & \text{if } F(u) \in L^1 \\
+\infty & \text{otherwise}.
\end{cases}
\]
Denote by $\lambda_j$ the $j$-th eigenvalue of $(-\Delta)^m$ in $H^m_0$ (counted according to its multiplicity) and by $e_j$ a corresponding eigenfunction satisfying $(e_i, e_j) = \delta_{ij}$.

5.4. Theorem. — Suppose that $\lim \inf_{|t| \to \infty} (F(t) - G(t))/t^2 > -\frac{1}{2} \lambda_1$ and $\lim \sup_{t \to 0} (F(t) - G(t))/t^2 < -\frac{1}{2} \lambda_k$. Then $I$ has at least $k$ distinct pairs of nontrivial critical points.

Proof. — We employ Theorem 4.3. It follows from the growth restriction on $g$ that $\Phi$ is of class $C^1$ and from [3, Proposition II.2.8] that $\psi$ is l.s.c. and convex. So (H) is satisfied. Now we proceed to verify (PS). Choose $R > 0$ and $\lambda < \lambda_1$ such that $\lim_{|t| > R} (F(t) - G(t))/t^2 \geq -\frac{1}{2} \lambda$ for $|t| > R$. Then

$$I(u) = \frac{1}{2} \| u \|^2 + \int_{|u| > R} (F(u) - G(u))dx + \int_{|u| \leq R} (F(u) - G(u))dx \geq \frac{1}{2} \| u \|^2 - \frac{1}{2} \lambda \int_{\Omega} u^2 dx - C \geq \frac{1}{2} (1 - \lambda/\lambda_1) \| u \|^2 - C.$$ 

Since $\lambda < \lambda_1$, $I(u) \to +\infty$ as $\| u \| \to \infty$. It follows that if $I(u_n) \to c \in \mathbb{R}$, $(u_n)$ is a bounded sequence. We may therefore assume that $u_n \to \bar{u}$ weakly, a.e. in $\Omega$, $\Phi'(u_n) \to \Phi'(\bar{u})$ strongly and $\| u_n \| \to a$. Set $v = \bar{u}$ in (2). Then

$$\frac{1}{2} \| \bar{u} \|^2 - \frac{1}{2} \| u_n \|^2 + \int_{\Omega} F(\bar{u}) dx - \int_{\Omega} F(u_n) dx + (\Phi'(u_n), \bar{u} - u_n) \geq -\varepsilon_n \| \bar{u} - u_n \|.$$ 

Passing to the limit and using Fatou’s lemma, we see as in the proof of Theorem 5.2 that $u_n \to \bar{u}$ strongly.

We complete the proof by demonstrating that

$$c_j = \inf_{A \in \Gamma_j} \sup_{u \in A} I(u)$$

satisfies $-\infty < c_j < 0$ for $1 \leq j \leq k$. Since $\Phi$ is weakly continuous, $\psi \geq 0$ and $I(u) \to +\infty$ as $\| u \| \to \infty$, $I$ is bounded below. Thus $c_j > -\infty$. Let

$$A = \{ u = x_1 e_1 + \ldots + x_j e_j : \| u \|^2 = x_1^2 + \ldots + x_j^2 = \rho^2 \}.$$ 

Then $A \in \Gamma_j$ because $\gamma(A) = j$ according to ($\nu$) of Proposition 4.1. Choose $r > 0$ and $\lambda > \lambda_k$ so that $(F(t) - G(t))/t^2 \leq -\frac{1}{2} \lambda$ as $|t| \leq r$. Let $\rho$ in the
5.5. COROLLARY. — Let F be as above and let m = 1. Denote the sub-
differential of F by f and let λ ∈ ℝ. Suppose that lim inf F(t)/t^2 > 1/2(λ - λ_1) and lim F(t)/t^2 = 0. If λ > λ_k, the boundary value problem
\[ \begin{cases} -\Delta u + f(u) \equiv \lambda u & \text{in } \Omega \\ u \in H^1_0 \end{cases} \]
has at least k distinct pairs of nontrivial solutions u ∈ H^2 ∩ H^1_0.

Proof. — Let G(t) = 1/2 λt^2. Then F and G satisfy the hypotheses of Theorem 5.4, so that I has k pairs of nontrivial critical points. Extend ψ to a functional \( \tilde{\psi} : L^2 \to (-\infty, +\infty) \) by setting \( \tilde{\psi}(u) = \psi(u) \) if \( u \in D(\psi) \), \( \tilde{\psi}(u) = +\infty \) otherwise. Then u is a critical point of I if and only if \( \lambda u \in \partial \tilde{\psi}(u) \). Since \( \lambda u \in \partial \tilde{\psi}(u) \) is equivalent to \( u \in H^2 \cap H^1_0 \) and \( \lambda u \equiv -\Delta u + f(u) \) a.e. in \( \Omega \) [3, Proposition II.3.8], the result follows. □

5.6. COROLLARY. — Let \( f(t) \) and \( g(t) \) be two odd \( C^1 \) functions on ℝ such that \( f(0) = g(0) = 0 \), \( f \) is nondecreasing, \( |g(t)| \leq c_1 + c_2 |t|^{p-1} \) for some \( p \in [2, p^*] \) and \( \lim \inf f'(t) - g'(t) > -\lambda_1 \). If \( f'(0) - g'(0) < -\lambda_k \), the boundary value problem
\[ \begin{cases} (\Delta)^m u + f(u) = g(u) & \text{in } \Omega \\ u \in H^1_0 \end{cases} \]
has at least k distinct pairs of nontrivial solutions u such that \( uf(u) \in L^1 \).

Proof. — Setting \( F(t) = \int_0^t f(s)ds \) and \( G(t) = \int_0^t g(s)ds \) it is easy to see that F and G satisfy the hypotheses of Theorem 5.4. So I has k distinct pairs of nontrivial critical points. Let \( \tilde{\psi} : L^p \to (-\infty, +\infty) \) be given by \( \tilde{\psi}(u) = \psi(u) \) if \( u \in D(\psi) \) and \( \tilde{\psi}(u) = +\infty \) otherwise. Set \( q = p/(p-1) \). Since \( g(u) \in L^q \) whenever \( u \in L^p \), u is a critical point of I if and only if
Define an operator \( A : D(A) \subset L^p \to L^q \) by \( Au = (-\Delta)^mu + f(u) \) with

\[
D(A) = \{ u \in H^m_0: (-\Delta)^mu + f(u) \in L^q, uf(u) \in L^1 \}.
\]

It is easy to see from the definitions that \( A \) is monotone (cf. the proof of Corollary IV.3 in [7]) and \( A \in \partial \tilde{\psi} \) (in the sense of graph inclusion). We complete the proof by showing that \( A \) is maximal monotone. It will then follow that \( A = \partial \tilde{\psi} \), so for each critical point \( u, g(u) = Au \) and, since \( u \in D(A) \), \( uf(u) \in L^1 \).

The equation \( Au + u = h \) has a solution \( u \in D(A) \) for each \( h \in L^q \) [7, Proposition IV.2 and Remark IV.1]. Note that \( L^p \subset L^q \) because \( p \geq 2 \). We shall use an argument similar to that of [4, Proposition 2.2]. If \( A \subset B \) with \( B \) monotone and if \( h \in Bu \), then \( Av + v = h + u \) for some \( v \in D(A) \) (because \( h + u \in L^q \)). Consequently, \( h + u - v = Av \in Bv \) and by monotonicity of \( B \), \( \langle (h + u - v) - h, v - u \rangle \geq 0 \), where \( \langle , \rangle \) denotes the duality pairing between \( L^q \) and \( L^p \). Hence \( \langle u - v, u - v \rangle \leq 0 \), so \( u = v \) and \( h = Au \). Since \( u \) was chosen arbitrarily, \( A = B \) and \( A \) is maximal monotone. \( \square \)

5.7. REMARK. — If \( m = 1 \), the conclusion of Corollary 5.6 is essentially contained in [21, Theorem 3.4 and 24, Theorem 5.23]. The proof given there uses a truncation argument based on the maximum principle, so it does not extend to the case of \( m > 1 \).

Let \( B : \mathbb{R} \to [0, +\infty] \) be an even, l.s.c. and convex function with subdifferential \( \partial B = \beta \) and let \( p \in (2, p^*) \). Suppose that \( B(0) = 0 \) and \( \beta \) and \( B \) satisfy the following growth restrictions: there is a constant \( c > 0 \) such that

\[
B(2t) \leq cB(t) \quad \forall t \in \mathbb{R},
\]

and for some \( q \in (2, p) \),

\[
st \leq qB(t) \quad \forall s \in \beta(t), \quad t \in \mathbb{R}.
\]

Consider the functional \( I = \Phi + \psi : H^1 \to (-\infty, +\infty] \) with

\[
\Phi(u) = -\frac{1}{2} (\lambda + 1) \int_\Omega u^2dx - p^{-1} \int_\Omega |u|^pdx,
\]

\[
\psi(u) = \begin{cases} 
\frac{1}{2} \int_\Omega (|\nabla u|^2 + u^2)dx + \int_\Gamma B(u)d\sigma & \text{if } B(u) \in L^1(\Gamma) \\
\frac{1}{2} \int_\Omega (|\nabla u|^2 + u^2)dx + \int_\Gamma B(u)d\sigma \equiv \frac{1}{2} ||u||^2 + \int_\Gamma B(u)d\sigma & \text{otherwise.}
\end{cases}
\]

It follows from (29) that \( D(B) = \mathbb{R} \) and if \( u \in D(\psi) \), then also \( ku \in D(\psi) \) for any \( k \in \mathbb{R} \). Note that \( B(t) = r^{-1} |t|^p, 1 \leq r < p \), satisfies the restrictions (29) and (30).
5.8. THEOREM. — The functional $I$ has infinitely many distinct pairs of nontrivial critical points.

Proof. — It is easy to see that $I$ satisfies (H) (cf. [3, p. 63]). Let $(u_n)$ be a sequence such that $I(u_n) \to c$ and $(\Phi'(u_n), v - u_n) + \psi(v) - \psi(u_n) \geq -\varepsilon_n \|v - u_n\| \quad \forall v \in H^1$. Set $v = u_n + tu_n$, $t > 0$, divide by $t$ and let $t \to 0$. This gives

$$
\|u_n\|^2 + \lim_{t \to 0} \int_{\Gamma} B(u_n + tu_n) - B(u_n))t^{-1}d\sigma - \frac{1}{\lambda} \int_{\Omega} u_n^2dx - \int_{\Omega} |u_n|^pdx \geq -\varepsilon_n \|u_n\|.
$$

By Lebesgue's monotone convergence theorem, we can take the limit under the integral sign. It follows that

$$
\|u_n\|^2 + \int_{\Gamma} w_n u_n d\sigma - \frac{1}{\lambda} \int_{\Omega} u_n^2dx - \int_{\Omega} |u_n|^pdx \geq -\varepsilon_n \|u_n\|,
$$

where $w_n(x) \in \beta(u_n(x))$ a.e. on $\Gamma$. Multiplying (31) by $-q^{-1}$ and adding it to the inequality $I(u_n) \leq c + 1$ (which holds for almost all $n$) gives

$$
c + 1 + \|u_n\| \geq \left(\frac{1}{2} - q^{-1}\right) \|u_n\|^2 + \int_{\Gamma} (B(u_n) - q^{-1}w_n u_n)d\sigma
$$

$$
+ \int_{\Omega} \left\{ (q^{-1} - p^{-1}) |u_n|^p + \frac{1}{2} \left( q^{-1} - \frac{1}{2} \right) u_n^2 \right\} dx.
$$

Note that the first integral is nonnegative according to (30). Since $q^{-1} > p^{-1}$ and $p > 2$, the second integrand is positive for large $|u_n|$. We can therefore find a constant $C$ such that

$$
c + 1 + \|u_n\| \geq \left(\frac{1}{2} - q^{-1}\right) \|u_n\|^2 - C.
$$

It follows that the sequence $(u_n)$ is bounded. Using Fatou's lemma in the same way we did before we deduce that $(u_n)$ possesses a convergent subsequence. Hence $I$ satisfies (PS).

We complete the proof by showing that the hypotheses i) and ii') of Corollary 4.8 are satisfied. Suppose $\lambda < \lambda_m$ and let $X_1 = \text{span} \{ e_1, \ldots, e_m \}$. If $u \in X_1$, $\frac{1}{2} \|u\|^2 - \frac{1}{2} \lambda \int_{\Omega} u^2dx \geq \alpha_1 \|u\|^2$ for some constant $\alpha_1 > 0$. It follows from a well known argument [1, proof of Lemma 3.3; 21, proof of Theorem 3.19] that $\int_{\Omega} |u|^pdx = o(\|u\|^2)$ as $u \to 0$. Hence we can find $\varepsilon, \rho > 0$ such that $I_{\delta B_\rho \cap X_1} \geq \varepsilon$. Finally, let $k$ be an arbitrary positive integer, $\varphi_1, \ldots, \varphi_k$ linearly independent functions in $C_0' (\Omega)$ and $X_2 = \text{span} \{ \varphi_1, \ldots, \varphi_k \}$. Note that $\int_{\Gamma} B(u)d\sigma = 0 \forall u \in X_2$. Since $\delta B_1 \cap X_2$
is compact, there is a constant \( \alpha_2 > 0 \) such that \( \frac{1}{p-1} \int_{\Omega} |v|^{p} dx \geq \alpha_2 \) \( \forall v \in \partial B_1 \cap X_2 \). Let \( u = tv \), where \( t > 0 \) and \( v \in \partial B_1 \cap X_2 \). Then

\[
I(u) \leq \frac{1}{2} ||u||^2 - \frac{1}{p-1} \int_{\Omega} |u|^{p} dx \leq \frac{1}{2} t^2 - \alpha_2 t^p.
\]

Hence \( I(u) \to -\infty \) as \( ||u|| \to \infty \), \( u \in X_2 \). □

Denote by \( \partial/\partial n \) the outward normal derivative.

5.9. COROLLARY. — In addition to the above hypotheses, suppose that

\[
2 < p \leq \frac{2N - 2}{(N - 2)} \text{ if } N > 2 \text{ and } 2 < p < \infty \text{ otherwise.}
\]

Then the boundary value problem

\[
\begin{align*}
-\Delta u - \lambda u - |u|^{p-2} u &= 0 & \text{in } \Omega \\
\partial u / \partial n &\in \beta(u) & \text{on } \Gamma
\end{align*}
\]

has infinitely many distinct pairs of nontrivial solutions \( u \in H^2 \).

Proof. — Let \( \tilde{\psi} \) be the extension of \( \psi \) to \( L^2 \) defined by \( \tilde{\psi}(u) = \psi(u) \) if \( u \in D(\psi) \), \( \tilde{\psi}(u) = +\infty \) otherwise. By Sobolev's embedding theorem, each element of \( H^1 \) is in \( L^p \). So if \( u \in H^1 \), \( |u|^{p-2} u \in L^{p/(p-1)} \). Since \( p - 1 \leq N/(N - 2) \), \( p^*/(p - 1) \geq 2 \). Hence \( |u|^{p-2} u \in L^2 \). It follows that \( u \) is a critical point of \( I \) if and only if \( \lambda u + |u|^{p-2} u \in \partial \tilde{\psi}(u) \). By [3, Proposition II.2.9 or 5, Theorem 12], \( \partial \tilde{\psi}(u) = -\Delta u + D(\partial \tilde{\psi}) = \{ u \in H^2 : -\partial u / \partial n \in \beta(u) \text{ a.e. on } \Gamma \} \).

So each critical point of \( I \) is a solution of the boundary value problem. Now the result follows from Theorem 5.8. □

Observe that the growth restrictions (29) and (30) were used only in order to verify (PS). So if one removes them, the conclusions of Theorem 5.8 and Corollary 5.9 remain true as long as (PS) is satisfied.

5.10. COROLLARY. — Suppose that the hypotheses of Theorem 5.8 and Corollary 5.9, with possible exception of (29) and (30), are satisfied. If the domain of \( B, D(B) \), is a proper subset of \( \mathbb{R} \), the conclusions remain true.

Proof. — As we have already observed, we need only verify (PS). An argument similar to that of [15, p. 75] shows that if

\[
D = \{ v \in H^1 : -\Delta v + v = 0 \text{ in } \Omega \text{ in the sense of distributions} \},
\]

then \( H^1 = H^1_0 \oplus D \) and \( D \) is orthogonal to \( H^1_0 \). (Given \( u \in H^1 \), let \( v \) be the minimizer of \( ||w||^2 \) on the closed convex set \( \{ w \in H^1 : u - w \in H^1_0 \} \). Then \( (v, \varphi) = 0 \) \( \forall \varphi \in H^1_0 \). Thus \( v \in D, u - v \in H^1_0 \) and \( (v, u - v) = 0 \).) Since \( D(B) \) is properly contained in \( \mathbb{R} \) and \( B \) is even, there exists a constant \( \alpha \) such that \( ||u||_{L^\infty(\Gamma)} \leq \alpha \) whenever \( u \in D(\psi) \). Let \( u = u' + u'' \) with \( u' \in H^1_0 \),
It follows from the maximum principle [17, Theorem 11.5.5] that
\[
\| u'' \|_{L^\infty(\Omega)} \leq \| u'' \|_{L^\infty(\Gamma)} = \| u \|_{L^\infty(\Gamma)} \leq \alpha
\]
(we have used the fact that \( \sup_{\Gamma} | u | = \| u \|_{L^\infty(\Gamma)} \) for \( \Gamma \) smooth).

Let now \((u_n)\) be a sequence satisfying the hypotheses of (PS). Write
\[
u_n = \nu_n' + \nu_n'' \quad \text{with} \quad \nu_n' \in H_0^1 \Omega. \quad \nu_n'' \in D \quad \text{and set} \quad \nu = u_n + t\nu_n', \quad t > 0.\]
Then
\[
t(\Phi'(u_n), \nu_n') + \psi(u_n + tu_n') - \psi(u_n) \geq - \varepsilon_n t \| u_n' \|. \]  Divide by \( t \) and let \( t \to 0. \)
Since
\[
\int_{\Gamma} B(u_n + tu_n')d\sigma = \int_{\Gamma} B(u_n)d\sigma \quad \text{and} \quad (u_n', u_n'') = 0, \]
\[
\| u_n' \|^2 - \lambda \int_{\Omega} u_n u_n' dx - \int_{\Omega} | u_n |^{p-2} u_n u_n' dx \geq - \varepsilon_n \| u_n' \|. \]
Let \( d \in \left( p^{-1}, \frac{1}{2} \right) \), multiply the above inequality by \(-d\) and add to \( I(u_n) \leq c + 1. \) Then
\[
c + 1 + \| u_n' \| \geq \left( \frac{1}{2} - d \right) \| u_n' \|^2 + \frac{1}{2} \| u_n'' \|^2
\]
\[
+ \int_{\Omega} \left\{ d | u_n |^{p-2} u_n (u_n' - d^{-1} p^{-1} u_n) + \lambda u_n \left( d u_n' - \frac{1}{2} u_n \right) \right\} dx. \]
Since \( | u_n - u_n' | \leq \alpha \) according to (32) and \( d^{-1} p^{-1} < 1, \) the integrand is positive if \( | u_n | \geq R \) and \( R \) is sufficiently large. Hence
\[
c + 1 + \| u_n' \| \geq \left( \frac{1}{2} - d \right) \| u_n' \|^2 + \frac{1}{2} \| u_n'' \|^2 - C. \]
So the sequence \((u_n)\) is bounded and a familiar argument shows that it possesses a strongly convergent subsequence. □

Note that Corollaries 5.9 and 5.10 partially generalize a result of Ambrosetti and Rabinowitz [I, Theorem 3.32, see also 21, 24].

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