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An existence result for nonlinear elliptic problems involving critical Sobolev exponent  


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An existence result
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by

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\textbf{Abstract.} — In this paper we consider the following problem:

\begin{equation}
\begin{cases}
- \Delta u - \lambda u = |u|^{2^*-2} \cdot u \\
u = 0 \quad \text{on } \partial \Omega \quad 2^* = \frac{2n}{n-2}
\end{cases}
\end{equation}

where $\Omega \subset \mathbb{R}^n$ is a bounded domain and $\lambda \in \mathbb{R}$.

We prove the existence of a nontrivial solution of (1) for any $\lambda > 0$, if $n \geq 4$.

\textbf{Résumé.} — Soient $\Omega$ un sous-ensemble ouvert borné de $\mathbb{R}^n$ et $\lambda$ un nombre positif, le but de cette note c'est de montrer que le problème suivant :

\begin{equation}
\begin{cases}
- \Delta u - \lambda u = |u|^{2^*-2} \cdot u \\
u |_{\partial \Omega} = 0 \quad 2^* = \frac{2n}{n-2}
\end{cases}
\end{equation}

admet, au moins, une solution non triviale, si $n \geq 4$. 

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0. INTRODUCTION

Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be an open bounded set with smooth boundary. Consider the problem

\[
\begin{cases}
- \Delta u - \lambda u - u \cdot |u|^{2^* - 2} = 0 \\
\quad u \in H^1_0(\Omega)
\end{cases}
\]

(0.1)

where $\lambda$ is a real parameter and $2^* = \frac{2n}{n - 2}$ is the critical Sobolev exponent for the Sobolev embedding $H^1_0(\Omega) \hookrightarrow L^{2^*}(\Omega)$.

The solutions of (0.1) are the critical points of the energy functional

\[
f_\lambda(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \frac{\lambda}{2} \int_\Omega |u|^2 dx - \frac{1}{2^*} \int_\Omega |u|^{2^*} dx.
\]

(0.2)

Since the embedding $H^1_0(\Omega) \hookrightarrow L^{2^*}(\Omega)$ is not compact the functional $f_\lambda$ does not satisfy the Palais-Smale condition in the energy range $[-\infty, +\infty]$ (cfr. remark 2.3 of [4]).

Moreover if $\lambda \leq 0$ and $\Omega$ is starshaped (0.1) has only the trivial solution (cf. [6]).

Recently Brezis and Nirenberg in [2] have proved that if $n > 4$ and $0 < \lambda < \lambda_1$ ($\lambda_1$ is the first eigenvalue of $-\Delta$) then (0.1) has a positive solution. In [4] Cerami, Fortunato and Struwe have obtained multiplicity results for (0.1) in the case in which $\lambda$ belongs to a suitable left neighbourhood of an arbitrary eigenvalue of $-\Delta$ (cf. also [3]).

In this paper we prove the following theorem:

**THEOREM 0.1.** If $n \geq 4$ the problem (0.1) possesses at least one non trivial solution for any $\lambda > 0$.

A weaker result related to theorem 0.1 has been announced in [5].

We observe that if $n = 3$ and $\Omega$ is a ball, Brezis and Nirenberg [2] have proved that the problem (0.1) does not have nontrivial radial solutions if $0 < \lambda < \frac{\lambda_1}{4}$.

1. SOME PRELIMINARIES

Let $\| \cdot \|_p$ denote respectively the norms in $H^1_0(\Omega)$ and $L^p(\Omega)$ ($1 \leq p \leq \infty$), and let

\[
S = \inf \{ \| u \|_p^2 / \| u \|_{2^*}^2 : u \in H^1_0(\Omega) \setminus \{0\} \}
\]

denote the best constant for the embedding $H^1_0(\Omega) \hookrightarrow L^{2^*}(\Omega)$.

The following lemma shows that $f_\lambda$ satisfies a local P.S. condition.

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LEMMA 1.1. — For any \( \lambda \in \mathbb{R} \) the functional \( f_\lambda \) (see (0.2)) satisfies the Palais-Smale condition in \( -\infty, -\infty \) in the following sense:

If \( c < \frac{1}{n} S_n^{1/2} \) and \( \{ u_m \} \) is a sequence in \( H_0^1(\Omega) \) such that

\[
\text{as } m \to \infty, f_\lambda(u_m) \to c, f'_\lambda(u_m) \to 0 \text{ strongly in } H^{-1}(\Omega), \text{then } \{ u_m \}\]

(P.S.) \( \{ u_m \} \) contains a subsequence converging strongly in \( H_0^1(\Omega) \).

The proof of this lemma is in [2] and in [4]. We recall that a deeper compactness result has been proved in [7].

We recall a critical point Theorem (cf. [1, Theorem 2.4]) which is a variant of some results contained in [0].

THEOREM 1.2. — Let \( H \) be a real Hilbert space and \( f \in C^1(H, \mathbb{R}) \) be a functional satisfying the following assumptions:

\((f_1)\) \( f(u) = f(-u), f(0) = 0 \) for any \( u \in H \)

\((f_2)\) there exists \( \beta > 0 \) such that \( f \) satisfies (P. S.) in \( ]0, \beta[ \)

\((f_3)\) there exist two closed subspaces \( V, W \subset H \) and positive constants \( \rho, \delta \) such that

(i) \( f(u) < \beta \) for any \( u \in W \)

(ii) \( f(u) \geq \delta \) for any \( u \in V, \| u \| = \rho \)

(iii) \( \text{codim } V < +\infty. \)

Then there exist at least \( m \) pairs of critical points, with

\[
m = \dim(V \cap W) - \text{codim}(V + W).
\]

2. PROOF OF THEOREM 0.1

Our aim is to define two suitable closed subspaces \( V \) and \( W \), with \( V \cap W \neq \{ 0 \} \) and \( V + W = H \), such that \( f_\lambda \) satisfies the assumptions \( f_2) \) and \( f_3) \) of Theorem 1.2 with \( \beta = \frac{1}{n} S_n^{1/2} \).

In the sequel we denote by \( \lambda_j \) the eigenvalues of \( -\Delta \) and by \( M(\lambda_j) \) the corresponding eigenspaces.

Given \( \lambda > 0 \), we set

\[
\lambda^+ = \min \{ \lambda_j | \lambda < \lambda_j \}
\]

\[
H_1 = \bigoplus_{\lambda_j \geq \lambda^+} M(\lambda_j) \quad H_2 = \bigoplus_{\lambda_j \leq \lambda^+} M(\lambda_j)
\]

where the closure is taken in \( H_0^1(\Omega) \).

If \( r > 0 \) we set

\[
N_r(0) = \{ x \in \mathbb{R}^n | \| x \| < r \}.
\]
Without loss of generality we can suppose that $0 \in \Omega$ and that $N_1(0) \subset \Omega$. Given $\mu > 0$ we set (cf. [2] [7])

$$\psi_\mu(x) = \phi(x) \cdot u_\mu(x)$$

where $\phi \in C_0^\infty(N_1(0))$, $\phi(x) = 1$ on $N_1(0)$, and

$$u_\mu(x) = \frac{|n(n-2)\mu|^{(n-2)/4}}{|\mu + |x|^2|^{(n-2)/2}}.$$

The following lemma holds:

**Lemma 2.1.** If $\psi_\mu(x)$ is defined as in (2.1), then for any $\mu$

\begin{align*}
(2.2) & \quad \| \psi_\mu \|_2^2 = S^{n/2} + O(\mu^{(n-2)/2}) \quad (1) \\
(2.3) & \quad |\psi_\mu|_{2^*}^2 = S^{n/2} + O(\mu^{n/2}) \\
(2.4) & \quad |\psi_\mu|_{2^*}^2 = \begin{cases} 
K_1 \mu + O(\mu^{(n-2)/2}) & \text{if } n \geq 5 \\
K_1 \mu |\log \mu| + O(\mu) & \text{if } n = 4 
\end{cases} \\
(2.5) & \quad |\psi_\mu|_{2^* - 1}^2 \leq K_2 \mu^{(n-2)/4} \\
(2.6) & \quad |\psi_\mu|_{2^* - 1}^2 \leq K_3 \mu^{(n-2)/4}
\end{align*}

where $K_1$, $K_2$, $K_3$ are suitable positive constants.

**Proof.** The proof of (2.2), (2.3), (2.4) is contained in [2], moreover (2.5) and (2.6) can be straightforward verified.

Now we shall prove some technical lemmas. We set

$$\bar{W}(\mu) = \left\{ u \in H_0^1 \mid u = u^- + t\psi_\mu, u^- \in H_2, t \in \mathbb{R} \right\}.$$

The following lemma holds:

**Lemma 2.2.** If $u \in \bar{W}(\mu)$, then for any $\mu > 0$

\begin{align*}
(2.7) & \quad |u|_{2^*}^{2*} \geq |t\psi_\mu|_{2^*}^{2*} + \frac{1}{2} |u^-|_{2^*}^{2*} - K_4 t^{2*} \mu^{n(n-2)/(2n+4)} \quad \text{for any } t \in \mathbb{R}.
\end{align*}

**Proof.** By the identity

\begin{align*}
(2.8) & \quad |u|_{2^*}^{2*} = 2^* \int_\Omega dx \int_0^u |s|^{2^* - 2} s ds
\end{align*}

\(^{(1)}\) In the sequel we denote by $O(\mu^a)$, $a > 0$ a function $|f(\mu)| \leq \text{const} \mu^a$ near $\mu = 0$, and by $O(\mu)$, a function such that $f(\mu)/\mu \to 0$ as $\mu \to 0$.

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it follows that

\[ (2.9) \quad \| u^- + t\psi_\mu \|_{2^*}^2 - | t\psi_\mu |_{2^*}^2 - | u^- |_{2^*}^2 = \]

\[ = 2^* \left( \int_0^1 dt \int_{\Omega} [ | tu^- + t\psi_\mu |_{2^*}^2 - (t\psi_\mu + tu^-) - | tu^- |_{2^*}^2 \cdot tu^- ] u^- dx \right) = \]

\[ = 2^*(2^* - 1) \int_0^1 dt \int_{\Omega} | tu^- + t\psi_\mu \theta |_{2^*}^2 \cdot t\psi_\mu \cdot u^- dx \]

where \( \theta = \theta(x) \) is a measurable function such that \( 0 < \theta(x) < 1 \).

By (2.9) and by (2.5), (2.6) we have that

\[ \leq c_1 \int_0^1 dt \int_{\Omega} \left\{ | u^- | \cdot | t\psi_\mu |_{2^*}^{2^* - 1} + \tau^{2^* - 2} \cdot | t\psi_\mu | \cdot | u^- |_{2^*}^{2^* - 1} \right\} dx \leq \]

\[ \leq c_2 \left\{ | t\psi_\mu |_{2^* - 1} \cdot | u^- |_{\infty} + | t\psi_\mu |_{1} \cdot | u^- |_{2^* - 1} \right\} \leq \]

\[ \leq c_3 \left\{ | t\psi_\mu |_{2^* - 1} \cdot | u^- |_{L^2} + | t\psi_\mu |_{1} \cdot | u^- |_{2^* - 1} \right\} \leq \]

\[ \leq e_3 \cdot t^{2^* - 1} \cdot \mu^{(n-2)/4} | u^- |_{L^2} + \frac{1}{4} | u^- |_{2^*}^2 + c_4 \cdot t^{2^*} \cdot \mu^{n/2} \leq \]

\[ \leq \frac{1}{2} | u^- |_{2^*}^2 + k_4 t^{2^*} \cdot \mu^{(n-2)/2n+4} \]

and the lemma is proved.

**Lemma 2.3.** — If \( \mu \) is sufficiently small, then

\[ (2.10a) \quad \frac{\| \psi_\mu \|^2 - \lambda | \psi_\mu |_{2^*}^2}{| \psi_\mu |_{2^*}^2} = \begin{cases} S - K_5 \mu + O(\mu^{n/2}) & \text{if } n \geq 5 \\ S + K_5 \mu \log \mu + O(\mu) & \text{if } n = 4 \end{cases} \]

**Proof.** — The evaluation (2.11) follows immediately by (2.2), (2.3) and (2.4).

**Remark 2.4.** — Suppose that \( \lambda = \lambda_j \) with \( \lambda_j \in \sigma(-\Delta) \) and denote by \( P_j \) the projector on the eigenspace \( M_j \) corresponding to \( \lambda_j \).

We set

\[ (2.12) \quad \tilde{\psi}_\mu = \psi_\mu - P_j \psi_\mu. \]

Let \( \{ v_k \} \) an orthonormal family spanning \( M_j \), then by (2.5) we have

\[ (2.13) \quad \left| P_j \psi_\mu \right|_{2^*}^2 = \sum_k \left( \int_{\Omega} \psi_\mu v_k dx \right)^2 \leq \text{const} \ | \psi_\mu |_1^2 \leq K_6 \mu^{\frac{n-2}{2}} \]

then

\[ (2.14) \quad \left| P_j \psi_\mu \right|_{L^\infty} \leq K_7 \mu^{\frac{n-2}{4}}. \]
Moreover we have
\[ \left| \int \Omega \left\{ |\tilde{\psi}_\mu|^{2^*} - |\psi_\mu|^{2^*} \right\} \, dx \right| = 2^* \int_0^1 d\tau \int \Omega \left| \psi_\mu - \tau P_j \psi_\mu \right|^{2^*-2} (\psi_\mu - \tau P_j \psi_\mu) P_j \psi_\mu \, dx \leq \]
\[ \leq 2^* \cdot 2^*-1 \int_0^1 d\tau \int \Omega \left\{ \left| \psi_\mu \right|^{2^*-1} + \tau^{2^*-1} \left| P_j \psi_\mu \right|^{2^*-1} \right\} \left| P_j \psi_\mu \right| \, dx \leq \text{const} \left\{ \left| \psi_\mu \right|^{2^*-1} + \left| P_j \psi_\mu \right|_\infty + \left| P_j \psi_\mu \right|^{2^*} \right\} . \]

Then by (2.14) and (2.6) it follows that
\[ (2.15) \quad \left| \left| \tilde{\psi}_\mu \right|^{2^*} - \left| \psi_\mu \right|^{2^*} \right| \leq c_1 \mu^{-\frac{n-2}{4}}. \]

Moreover by (2.14) and (2.6) we have
\[ (2.16) \quad \left| \left| \tilde{\psi}_\mu \right|^{2^*-1} \right| = \left| \psi_\mu - P_j \psi_\mu \right|^{2^*-1} \leq \text{const} \left\{ \left| \psi_\mu \right|^{2^*-1} + \left| P_j \psi_\mu \right|^{2^*-1} \right\} \leq \text{const} \mu^{-\frac{n-2}{4}}. \]

Analogously by (2.14) and (2.5) we have
\[ (2.17) \quad \left| \tilde{\psi}_\mu \right|_1 \leq \text{const} \mu^{-\frac{n-2}{4}}. \]

By (2.15), (2.16), (2.17) it easily follows that (2.11) holds if we replace \( \psi_\mu \) with \( \tilde{\psi}_\mu \).

Moreover, by (2.15), (2.16), (2.17), also (2.7) holds (for \( \mu \) small) if we replace \( \psi_\mu \) with \( \tilde{\psi}_\mu \) and \( \overline{W} (\mu) \) with
\[ \overline{W} (\mu) = \{ u \in H^1_{0} \mid u = u^- + t \tilde{\psi}_\mu, u^- \in H_{2}, t \in \mathbb{R} \}. \]

Now we can prove a crucial lemma:

**Lemma 2.5.** — For \( \mu \) sufficiently small
\[ (2.18) \quad \sup_{\mathcal{W}} f(u) < \frac{1}{n} S^{n/2} \]
where \( \mathcal{W} = \overline{W} (\mu) \) (resp. \( \overline{W} (\mu) \)) if \( \lambda \notin \sigma (-\Delta) \) (resp. \( \lambda \in \sigma (-\Delta) \)).

**Proof.** — Observe that if we fix \( u \in H^1_{0} (\Omega) \), \( u \neq 0 \), then
\[ (2.19) \quad \max_{t} f_u (tu) = \frac{1}{n} \left( \| u \|^{2} - \lambda \left| u \right|_{2}^{2} \right)^{n/2}. \]

Then in order to prove (2.18) we need to evaluate
\[ (2.20) \quad \sup_{\mathcal{W} (\mu) \setminus \{ u \mid \lambda u \mid_{2} = 1 \}} \{ \| u \|^{2} - \lambda \left| u \right|_{2}^{2} \}. \]

We distinguish two cases:

**Case i**) \( \lambda \notin \sigma (-\Delta) \).

Let \( u = u^- + t \psi_\mu \in \overline{W} (\mu) \) with \( |u|_{2^*} = 1 \).
Observe that $t$ is bounded if $\mu$ is small, in fact by (2.7) and (2.3) we get
\[ 1 = |u|^2 - \frac{2n}{n-2} |\nabla u|^2 - K_4 t^{2n/2} + \frac{1}{2} |u^-|^2 = t^{2n/2} + O(\mu^{n/2}) + \frac{1}{2} |u^-|^2. \]

Then by (2.5) we have that
\[ (2.21) \quad \|u\|^2 - \mu |u|_2^2 = |\nabla u|^2 - \frac{2n}{n-2} |\nabla u|^2 - \lambda |u|^2 + |\nabla t\psi_\mu|^2 - \lambda |t\psi_\mu|^2 - \mu \lambda \|u\|^2. \]
\[ \leq |\nabla u|^2 - \frac{2n}{n-2} |\nabla u|^2 - \lambda |u|^2 + |\nabla t\psi_\mu|^2 - \lambda |t\psi_\mu|^2 - \mu \lambda \|u\|^2. \]
\[ \leq (\lambda - \frac{2n}{n-2}) |u|_2^2 - \frac{2n}{n-2} |\nabla t\psi_\mu|^2 - \lambda |t\psi_\mu|^2 - \mu \lambda \|u\|^2. \]

where $\lambda = \max \{ \lambda_j | \lambda_j < \lambda \}$. 

We set $A(u^-, \mu, c) = (\lambda - \frac{2n}{n-2}) |u^-|^2 + C |u^-|^2 - \mu \lambda$ and observe that
\[ (2.22) \quad A(u^-, \mu, c) \leq 0 \quad \text{or} \quad A(u^-, \mu, c) \leq c^2/(\lambda - \frac{2n}{n-2}). \]

If $|u^-|^2 < 2K_4 t^{2n/2} + \mu^{n-2} + 4$, by (2.10) and the boundness of $t$,
\[ |t\psi_\mu|^2 \leq \left( 1 - \frac{3}{4} |u|^2 + c_3 \mu |u|_2^2 + c_4 \right)^2 \]
\[ \leq 1 + \frac{2}{2^*} (c_3 \mu |u|_2^2 + c_4 \mu^2)^2. \]

then, if $n \geq 5$, by (2.11), (2.21)
\[ (2.23) \quad \|u\|^2 - \lambda |u|_2^2 \leq (S - K_5 \mu + O(\mu^{n/2})) + A(u^-, \mu, c). \]

If $|u^-|^2 > 2K_4 t^{2n/2} + \mu^{n-2} + 4$, by (2.7), $|t\psi_\mu|^2 < 1$, then, by (2.21)
\[ (2.24) \quad \|u\|^2 - \lambda |u|_2^2 \leq (S - K_5 \mu + O(\mu^{n/2})) + A(u^-, \mu, c). \]

then, by (2.22), the conclusion follows in the case $n \geq 5$.

If $n = 4$ the proof is the same. In this case (2.11) replaces (2.11) in (2.22).

Case ii) $\lambda = \lambda_j \in \sigma(-A)$.

Let $u = u^- + t\psi_\mu \in W(\mu)$ with $|u|_{2^*} = 1$. We set $u = u^- + t\psi_\mu = \hat{u} + P_j u^- + t\psi_\mu$, then
\[ \|u\|^2 - \lambda_j |u|_2^2 \]
\[ = |\nabla \psi_\mu|^2 - \lambda_j |t\psi_\mu|^2 + |\nabla u^-|^2 - \lambda_j |u^-|^2 - 2 \int \psi_\mu \Delta u^- + \lambda_j \psi_\mu u^-) dx. \]

Observe that
\[
\int_\Omega (t\tilde{\psi}_\mu \Delta u^- + \lambda \tilde{\psi}_\mu u^-)dx = \int_\Omega (t\tilde{\psi}_\mu \Delta \tilde{u}^- + \lambda \tilde{\psi}_\mu \tilde{u}^-)dx \leq
\]
\[
\leq |\Delta \tilde{u}^-|_\infty |t\tilde{\psi}_\mu|_1 + \lambda \tilde{\psi}_\mu |\tilde{u}^-|_1 \leq c_3 |\tilde{u}^-|_1 \quad n-2 \mu^\frac{n-2}{4}.
\]
Now the proof follows by using the previous arguments.

**Proof of theorem 0.1.** — If \( \lambda \notin \sigma(-\Delta) \) (\( \lambda > 0 \)) we set \( V = H_1 \) and \( W = \overline{W(\mu)} \) with \( \mu \) suitably small in order that \((2.18)\) is verified. We see that the assumptions of Theorem 1.2 are satisfied. Obviously \((f_1)\) and \((f_3.iii)\) are verified. Moreover \((f_2)\) is verified with \( \beta = \frac{1}{n} S_n^{\alpha_2} \) by lemma 1.1 and \((f_3.i)\) \( \left( \text{with } \beta = \frac{1}{n} S_n^{\alpha_2} \right) \) is verified by lemma 2.5.

Finally observe that if \( u \in H_1 \), then
\[
(2.25) \quad f_3(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \frac{\lambda}{2} \int_\Omega |u|^2 dx - \frac{1}{2*} \int_\Omega |u|^{2*} dx \geq
\]
\[
\geq \frac{1}{2} \left( 1 - \frac{\lambda}{\lambda^+} \right) ||u||^2 - \frac{1}{2*} ||u||^{2*} \geq \frac{1}{2} \left( 1 - \frac{\lambda}{\lambda^+} \right) ||u||^2 - \text{const} ||u||^{2*} \geq \delta > 0
\]
if \( ||u|| = \rho \) with \( \rho \) suitably small.

Hence by \((2.27)\) also \((f_3.ii)\) is verified. Since \( \text{dim } V \cap W = 1 \) and \( V + W = H_0^1(\Omega) \), then by Theorem 1.2, we deduce that problem \((0.1)\) has at least one non trivial solution.

If \( \lambda \in \sigma(-\Delta) \) we set \( W = \overline{W(\mu)} \) with \( \mu \) suitably small in order that \((2.18)\) is verified and, by repeating the above arguments, the conclusion follows.

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