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An existence result for nonlinear elliptic problems involving critical Sobolev exponent


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by

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ABSTRACT. — In this paper we consider the following problem:

\[\begin{align*}
-\Delta u - \lambda u &= |u|^{2^*-2} \cdot u \\
u &= 0 \quad \text{on } \partial \Omega
\end{align*}\]

where \(\Omega \subset \mathbb{R}^n\) is a bounded domain and \(\lambda \in \mathbb{R}\).

We prove the existence of a nontrivial solution of (1) for any \(\lambda > 0\), if \(n \geq 4\).

RÉSUMÉ. — Soient \(\Omega\) un sous-ensemble ouvert borné de \(\mathbb{R}^n\) et \(\lambda\) un nombre positif, le but de cette note c'est de montrer que le problème suivant :

\[\begin{align*}
-\Delta u - \lambda u &= |u|^{2^*-2} \cdot u \\
u |_{\partial \Omega} &= 0
\end{align*}\]

admet, au moins, une solution non triviale, si \(n \geq 4\).

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0. INTRODUCTION

Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be an open bounded set with smooth boundary. Consider the problem

\begin{equation}
\begin{aligned}
- \Delta u - \lambda u - u \cdot |u|^{2^* - 2} &= 0 \\
&\text{in } \Omega
\end{aligned}
\end{equation}

where $\lambda$ is a real parameter and $2^* = \frac{2n}{n - 2}$ is the critical Sobolev exponent for the Sobolev embedding $H^1_0(\Omega) \subset L^{2^*}(\Omega)$.

The solutions of (0.1) are the critical points of the energy functional

\begin{equation}
f_\lambda(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \frac{\lambda}{2} \int_\Omega |u|^2 dx - \frac{1}{2^*} \int_\Omega |u|^{2^*} dx.
\end{equation}

Since the embedding $H^1_0(\Omega) \subset L^{2^*}(\Omega)$ is not compact the functional $f_\lambda$ does not satisfy the Palais-Smale condition in the energy range $]-\infty, +\infty[$ (cfr. remark 2.3 of [4]).

Moreover if $\lambda \leq 0$ and $\Omega$ is starshaped (0.1) has only the trivial solution (cf. [6]).

Recently Brezis and Nirenberg in [2] have proved that if $n > 4$ and $0 < \lambda < \lambda_1$ ($\lambda_1$ is the first eigenvalue of $-\Delta$) then (0.1) has a positive solution. In [4] Cerami, Fortunato and Struwe have obtained multiplicity results for (0.1) in the case in which $\lambda$ belongs to a suitable left neighbourhood of an arbitrary eigenvalue of $-\Delta$ (cf. also [3]).

In this paper we prove the following theorem:

**Theorem 0.1.** — If $n \geq 4$ the problem (0.1) possesses at least one non trivial solution for any $\lambda > 0$.

A weaker result related to theorem 0.1 has been announced in [5].

We observe that if $n = 3$ and $\Omega$ is a ball, Brezis and Nirenberg [2] have proved that the problem (0.1) does not have nontrivial radial solutions if $0 < \lambda < \frac{\lambda_1}{4}$.

1. SOME PRELIMINARIES

Let $\| \cdot \|, \| \cdot \|_p$ denote respectively the norms in $H^1_0(\Omega)$ and $L^p(\Omega)$ $(1 \leq p \leq \infty)$, and let

\[ S = \inf \{ \| u \|^2_2 / \| u \|_{2^*}^2 : u \in H^1_0(\Omega) \setminus \{ 0 \} \} \]

denote the best constant for the embedding $H^1_0(\Omega) \subset L^{2^*}(\Omega)$.

The following lemma shows that $f_\lambda$ satisfies a local P.S. condition.
LEMMA I.1. — For any \( \lambda \in \mathbb{R} \) the functional \( f_\lambda \) (see (0.2)) satisfies the Palais-Smale condition in 
\[ -\infty, -\frac{1}{n}S^{n/2} \] in the following sense:

If \( c < -\frac{1}{n}S^{n/2} \) and \( \{ u_m \} \) is a sequence in \( H_0^1(\Omega) \) such that

(P. S.) \( \{ \) as \( m \to \infty f_\lambda(u_m) \to c, f_\lambda'(u_m) \to 0 \) strongly in \( H^{-1}(\Omega) \), then \( \{ u_m \} \) contains a subsequence converging strongly in \( H^1_0(\Omega) \).

The proof of this lemma is in [2] and in [4]. We recall that a deeper compactness result has been proved in [7].

We recall a critical point Theorem (cf. [1, Theorem 2.4]) which is a variant of some results contained in [8].

THEOREM 1.2. — Let \( H \) be a real Hilbert space and \( f \in C^1(H, \mathbb{R}) \) be a functional satisfying the following assumptions:

\( (f_1) \) \( f(u) = f(-u), f(0) = 0 \) for any \( u \in H \)

\( (f_2) \) there exists \( \beta > 0 \) such that \( f \) satisfies (P. S.) in \( ]0, \beta[ \)

\( (f_3) \) there exist two closed subspaces \( V, W \subset H \) and positive constants \( \rho, \delta \) such that

(i) \( f(u) \leq \beta \) for any \( u \in W \)

(ii) \( f(u) \geq \delta \) for any \( u \in V, \| u \| = \rho \)

(iii) \( \text{codim } V < + \infty \).

Then there exist at least \( m \) pairs of critical points, with

\[ m = \text{dim } (V \cap W) - \text{codim } (V + W) \]

2. PROOF OF THEOREM 0.1

Our aim is to define two suitable closed subspaces \( V \) and \( W \), with \( V \cap W \neq \{ 0 \} \) and \( V + W = H \), such that \( f_\lambda \) satisfies the assumptions \( f_2) \) and \( f_3) \) of Theorem 1.2 with \( \beta = \frac{1}{n}S^{n/2} \).

In the sequel we denote by \( \lambda_j \) the eigenvalues of \( -\Delta \) and by \( M(\lambda_j) \) the corresponding eigenspaces.

Given \( \lambda > 0 \), we set

\[ \begin{array}{c}
\lambda_j^+ = \min \{ \lambda_j \mid \lambda_j < \lambda \} \\
H_1 = \bigoplus_{\lambda_j^+ \geq \lambda} M(\lambda_j) \\
H_2 = \bigoplus_{\lambda_j < \lambda} M(\lambda_j)
\end{array} \]

where the closure is taken in \( H_0^1(\Omega) \).

If \( r > 0 \) we set

\[ N_r(0) = \{ x \in \mathbb{R}^n \mid \| x \| < r \} \].
Without loss of generality we can suppose that $0 \in \Omega$ and that $N_1(0) \subset \Omega$. Given $\mu > 0$ we set (cf. [2] [7])

$$\psi_{\mu}(x) = \phi(x) \cdot u_{\mu}(x)$$

where $\phi \in C_0^\infty(N_1(0))$, $\phi(x) = 1$ on $N_1(0)$, and

$$u_{\mu}(x) = \frac{|n(n-2)\mu|^{(n-2)/4}}{|\mu + |x|^2|^{(n-2)/2}}.$$

The following lemma holds:

**Lemma 2.1.** If $\psi_{\mu}(x)$ is defined as in (2.1), then for any $\mu$

\begin{align*}
(2.2) & \quad \| \psi_{\mu} \|^2 = S^{n/2} + O(\mu^{(n-2)/2}) \quad (1) \\
(2.3) & \quad \| \psi_{\mu} \|^2 \leq S^{n/2} + O(\mu^{n/2}) \\
(2.4) & \quad \| \psi_{\mu} \|^2 = \begin{cases} 
K_1\mu + O(\mu^{(n-2)/2}) & \text{if } n \geq 5 \\
K_1\mu \log \mu + O(\mu) & \text{if } n = 4 
\end{cases} \\
(2.5) & \quad \| \psi_{\mu} \|_1 \leq K_2\mu^{(n-2)/4} \\
(2.6) & \quad \| \psi_{\mu} \|_{2^* - 1} \leq K_3\mu^{(n-2)/4}
\end{align*}

where $K_1$, $K_2$, $K_3$ are suitable positive constants.

**Proof.** The proof of (2.2), (2.3), (2.4) is contained in [2], moreover (2.5) and (2.6) can be straightforward verified.

Now we shall prove some technical lemmas. We set

$$\overline{W}(\mu) = \{ u \in H^1_0 \mid u = u^- + t\psi_{\mu}, \ u^- \in H_2, t \in \mathbb{R} \}.$$ 

The following lemma holds:

**Lemma 2.2.** If $u \in \overline{W}(\mu)$, then for any $\mu > 0$

\begin{align*}
(2.7) & \quad \| u \|_{2^*}^2 \geq \| t\psi_{\mu} \|_{2^*}^2 + \frac{1}{2} \| u^- \|_{2^*}^2 - K_4 t^{2^*} \mu^{\alpha(n-2)/(2n+4)} \quad \text{for any } t \in \mathbb{R}.
\end{align*}

**Proof.** By the identity

\begin{align*}
(2.8) & \quad \| u \|_{2^*}^2 = 2^* \int \int_0^u dx \int_0^s |s|^{2^* - 2} ds
\end{align*}

\(^{(1)}\) In the sequel we denote by $O(\mu^a)$, $a > 0$ a function $|f(\mu)| \leq \text{const } \mu^a$ near $\mu = 0$, and by $o(\mu)$, a function such that $f(\mu)/\mu \to 0$ as $\mu \to 0$. 

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it follows that
\[(2.9) \quad \left| u^- + t\psi_\mu \right|^{2^*_\mu} - |t\psi_\mu|^{2^*_\mu} - |u^-|^{2^*_\mu} =\]
\[= 2^* \int_0^1 \int_\Omega \left[ |t\psi_\mu + \tau u^-|^{2^* - 2} \cdot (t\psi_\mu + \tau u^-) - |\tau u^-|^{2^* - 2} \cdot \tau u^- \right] u^- dx =\]
\[= 2^*(2^* - 1) \int_0^1 \int_\Omega \left| \tau u^- + t\psi_\mu \theta \right|^{2^* - 2} \cdot t\psi_\mu \cdot u^- dx\]

where \( \theta = \theta(x) \) is a measurable function such that \( 0 < \theta(x) < 1 \).

By (2.9) and by (2.5), (2.6) we have that
\[(2.10) \quad \left| \left| u^- \right|^{\frac{2^*_\mu}{2}} - \left| t\psi_\mu \right|^{\frac{2^*_\mu}{2}} - \left| u^- \right|^{\frac{2^*_\mu}{2}} \right|
\[\leq c_1 \int_0^1 \int_\Omega \left\{ \left| u^- \right| \cdot |t\psi_\mu|^{2^* - 1} + \tau^{2^* - 2} \cdot |t\psi_\mu| \cdot |u^-|^{2^* - 1} \right\} dx \leq\]
\[\leq c_2 \left\{ \left| t\psi_\mu \right|^{\frac{2^*_\mu}{2} - 1} \cdot |u^-|_{\infty} + |t\psi_\mu|_1 \cdot |u^-|^{\frac{2^*_\mu}{2} - 1} \right\} \leq\]
\[\leq c_3 \left\{ \left| t\psi_\mu \right|^{\frac{2^*_\mu}{2} - 1} \cdot |u^-|_2 + |t\psi_\mu|_1 \cdot |u^-|^{\frac{2^*_\mu}{2} - 1} \right\} \leq\]
\[(2.10) \quad e_3 \cdot t^{2^* - 1} \cdot \mu^{(n-2)/4} |u^-|_2 + \frac{1}{4} u^- \cdot |t\psi_\mu|^{\frac{2^*_\mu}{2}} + c_4 \cdot t^{2^*} \cdot \mu^{n/2} \leq\]
\[\leq \frac{1}{2} |u^-|^{\frac{2^*_\mu}{2}} + k_4 t^{2^*} \cdot \mu^{(n-2)/2n + 4}\]
and the lemma is proved.

**Lemma 2.3.** — If \( \mu \) is sufficiently small, then
\[(2.11) \quad \frac{\left| |\psi_\mu|^2 - \lambda \right| |\psi_\mu|^2}{|\psi_\mu|^{2^*_\mu}} = \begin{cases} S - K_5 \mu + O(\mu^{\frac{n-2}{4}}) & \text{if } n \geq 5 \\ S + K_5 \log \mu + O(\mu) & \text{if } n = 4 \end{cases} \]

**Proof.** — The evaluation (2.11) follows immediately by (2.2), (2.3) and (2.4).

**Remark 2.4.** — Suppose that \( \lambda = \lambda_j \), with \( \lambda_j \in \sigma(- \Delta) \) and denote by \( P_j \) the projector on the eigenspace \( M_j \) corresponding to \( \lambda_j \).

We set
\[(2.12) \quad \widetilde{\psi}_\mu = \psi_\mu - P_j \psi_\mu.\]

Let \( \{ v_k \} \) an orthonormal family spanning \( M_j \), then by (2.5) we have
\[(2.13) \quad |P_j \psi_\mu|_2^2 = \sum_k \left( \int_\Omega \psi_\mu v_k dx \right)^2 \leq \text{const} \quad |\psi_\mu|^2 \leq K_6 \mu^{\frac{n-2}{2}}\]
then
\[(2.14) \quad |P_j \psi_\mu|_{\infty} \leq K_7 \mu^{\frac{n-2}{4}}.\]
Moreover we have
\[\left| \int_{\Omega} \left( |\tilde{\psi}_\mu|^2 - |\psi_\mu|^2 \right) dx \right| = 2^* \int_0^1 d\tau \int_{\Omega} \left| \psi_\mu - \tau P_j \psi_\mu \right|^{2^* - 2} (\psi_\mu - \tau P_j \psi_\mu) P_j \psi_\mu dx \leq \right.
\[\leq 2^* \cdot 2^{2^* - 1} \int_0^1 d\tau \int_{\Omega} \left( |\psi_\mu|^{2^* - 1} + \tau^{2^* - 1} |P_j \psi_\mu|^{2^* - 1} \right) |P_j \psi_\mu| dx \leq\]
\[\leq \text{const} \left\{ |\psi_\mu|^{\frac{2^*-1}{2}} |P_j \psi_\mu|_\infty + |P_j \psi_\mu|^{\frac{2^*}{2}} \right\}.
\]
Then by (2.14) and (2.6) it follows that
\[|\tilde{\psi}_\mu|^{\frac{2^*}{2}} - |\psi_\mu|^{\frac{2^*}{2}} \leq c_1 \mu^{\frac{n-2}{4}}.
\]
Moreover by (2.14) and (2.6) we have
\[|\tilde{\psi}_\mu|^{\frac{2^*}{2}} - \frac{1}{2} |\psi_\mu - P_j \psi_\mu|^{\frac{2^*}{2}} \leq \text{const} \left\{ |\psi_\mu|^{\frac{2^*}{2}} + |P_j \psi_\mu|^{\frac{2^*}{2}} \right\}
\leq\]
\[\leq \text{const} \mu^{\frac{n-2}{4}}.
\]
Analogously by (2.14) and (2.5) we have
\[|\tilde{\psi}_\mu|_1 \leq \text{const} \mu^{\frac{n-2}{4}}.
\]
By (2.15), (2.16), (2.17) it easily follows that (2.11) holds if we replace \(\psi_\mu\) with \(\tilde{\psi}_\mu\).
Moreover, by (2.15), (2.16), (2.17), also (2.7) holds (for \(\mu\) small) if we replace \(\psi_\mu\) with \(\tilde{\psi}_\mu\) and \(\bar{W}(\mu)\) with
\[\bar{W}(\mu) = \{ u \in H^1_0 \mid u = u^- + t \tilde{\psi}_\mu, u^- \in H_2, t \in \mathbb{R} \}.
\]
Now we can prove a crucial lemma:

**Lemma 2.5.** — *For \(\mu\) sufficiently small*

\[\sup_{\bar{W}} f(u) < \frac{1}{n} S^{n/2}
\]
where \(W = \bar{W}(\mu)\) (resp. \(\bar{W}(\mu)\)) if \(\lambda \notin \sigma(- \Delta)\) (resp. \(\lambda \in \sigma(- \Delta)\)).

**Proof.** — Observe that if we fix \(u \in H^1_0(\Omega)\), \(u \neq 0\), then
\[\max_t f_\lambda(tu) = \frac{1}{n} \left( \|u\|^2 - \lambda \|u\|_2^2 \right)^{n/2}.
\]
Then in order to prove (2.18) we need to evaluate
\[\sup_{u \in \bar{W}(\mu)} \{ \|u\|^2 - \lambda \|u\|_2^2 \} \]
We distinguish two cases:

*Case i) \(\lambda \notin \sigma(- \Delta)\).*

Let \(u = u^- + t \psi_\mu \in \bar{W}(\mu)\) with \(\|u\|_2 = 1\).
Observe that $t$ is bounded if $\mu$ is small, in fact by (2.7) and (2.3) we get

$$1 = |u|^2 + |\nabla u|^2 - K_n t^{2^* \mu^{n/2}} + \frac{1}{2} |u^-|^2 - t^{2^*} [S^{n/2} + O(\mu^{n/2})] + \frac{1}{2} |u^-|^2.$$

Then by (2.5) we have that

$$(2.21) \quad \|u\|^2 - \lambda |u|^2 = \|\nabla u\|^2 - \lambda |\nabla u|^2 + |\nabla t\psi|_2^2 - \lambda |\nabla u|^2 - 2 \int_\Omega \{ |t\psi|_1^2 + \Delta u^- + \lambda |u^-|^2 \} \, dx \leq$$

$$\leq |\nabla u|^2 - \lambda |u|^2 + |\nabla t\psi|_2^2 - \lambda |\nabla u|^2 + c_1 \{ |\Delta u|^2 \} + |u^-|^2 \} + |u^-|^2 \} \leq$$

$$\leq (\lambda - \bar{\lambda}) |u|^2 + \|\nabla t\psi|_2^2 - \lambda |\nabla t\psi|_2^2 + c_2 |u^-|^2 \leq 2 \mu^{n/4}$$

where $\bar{\lambda} = \max \{ \lambda_j | \lambda_j < \lambda \}$.

We set $A(u^-, \mu, c) = (\lambda - \bar{\lambda}) |u^-|^2 + C |u^-|^2 \mu^{n/4}$ and observe that

$$(2.22) \quad A(u^-, \mu, c) \leq 0 \quad \text{or} \quad A(u^-, \mu, c) \leq c^2 / (\lambda - \bar{\lambda}) \mu^{n/2}$$

If $|u^-|^2 \leq 2K_nt^{2^* \mu^{2n/4}}$, by (2.10) and the boundness of $t$,

$$|t\psi|_2^2 \leq \left( 1 - \frac{3}{4} |u^-|^2 + c_3 \mu^{n/4} |u^-|^2 + c_4 \mu^n \right)^{2^*}$$

$$\leq 1 + \frac{2}{2^*} (c_3 \mu^{n/4} |u^-|^2 + c_4 \mu^n)^{2^*},$$

then, if $n \geq 5$, by (2.11) and (2.21)

$$(2.23) \quad \|u\|^2 - \lambda |u|^2 \leq (S - K_n \mu + 0(\mu^{n/2})) (1 + c_5 \mu^{n/2}) + A(u^-, \mu, c).$$

If $|u^-|^2 > 2K_nt^{2^* \mu^{2n/4}}$, by (2.7), $|t\psi|_2^2 < 1$, then, by (2.21)

$$(2.24) \quad \|u\|^2 - \lambda |u|^2 \leq (S - K_n \mu + 0(\mu^{n/2})) + A(u^-, \mu, c),$$

then, by (2.22), the conclusion follows in the case $n \geq 5$.

If $n = 4$ the proof is the same. In this case (2.11) replaces (2.11) in (2.22).

Case ii) $\lambda = \lambda_j \in \sigma(- \Delta)$

Let $u = u_- + t\tilde{\psi} \in \mathcal{W}(\mu)$ with $|u| = 1$. We set $u = u_- + t\tilde{\psi} = \tilde{u} + P_j u^- + t\tilde{\psi}_\mu$, then

$$\|u\|^2 - \lambda_j |u|^2$$

$$= |\nabla \tilde{\psi}|_2^2 - \lambda_j |t\psi|_2^2 + |\nabla u^-|_2^2 - \lambda_j |u^-|_2^2 - 2 \int_\Omega \{ t\psi \Delta u^- + \lambda_j t\psi u^- \} \, dx.$$
Observe that
\[
\int_{\Omega} (t \tilde{\psi}_\mu \Delta u^- + \lambda \tilde{\psi}_\mu u^-)dx = \int_{\Omega} (t \tilde{\psi}_\mu \Delta \tilde{u}_- + \lambda \tilde{\psi}_\mu \tilde{u}_-)dx \leq \left| \Delta \tilde{u}_- \right|_{\infty} |t\tilde{\psi}_\mu|_1 + \lambda \left| \tilde{\psi}_\mu \right|_1 \left| \tilde{u}_- \right|_{2\mu}^{n-2}.
\]
Now the proof follows by using the previous arguments.

Proof of theorem 0.1. — If \( \lambda \notin \sigma(-\Delta) (\lambda > 0) \) we set \( V = H_1 \) and \( W = \overline{W}(\mu) \) with \( \mu \) suitably small in order that (2.18) is verified. We see that the assumptions of Theorem 1.2 are satisfied. Obviously (f1) and (f3.iii) are verified. Moreover (f2) is verified with \( \beta = \frac{1}{n} S^{n/2} \) by lemma 1.1 and (f3.i) \( \left( \text{with } \beta = \frac{1}{n} S^{n/2} \right) \) is verified by lemma 2.5.

Finally observe that if \( u \in H_1 \), then
\[
(2.25) \quad f_2(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \frac{\lambda}{2} \int_{\Omega} |u|^2 dx - \frac{1}{2^*} \int_{\Omega} |u|^{2^*} dx \geq \frac{1}{2} \left( 1 - \frac{\lambda}{\lambda^*} \right) \|u\|^2 - \frac{1}{2} \left( \lambda - \frac{\lambda}{\lambda^*} \right) \|u\|^2 - \text{const} \|u\|^{2^*} \geq \delta > 0
\]
if \( \|u\| = \rho \) with \( \rho \) suitably small.

Hence by (2.27) also (f3.ii) is verified. Since \( \dim V \cap W = 1 \) and \( V + W = H^1_0(\Omega) \), then by Theorem 1.2, we deduce that problem (0.1) has at least one non trivial solution.

If \( \lambda \in \sigma(-\Delta) \) we set \( W = \overline{W}(\mu) \) with \( \mu \) suitably small in order that (2.18) is verified and, by repeating the above arguments, the conclusion follows.

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