HELMUT HOFER
Lagrangian embeddings and critical point theory
<http://www.numdam.org/item?id=AIHPC_1985__2_6_407_0>
Lagrangian embeddings and critical point theory

by

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ABSTRACT. — We derive a lower bound for the number of intersection points of an exact Lagrangian embedding of a compact manifold into its cotangent bundle with the zero section. To do this the intersection problem is converted into the problem of finding solutions of a Hamiltonian system satisfying canonical boundary conditions. The dynamical problem is then solved by global variational methods on a Hilbert manifold.

Key-words: Lagrangian Intersection theory, symplectic geometry, Hamiltonian systems, variational methods, strongly indefinite functionals.

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I. INTRODUCTION AND STATEMENT
OF THE MAIN RESULTS

1.1. Lagrangian embeddings and Hamiltonian systems.

Let $M$ be a compact connected differentiable manifold and $\tau^*_M : T^*M \to M$ its cotangent bundle. Denote by $s : M \to T^*M$ the zero section and let $\Sigma = s(M)$. On $T^*M$ there exists a unique 1-form $\lambda$, called the Liouville-form, such that $\beta^*\lambda = \beta$ for all 1-forms $\beta$ on $M$ considered as maps $M \to T^*M$. The associated 2-form $\omega = d\lambda$ is called the canonical symplectic form on $T^*M$. An immersion $\phi : M \to T^*M$ is called « Lagrangian » if $\phi^*\omega = 0$. If $\phi^*\lambda$ is exact we call it « exact Lagrangian ».

**Definition 1.** — A Lagrangian embedding $\phi : M \to T^*M$ is called « nice » if

(i) $\phi$ is exact.

(ii) There exists a differentiable map $\tilde{\phi} : [0, 1] \times M \to T^*M$ such that $\tilde{\phi}(\{0\} \times \Sigma) = \Sigma$, $\tilde{\phi}(1, .) = \phi$ and $\tilde{\phi}(t, .)$ is a Lagrangian embedding for all $t \in [0, 1]$.

In order to give the statement of the main result we use the notion of cohomological category.
DEFINITION 2. — The cohomological category of a topological space $X$, denoted by $\mathcal{C}(X)$, is the maximal number $k$ such that there exists a ring $R$ and cohomology classes $x_j \in H^{n(j)}(X, R)$, $n(j) \geq 1$, for $j = 1, \ldots, k - 1$, such that

$$x_1 \cup x_2 \cup \ldots \cup x_{k-1} \neq 0.$$ 

If there exist arbitrarily long products of the above type one puts $\mathcal{C}(X) = \infty$, and if $X = \emptyset$ one define $\mathcal{C}(X) = 0$.

The main result is the following:

**THEOREM 1.** — Let $\phi : M \to T^*M$ be a nice Lagrangian embedding. Then $\phi(M) \cap \Sigma$ contains at least $\mathcal{C}(M)$ points.

Let us state a theorem which implies Theorem 1. Denote by $h^* : [0, 1] \times T^*M \to \mathbb{R}$ a smooth map with compact support and let $h_t^* = h^*(t, \cdot)$. We introduce the associated (exact) Hamiltonian vectorfield $X_t$ by

$$i_{X_t} \omega = dh_t^*$$

and study the time-dependent Hamiltonian system with boundary conditions

$$\text{(HS)} \quad \dot{x} = X_t(x), \quad x(0), \quad x(1) \in \Sigma.$$

We have

**THEOREM 2.** — Let $h^*$ be as described above. Then (HS) possesses at least $\mathcal{C}(M)$ different solutions.

That Theorem 2 implies Theorem 1 was observed by M. Chaperon, who proved Theorem 2 for the special case $M = T^n$, which has been conjectured by Arnold [2]. The more general statement in Theorem 1 had been conjectured by M. Chaperon [5]. In order to reduce Theorem 1 to Theorem 2 we need the following lemma due to Chaperon ([5, 0.4.2 Theorem or (for the case $M = T^n$) 6, Lemme 2]).

**LEMMA.** — Let $\phi : M \to T^*M$ be a nice Lagrangian embedding. Then there exists a smooth map $h^* : [0, 1] \times T^*M \to \mathbb{R}$ with compact support, such that the points in $\phi(M) \cap \Sigma$ are in bijective correspondance to the solutions of (HS), where $X_t$ is the time-dependent Hamiltonian vectorfield associated to $h^*$.

1.2. Sketch of the proof.

Following the ideas of M. Chaperon we reduce Theorem 1 to Theorem 2. The problem to find solutions of (HS) is variational (the well-known degenerate classical variational principle). In order to give a good variational formulation fix a Riemannian metric $\langle \cdot, \cdot \rangle$ on $M$. Using the cano-
nical map $TM \to T^*M : x \to \langle x, \cdot \rangle$ the symplectic structure on $T^*M$ induces one on $TM$. Therefore, without loss of generality, we may assume that $\dot{x} = X_f(x)$ is a Hamiltonian system on $TM$ defined by a smooth map with compact support, say $h : [0, 1] \times TM \to \mathbb{R}$.

Denote by $\Lambda$ the Hilbert manifold consisting of absolutely continuous curves $q : [0, 1] \to M$ with square-integrable derivatives. Moreover denote by $\Lambda A$ the vector bundle over $\Lambda$ consisting of $L^2$-sections along $H^1$-curves. Let $\pi : \Lambda A \to \Lambda$ be the canonical projection. Define a $C^1$-map $\Psi_x : \Lambda A \to \mathbb{R}$ by

$$\Psi_x(q) = \int_0^1 \langle \dot{q}, x \rangle \, dt - \int_0^1 h_\ast(x(t)) \, dt$$

where $q = \pi x$. The solutions of (HS) are exactly the critical points of $\Psi_x$. $\alpha_\infty$ is a bounded perturbation of $\Psi$. So one might expect that the behaviour of $\Psi_\infty$ is similar to that of $\Psi$ as far as critical points are concerned. The first indication that this is true is the fact that $\Psi$ and $\Psi_\infty$ satisfy the so-called Palais-Smale condition. On the other hand $\Psi$ is of class $C^\infty$ and the linearisation at a critical point has an infinite positive and negative Morse-index. This implies that passing a critical level one has to attach infinite-dimensional cells which are invisible from the topological point of view. Clearly this will cause some difficulties. This difficulty with Morse theory or more generally with variational techniques for Hamiltonian systems is of course not new and well-known. One should mention here that in the framework of convex Hamiltonian systems a Morse theory, due to I. Ekeland [21], exists, which however cannot be applied in our case.

Now let us have a look at $\Psi$. Its critical points are exactly the constant maps $t \to O_m \in TM$, $m \in M$, where $O_m$ denotes the zero-element in $T_m M$. In the following we shall identify $M$ with the zero section in $TM$ and $TM$ with the constant curves in $\Lambda A$. Define the sets

$$S_1 = \{ \dot{q} \mid q \in \Lambda \}$$

$$S_2 = \{ - \dot{q} \mid q \in \Lambda \} .$$

We have

$$\Psi(S_1 \setminus M) \subset (0, + \infty)$$

$$\Psi(S_2 \setminus M) \subset (- \infty, 0)$$

$$\Psi(M) = 0$$

$$S_1 \cap S_2 = M .$$

Moreover $\Psi(\dot{q}) \to + \infty$ and $\Psi(- \dot{q}) \to - \infty$ as $\int | \dot{q} |^2 \to + \infty$. This shows that we have a « hyperbolic structure ». A feature already clearly exhibited in the seminal paper by Conley and Zehnder [7], where for the first time a global problem of symplectic geometry on a manifold was
solved by means of a classical variational principle in the loop space over the manifold. This hyperbolic structure will be preserved under the perturbation $\alpha_x$. The natural procedure to find critical points of $\Psi_x$ would be to apply the minus-gradient flow (which we assume to exist) $LA \times \mathbb{R} \rightarrow \mathbb{L}A : (x, t) \rightarrow x * t$ to the set $S_2$ and to show that the infinite-dimensional intersection problem $S_2 * t \cap S_1 \neq \emptyset$ has a solution for every $t \geq 0$. Then the number $c$ given by

$$ + \infty > c := \lim_{t \to +\infty} \sup \Psi_x(S_2 * t) \geq \inf \Psi_x(S_1) > - \infty $$

would be a critical level, since the Palais-Smale condition holds. A more sophisticated procedure, taking into account the size of the intersection, would give at least $c(M)$ critical points.

What we have just described will be in fact the underlying idea of our procedure. However, there are several underlying difficulties. The study of the intersection problem $S_2 * t \cap S_1 \neq \emptyset$ leads to a fixed point problem. In order to show the existence of fixed points one needs topological tools, which however are only applicable if some form of compactness is available in the problem, for example if the fixed point set is compact. Unfortunately, this cannot be shown. The reason for this is the fact that the gradient of $\alpha_x$ is not small as far as compactness is concerned: the vertical component of the gradient will not be compact (in local coordinates). To avoid this difficulty one approximates $\alpha_x$ by functionals $\alpha_n$, which have in some sense a compact gradient. This approximation will be carried out in III. 4 and IV. Instead of $\Psi_x$ one studies the functionals $\Psi_n = \Psi - \alpha_n$. The question of course is how good is the behaviour of $\Psi_n$ described by the behaviour of the family of functionals $(\Psi_n)$. Here, an abstract critical point theorem proved in chapter II. (Theorem 3) reduces the study of $\Psi_n$ essentially to the study of a single $\Psi_n$. Having this abstract result we apply the procedure outlined already for $\Psi_x$ to the functional $\Psi_n$ (for some $n$ large). It turns out that the corresponding intersection problem $S_2 * t \cap S_1 \neq \emptyset$ can be studied by converting it to a fixed point problem for a fibre-preserving map in some infinite-dimensional vector-bundle over $M$, where the maps in the fibres are compact. Hence the topological machinery is applicable. In order to carry out the conversion intersection problem $\leftrightarrow$ fixed point problem for a compact map, one derives a representation for the flow associated to $\Psi_n$ which relates it in some sense to the flow of the unperturbed problem. The representation for the flow and some compactness estimates will be derived in chapter V. Our procedure shows as in [7] quite clearly, in fact in contrast to the coercive closed geodesic problem, that for the variational problems for Hamiltonian systems only the topology of the underlying manifold itself is reflected in the topology of the critical points. It also shows that these can be found by studying the gradient flow in relation to the hyperbolic flow of the unperturbed problem.

1.3. Concluding remarks.

The first who employed global variational methods to solve global problems in symplectic geometry were Conley and Zehnder [7], in their astounding solution of Arnold’s Conjecture on the number of fixed points for symplectic self-maps of Tori. Their method was adapted by M. Chaperon [5], to solve our Theorem 2 for the case $M = T^n$ and to prove Theorem 1 for $T^4$. Motivated by [7] Weinstein proved Theorem 2 for all compact manifolds $M$, requiring however the $C^1$-smallness of $h^*$ [19]. From that point of view we find in fact not a new phenomenon, but we remove this smallness-condition imposed by Weinstein. Other related results are concerned with fixed point theorems for symplectic maps on compact manifolds. For example, Fortune and Weinstein [12], show that a symplectic map $\mathbb{P}^n \to \mathbb{P}^n$ homologous to the identity has at least $n + 1$ fixed points. Floer [11], recently proved that a symplectic map $M \to M$ homologous to the identity, where $M$ is a compact Kähler manifold with a vanishing second homotopy group, Abelian Holonomy, and non-positive sectional curvature, has at least $c(M)$ fixed points. Fortune and Weinstein extend Conley and Zehnder’s idea and lift the problem into an Euclidean space invariant under symmetries. Then they use different methods in the spirit of [4] [14-15]. Floer carries out a nonlinear variant of the Liapunov-Schmidt reduction. The obtained finite-dimensional problem is then solved in the spirit of [7].

Now a few remarks concerning the method employed to solve Theorem 2. The first who used the classical variational principle to study Hamiltonian systems in the large was P. Rabinowitz [20]. His ideas were later on abstracted, extended and simplified (see for example [3] [4] [14] [15]). Our approach here is motivated by the results in [3] [15]. In fact, as in this papers we attack the variational problem without carrying out a finite-dimensional reduction, except at the very end where we have to carry out some reduction—not in the variational problem, but—in the fixed point problem representing the intersection problem for the sets $S_1$ and $S_2$. This is in contrast to the reductions which have been carried out in [5] [7] [11].

II. AN ABSTRACT CRITICAL POINT THEOREM

We shall give an abstract result concerned with the behaviour of a functional and certain approximations.

DEFINITION 3. — Let $(L, (., .))$ be a connected metrically complete space.
Hilbert manifold and $\Psi_\infty \in C^1(L, \mathbb{R})$. An $\mathbb{N}$-family for $\Psi_\infty$ is a sequence of maps $\Psi_n : L \to \mathbb{R}$, $n \in \mathbb{N}$, such that

(i) $\Psi_n \in C^\infty(L, \mathbb{R})$ for all $n \in \mathbb{N}$.

(ii) For all sequences $(x_k) \subset L$, $x_k \to x$, and $(n_k) \subset \mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$, $n_k \to +\infty$, we have $\Psi_{n_k}(x_k) \to \Psi_\infty(x)$. Moreover, $\Psi_\infty'(x) = 0$ implies $\|\Psi_{n_k}'(x_k)\| \to 0$ and $\|\Psi_{n_k}'(x_k)\| \to 0$ implies $\Psi_\infty'(x) = 0$.

(iii) If for some sequences $(x_k) \subset L$, $(n_k) \subset \mathbb{N}_\infty$, $n_{k+1} \geq n_k$, we have $\|\Psi_{n_k}'(x_k)\| \to 0$ and $\Psi_{n_k}(x_k) \to d$, then $(x_k)$ is precompact.

In the above definition of course $\Psi_n'$ denotes the gradient and $L$ is equipped with the metric $d_L : L \times L \to \mathbb{R}_+$ derived from $(\cdot, \cdot)$ in the usual way.

Let $(\Psi_n)$ be a $\mathbb{N}$-family for $\Psi_\infty$ and let $\beta : \mathbb{R} \to \mathbb{R}$ be a smooth map such that

- $\beta$ is monotone decreasing (not necessarily strict).
- $\beta(s) = 1$ for all $s \leq 1$ and $\beta(s) = s^{-1}$ for all $s \geq 2$.

We introduce vectorfields $G_n : L \to TL$ by

$$G_n(x) = -\beta(\|\Psi_n'(x)\|)\Psi_n'(x).$$

Clearly the $G_n$ are smooth. Since $\|G_n(x)\| \leq 2$ we have global existence for the corresponding flows $\phi_n : L \times \mathbb{R} \to L$ defined by

$$\phi_{n,x}^t = G_n(\phi_{n,x}), \quad \phi_{n,x}(0) = x.$$

In order to simplify the notation we shall write

$$x \ast_n s := \phi_n(x, s).$$

Denote by $CL$ the set consisting of all closed subsets of $L$.

**Definition 4.** — The Lyusternik-Schnirelman category on $L$ is a map

$$\text{cat} : L \to \mathbb{N}_{0,\infty} = \mathbb{N}_\infty \cup \{0\}$$

satisfying the following

(i) $\text{cat}(\emptyset) = 0$.

(ii) $\text{cat}(D) = (k \in \mathbb{N}$ if there exist $k$ open sets $U_1, \ldots, U_k$ in $L$, each contractible to a point in $L$, such that their union covers $D$, and $D$ cannot be covered by a collection of $k - 1$ contractible (in $L$) open sets.

(iii) $\text{cat}(D) = \infty$ if there exists no finite open covering as above.

One calls $\text{cat}(D)$ the Lyusternik-Schnirelman category of the set $D$ in $L$.

We have the following result.

**Theorem 3.** — Assume $\Psi_\infty \in C^1(L, \mathbb{R})$ and $(\Psi_n)$ is a $\mathbb{N}$-family for $\Psi_\infty$. Suppose $S_1$ and $S_2$ are closed subsets of $L$ such that for some number $d \in \mathbb{R}$, $d > 0$

$$d \leq \inf \Psi_n(S_1) \leq \sup \Psi_n(S_2) \leq d$$

for all $n \in \mathbb{N}$. Define maps $i_n : \text{CL} \to \mathbb{N}_{0, \infty}$ by $(n \in \mathbb{N})$

$$i_n(D) = \inf_{t \in \mathbb{R}} \text{cat} ((D \ast_n t) \cap S_1)$$

and assume

$$i_n(S_2) \geq N \geq 1$$

for all $n \in \mathbb{N}$ for some $N \in \mathbb{N}$. Then $\Psi_\infty$ has at least $N$ critical points with corresponding every levels in the interval $[-d, d]$.

Index maps like the $i_n$ were introduced by Benci [4], and in a weak form by the author [14], to overcome the difficulties of the infinite Morse-index.

Let us collect first some properties of $\text{cat}$ and the index maps $i_n$. Let $D, E \in \text{CL}$.

. If $D \subseteq E$ then $\text{cat}(D) \leq \text{cat}(E)$.
. If $H : [0, 1] \times I \to I$ is continuous and $H(0, .) = \text{Id}$ and $H(t, .)$ is a homeomorphism for all $t \in [0, 1]$ then $\text{cat}(H(\cdot t \times D)) = \text{cat}(D)$ for all $t \in [0, 1]$.
. If $\text{cat}(D) \geq 2$ then $D$ contains infinitely many points.
. If $D$ is compact then $\text{cat}(D)$ is finite and there exists an open neighbourhood $U$ of $D$ such that $\text{cat}((\text{cl}(U))) = \text{cat}(D)$.
. If $D \subseteq E$ then $i_n(D) \leq i_n(E)$.
. $i_n(D \cup E) \leq i_n(D) + \text{cat}(E)$.
. $\text{in}(D \ast_s) \geq i_n(D)$ for all $s \in \mathbb{R}^+$.

The above properties are trivial consequences of the definitions of $\text{cat}$ and $i_n$.

Now define for $n \in \mathbb{N}$ and $j \in \{1, \ldots, N\}$.

$$(1) \quad c_j(n) = \inf_{D \in \text{CL}, i_n(D) \geq j} \sup \Psi'_n(D).$$

Let us show

$$(2) \quad -d \leq c_1(n) \leq \ldots \leq c_N(n) \leq d$$

for all $n \in \mathbb{N}$. If $D \in \text{CL}$ and $i_n(D) \geq 1$ we infer from the definition of the category that $D \cap S_1 \neq 0$. This implies

$$(3) \quad \sup \Psi'_n(D) \geq \inf \Psi'_n(S_1) \geq -d$$

for all $n \in \mathbb{N}$. On the other hand since $i_n(S_2) \geq N \geq 1$ we conclude for $n \in \mathbb{N}$ and $j \in \{1, \ldots, N\}$

$$(4) \quad c_j(n) \leq \sup \Psi'_n(S_2) \leq d.$$ 

Hence combining (3) and (4) and using the monotonicity of $i_n$ we find (2). Eventually dropping some of the $\Psi'_n$ and making some renumbering we may assume that $\lim_{n \to +\infty} c_j(n) = c_j$ exists for all $j \in \{1, \ldots, N\}$. Clearly we must have

$$(5) \quad -d \leq c_1 \leq c_2 \ldots \leq c_N \leq d.$$ 

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We shall show the following

\( (6) \) If \( c_j = \ldots = c_{j+k} \) for some \( k \in \{0, \ldots, N-1\} \) then the set

\[ C_r \cap \{ x \in L \mid \Psi'_\infty(x) = 0, \Psi_\infty(x) = c_j \} \]

has at least category \( k+1 \) in \( L \).

In particular this implies that the \( c_j \) are critical levels for \( \Psi_\infty \). Since

the \( (\Psi_n) \) are a \( \mathbb{N} \)-family for \( \Psi_\infty \), condition \((iii)\) implies that \( C_r \) is compact.

Assume \( \text{cat}(C_r) \leq k \). We shall prove \((6)\) by deriving a contradiction. There exists an open neighbourhood \( U \) of \( C_r \) such that \( \text{cat}(\overline{U}) = \text{cat}(C_r) \).

Clearly if \( C_r = \emptyset \) we have \( U = \emptyset \). Define

\[ \rho = \frac{1}{2} \text{dist}(\partial U, C_r) \]

and

\( (7) \)

\[ V = \{ x \in L \mid \text{dist}(x, C_r) < \rho \} \]

Then \( \overline{V} \subset U \) and \( \text{dist}(\partial U, V) \geq \rho \).

Next we shall show

\( (8) \) There exist \( \varepsilon_0 > 0 \), \( \tau > 0 \), and \( n_0 \geq 1 \) such that \( ||\Psi'_n(x)|| \geq \tau \) for all \( x \in L \setminus V \) with \( \Psi_n(x) \in [c_j - \varepsilon_0, c_j + \varepsilon_0] \) provided \( n \geq n_0 \) or \( n = +\infty \).

Arguing indirectly we find sequences \( (x_e) \subset L \setminus V \), \( (n_e) \subset \mathbb{N} \), \( n_e < n_{e+1} \), with

\[ ||\Psi'_n(x_e)|| \leq \frac{1}{\varepsilon} \]

and

\[ |\Psi'_n(x_e) - c_j| \leq \frac{1}{\varepsilon}. \]

By \((iii)\) \((x_e)\) is precompact. Hence eventually taking a subsequence we may assume \( x_e \to x \in L \setminus V \). By \((ii)\) \( \Psi_\infty(x) = c_j \) and \( \Psi'_\infty(x) = 0 \) giving a contradiction. Hence \((8)\) must hold for all natural numbers \( n \geq n_0 \) for some \( n_0 \in \mathbb{N} \). Using \((iii)\), eventually replacing \( \varepsilon_0 \) and \( \tau \) by smaller numbers, we find that it also holds for \( n = +\infty \). Now fix \( \varepsilon > 0 \) such that

\( (9) \)

\[ 0 < \varepsilon < \frac{1}{8} \min \{ \varepsilon_0, \tau, \tau^2, \rho \tau, \rho \tau^2 \} \]

Following the arguments given in [22] one easily obtains

\( (10) \) For all \( x \in L \setminus U \) such that

\[ \Psi'_n(x) \leq c_j + \varepsilon \text{ we have } \Psi'_n(x * n 1) \leq c_j - \varepsilon. \]

Now we find \( n_1 \geq n_0 \) such that for all \( n \geq n_1 \) and all \( i \in \{1, \ldots, N\} \) we have

\( (11) \)

\[ |c_i(n) - c_i| \leq \frac{\varepsilon}{2} \]
and

\[(12) \quad \text{Cr}_i(n) = \{ x \mid \Psi'_n(x) = 0, \Psi_n(x) = c_i(n) \}\]

is contained in $V$ for $i = j, \ldots, j + k$.

In fact (12) follows immediately from (ii) and (iii). We find a set $D \in \mathcal{C}L$, $i_n(D) \geq j + k$ and

\[
\Psi_n(D) \subset (-\infty, c_{j+k}(n) + \varepsilon/2) \subset (-\infty, c_j + \varepsilon].
\]

Define $\tilde{D} = D \setminus U$. By the previous discussion we have using (10)

\[
\Psi_n(\tilde{D} \star 1) \subset (-\infty, c_j - \varepsilon] \subset (-\infty, c_j(n) - \varepsilon/2].
\]

By the definition of $i_n$ and $c_j(n)$ this implies

\[(13) \quad i_n(\tilde{D}) < j.\]

On the other hand we infer

\[(14) \quad k + j \leq i_n(D) = i_n(\tilde{D} \cup (\text{cl}(U) \cap D)) \leq i_n(\tilde{D}) + \text{cat} (\text{cl}(U) \cap D) \leq i_n(\tilde{D}) + \text{cat} (\text{cl}(U)) = i_n(\tilde{D}) + \text{cat} (\text{cr}) \leq i_n(\tilde{D}) + k.
\]

Hence

\[j \leq i_n(\tilde{D})\]

which contradicts (14). This proves (6). Now (6) implies the following: If all the $c_j$ are different we have $N$ different critical levels and consequently at least $N$ different critical point. On the other hand, if two numbers $c_j$ and $c_{j'}$, $j \neq j'$, coincide the corresponding critical set has at least category two and therefore contains infinitely many critical points.

III. NOTATIONS AND PRELIMINARY RESULTS

In this chapter we shall introduce the Hilbert and Banach manifolds which will be used in the proof of the main results. In general we shall use the notation in Klingenberg’s closed geodesic book [16], we also borrow some results from A. Floer [11], who was the first to study a related variational problem in this general setting. Further we study certain fibre-preserving maps in vectorbundles over spaces of curves. The main part,
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however, will be the study of the behaviour of compact subsets of fibres under transport (for example, parallel transport). Finally, we introduce a family of smoothing operators which make the compactness concept work.

III.1. The spaces of \( H^1 \)-curves and corresponding vectorbundles.

We equip the compact connected manifold \( M \) with a Riemannian metric \( \langle \cdot, \cdot \rangle : TM \oplus TM \rightarrow \mathbb{R} \) and denote by \( \tau_M : TM \rightarrow M \) the tangent bundle. Moreover, we denote by \( \exp : TM \rightarrow M \) the exponential map associated to the Riemannian metric on \( M \). By \( \Lambda \) we denote the set \( H^1([0, 1], M) \) consisting of absolutely continuous curves \( q : [0, 1] \rightarrow M \) such that

\[
E(q) := \int_0^1 |\dot{q}(t)|^2 dt < + \infty.
\]

We call \( E(q) \) the energy of \( q \). Further let \( C \) be the Banach manifold consisting of all continuous maps \( q : [0, 1] \rightarrow M \). We equip \( C \) with the metric

\[
d_c(q, \tilde{q}) = \max_{t \in [0,1]} d_M(q(t), \tilde{q}(t)).
\]

There is a standard metric on \( TM \) derived from \( \langle \cdot, \cdot \rangle \) and the associated Levi-Civita connection \( K \), turning \( TM \) into a Riemannian manifold. Namely

\[
\langle a, b \rangle_{TM} = \langle Ka, Kb \rangle + \langle T\tau_M a, T\tau_M b \rangle.
\]

For \( q \in \Lambda \) we define

\[
T_q\Lambda = \{ y \in H^1([0, 1], TM) \mid \tau_M y = q \}.
\]

We equip \( T_q\Lambda \) with the structure of a Hilbert space by defining the inner product

\[
(x, y)_\Lambda = \int_0^1 (\langle x(t), y(t) \rangle + \langle \nabla x(t), \nabla y(t) \rangle) dt.
\]

Here \( \nabla x \) denotes the covariant derivative along the curve \( q \) associated to \( K \), which is almost everywhere defined. It is well-known that \( T\Lambda = \cup_{q \in \Lambda} T_q\Lambda \) can be canonically identified with the tangent space of \( \Lambda \) \([16]\) \([9]\). Therefore the notation is justified. We have a canonical embedding \( p : M \rightarrow \Lambda \) by \( p(m)(t) = m \) for all \( t \in [0, 1] \). To simplify notations we shall write \( m \) for \( p(m) \) and \( M \) for \( p(M) \) if there is no danger of confusion. We denote by \( \pi_M : B \rightarrow M \) the pullback of the tangent bundle \( \tau_\Lambda : T\Lambda \rightarrow \Lambda \) via \( p \). \( \pi_M \) possesses an important smooth sub-bundle \( \pi_M^0 : B^0 \rightarrow M \), where \( B^0 \) is the kernel of the smooth fibre-preserving map over \( M \) defined by

\[
B \rightarrow TM : x \rightarrow \int_0^1 x(t) dt.
\]

Here, of course, we consider $\mathcal{B}$ to be $\mathcal{T} \Lambda |_M$. The vector bundles $\pi^0_M$ and $\pi_M$ possess smooth Riemannian metrics induced from the inner product $(., .)_\Lambda$. We can equip $\mathcal{T}_q \Lambda$ with a different inner product

$$
(x, y) = \int_0^1 \langle x(t), y(t) \rangle \, dt .
$$

Denoting by $L_q \Lambda$ the completion of $\mathcal{T}_q \Lambda$ with respect to the norm $\| \cdot \|$ associated to $(., .)$ we obtain a Hilbert space. It is well-known that $\mathcal{L} \Lambda = \bigcup_{q \in \Lambda} L_q \Lambda$ carries in a natural way the structure of a smooth vector-bundle over $\Lambda$ and moreover $(., .)$ defines a smooth Riemannian metric for the bundle $\pi : \mathcal{L} \Lambda \to \Lambda$.

We introduce a continuous map $\| \|_\infty : \mathcal{T} \Lambda \to \mathbb{R}$ by

$$
\| x \|_\infty = \max_{t \in [0, 1]} | x(t) | .
$$

In the following we need several standard estimates (see [16])

$$
\| x \| \leq \| x \|_\infty \leq 2 \| x \|_\Lambda
$$

for all $x \in \mathcal{T} \Lambda$. Moreover for all $q \in \Lambda$ we have the estimate

$$
d_{\mathcal{L}}(q(t), q(s)) \leq \sqrt{E(q)} \, | t - s |^{\frac{1}{2}} .
$$

Denote by $d_\Lambda : \Lambda \times \Lambda \to \mathbb{R}$ the metric induced by the Riemannian metric $(., .)_\Lambda : \mathcal{T} \Lambda \oplus \mathcal{T} \Lambda \to \mathbb{R}$. By a result in [16] we have the estimate

$$
E(q)^{\frac{1}{2}} - E(\tilde{q})^{\frac{1}{2}} \leq d_\Lambda(q, \tilde{q}) .
$$

Using the compactness of $M$ the Ascoli-Arzela-Theorem can be applied to conclude from (5) and (6) that the embedding $(\Lambda, d_\Lambda) \hookrightarrow (C, d_C)$ is compact. Before we give some canonical charts for the manifolds just introduced we state a simple interpolation inequality which will be very useful later on.

$$
\| x \|_{\infty}^2 \leq \| x \|^2 + 2 \| x \| \| \nabla x \|
$$

for all $x \in \mathcal{T} \Lambda$. The proof of (7) is very simple. Fix $t_0 \in [0, 1]$ such that $| x(t_0) | = \| x \|_{\infty}$. Then

$$
| x(t) |^2 = | x(t_0) |^2 + 2 \int_{t_0}^t \langle x(s), \nabla x(s) \rangle \, ds
$$

$$
\geq \| x \|_{\infty}^2 - 2 \| x \| \| \nabla x \| .
$$

Integrating over $[0, 1]$ gives (7).

To introduce local trivializations denote by $\exp^m : \mathcal{T}_m M \to M$ the restriction of the exponential map. For $q \in \Lambda$ we define $\exp_q$ by

$$
\exp_q : \mathcal{T}_q \Lambda \to \Lambda : \exp_q(x)(t) = \exp_{q(t)}(x(t)) .
$$

This map is injective on an open neighbourhood $V$ of $O \in \mathcal{T}_q \Lambda$ and can be used to define the differentiable structure on $\Lambda$. Hence

$$
\exp_q : V \to U
$$
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is a diffeomorphism, where $U$ is a suitable open neighbourhood in $\Lambda$. In fact, we can take rather big neighbourhoods $V$. For example $V$ can be taken of the form $\{ x \in T_qA \mid x(t) \in W \text{ for all } t \in [0, 1] \}$, where $W$ is an open neighbourhood of the zero-section of $TM \rightarrow M$ (see [16]). Note that by the compact embedding $\Lambda \subset C$ already finitely many of such « big » neighbourhoods will cover a bounded set in $\Lambda$.

For $m \in M$ and $x \in T_mM$ let

$$D \exp_m(x) : T_mM \rightarrow T_{\exp(x)}M$$

denote the linearisation of $\exp_m$ at $x$. Then with $\exp_q$ as defined above we define a map $\Phi_q$ by

$$\Phi_q : V \times L_qA \rightarrow LA|_U$$

where

$$\Phi_q(x, y)(t) = D \exp_{q(t)}(x(t)). y(t).$$

Similar maps give the local trivializations of the tangent bundle of $A$. In order to give local trivializations of $\pi_M$ and $\pi^0_M$ let $V_m \subset T_mM$ be an open neighbourhood of zero in $T_mM$ such that $\exp_m : V_m \rightarrow U$ establishes a diffeomorphism for a suitable $U \subset M$.

Define

$$\Phi_m : V_m \times T_mA \rightarrow B|_U$$

by

$$\Phi_m(a, x)(t) = D \exp_m(a). x(t).$$

Clearly a local trivialization in $\pi^0_M$ is given by

$$V_m \times \left\{ x \in T_mA \mid \int_0^1 x(t)dt = 0 \right\} \rightarrow B^0|_U$$

$$(a, x) \rightarrow D \exp_m(a). x$$

It is well-known that the bundle $\pi : LA \rightarrow A$ possesses a connection $\hat{K} : TL \rightarrow LA$ induced by the Levi-Civita connection $K$ on $\tau_M$ [16] 1.3.4. Using $\hat{K}$ and $\pi$ we can turn $LA$ into a Hilbert manifold by defining

$$\langle a, b \rangle_L = (\hat{K}a, \hat{K}b) + (T\pi a, T\pi b)_A$$

for $a, b \in T_ALA$. The metric $d_L : LA \times LA \rightarrow \mathbb{R}$ induced by $(., .)_L$ turns $LA$ into a complete metric space. Hence $(LA, (., .)_L)$ is a metrically complete Hilbert manifold. Recall the definition of the bundle $\pi^0_M : B^0 \rightarrow M$. Denote by $B^0_\rho$ for some $\rho > 0$ the subset of $B^0$ consisting of all $x \in B^0$ such that $\|x\|_x < \rho$ (again we consider $B \approx TA|_M$).

LEMMA 1. — There exists $\rho > 0$ such that the map

$$(16) \quad \Phi : B^0 \rightarrow \Lambda : \Phi(x(t)) = \exp(x(t))$$

induces a diffeomorphism of $B^0_\rho$ onto an open neighbourhood of $\Lambda_0 = \rho(\Lambda)$ in $\Lambda$.

Proof. — Fix $0_m \in B^0$. We show that $T\Phi(0_m) : T_{0_m}B^0 \rightarrow T_{m}\Lambda$ is an isomorphism. In local co-ordinates we have with

$$m = \exp_m(a)$$
$$x(t) = D \exp_m(a) . \xi(t)$$

the following representation of the local representative of $\Phi$

$$\Phi_m(a, \xi)(t) = \exp_m^{-1} \left( \exp \left( D \exp_m(a) . \xi(t) \right) \right).$$

Here, of course, $\Phi_m : V_m \times (B^0)^m \rightarrow T_{m}\Lambda$.

Since

$$\Phi_m(0, \xi)(t) = \xi(t), \Phi_m(a, 0)(t) = a$$

we infer

$$D\Phi_m(0, 0)(h, k)(t) = h + k(t).$$

Clearly this means that $D\Phi_m(0, 0)$ establishes an isomorphism

$$T_{m}\Lambda \times (B^0)_m \rightarrow T_m\Lambda.$$  

Therefore, $T\Phi(0_m)$ is a diffeomorphism for all $m \in M$. By the inverse function theorem $\Phi$ establishes a diffeomorphism from an open neighbourhood of $0_m$ onto an open neighbourhood of $\rho(m)$ for all $m \in M$. By the compactness of $M$ we find $\rho > 0$ and $U \subset \Lambda$ open, $\Lambda_0 \subset U$, such that the map

$$\Phi : B^0_\rho \rightarrow U$$

is onto and a local diffeomorphism. Therefore it is enough to show that $\Phi | B^0_\rho$ is injective in order to complete the proof of Lemma 1. By the previous discussion the number of points in $(\Phi | B^0_\rho)^{-1}(q)$ is finite for all $q \in U$ and constant. The injectivity of $\Phi | B^0_\rho$ will follow if we can show that $(\Phi | B^0_\rho)^{-1}(m) = \{ 0_m \}$. Arguing indirectly assume for some $x \neq 0_m, x \in B^0_\rho$

$$\exp(x(t)) = m$$

for all $t \in [0, 1]$. Then $x(t) = x_0$ for all $t \in [0, 1]$ for some constant curve in $B^0$. Since $\int_0^1 x_0 dt = 0_\tilde{m}$ for some $\tilde{m} \in \tilde{M}$ we must have $x_0 = 0_\tilde{m}$. Hence

$$m = \exp(x_0) = \exp(0_\tilde{m}) = \tilde{m}$$

which implies $\tilde{m} = m$ giving a contraction.

The vectorbundle $\pi_B^M : B^0 \rightarrow M$ will be crucial in the construction of the intersection pairs.
III.2. Covariant derivatives and curvature.

Denote by $\delta : TA \to LA$ the smooth fibre preserving map defined by

$$ (\delta x)(t) = \nabla x(t) . $$

It is well-known that $\delta$ is the covariant derivative of the smooth section $\partial : \Lambda \to LA$ defined by

$$ \partial q = \dot{q} , $$

see [16], Proposition 1.3.5.

Let $G$ denote one of the bundles $TA$ or $LA$ over $\Lambda$. If $\xi : \Lambda \to G$ is a smooth section and $x \in T_q\Lambda$ we define the covariant derivative of $\xi$ at $q \in \Lambda$ in the direction of $x$ by

$$ (D_x \xi)(t) = \nabla_s \xi_s(s, t) |_{s=0} , $$

where $\xi_s(s, t) = \xi(\exp_q(s.x))(t)$.

Clearly we have

$$ (D_x \xi)(t) = (\tilde{R}(T\xi, x))(t) $$

if $\xi$ is a section of $LA \to \Lambda$.

Since we consider $T(TA)$ in a natural way as a subset of $T(LA)$, formula (4) remains valid for sections of $TA \to \Lambda$. Now let $G_1$ and $G_2$ denote bundles of type $LA \to \Lambda$ or $TA \to \Lambda$ and assume $\rho : G_1 \to G_2$ is a fibre-preserving map, we define the covariant derivative at $q$ in the direction $x \in T_q\Lambda$ by

$$ (D_x \rho)^\xi = D_x(\rho \xi) - \rho(D_x \xi) $$

where $\xi$ is a smooth section of $G_1 \to \Lambda$.

Denote by $R : TM \oplus TM \oplus TM \to TM$ the curvature tensor of $(M, \langle \cdot, \cdot \rangle)$. For $q \in \Lambda, \eta \in T_q\Lambda$ we define a map

$$ R(\eta, \dot{\eta}) : T_q\Lambda \to L_q\Lambda $$

by

$$ (R(\eta, \dot{\eta})x)(t) = R(\eta(t), \dot{\eta}(t))x(t) . $$

**Lemma 2.** $D_x \delta = R(x, \dot{q})$, where $x \in T_q\Lambda$.

A proof can be found in [11].

Before we proceed further, let us note that if $\rho_1 : G_1 \to G_2$ and $\rho_2 : G_2 \to G_3$ are fibre-preserving maps, where the $G_i$ are as before we must have

$$ D_x(\rho_2 \rho_1) = (D_x \rho_2)\rho_1 + \rho_2(D_x \rho_1) . $$
LEMMA 3. — For all \( q \in \Lambda \) \( \delta \) induces a surjective Fredholm operator
\( \delta_q : T_q \Lambda \to L_q \Lambda \) of index \( i(\delta_q) = \dim (M) \). Moreover if \( S \) is a bounded subset of \( \Lambda \) there exists a positive constant \( c = c(S) \) such that
\[
\| \delta x \| \geq c \| x \|_\Lambda
\]
for all \( x \in T_{\Lambda} |_{S} \) such that \( x \) is \((\cdot, \cdot)\)-orthogonal to \( \ker(\delta_q) \), where \( q = \pi x \).

Proof. — Arguing indirectly we find a sequence \( (x_n) \subseteq T_{\Lambda} \) such that
\[
q_n = \pi x_n \in S \text{ and } \quad \| x_n \|_\Lambda = 1, \quad \| \delta x_n \| \leq \frac{1}{n}, \quad x_n \perp \ker(\delta_{q_n}).
\]
Since \( (q_n) \) is bounded we may assume, without loss of generality that
\[
q_n \to q_0 \text{ in } C
\]
where \( q_0 \in C \). Taking a chart centred at some \( q \in \Lambda \) close to \( q_0 \) we have
\[
q_n = \exp_q (\xi_n).
\]
for all \( n \) large enough. It is easy to see that \( (\| \xi_n \|_\Lambda) \) must be bounded. Hence \( (\xi_n) \) possesses a weakly convergent subsequence in \( T_q \Lambda \). This, of course, implies that \( q_0 \) is, in fact, an element in \( \Lambda \). Let \( q = q_0 \). We have
\[
x_n = \Phi_q (\xi_n, \eta_n).
\]
Using the notation in \([16], p. 7-22\), (9) looks, in local co-ordinates, like
\[
(G(\xi_n)(\nabla \eta_n) + D_2 \Theta_q (\xi_n) \eta_n + \Gamma_q (\xi_n)(\eta_n, \partial_q \xi_n)),
\]
\[
\nabla \eta_n + D_2 \Theta_q (\xi_n) \eta_n + \Gamma_q (\xi_n)(\eta_n, \partial_q \xi_n)) \leq \frac{1}{n}.
\]
Since \( G(\xi_n) \to G(0) = \text{Id} \), \( D_2 \Theta_q (\xi_n) \to 0 \) and \( \Gamma_q (\xi_n) \to 0 \) uniformly as \( n \to + \infty \) we obtain
\[
\lim \| \nabla \eta_n \| = 0.
\]
Since \( T_q \Lambda \) is compactly embedded in \( L_q \Lambda \) and \( \| x_n \|_\Lambda = 1 \), we infer eventually taking a subsequence
\[
\eta_n \to \eta \text{ in } L_q \Lambda
\]
\[
\eta_n \to \eta \text{ in } T_q \Lambda.
\]
Hence by (11) \( \nabla \eta = 0 \). Therefore \( \eta \in \ker(\delta_q) = \ker(\nabla) \). On the other hand, we know that
\[
x_n \perp \ker(\delta_{q_n}).
\]
Using the standard \( L^2 \)-theory for ordinary differential equations we find for all \( n \in \mathbb{N} \) a unique solution \( \tilde{\eta}_n \) on \( H^1([0, 1], TM) \), \( \tilde{\eta}_n(t) \in T_{\pi q(t)}M \) such that
\[
\nabla \tilde{\eta}_n + D_2 \Theta_q (\xi_n) \tilde{\eta}_n + \Gamma_q (\xi_n)(\tilde{\eta}_n, \partial_q \xi_n) = 0
\]
\[
\tilde{\eta}_n(0) = \eta(0).
\]
Let $\tilde{x}_n = \Phi_q(\xi_n, \tilde{\eta}_n)$. Then $\delta \tilde{x}_n = 0$. Hence $|\tilde{x}_n(t)| = \text{const.}$ Since $(\|\partial_q \tilde{x}_n\|)$ is bounded we find that
\[
D_2 \theta_q(\xi_n) \tilde{\eta}_n + \Gamma_d(\xi_n)(\tilde{\eta}_n, \partial_q \tilde{x}_n) \to 0
\]
in $L_q\Lambda$. Hence eventually taking a subsequence of $(\tilde{\eta}_n)$ we may assume
\[
\tilde{\eta}_n \to \tilde{\eta} \quad \text{in} \quad T_q\Lambda
\]
\[
\eta_n \to \tilde{\eta} \quad \text{uniformly.}
\]
Taking the limit in (12) gives
\[
\nabla \tilde{\eta} = 0
\]
\[
\tilde{\eta}(0) = \eta(0).
\]
By the uniqueness of solutions for given initial value we must have $\tilde{\eta} = \eta$. We have by our assumption
\[
(x_n, \tilde{x}_n) = 0.
\]
Hence in local co-ordinates
\[
0 = (G(\xi_n)\eta_n, \tilde{\eta}_n) \to (\eta, \eta) = 1
\]
giving a contradiction. \hfill \square

Denote by $A : L\Lambda \to L\Lambda$ the fibre-preserving map defined by
\[
A_x = (\delta_{px} \mid \text{kern} \ (\delta_{px})^\perp \cap T\Lambda)^{-1} x.
\]

**Lemma 4.** — $A$ is smooth.
The easy proof is left to the reader.
Denote by $\|A_q\|$ the norm of the operator $A : L_q\Lambda \to L_q\Lambda$. By Lemma 3 the map
\[
q \to \|A_q\|
\]
maps bounded sets into bounded sets. Denote by $A^* : L\Lambda \to L\Lambda$ the L-adjoint of $A$, i. e.
\[
(Ax, y) = (x, A^*y)
\]
for all $(x, y) \in L\Lambda \oplus L\Lambda$. Clearly $A^*$ is smooth and $\|A_q^*\| = \|A_q\|.$

**Lemma 5.** — Let $x \in L_q\Lambda$ and assume
\[
(x, \delta y) \leq c \|y\|
\]
for all $y \in T_q\Lambda$ for some constant $c$ independent of $y$. Then $x \in T_q\Lambda$.
This is an easy consequence of the $H = W$ result of Serrin and Meyers [1].

III.3. Compactness and transport equations.

The compactness concept we shall introduce in this section is prompted by the corresponding theorem by M. Riesz in $L^p$. The following results provide the necessary background. Given $q \in \Lambda$ we denote by

$$A_{t,s}^q : T_{q(t)}M \to T_{q(s)}M$$

for $t, s \in [0, 1]$ the parallel transport along $q$. Since $q$ is of class $H^1$ the standard $L^2$-theory of the Cauchy problem $\nabla_q x = 0$ then shows uniqueness and existence of $A_{t,s}^q$. We define a one parameter family $(Z(\tau))_{\tau \in [-1, 1]}$ of fibre preserving maps $Z(\tau) : \Lambda \to \Lambda$ by

$$(Z(\tau)x)(t) = \begin{cases} A_{t,t+\tau}^q x(t + \tau) & \text{if } t + \tau, t \in [0, 1] \\ 0 & \text{if } t + \tau \notin [0, 1], t \in [0, 1] \end{cases}$$

where $x \in T_q\Lambda$. Since $\| Z(\tau)x \| \leq \| x \|$ and $T_q\Lambda$ is dense in $L_q\Lambda$ it determines a unique map $\Lambda \to \Lambda$. Note that

$$(Z(\tau)x, y) = (x, Z(\tau)y)$$

for all $\tau \in [-1, 1]$ and all $(x, y) \in \Lambda \oplus \Lambda$. In fact we calculate assuming that $\tau \geq 0$

$$(Z(\tau)x, y) = \int_0^{\tau-\tau} \langle A_{t,t+\tau}^q x(t + \tau), y(t) \rangle \, dt$$

$$= \int_0^{\tau-\tau} \langle x(t + \tau), A_{t,t+\tau}^q y(t) \rangle \, dt$$

$$= \int_0^1 \langle x(t), A_{t,t-\tau}^q y(t - \tau) \rangle \, dt$$

$$(x, Z(\tau)y).$$

Moreover,

$$(Z(\tau)x, y) = (x, Z(\tau)y)$$

Lemmas 6. — For a bounded set $S$ in $L_q\Lambda$ the following statements are equivalent

(i) $S$ is precompact

(ii) Given any $\varepsilon > 0$ there exists $\rho \in (0, 1]$ such that $\| Z(\tau)x - x \| \leq \varepsilon$

for all $x \in S$ and $| \tau | \leq \rho$.

For the moment we shall postpone the proof. Later on we shall reduce Lemma 6 to a standard result in $L^p$-theory. Next we give a useful estimate relating to the $T_q\Lambda$-norm and the quantitative expression in (ii).

Lemmas 7. — For all $x \in T\Lambda$ we have the estimate

$$\| Z(\tau)x - x \| \leq 2 \| x \|_\Lambda | \tau |^{\frac{1}{2}}.$$
Proof. — Assume first $t$ and $t + \tau \in [0, 1]$. Since

$$|(Z(\tau)x)(t) - x(t)| = |A_{t,t+\tau}^{\rho}x(t + \tau) - x(t)| = |x(t + \tau) - A_{t,t+\tau}^{\rho}x(t)|.$$ 

we infer

$$\frac{d}{dt}|(Z(\tau)x)(t) - x(t)|^2 = 2 \langle x(t + \tau) - A_{t,t+\tau}^{\rho}x(t), \nabla x(t + \tau) \rangle \leq 2 |x(t + \tau) - A_{t,t+\tau}^{\rho}x(t)| |\nabla x(t + \tau)|$$

$$= 2 |A_{t,t+\tau}^{\rho}x(t + \tau) - x(t)| |\nabla x(t + \tau)|$$

$$= 2 |(Z(\tau)x)(t) - x(t)| |\nabla x(t + \tau)|.$$ 

Consequently

$$|x(t)|^2 = |(Z(\tau)x)(t) - x(t)|^2.$$ 

Hence

$$|(Z(\tau)x)(t) - x(t)| \leq \int_t^{t+\tau} |\nabla x(s)| \, ds \leq \|\delta x\| \tau \frac{1}{2}.$$ 

If $t + \tau \notin [0, 1]$, $t \in [0, 1]$ we have

$$(Z(\tau)x)(t) = 0.$$ 

This yields (4). □

Let $\phi \in C^1(\mathbb{R}, \Lambda)$ and $\beta : \mathbb{R} \to [-1,1]$ smooth. Denote by $V_\phi$, the covariant derivative along the curve $\phi$ in the bundle $L\Lambda \to \Lambda$ associated to $\hat{K}$. Consider the differential equation

$$(6) \quad V_\phi^*a = \beta(s)a.$$ 

We denote by $C(s_1, s_0) : L_{\phi(s_0)}\Lambda \to L_{\phi(s_2)}\Lambda$ the induced linear « transport map ». Clearly we have the estimate

$$(7) \quad \|C(s_1, s_0)\| \leq \exp(|s_1 - s_0|)$$

for the operator norm.

The following result is crucial.

PROPOSITION 1. — Let $\phi$ be as defined above. We have the following estimate

$$
\| Z(\tau) C(s_1, s_0) x - C(s_1, s_0) x \| \leq \tilde{M} \left[ \phi, s_1, s_0 \right] \left( \| Z(\tau) x - x \| + \| x \| + | \tau |^2 \right)
$$

for all $x \in L_{\phi(s_0)}$, all $\tau \in [-1, 1]$, for a suitable constant $\tilde{M}$ only depending on $(M, \langle \ldots \rangle)$. Here $[\phi, s_1, s_0]$ denotes the number

$$
(1 + \max \left\{ \| \partial_\phi(h) \| , \| \phi'(h) \|_{\infty} \ \mid h \in \left[ \min \left\{ s_1, s_0 \right\}, \max \left\{ s_1, s_0 \right\} \right] \right\}) \cdot \exp \left( 3 | s_1 - s_0 | \right).
$$

Proof. — Define a map $\hat{\phi} : \mathbb{R} \times [0, 1] \to M$ by $\hat{\phi}(s, t) = \phi(s)(t)$.

First we shall prove the Proposition under the additional assumption that $\hat{\phi}$ is smooth. Denote for $t_0, t_1 \in [0, 1]$ and $s_0, s_1 \in \mathbb{R}$ by

$$
A(s_0 ; t_1, t_0) : T_{\hat{\phi}(s_0, t_0)} M \to T_{\hat{\phi}(s_1, t_1)} M
$$

the parallel transport along the path $t \to \hat{\phi}(s_0, t)$. Moreover denote by

$$
B(t_0; s_1, s_0) : T_{\hat{\phi}(s_0, t_0)} M \to T_{\hat{\phi}(s_1, t_0)} M
$$

the map induced by the differential equation $\nabla_{\hat{\phi}(s_0, t_0)} a = \beta(s)a$ where $\hat{\phi}'(\cdot, t_0)$ denotes the partial differential with respect to $s$. Clearly we have for $x \in T_{\phi(s_0)} M$

$$
(C(s_1, s_0)x)(t) = B(t ; s_1, s_0)(x(t))
$$

Moreover we compute for $x \in T_{\hat{\phi}(s_0, t)} M$ and $y \in T_{\hat{\phi}(s_0, t)} M$ with the abbreviations

$$
av(s, t) = B(t; s, s_0)x - A(s; t, t_0)B(t_0; s, s_0)y
$$

$$
b(s, t) = A(s; t, s_0)B(t_0; s, s_0) y
$$

the following

$$
\left( \frac{\partial}{\partial s} | a |^2 \right)(s, t) = 2 \langle a(s, t), \beta(s)B(t ; s, s_0)x - \nabla_s b(s, t) \rangle
$$

Denote by $\hat{\phi}_t$ and $\hat{\phi}_s$ the partial derivatives with respect to $t$ and $s$, respectively. We infer

$$
\nabla_t \nabla_s b(s, t) = \nabla_s \nabla_t b(s, t) + R(\hat{\phi}_s(s, t), \hat{\phi}_s(s, t)) b(s, t) = R(\hat{\phi}_t(s, t), \hat{\phi}_t(s, t)) b(s, t).
$$

Here $R$ denotes the curvature tensor. Now using the variation of constant formula we deduce

$$
\nabla_s b(s, t) = A(s; t, t_0)(\nabla_s b(s, t_0)) + \int_{t_0}^{t} A(s; t, h) R(\hat{\phi}_t(s, h), \hat{\phi}_t(s, h)) A(s; h, t_0) dh \cdot B(t_0; s, s_0) y
$$

$$
= A(s; t, t_0)(\beta(s) B(t_0; s, s_0) y) + K(s; t, t_0) B(t_0; s, s_0) y.
$$
Here

\[ K(s; t, t_0) = \int_{t_0}^{t} A(s; t, h) R(\hat{\phi}(s, h), \hat{\phi}(s, h)) A(s; h, t_0) dh. \]

Now combining (10) and (12) yields

(13) \[ \left( \frac{\partial}{\partial s} | a |^2 \right)(s, t) = 2 \langle a(s, t), \beta(s) a(s, t) \rangle - 2 \langle a(s, t), K(s; t, t_0) B(t_0; s, s_0) y \rangle = 2\beta(s) | a(s, t) |^2 - 2 \langle a(s, t), K(s; t, t_0) B(t_0; s, s_0) y \rangle \]

Now let \( x \in T_{\phi(s_0)} \Lambda \). Then

(14) \( (C(s, s_0) x)(t) = B(t; s, s_0)(x(t)), \ t \in [0, 1]; \)

\( (Z(\tau) C(s, s_0) x)(t) = A(s; t, t + \tau) B(t + \tau; s, s_0)(x(t + \tau)) \) if \( t + \tau \in [0, 1]; \)

\( (Z(\tau) C(s, s_0) x)(t) = 0 \) if \( t + \tau \notin [0, 1] \).

Now by the previous discussion we infer if \( t + \tau \in [0, 1] \)

\[ \frac{\partial}{\partial s} | C(s, s_0) x(t) - (Z(\tau) C(s, s_0) x)(t) |^2 \leq 2\beta(s) | (C(s, s_0) x)(t) - (Z(\tau) C(s, s_0) x)(t) |^2 + 2 | (C(s, s_0) x)(t) - (Z(\tau) C(s, s_0) x)(t) | \tilde{M} \int_{t + \tau}^{t} | \hat{\phi}(s, h) | d h | x(t + \tau) | \exp(|s - s_0|). \]

Here \( \tilde{M} \) is a constant only depending on \( M, \langle \cdot, \cdot \rangle \) such that

\[ | R(a, b, c) | \leq \tilde{M} | a | | b | | c | \] for all \( (a, b, c) \in TM \oplus TM \oplus TM \).

Further we have used that parallel transport preserves the inner product on \( M \).

We introduce the abbreviation

\[ c(s, t) = | (C(s, s_0) x)(t) - (Z(\tau) C(s, s_0) x)(t) |. \]

Hence we have provided that \( t + \tau \in [0, 1] \)

(14') \[ \left( \frac{\partial}{\partial s} | c |^2 \right)(s, t) \leq 2 | c(s, t) |^2 + 2 | c(s, t) | \tilde{M} \| \phi'(s) \|_\infty \| \hat{\phi}(s) \| | \tau |^4 | x(t + \tau) | \exp(|s - s_0|) = 2 | c(s, t) |^2 + (2\tilde{M} \exp(|s - s_0|)) | \tau |^4 \| \hat{\phi}(s) \| \| \phi'(s) \|_\infty | c(s, t) | | x(t + \tau) |. \]

If \( t + \tau \notin [0, 1] \) we find simply

(15) \[ \left( \frac{\partial}{\partial s} | c |^2 \right)(s, t) = \frac{\partial}{\partial s} | (C(s, s_0) x)(t) |^2 \leq 2 | (C(s, s_0) x)(t) |^2 = 2 | c(s, t) |^2. \]

combining (14) and (15) yields
\[ \frac{d}{ds} \| Z(\tau)C(s, s_0)x - C(s, s_0)x \| ^2 \leq 2 \| Z(\tau)C(s, s_0)x - C(s, s_0)x \| ^2 + (2\tilde{M} \exp(|s-s_0|) \| \partial \phi(s) \| \| \phi'(s) \|_\infty | \tau |^{\frac{1}{2}} \| x \| \cdot \| Z(\tau)C(s, s_0)x - C(s, s_0)x \| . \]

Now applying Gronwall's Lemma yields for a suitable constant \( M^* \) only depending on \( (M, \langle \cdot, \cdot \rangle) \)
\[ (16) \quad \| Z(\tau)C(s, s_0)x - C(s, s_0)x \| \leq (\| Z(\tau)x - x \| + M^* \| x \| \| \tau \|^{\frac{1}{2}} | s-s_0 | \exp(|s-s_0|) \| \gamma(s, s_0) \| \cdot \exp(|s-s_0|)) \right) \]

where \( \gamma(s, s_0) = \max \{ \| \partial \phi(h) \| \| \phi'(h) \|_\infty | h \in [\min \{ s, s_0 \}, \max \{ s, s_0 \}] \} \).

Now taking a suitable constant \( \tilde{M} \) large enough only depending on \( (M, \langle \cdot, \cdot \rangle) \) we obtain
\[ (17) \quad \| Z(\tau)C(s, s_0)x - C(s, s_0)x \| \leq \tilde{M}[\phi, s, s_0](\| Z(\tau)x - x \| + \| x \| | \tau |^{\frac{1}{2}}) \]
as claimed. Next we remove the assumption that the induced map \( \tilde{\phi} : \mathbb{R} \times [0, 1] \rightarrow M \) is smooth. Assume we can show for given \( s_1, s_0, \) say \( s_0 \leq s_1 \), the existence of a sequence of maps \( \phi_n \in C^1(\mathbb{R}, \Lambda) \) such that
\[ d_{\Lambda}(\phi_n'(s), \phi'(s)) \leq \frac{1}{n} , \]
for all \( s \in [s_0, s_1] \) and that moreover the induced maps \( \tilde{\phi}_n : [s_0, s_1] \times [0, 1] \rightarrow M \) are smooth. Clearly \( [\phi_n, s_1, s_0] \rightarrow [\phi, s_1, s_0] \) as \( n \rightarrow + \infty \). Moreover, given \( x \in \Lambda_\phi \Lambda \) we find \( (x_n) \subset \Lambda_\phi \Lambda \) such that \( x_n \rightarrow x \).

By the continuous parameter dependence theory we infer that \( C^n(s_1, s_0)x_n \rightarrow C(s_1, s_0)x \) where \( C^n \) is the family of linear maps corresponding to \( \phi_n \). Since for fixed \( \tau \in [-1, 1] \) \( Z(\tau) : \Lambda \rightarrow \Lambda \) is continuous (we leave the proof as an easy exercise to the reader) we infer
\[ \| Z(\tau)C(s_1, s_0)x - C(s_1, s_0)x \| = \lim \| Z(\tau)C^n(s_1, s_0)x_n - C^n(s_1, s_0)x_n \| \leq \tilde{M}[\phi, s_1, s_0](\| Z(\tau)x - x \| + \| x \| | \tau |^{\frac{1}{2}}) \]
which completes the proof. So it remains to show the approximation result.

\[
\text{Lemma 8.} \quad \text{Given } \phi \in C^1(\mathbb{R}, \Lambda) \text{ and } s_0, s_1 \in \mathbb{R} \text{ (say } s_0 \leq s_1 \text{) there exists a sequence } (\phi_n) \subset C^1(\mathbb{R}, \Lambda) \text{ such that } d_{\Lambda}(\phi_n'(s), \phi'(s)) \leq \frac{1}{n} \text{ for all } s \in [s_0, s_1] \text{ and } \tilde{\phi}_n : \mathbb{R} \times [0, 1] \rightarrow M \text{ is smooth.}
\]

\[
\text{Proof.} \quad \text{We embed } (M, \langle \cdot, \cdot \rangle) \text{ isometrically in some } \mathbb{R}^m \text{ and denote the image by } M. \text{ Denote by } H \text{ the space } H^1([0, 1], \mathbb{R}^m). \text{ We can consider } \phi \text{ as a map in } C^1(\mathbb{R}, H). \text{ Let } U \subset \mathbb{R}^m \text{ open, } M \subset U \text{ be a tubular neighbourhood of } M \text{ and denote by } p : U \rightarrow M \text{ the projection. Let }
\]
\[ H_U = \{ q \in H | q([0, 1]) \subset U \}.
\]
Then $H_U$ is an open subset of $H$. Moreover $p$ induces a smooth map $\tilde{p} : H_U \to \Lambda$ by $\tilde{p}(q) = p \circ q$. If we approximate in $C^1([s_0, s_1], H)$, the composition of the approximation with $\tilde{p}$ will be an approximation of $\phi$ with image in $\Lambda$. By the previous remark we have only to show the following: given a map $\tilde{\phi} \in C^1(\mathbb{R}, H)$ and numbers $s_0, s_1$; $s_0 \leq s_1$, $\varepsilon > 0$ there exists $\tilde{\phi} \in C^1(\mathbb{R}, H)$ such that
\[
\| \phi - \tilde{\phi} \|_{C^1([s_0, s_1], H)} \leq \varepsilon
\]
and $\tilde{\phi}$ is smooth.

This approximation can be carried out using mollifiers. First we define
\[
\phi_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^m
\]
by
\[
\phi_1(s)(t) = \begin{cases} 
\phi(s)(0) & t < 0 \\
\phi(s)(t) & t \in [0, 1] \\
\phi(s)(1) & t > 1
\end{cases}
\]
Then one mollifies in the $t$-variable. One gets for $\varepsilon > 0$
\[
\phi_2 \in C^1(\mathbb{R}, H)
\]
by
\[
\phi_2(s)(t) = (\theta_{\varepsilon} \star_t (\phi_1(s)))(t), \quad t \in [0, 1]
\]
where $\theta_\varepsilon$ denotes the standard mollifier say $\theta_\varepsilon(t) = \frac{1}{\varepsilon} \theta\left( \frac{t}{\varepsilon} \right)$, and
\[
\theta(t) = \begin{cases} 
C \exp\left(\frac{-1}{1 - t^2}\right) & \text{if } |t| < 1 \\
0 & \text{if } |t| \geq 1
\end{cases}
\]
Here $C > 0$ is a constant such that $\int_{-\infty}^{\infty} \theta = 1$. If $\varepsilon \to 0 \phi_2 \to \phi$ in $C^1$.

Moreover $t \to \phi_2(s)(t)$ is of class $C^\infty$. Next one mollifies $\phi_2$ in the $s$-variable getting $\phi^{s, \delta} \in C^\infty(\mathbb{R}, H)$, where $\phi^{s, \delta}(s)$ is of class $C^\infty$. Moreover $\phi^{s, \delta} \to \phi$ as $\varepsilon, \delta \to 0$ in $C^1$ on $[s_0, s_1]$.

Now note that
\[
\tilde{\phi}^{s, t}(s, t) = \int_{-\infty}^{+\infty} \theta_\varepsilon(s - \tau)\left(\int_{-\infty}^{+\infty} \theta_\delta(t - y)\phi_2(\tau)(y)dy\right)d\tau.
\]
Clearly $\tilde{\phi}^{s, t}$ is smooth. □

Proof of Lemma 6. — Since $\Lambda$ is connected we find a smooth path $\phi : [0, 1] \to \Lambda$ such that $\phi(0) = m$, $\phi(s_1) = q$. Here $m \in M$ is an arbitrarily fixed element. Applying Proposition 1 to the parallel transport $B : L_u \Lambda \to L_q \Lambda$ we find a suitable constant $a > 0$ such that
\[
a^{-1} \| Z(\tau)x - x \| \leq \| Z(\tau)Bx - Bx \| + \| x \| \| \tau \|^{\frac{1}{2}}
\leq a(\| Z(\tau)x - x \| + \| x \| \| \tau \|^{\frac{1}{2}}).
\]
Hence the statements in Lemma 6 are equivalent provided they are equivalent in $L_m\Lambda$ for $B^{-1}(S)$. In $L_m\Lambda$ we have for all $x$

$$(Z(\tau)x)(t) = \begin{cases} 0 & \text{if } t + \tau \notin [0, 1] \\ x(t + \tau) & \text{if } t + \tau \in [0, 1]. \end{cases}$$

Define $\tilde{x} : \mathbb{R} \to T_m^\Lambda$ by $x(s) = \tilde{x}(s)$ if $s \in [0, 1]$ and 0 otherwise. By a result in [1], Theorem 2.21, we have the following equivalence for a bounded set $S$ of $L^2(0, 1; \mathbb{R}^n)$

$$(17) \quad S \text{ is precompact iff for all } \varepsilon > 0 \text{ there exists } \rho > 0 \text{ such that }$$

$$\int_{-\infty}^{+\infty} |\tilde{x}(t + \tau) - \tilde{x}(t)|^2 dt \leq \varepsilon^2 \quad \text{for all } x \in S \text{ and } |\tau| \leq \rho.$$ 

Taking $\mathbb{R}^n \simeq T_m^\Lambda$ we get a criterion on $L_m\Lambda$. A straightforward calculation shows for $t \in [-1, 1]$.

$$||Z(\tau)x - x||^2 + ||Z(-\tau)x - x||^2$$

$$\leq \int_{-\infty}^{+\infty} |\tilde{x}(t + \tau) - \tilde{x}(t)|^2 dt + \int_{-\infty}^{+\infty} |\tilde{x}(t - \tau) - \tilde{x}(t)|^2 dt$$

$$\leq 3(||Z(\tau)x - x||^2 + ||Z(-\tau)x - x||^2).$$

Hence (17) implies our assertion. □

**Definition 5.** — A subset $S \subset LA$ is called uniformly fibre-compact (ufpc) if the following holds:

(i) $S$ is bounded.

(ii) Given any $\varepsilon > 0$ there exists $\rho \in (0, 1]$ such that $||Z(\tau)x - x|| \leq \varepsilon$ for all $x \in S$ and $|\tau| \leq \rho$.

If, in addition, $S$ is closed we call $S$ uniformly fibre-compact (ufc).

In the following we denote by $\lfloor \cdot \rfloor : LA \to \mathbb{R}^+$ the map defined by

$$(18) \quad \lfloor x \rfloor = ||x||^2 + E(\pi x).$$

**Definition 6.** — A smooth map $D : LA \to LA$ is called fibre-compact if the following holds:

(i) $D$ maps bounded sets into bounded sets.

(ii) There exists a number $\sigma \in (0, 1]$ and a monotone increasing map $\mu_D : \mathbb{R}^+ \to \mathbb{R}^+$ such that $||Z(\tau)Dx - Dx|| \leq \mu_D(\lfloor x \rfloor) |\tau|^\sigma$ for all $x \in LA$ and $\tau \in [-1, 1]$.

For example the fibre-preserving map $A : LA \to LA$ is fibre compact. In fact, by Lemma 3 we find a monotone increasing map $\tilde{\mu}_A$ such that

$$\tilde{\mu}_A(\|Dq\|) \|x\| \geq \|Ax\|_A.$$
Combining this with Lemma 7 we find for a suitable monotone map
\[ \mu_\Lambda : \mathbb{R}^+ \to \mathbb{R}^+ \]
\[ ||Z(\tau)A x - A x|| \leq \mu_\Lambda([x]) |\tau|^{1/2}. \]

**Lemma 9.** Assume \((q_k) \subset \Lambda\) is a sequence such that \((\partial q_k) \subset L\Lambda\) is uniformly fibre-compact. Then \((q_k)\) is precompact.

**Proof.** We have to construct a convergent subsequence of \((q_k)\). Since \((\partial q_k)\) is bounded, we find eventually, taking a subsequence, that 
\[ q_k \to q \text{ in } C \text{ for some } q \in \Lambda. \]

Taking local co-ordinates based at \(q\) let
\[ q_k = \exp_q(\xi_k) \text{ and } \partial q_k = \Phi_q(\xi_k, \eta_k). \]

This is well-defined provided \(k\) is large enough.

Using the notations in [16] we have
\[ \eta_k = \nabla \xi_k + \theta_q(\xi_k). \]

Define a family of curves \(\phi_k : [0, 1] \to \Lambda\) by
\[ \phi_k(s) = \exp_q((1 - s)\xi_k). \]

Note that \((\xi_k)\) is bounded in \(T_q\Lambda\). So by the special form of the \(\phi_k\) the numbers \([\phi_k, 0, 1]\) will be uniformly bounded. Moreover we must have \(\xi_k \to 0\) uniformly.

We carry out the parallel transport along \(\phi_k\) denoting by \(x_k \in L_q\Lambda\) the image of \(\partial q_k\). By Proposition 1 there exists a constant \(c_2 > 0\) such that
\[ ||Z(\tau)x_k - x_k|| \leq C_2(||Z(\tau)\partial q_k - \partial q_k|| + |\tau|^{1/2}) \]
for all \(k \in \mathbb{N}\) and all \(\tau \in [-, 1]\). Here, of course, the constant bounding \(||\partial q_k||\) is contained in \(C_2\). Since \(\partial q_k\) is \((u f p c)\) we infer that \((x_k)\) is precompact in \(L_q\Lambda\). Eventually, taking a subsequence we may assume 
\[ x_k \to x \text{ in } L_q\Lambda, \]

for some \(x \in L_q\Lambda\). In local co-ordinates the transport equation is given by
\[ \dot{a} = \Gamma_q((1 - s)\xi_k)(\xi_k, a) \]
\[ a(0) = \eta_k. \]

Since \(\xi_k \to 0\) uniformly and \(s_k \geq 1\), we obtain \(||\Gamma_q((1 - s)\xi_k)(\xi_k, a)|| \leq \varepsilon_k||a||\), where \(\varepsilon_k \to 0\). For \(\sigma_k\) being the local representative of \(x_k\) this implies
\[ ||\sigma_k - \eta_k|| \to 0. \]

Since \(\sigma_k \to \sigma\), where \(\sigma\) is the representative of \(x\), we infer \(\eta_k \to \sigma\). Hence
\[ \nabla \xi_k + \theta_q(\xi_k) \to \sigma \text{ in } L_q\Lambda \]
\[ \xi_k \to 0 \text{ uniformly.} \]
Therefore
\[ \xi_k \to 0 \quad \text{in} \quad T_q \Lambda. \]

This implies
\[ q_k \to q \quad \text{in} \quad \Lambda. \]

Next we study the question how uniformly fibre-precompact sets are mapped by certain operators \( \Lambda \to \Lambda \).

**Lemma 10.** — Let \( f : TM \to TM \) be a continuous fibre-preserving map and assume there exists a constant \( C > 0 \) such that \( |f(x)| \leq C(1 + |x|) \).

Then the map \( \tilde{f} : \Lambda \to \Lambda \) defined by \( \tilde{f}(x)(t) = f(x(t)) \) is continuous and maps uniformly fibre-precompact sets into uniformly fibre-precompact sets.

**Proof.** — The continuity is a well-known fact from nonlinear analysis. For the second assertion note that we can restrict ourselves to a chart since a \((u f p c)\) set can be covered by finitely many « big » exponential charts. So assume
\[ \pi(S) \subset \exp_q \left( \frac{1}{2} V \right), \]
where \( \exp_q : V \to U \) is a local co-ordinate system on \( \Lambda \). Moreover we may assume \( V \) is convex. Let \((\xi, \eta) \in V \times L_q \Lambda \) be the representative of some \( x \in S \).

Let
\[ S_q = \{ \eta \in L_q \Lambda \mid \exists \xi \in V \text{ such that } \Phi_q(\xi, \eta) \in S \} \]

In view of the technique used in the proofs of Lemma 6 and Lemma 9 it is clear that the set \( S_q \) is precompact. Denote by \( \tilde{f}_q \) the representative of \( \tilde{f} \).
Then
\[ \tilde{f}_q(\xi, \eta) = \Phi_q^{-1} \circ \tilde{f}(\Phi_q(\xi, \eta)). \]

It is enough to show that \( pr_2 \circ \tilde{f}(\Phi_q^{-1}(S)) \) is precompact (using Proposition 1 and the previous remark). Let \((\tilde{\eta}_k) \subset pr_2 \circ \tilde{f}_q(\Phi_q^{-1}(S)) \). Taking \( \tilde{\eta}_n = pr_2 \circ \tilde{f}_q(\xi, \eta) \) where \( \eta \in S_q \) we infer eventually taking a subsequence \( \xi_n \to \xi \) uniformly and \( \eta_n \to \eta \) in \( L_q \Lambda \) for some \( \xi \in V \) and \( \eta \in L_q \Lambda \). Hence \( \tilde{\eta}_n \to pr_2 \circ f_q(\xi, \eta) \). This proves Lemma 10. \( \square \)

**III.4. A family of smoothing operators.**

One of the difficulties in the proof of the main result is the fact—as already explained—that the vertical component of the gradient of \( x \) is not compact. The idea is now to replace \( x \) by \( x = x \circ F_n \), where the \( F_n \) are the smoothing operators introduced below.

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Define for all \( n \in \mathbb{N} \) a fibre-preserving map \( F_n : \Lambda \rightarrow TA \) by
\[
(1) \quad \frac{1}{n} (\delta F_n x, \delta y) + (F_n x, y) = (x, y)
\]
for \( x \in L_q \Lambda \) and all \( y \in T_q \Lambda \).

**Lemma 11.** For all \( n \in \mathbb{N} \), \( F_n \) is a smooth map. Moreover we have
\[
(2) \quad (F_n x, y) = (x, F_n y) \quad \text{for all} \quad (x, y) \in \Lambda \Lambda \oplus \Lambda \Lambda .
\]

If we consider \( F_n \) as a map \( \Lambda \Lambda \rightarrow \Lambda \Lambda \) it is fibre-compact and we have the estimates
\[
(3) \quad || Z(\tau) F_n x - F_n x || \leq 2 \sqrt{n} || x || || \tau || \frac{1}{2} \\
|| F_n x || \leq || x || .
\]

The proof uses Lemma 7 and is straightforward.

**Lemma 12.** If \( (x_k) \subset \Lambda \Lambda \) is \((u f p c)\) and \( (n_k) \subset \mathbb{N} \), \( n_k \rightarrow + \infty \), then \( (F_{n_k} x_k) \) is \((u f p c)\). Moreover if \( x_k \rightarrow x \) then \( F_{n_k} x_k \rightarrow x \) in \( \Lambda \Lambda \).

**Proof.** Let \( q_k = \pi x_k \). Since \( (x_k) \) is bounded \( (q_k) \) is bounded. So we find finitely many exponential charts \( \exp_{p_i} : V_{p_i} \rightarrow U_{p_i} \) such that \( V_{p_i} \) is convex in \( T_{p_i} \Lambda \) and the family \( \left( \exp_{p_i} \left( \frac{1}{2} V_{p_i} \right) \right) \) cover the whole sequence \( (q_k) \).

Clearly it is enough by the previous construction to show that the subsequence of \( (x_k) \) consisting of those elements that the corresponding \( q_k \) lie in a specific chart \( \exp \left( \frac{1}{2} V_{p_i} \right) \) is \((u f p c)\). So we may assume without loss of generality that \( (q_k) \subset \exp \left( \frac{1}{2} V \right) \) for some convex open zero-neighborhood \( V \) in \( T_q \Lambda \).

We carry out the parallel transport along the curves \( \phi_k : s \rightarrow \exp_{q_k}(1-s)(\tilde{z}_k) \). Note that \( (\tilde{z}_k) \) is bounded in \( T_q \Lambda \). Denote the image of \( F_{n_k} x_k \) by \( \tilde{z}_k \in L_q \Lambda \).

If we can show that \( (\tilde{z}_k) \) is precompact we are done by Proposition 1.

So we shall show that \( (\tilde{z}_k) \) has a convergent subsequence. Eventually taking a subsequence of \( (q_k) \) we may assume \( q_k \rightarrow q_0 \) in \( C \) for some \( q_0 \in \Lambda \). Denoting by \( z_k \) the local representation of \( F_{n_k} x_k \) we have for all \( y \in T_q \Lambda \) using the notation in \([16]\).

\[
(4) \quad \frac{1}{n_k} \left( G(\tilde{z}_k)(V z_k + D_2 \theta_{q}(\tilde{z}_k) z_k + \Gamma_q(x_k)(z_k, \delta_q \tilde{z}_k)), \right.
\]
\[
\left. \quad \forall y + D_2 \theta_{q}(\tilde{z}_k)y + \Gamma_q(\tilde{z}_k)(y, \tilde{c}_q \tilde{z}_k)) \right)
\]
\[
+ (G(\tilde{z}_k) z_k, y)
\]
\[
= (G(\tilde{z}_k) n_k, y),
\]

where $\eta_k$ is the local representative of $x_k$. We have $\zeta_k \to \zeta_0$ uniformly. Denote by $\hat{x}_k$ the image of $x_k$ under the parallel transport along $\phi_k$. By our assumption we may assume eventually taking a subsequence $\hat{x}_k \to \hat{x}_0$ for some $\hat{x}_0 \in L_q\Lambda$. In local coordinates

$$\hat{x}_k = a_k(1), \quad \eta_k = a_k(0),$$

where

$$\dot{a}_k = \Gamma_q((1 - s)\zeta_k)(\zeta_k, a_k).$$

Since $\zeta_k \to \zeta_0$ uniformly we infer that $\eta_k \to \eta_0$, where $\eta_0 = a_0(0)$, $\hat{x}_0 = a_0(1)$ and $\dot{a}_0 = \Gamma_q((1 - s)\zeta_0)(\zeta_0, a_0)$. By the definition of $F_n$ it is clear that $\|\delta F_n x_k\| \leq \sqrt{\eta_k}\|x_k\|$.

Hence, taking the limit in (4) for fixed $y \in T_q\Lambda$ we infer

$$\lim (G(\zeta_0)z_k, y) = \lim (G(\zeta_k)z_k, y) = (G(\zeta_0)\eta_0, y).$$

Since $(z_k)$ is bounded and $T_q\Lambda$ dense in $L_q\Lambda$ we deduce that (5) holds for all $y \in L_q\Lambda$. This implies

$$z_k \rightharpoonup \eta_0 \quad \text{weakly in } L_q\Lambda.$$

Moreover $\lim \inf \|z_k\| \|G(\zeta_0)y\| \geq (\eta_0, G(\zeta_0)y)$. This implies

$$\lim \inf \|z_k\| \geq \|\eta_0\|.$$

On the other hand $\|F_n x_k\| \leq \|x_k\|$. This implies

$$\lim \sup \|z_k\| \leq \|\eta_0\|.$$

Therefore

$$z_k \rightharpoonup \eta_0 \quad \text{weakly in } L_q\Lambda$$

$$\|z_k\| \rightharpoonup \|\eta_0\|.$$

This implies

$$z_k \to \eta_0 \quad \text{strongly in } L_q\Lambda.$$

Now

$$\hat{z}_k = b_k(1), \quad z_k = b_k(0),$$

where

$$\dot{b}_k = \Gamma_q((1 - s)\zeta_k)(\zeta_k, b_k).$$

Since $\zeta_k \to \zeta_0$ uniformly and $z_k \to \eta_0$ we obtain $\hat{z}_k \to \hat{z}_0$, where

$$\hat{z}_0 = b_0(1), \quad \eta_0 = b_0(0)$$

and

$$\dot{b}_0 = \Gamma_q((1 - s)\zeta_0)(\zeta_0, b_0).$$

This shows that $(\hat{z}_k)$ is precompact. Hence $(F_n x_k)$ is (ufp c). The proof of the second part is implicitly contained in the previous arguments. The details are left to the reader. □
Lemma 13. — For fixed $x \in L_q^\Lambda; y, z \in T_q^\Lambda$ we have the identity

$$
\frac{1}{n} (R(y, \dot{q})F_n x, \delta z) + \frac{1}{n} (\delta F_n x, R(y, \dot{q})z) + \frac{1}{n} (\delta(D_yF_n)x, \delta z) + ((D_yF_n)x, z) = 0
$$

Moreover, the following estimate holds for some constant $\tilde{M}$ only depending on $M$

$$
\left( \frac{1}{n} \| (D_yF_n)x \|^2 + \left( 1 - \frac{1}{n} \right) \| D_yF_n x \|^2 \right) \leq \tilde{M} n^{-\frac{1}{4}} \| x \| \| \delta q \| \| y \|_x
$$

Proof. — (6) follows from a straightforward application of the covariant derivative $D_y$ to the definition of $F_n$, (1), using Lemma 2. In order to prove (7) let $z = (D_yF_n)x$. Moreover note that there exists a constant $c_1 > 0$ only depending on $M$ such that $|R(a, b, c)| \leq c_1 |a| |b| |c|$ for all $(a, b, c) \in TM \oplus TM \oplus TM$. We obtain from (6)

$$
\frac{1}{n} \| z \|_A^2 + \left( 1 - \frac{1}{n} \right) \| z \|_A^2 \leq -c_1 \| y \|_\infty \| F_n x \|_\infty \| \dot{q} \| \| \delta z \|
\quad + \frac{1}{n} c_1 \| y \|_\infty \| z \|_\infty \| \delta F_n x \| \| \dot{q} \|
$$

Applying the interpolation inequality III.1 (7) to $F_n x$ and $z$ we infer

$$
\| F_n x \|_A^2 \leq \| F_n x \|^2 + 2 \| F_n x \| \| \delta F_n x \| \leq (1 + 2\sqrt{n}) \| x \|^2
$$

and with

$$
a = \left( \frac{1}{n} \| z \|_A^2 + \left( 1 - \frac{1}{n} \right) \| z \|_A^2 \right)^{\frac{1}{2}}
$$

$$
\| z \|_\infty^2 \leq \| z \|^2 + 2 \| z \| \| \delta z \| \leq a^2 + 2\sqrt{na^2} = (1 + 2\sqrt{n})a^2.
$$

Combining (8)-(10) gives

$$
a^2 \leq 2c_1 || \dot{q} || || y ||_\infty || x || ((1 + 2\sqrt{n})n^{-1}) \frac{1}{2} a \\
\leq 8c_1 n^{-\frac{1}{4}} || \dot{q} || || y ||_\infty || x || || a = \tilde{M} n^{-\frac{1}{4}} || \dot{q} || || x || || y ||_\infty a.
$$

IV. CONSTRUCTION OF A $\mathbb{N}$-FAMILY

In this chapter we shall introduce the $\Psi_n$ described in 1.2. Here the smoothing operators $F_n$ and the compactness concept introduced in III will be crucial.
IV.1. The set-up.

We introduce a functional \( \Psi \in C^\infty(\Lambda_\alpha, \mathbb{R}) \) by
\[
\Psi(x) = (\partial \pi x, x).
\]

For \( a \in T_x \Lambda_\alpha \) we compute
\[
d\Psi(x)a = (\partial \pi x, \partial a) + (\partial(T\pi)a, x).
\]

We introduce the smooth gradient vectorfield \( \Psi' : \Lambda_\alpha \to T\Lambda_\alpha \) by
\[
d\Psi(x)a = (\Psi'(x), a)_L.
\]

Let
\[
\Sigma = (T\pi)\Psi'
\]
and put \( y = (T\pi)a \) and \( z = \partial a \) for some \( a \in T_x \Lambda_\alpha \). We find using (2)
\[
(\partial \Psi'(x), z) + (\Sigma x, y) + (\partial \Sigma x, \partial y) = (\partial \pi x, z) + (\partial y, x).
\]

This is true for all \( y \in T_{q\Lambda} \) and all \( z \in L_{q\Lambda} \).

Hence
\[
\partial \pi = \partial \Psi'
\]
and
\[
(\Sigma x, y) + (\partial \Sigma x, \partial y) = (x, \partial y).
\]

Using the fibre preserving map \( A^* : \Lambda_\alpha \to \Lambda_\alpha \) we deduce from (5)
\[
A^* \Sigma + \partial \Sigma = \text{Id}.
\]

Further (5) implies
\[
\| \Sigma x \|_{\Lambda} \leq \| x \|.
\]

Define a smooth fibre-preserving map \( \phi : T^*\mathbb{M} \to \mathbb{M} \) by \( f = \langle \phi(f), \cdot \rangle \).
Introduce a smooth map with compact support \( h : [0, 1] \times T\mathbb{M} \to \mathbb{R} \) by
\[
h(t, x) = h^*(t, \phi^{-1}(x))
\]
where \( h^* \) is the map introduced in Chapter I before Theorem 2. We define a map \( \alpha_\infty : \Lambda_\alpha \to \mathbb{R} \) by
\[
\alpha_\infty(x) = \int_0^1 h(t, x(t))dt.
\]

Then \( \alpha_\infty \in C^1(\Lambda_\alpha, \mathbb{R}) \), but in general it has no better regularity properties. However, if we consider \( \alpha_\infty \circ T\Lambda : T\Lambda \to \mathbb{R} \) for the differentiable structure on \( T\Lambda \) it is of class \( C^\infty \). We define now \( \alpha_n : \Lambda_\alpha \to \mathbb{R} \) by
\[
\alpha_n = \alpha_\infty \circ F_n.
\]
Since $F_n : \mathcal{A} \rightarrow \mathcal{T} \mathcal{A}$ is smooth we infer $\alpha_n \in C^\infty (\mathcal{A}, \mathbb{R})$. Finally, define for $n \in \mathbb{N}_\infty$

$$\Psi_n = \Psi - \alpha_n.$$ 

Clearly $\Psi_n \in C^\infty (\mathcal{A}, \mathbb{R})$ for $n \in \mathbb{N}$. With $a \in T_x \mathcal{A}$ we have

$$(\alpha'_n(x), a)_\mathcal{A} = d\alpha_n (F_n x)(T\pi a) = (\alpha'_n(F_n x), TF_n a)_\mathcal{A}.$$ 

Further

$$(\alpha'_n(x), a)_\mathcal{A} = (\tilde{K} H(x), \tilde{K} a) + ((T\pi) H(x), (T\pi) a)$$

where $H : \mathcal{A} \rightarrow \mathcal{A}$ is defined by

$$H(x)(t) = (\text{grad } h_i)(x(t))$$

and

$$\langle \text{grad } h_i x, b \rangle_{\mathcal{T} \mathcal{M}} = dh_i(x). b.$$ 

Hence

$$\langle \alpha'_n(x), a \rangle_{\mathcal{A}} = (\tilde{K} \text{HF}_n x, (T\pi) \text{HF}_n x) + ((T\pi) \text{HF}_n x, (T\pi) \text{HF}_n x)$$

$$= (\tilde{K} \text{HF}_n x, K_a) + (\tilde{K} \text{HF}_n x, (D_{T\pi a} F_n) x) + ((T\pi) \text{HF}_n x, (T\pi) a)$$

$$= (F_n \tilde{K} \text{HF}_n x, K a) + (\tilde{K} \text{HF}_n x, (D_{T\pi a} F_n) x) + ((T\pi) \text{HF}_n x, (T\pi) a).$$

IV.2. Compactness estimates.

We shall now derive some estimates which will be used two times. In the proof that the family $(\Psi_n)$ constructed in the previous paragraph is in fact an $\mathbb{N}$-family and in the representation result for the gradient flows.

**Lemma 14.** — There exists a constant $\tilde{H}$ only depending on $M$ and $h$ such that

$$\| \alpha'_n(x) \|_\mathcal{A} \leq \tilde{H} (1 + n^{-\frac{1}{2}} \| x \| \| \dot{q} \|) \quad \| K \alpha'_n(x) \| \leq \tilde{H}$$

for all $x \in \mathcal{A}$, where $q = \pi x$. The estimate is true for all $n \in \mathbb{N}_\infty$ if we put $n^{-\frac{1}{4}} = 0$.

**Proof.** — From IV.1 (12) we infer

$$\| K \alpha'_n(x) \| = \| F_n \tilde{K} \text{HF}_n x \| \leq \| \tilde{K} \text{HF}_n x \| \leq \tilde{H}$$

for a suitable constant $\tilde{H} > 0$ since $h$ has compact support. Moreover using IV.1 (12) again, combining it with Lemma 13, we infer for arbitrary $a \in T_x \mathcal{A}$

$$\| (\alpha'_n(x), a) \| = \tilde{H} \| \tilde{K} a \| + \tilde{H} M n^{-\frac{1}{2}} \| x \| \| \dot{q} \| \| (T\pi) a \|_\mathcal{X} + \tilde{H} \| (T\pi) a \|[.]$$

Hence, eventually taking a greater constant
\[ |(\alpha'_n(x), a)_H| \leq \tilde{H}(1 + n^{-\frac{1}{2}} ||x|| ||\dot{q}||) ||a||_L. \]
Taking \( a = \alpha'_n(x) \) yields (1). The claim for \( n = \infty \) is trivial. \( \square \)

For the following we define smooth maps \( B^n : LA \to LA \) for \( n \in \mathbb{N} \) by
\[ (B^n x, y) = ((T\pi)HF_{n,x}, Ay) + (\tilde{K}HF_n, (D_AF_n)x) \]
where \( A \) was defined in III.2. Denote by \( A^* \) the \( LA \)-adjoint of \( A \), i.e.
\[ (Ax, y) = (x, A^*y). \]

Further denote by \( Q : LA \to LA \) the fibre-preserving maps which induces on each fibre the orthogonal projection onto the orthogonal space in \( LA \) of \( \ker(\delta) \). Recalling that \( \ker(\delta) \) was a smooth bundle over \( \Lambda \) we infer that \( Q \) is smooth. However, this property is not needed for the following estimate. We have
\[ (A^*x, \delta y) = (x, A\delta y) = (x, A\delta Qy) = (x, Qy) = (Qx, y). \]

Since this is true for all \( y \in T_qA \) we infer that \( A^*x \in T_qA \), where \( q = \pi x \), by Lemma 5. Hence
\[ ||\delta A^*x|| \leq ||x||. \]
Moreover \( ||A^*_q|| = ||A_q||. \) This of course implies that \( A^* \) is fibre-compact with factor \( \frac{1}{2} \), i.e.
\[ ||Z(\tau)A^*x - A^*x|| \leq \mu_A([x]) |\tau|^{\frac{1}{2}} \]

**LEMMA 15.** — For all \( n \in \mathbb{N} \) \( B^n \) is fibre-compact. Moreover there exists a monotone increasing map \( \mu_B : \mathbb{R}^+ \to \mathbb{R}^+ \) independent of \( n \in \mathbb{N} \) such that
\[ ||Z(\tau)B^n x - B^n x|| \leq \mu_B([x]) |\tau|^{\frac{1}{2}} \]
for all \( x \in LA \) and \( \tau \in [-1, 1]. \)

**Proof.** — In fact, using Lemma 7, Lemma 13 and Lemma 14, we infer
\[ (Z(\tau)B^n x - B^n x, y) = (B^n x, Z(-\tau)y - y) \]
\[ = ((T\pi)HF_{n,x}, A(Z(-\tau)y - y)) \]
\[ + (\tilde{K}HF_{n,x}, (D_A(Z(-\tau)y - y)F_n)x) \]
\[ \leq (A^*(T\pi)HF_{n,x}, Z(-\tau)y - y) \]
\[ + ||\tilde{K}HF_{n,x}|| ||(D_A(Z(-\tau)y - y)F_n)x|| \]
\[ \leq (Z(\tau)A^*(T\pi)HF_{n,x} - A^*(T\pi)HF_{n,x}, y) \]
\[ + \|\tilde{H}Mn^{-\frac{1}{4}} ||x|| ||\dot{q}|| ||A(Z(-\tau)y - y)||_x \]
\[ \leq \mu_A([T\pi)HF_{n,x})] |\tau|^{\frac{1}{2}} ||y|| \]
\[ + \|\tilde{H}M ||x|| ||\dot{q}|| ||A(Z(-\tau)y - y)||_x \]

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Next we estimate
\[(A(Z(- \tau)y - y), z) = (Z(- \tau)y - y, A^*z)\]
\[= (y, Z(\tau)A^*z - A^*z)\]
\[\leq \| y \| 2 \| A^*z \|_\Lambda | \tau |^{\frac{1}{2}}\]
\[\leq 2 \| y \| (\| z \|^2 + \| A_q \|^2 \| z \|^2)^{\frac{1}{2}} | \tau |^{\frac{1}{2}}.\]

This implies
\[(5) \quad \| A(Z(- \tau)y - y) \| \leq 2(1 + \| A_q \|^2)^{\frac{1}{2}} \| y \| | \tau |^{\frac{1}{2}}.\]

Now we use the interpolation inequality III.1 (7) and find using that
\[| \tau | \leq 1\]
\[(6) \quad \| A(Z(- \tau)y - y) \|_\infty \leq 3(1 + \| A_q \|^2)^{\frac{1}{2}} \| y \| | \tau |^{\frac{1}{2}}.\]

Combining (4) and (7) yields
\[(Z(\tau)B^x - B^x z, y) \leq (\mu A^*([T\tau]HF_{\mu}x)) | \tau |^{\frac{1}{2}}\]
\[+ 3HM \| x \| \| \dot{q} \| (1 + \| A_q \|^2)^{\frac{1}{2}} | \tau |^{\frac{1}{2}} \| y \|.\]

Note that the expression on the right hand side is independent of \(n \in \mathbb{N}.\)

Now for a suitable monotone increasing map \(\mu_B\) our assertion follows.

\[\square\]

IV. 3. Verification of \(N\)-family properties.

The following result is one in the key steps of reducing Theorem 2 to Theorem 3.

**Proposition 2.** There exists \(n_0 \geq 1\) such that \((\Psi_n)_{n \geq n_0}\) is an \(N\)-family for \(\Psi_\infty.\)

**Proof.** We have to check (i)-(iii) of Definition 3. Clearly (i) holds by our construction. Next we prove (ii). Let \((x_k) \subset L\Lambda\) such that \(x_k \to x\) and \((n_k) \subset \mathbb{N}_\infty\) with \(n_k \to + \infty.\) Define \(F_\infty = \text{Id}.\) By Lemma 12 we have
\[F_{n_k}x_k \to x.\]
This implies \(\alpha_{n_k}(x_k) \to \alpha_x(x).\)

Hence
\[\lim \Psi_{n_k}(x_k) = \Psi_x(x).\]

Next assume \(\Psi'_\infty(x) = 0.\) We have
\[\| \Psi'_\infty(x_k) \|_{L} \leq \| \Psi'_x(x_j) \|_{L} + \| \alpha'_{n_k}(x_k) - \alpha'_{x}(x_k) \|_{L}.\]
In order to show that the left hand side tends to zero we have only to show that the second term on the right hand side tends to zero. Using Lemma 13 and Lemma 14 we infer for a suitable constant $\tilde{H}$ only depending on $M$ and $h$

$$||\alpha'_n(x_k) - \alpha'_n(x_k)||_L \leq ||\tilde{K}Hx_k - F_{n_k}\tilde{K}HF_{n_k}x_k|| + ||(T\pi)Hx_k - (T\pi)HF_{n_k}x_k|| + 2\tilde{M}\tilde{H}||x_k||\tilde{q}_k||n_k^{-\frac{1}{2}} = I_k + II_k + III_k.$$ 

Since $F_{n_k}x_k \to x$ clearly $I_k + II_k \to 0$. Since $(x_k)$ is bounded in $LA$ and $n_k \to +\infty$ we infer that $III_k \to 0$. Similarly if $||\Psi'_{n_k}(x_k)||_L \to 0$ we infer

$$||\Psi'_{n_k}(x_k)||_L \leq ||\Psi'_{n_k}(x_k)||_L + ||\alpha'_n(x_k) - \alpha'_n(x_k)||_L =: I_k + II_k.$$ 

By our assumption $I_k \to 0$ and similar to the argument used before $II_k \to 0$. Hence $||\Psi'_{n_k}(x_k)|| \to 0$. Since $x_k \to x$ this implies $\Psi'_{n_k}(x) = 0$. The proof of $(ii)$ is complete.

Next we show that $(iii)$ holds. Let $(x_k) \subset LA$ and $(n_k) \subset \mathbb{N}_\infty$, $n_{k+1} \geq n_k$, such that

$$||\Psi'_{n_k}(x_k)||_L \to 0, \quad \Psi_{n_k}(x_k) \to d.$$ 

With $q_k = \pi x_k$ we infer

$$(8) \quad ||\tilde{q}_k - F_{n_k}\tilde{K}HF_{n_k}x_k|| \to 0$$ 

and

$$(9) \quad ||\Sigma x_k - (T\pi)\alpha'_n(x_k)||_\Lambda \to 0.$$ 

Define $d_k^1 := ||\tilde{q}_k - F_{n_k}\tilde{K}HF_{n_k}x_k||$. By Lemma 14 we have

$$(10) \quad ||\tilde{q}_k|| \leq d_k^1 + \tilde{H}.$$ 

By Lemma 3 there exists a constant $c > 0$ only depending on $\tilde{H}$ such that for all $x \in T_\Lambda$ with $||\tilde{q}|| \leq 2\tilde{H} + 1$ we have

$$||\tilde{q}|| \geq c ||x||_\Lambda$$ 

provided $x \perp \text{ker}(\delta)$.

We have

$$(\delta\Sigma x, \delta y) + (\Sigma x, y) = (x, \delta y).$$ 

Therefore

$$||x||^2 = (\delta\Sigma x, x) + (\Sigma x, Ax) \leq ||\Sigma x||_w ||x|| + ||\Sigma x||_\Lambda c^{-1} ||x||.$$ 

Hence for a suitable constant $\tilde{c} > 0$ only depending on $\tilde{H}$ and therefore only depending on $M$ and $h$ we find

$$(11) \quad d_k^2 := ||\Sigma x_k - (T\pi)\alpha'_n(x_k)||_\Lambda 
\geq ||\Sigma x_k||_\Lambda - \tilde{H}(1 + n_k^{-\frac{1}{2}} ||x_k|| (\tilde{H} + d_k^1)) 
\geq \tilde{c} ||x_k|| - H(1 + n_k^{-\frac{1}{2}} ||x_k|| (\tilde{H} + d_k^1))$$
provided \( k \) is large enough since \( d_k^k \to 0 \). Clearly if a priori \( n_k \geq n_0 \) for a suitable \( n_0 \) only depending on \( M \) and \( h \) (11) will imply that the sequence \( \{ \| x_k \| \} \) is bounded. For the following we fix such a \( n_0 \). Hence if \( n_1 \geq n_0 \) and \( \| \Psi_n(x_k) \|_L \to 0 \) then \( \{ \| \partial q_k \| + \| x_k \| \} \) is bounded. It is easy to see that this implies that \( (x_k) \) is bounded in \( L^\Lambda \). So for the following let \( (n_k) \subset \mathbb{N} \), \( n_{k+1} \geq n_k \geq n_0 \), and \( (\lambda_k) \subset L^\Lambda \) such that \( \| \Psi_n(x_k) \|_L \to 0 \).

Define

\[
(12) \quad b_k = \sum x_k - (T\pi)x_n(x_k).
\]

We have \( b_k \|_\Lambda \to 0 \). Hence taking the inner product with \( Az \) where \( z \in L_q R^\Lambda \) we obtain

\[
(b_k, Az)_{\Lambda} = (x_k, z) - ((T\pi)HF_n x_k, Az) - (\hat{K}HF_n x_k, (D_Az F_n) x_k) = (x_k - B_n^nx_k, z).
\]

Since \( \{ \| \partial q_k \| \} \) is bounded we find by Lemma 3 a suitable constant \( c > 0 \) such that

\[
(13) \quad \| x_k - B_n^nx_k \| \leq c \| b_k \|_\Lambda \to 0.
\]

Moreover

\[
(14) \quad \| \partial q_k - F_n^\lambda \hat{K}HF_n x_k \| \to 0.
\]

Since \((x_k)\) is bounded using (13) and Lemma (15) we infer that \((x_k)\) is \((ufp c)\). If \((n_k)\) becomes stationary for \( k \) large enough we conclude from Lemma 9 and Lemma 11 that \((q_k)\) must be precompact. If \( n_k \to +\infty \) we infer from Lemma 12 that \((F_n x_k)\) is \((ufp c)\). Hence by Lemma 10 \((\hat{K}HF_n x_k)\) is \( ufp c \). Finally that implies that \((F_n^\lambda \hat{K}HF_n x_k)\) is \( ufp c \). Hence by (14) \((\partial q_k)\) is \( ufp c \) and by Lemma 9 \((q_k)\) is precompact. Hence \((q_k)\) is precompact and \((x_k)\) is \( ufp c \). Therefore \((x_k)\) is precompact. \( \square \)

V. ESTIMATES AND REPRESENTATIONS
FOR GRADIENT FLOWS

V.1. Parameterised families of transport equations
and a compactness estimate.

In the following we may assume, without loss of generality, that the number \( n_0 \) introduced in Proposition 2 is equals one. We fix for the following an \( n \in \mathbb{N}_0 = \mathbb{N} \cup \{ 0 \} \) where we denote by \( \Psi_0 \) the functional \( \Psi \). Denote by

\[
L^\Lambda \times \mathbb{R} \to L^\Lambda : (x, s) \to x \circ s
\]

the global flow corresponding to the differential equation

\[
x' = - \beta(\| \Psi_n''(x) \|_L) \Psi_n'(x) =: G_n(x)
\]

where $\beta$ is as described in 1.2. We define a smooth vector bundle $\gamma : \mathcal{L} \to \Lambda \times \Lambda$ where the fibre over $(q_2, q_1) \in \Lambda \times \Lambda$ is given by

$$\mathcal{L}_{(q_2, q_1)} = \mathcal{L}(L_q, L_{q_2}).$$

Here the right hand side denotes the Banach space of continuous linear operators $L_q : \Lambda \to L_{q_2}$. Further we denote by

$$\kappa : \Lambda \times \mathbb{R} \times \mathbb{R} \to \Lambda \times \Lambda$$

the smooth map defined by

$$\kappa(x, s, t) = (\pi(x \circ s), \pi(x \circ t)).$$

We define a map $\mu_n : \Lambda \times \mathbb{R} \times \mathbb{R} \to \mathcal{L}$ such that $\gamma \circ \mu_n = \kappa$ by

$$\mu_n(x, s_2, s_1) = \mu_n(x, s_2, s_1),$$

where $\mu_n(x, s_2, s_1) \in \mathcal{L}(L_{\pi(x \circ s_2)}, L_{\pi(x \circ s_1)})$ is defined by the Cauchy problem

$$(x, s, t) \to (\pi(x \circ s), \pi(x \circ t)).$$

**LEMMA 16.** $\mu_n$ is smooth.

**Proof.** Let us first prove the assertion for the case that $|s_2 - s_1|$ is small. Then the problem is localised in a co-ordinate chart.

Let $\exp_q : V_q \to U_q$ and $\exp_q : V_q \to U_q$ be co-ordinate systems such that for some $\varepsilon > 0$ we have

$$x \in U_q, \quad \pi(x \circ s) \in U_q \quad \text{for} \quad s \in (s_1 - \varepsilon, s_2 + \varepsilon).$$

We assume without loss of generality that $s_1 \leq s_2$.

Let

$$x = \Phi_q(\xi, \eta)$$

$$\pi(x \circ s) = \exp_q (b_q(s, (\xi, \eta))).$$

Define a map $G$ by

$$G : V_q \times L_q \Lambda \times (s_1 - \varepsilon, s_2 + \varepsilon) \times \mathcal{L}(L_q, L_{q_1} \Lambda) \to \mathcal{L}(L_q \Lambda, L_q \Lambda)$$

where $G(\xi, \eta, s, \sigma) = \Gamma_q(b_q(s, (\xi, \eta)))\left(\frac{\sigma}{\partial t} b_q(\xi, \eta), s\right) + \beta(\Psi^t(\Phi_q(\xi, \eta) \circ s) || \lambda, \sigma).$

Clearly $G$ is smooth. Then (3) written in local co-ordinates corresponds to the operator differential equation

$$\sigma' = G(\xi, \eta, s)\sigma$$

$$\sigma(s_1) = \text{Id}.$$

By the continuous and differentiable-parameter dependence theory for ordinary differential equations, we infer that the map which asso-
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associates to \((\xi, \eta, s, t) \in V_q \times L_q \Lambda \times (s_1 - \epsilon, s_1 + \epsilon) \times (s_2 - \epsilon, s_2 + \epsilon)\) the unique linear map \(\tilde{\sigma} \in \mathcal{L}(L_q \Lambda, L_q \Lambda)\) such that \(\tilde{\sigma} = \sigma(t)\), \(\sigma(s) = \text{Id}\) and \(\sigma\) satisfies (4) is a smooth assignment. Hence \(\mu_n\) is smooth provided \(|s_1 - s_2|\) is small.

In the general case we can write \(s \geq t\)

\[\mu_n(x, s, t) = \mu_n(x, s_N, s_{N-1}) \mu_n(x, s_{N-1}, s_{N-2}) \ldots \mu_n(x, s_1, s_0)\]

where \(s = s_N > s_{N-1} > \ldots > s_1 > s_0 = t\). The previous argument applies to all \(\mu_n(x, s_i, s_{i-1})\) provided \(|s_i - s_{i-1}|\) is small enough. Hence \(\mu_n(x, s, t)\) is smoothly depending on \(x, s, t\) in the general case. \(\square\)

\(\mu_n\) can be considered as a smooth parametrisation of certain linear transport equations of type (3). The importance will become clear in the next section.

Since \(\beta(||\Psi_n(x \circ s)||_L) \leq 1\) for all \(x \in L \Lambda, s \in \mathbb{R}\) we immediately obtain denoting by \(||.|| : \mathcal{L}^p \rightarrow \mathbb{R}^+\) the operator norm the following estimate

\[|| \mu_n(x, s_2, s_1) || \leq \exp(|s_2 - s_1|)\]  

Define a smooth fibre preserving map

\[D^n : L \Lambda \rightarrow L \Lambda\]

\[D^n x = - A^* \Sigma x - B^n x + F_n \hat{K} \hat{H} F_n x + A^*(T \pi) \psi'_n(x)\]

for \(n \in \mathbb{N}\) and

\[D^0 = - A^* \Sigma\]

where \(B^n\) was introduced in IV. 2 (2).

**Lemma 17.** — For all \(n \in \mathbb{N}_0\) there exists a monotone increasing map \(\mu_{D^n} : \mathbb{R}^+ \rightarrow \mathbb{R}^+\) such that

\[|| Z(\tau) D^n x - D^n x || \leq \mu_{D^n}([x]) |\tau|^{1/2}\]

for all \(x \in L \Lambda\) and \(\tau \in [-1, 1]\).

**Proof.** — We have using the compactness estimates for \(A^*\) (before Lemma 15, Lemmas 11, 13 and 14.

\[|| Z(\tau) D^n x - D^n x || \leq || Z(\tau) A^* \Sigma x - A^* \Sigma x || + || Z(\tau) B^n x - B^n x || + || Z(\tau) F_n \hat{K} \hat{H} F_n x - F_n \hat{K} \hat{H} F_n x || + || Z(\tau) A^*(T \pi) \psi'_n(x) - A^*(T \pi) \psi'_n(x) ||
\]

\[\leq \mu_A([x]) |\tau|^{1/2} + \mu_B([x]) |\tau|^{1/2} + 2 \sqrt{n \bar{H}} |\tau|^{1/2} + \mu_A([\sigma'(x)]) |\tau|^{1/2} \leq \mu_A([x]) |\tau|^{1/2} + \mu_B([x]) |\tau|^{1/2} + 2 \sqrt{n \bar{H}} |\tau|^{1/2} + \mu_A([\sigma([x])]) |\tau|^{1/2}
\]

where \(\sigma : \mathbb{R}^+ \rightarrow \mathbb{R}^+\) is monotone increasing. Hence for suitable monotone increasing maps \(\mu_{D^n} : \mathbb{R}^+ \rightarrow \mathbb{R}^+\)

\[|| Z(\tau) D^n x - D^n x || \leq \mu_{D^n}([x]) |\tau|^{1/2}\]

which is the desired result. \(\square\)
V. 2. The representation proposition.

Define for \( n \in \mathbb{N}_0 \) a map

\[
E_n : \Lambda \times \mathbb{R} \to \Lambda
\]

by

\[
E_n(x, s) = x \ast_n s - \delta \pi(x \ast_n s).
\]

**Proposition 3.** For all \( n \in \mathbb{N}_0 \) we have the following representation for \( E_n \)

\[
E_n(x, s) = \mu_n(x, s, 0)E_n(x, 0) + K_n(x, s)
\]

where

\[
K_n(x, s) = \int_0^s \mu_n(x, s, h) \beta\left( \| \Psi'_n(x \ast_n h) \|_{L^1} \right) d\theta(x \ast_n h).\]

Moreover there exists a monotone increasing map \( v_n : \mathbb{R}^+ \to \mathbb{R}^+ \) such that

\[
\| Z(\tau)K_n(x, s) - K_n(x, s) \| \leq v_n(|x| + |s|) |\tau|^{\frac{1}{2}}
\]

for all \( x \in \Lambda \), \( s \in \mathbb{R} \), and \( \tau \in [-1, 1] \).

**Proof.** We calculate

\[
\nabla_{d_{\pi(x \ast_n s)}} (E_n(x, s)) = \hat{K}G_n(x \ast_n s) - \hat{K}(T\delta)(T\pi)G_n(x \ast_n s)
\]

\[
= -\beta\left( \| \Psi'_n(x \ast_n s) \|_{L^1} \right)(\delta \pi(x \ast_n s) - F_n\hat{K}HF_n(x \ast_n s))
\]

\[
+ \beta\left( \| \Psi'_n(x \ast_n s) \|_{L^1} \right)(\delta(T\pi)\Psi'_0(x \ast_n s) - \delta(T\pi)\xi'_0(x \ast_n s))
\]

\[
= \beta\left( \| \Psi'_n(x \ast_n s) \|_{L^1} \right)(\delta \Sigma(x \ast_n s) - \delta \pi(x \ast_n s))
\]

\[
= \beta\left( \| \Psi'_n(x \ast_n s) \|_{L^1} \right)(F_n\hat{K}HF_n(x \ast_n s) - \delta(T\pi)\xi'_0(x \ast_n s)).
\]

Now using IV. 1 (12) and IV. 2 (2) we infer

\[
(\delta(T\pi)\xi'_0(x), z) = -(T\pi)\xi'_0(x, Az) + (T\pi)HF_n(x, Az) + (\hat{K}HF_n(x, (D_{Az}F_n)x)
\]

\[
= -(A^*(T\pi)\xi'_0(x, z)) + (B^n x, z).
\]

Hence

\[
F_n\hat{K}HF_n(x \ast_n s) - \delta(T\pi)\xi'_0(x \ast_n s) = F_n\hat{K}HF_n(x \ast_n s) + A^*(T\pi)\xi'_0(x \ast_n s)
\]

\[
- B^n(x \ast_n s) = D^n(x \ast_n s) + A^* \Sigma(x \ast_n s).
\]

Moreover

\[
(\delta \Sigma, x, y) = (x, dy).
\]

This implies

\[
\delta \Sigma + A^* \Sigma = \text{Id}.
\]
Combining (5), (6) and (7) gives

\[ \frac{d}{ds}(\pi(x_n(s))) = \beta(\| \Psi'_n(x_n s) \|_L)(x_n s - \partial \pi(x_n s)) + \beta(\| \Psi'_n(x_n s) \|_L)D^n(x_n s) \]
\[ = \beta(\| \Psi'_n(x_n s) \|_L)E_n(x, s) + \beta(\| \Psi'_n(x_n s) \|_L)D^n(x_n s). \]

Now applying the variation of constant formula to (8) using the definition of \( \mu_n \) gives

\[ E_n(x, s) = \mu_n(x, s, 0)E_n(x, 0) + \int_0^s \mu_n(x, s, h)\beta(\| \Psi'_n(x_n h) \|_L)D^n(x_n h)dh \]
\[ = \mu_n(x, s, 0)E_n(x, 0) + K_n(x, s). \]

This proves (2). This representation together with the fact that \( K_n(x, s) \) is uniformly fibre compact as we shall prove, will be one of the key ingredients of the proof. Next we prove (4). We have

\[ \| Z(\tau)K(x, s) - K(x, s) \| \leq \left| \int_0^s \| Z(\tau)\mu_n(x, s, h)D^n(x_n h) - \mu_n(x, s, h)D^n(x_n h) \| dh \right|. \]

Observe that \( \| \mathcal{G}_n(x) \|_L \leq 2 \). Hence

\[ \left| \int_0^s \| Z(\tau)\mu_n(x, s, h)D^n(x_n h) - \mu_n(x, s, h)D^n(x_n h) \| dh \right| \leq 2. \]

Using (10) we can apply Proposition 1 and find for a suitable constant \( \tilde{M} \) only depending on \( M \)

\[ \| Z(\tau)\mu_n(x_n s) \| \leq \tilde{M}[\mathcal{G}_n(x_n s), s, h] + \| D^n(x_n h) \| \| \tau \|. \]

Now using Lemma 17 and the fact that \( Z(-1)x = 0 \) (in LA) we infer

\[ \| Z(\tau)\mu_n(x_n s) \| \leq \tilde{M} \left( \mathcal{G}_n(x_n s), s, h \right) + \| D^n(x_n h) \| \| \tau \|. \]

Now let us investigate the expressions \( \mathcal{G}_n(x_n s), s, h \) and \( D^n(x_n h) \). We have by (10)

\[ \left| \int_0^s \| d\pi(x_n s) \|_L \leq 2 \right|. \]

Moreover

\[ \| x_n s \|^2 = 2(x_n s, \hat{K}_n(x_n s)) \leq 4 \| x_n s \|. \]

Hence

\[ \| x_n s \| \leq \| x_n \| + 2 \| t - s \|. \]
for all $t, s \in \mathbb{R}$. Further

$$\frac{d}{ds} \| \partial \pi(x \ast_n s) \|^2 = 2(\partial \pi(x \ast_n s), \delta(T\pi)G'_n(x \ast_n s))$$

$$\leq 4 \| \partial \pi(x \ast_n s) \|.$$ 

Hence

$$\| \partial \pi(x \ast_n s) \| \leq \| \partial \pi(x \ast_n t) \| + 2 |t - s|$$

for all $t, s \in \mathbb{R}$. Now using the definition of $[x \ast_n h]$ we infer

$$[x \ast_n h] = \| x \ast_n h \|^2 + \| \partial \pi(x \ast_n h) \|^2$$

$$\leq (\| x \|^2 + 2|h|)^2 + (\| \partial \pi x \|^2 + 2|h|)^2$$

$$\leq 2|x| + 16|h|^2$$

$$\leq 2|x| + 16|s|^2$$

$$\leq 16(|x| + |s|^2).$$

Using the definition of $[\pi(x \ast_n (.)), s, h]$ we infer

$$[(x \ast_n (.)), s, h] \leq 2 \exp(3|s|)(1 + \| \partial \pi x \| + 2|s|).$$

Hence combining (15), (16) and (12) we find for a suitable increasing map $\tilde{v}_n : \mathbb{R}^+ \rightarrow \mathbb{R}^+$

$$\| Z(\tau)\mu_n(s, x, h)D^n(x \ast_n h) - \mu_n(s, x, h)D^n(x \ast_n h) \| \leq \tilde{v}_n(|x| + |s|) |\tau|^\frac{1}{2}.$$

Now we combine (9) and (17) and find

$$\| Z(\tau)K_n(x, s) - K_n(x, s) \| \leq \tilde{v}_n(|x| + |s|) |\tau|^\frac{1}{2}$$

$$\leq \tilde{v}_n(|x| + |s|)(|x| + |s|) |\tau|^\frac{1}{2}$$

$$= v_n(|x| + |s|) |\tau|^\frac{1}{2},$$

completing the proof.

VI. CONSTRUCTION OF INTERSECTION PAIRS AND COMPUTATION OF INDICES

In this chapter we shall construct the sets $S_1, S_2$ occurring in the abstract critical point theorem and compute the indices $i_n$ for relevant sets

VI.1. Intersection pairs.

Recall the definition of the bundles $\pi_M : B \rightarrow M$ and $\pi^0_M : B^0 \rightarrow M$ introduced in III.1 as well as the contents of Lemma 1:

$$\Phi : B^0_\rho \rightarrow U$$

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is a diffeomorphism onto an open neighbourhood $U$ of $\Lambda_0 = p(M)$, where $p : M \rightarrow \Lambda$ was defined by $m \mapsto m$.

We find a number $\varepsilon > 0$ such that the set

$$\Lambda_\varepsilon = \{ q \in \Lambda | \ d_\Lambda(q, \Lambda_0) \leq \varepsilon \}$$

is completely contained in $U$. We denote by $bd(\Lambda_\varepsilon)$ the boundary of $\Lambda_\varepsilon$:

$$bd(\Lambda_\varepsilon) = \{ q \in \Lambda | \ d_\Lambda(q, \Lambda_0) = \varepsilon \}.$$

Denote by $\Lambda \times \mathbb{R} \to \Lambda \times \mathbb{R}^+ : (x, s) \mapsto x \circ s$ the flow associated to the differential equation

$$x' = -\beta(\| \Psi'(x) \|_L)\Psi'(x).$$

Define subsets of $\Lambda \times \mathbb{R}^+$, $\mathbb{R}^+ = [0, +\infty)$, by

$$\Omega = \{ ((-\partial q) \circ t, t) | q \in \Lambda_\varepsilon, t \in \mathbb{R}^+ \} \quad \tilde{\Omega} = \{ ((-\partial q) \circ t, t) | q \in bd(\Lambda_\varepsilon), t \in \mathbb{R}^+ \}.$$

For fixed $t \in \mathbb{R}^+$ define

$$\Omega_t = \{ x | (x, t) \in \Omega \} \quad \tilde{\Omega}_t = \{ x | (x, t) \in \tilde{\Omega} \}.$$

Moreover define

$$S_1 = \{ \partial q | q \in \Lambda \}.$$

Note that

$$\Omega_t \cap S_1 = p(M) \quad \tilde{\Omega}_t \cap S_1 = \phi \quad \text{for all} \ t \in \mathbb{R}^+.$$

(6) follows immediately from the fact that the map $t \mapsto \Psi(x \circ t)$ is decreasing for all $x \in \Lambda \Lambda$ and that $\Psi(\tilde{\Omega}_t) \subset (-\infty, 0)$, $\Psi(S_1) \subset [0, +\infty)$. Moreover the set $\Lambda_0 = p(M)$ is the set of stationary points of the flow.

**Lemma 18.** Given any constant $c > 0$ there exists $t_0 > 0$ such that for all $t \geq t_0$ the following holds

$$\Psi(\tilde{\Omega}_t) \subset (-\infty, -c] \quad \Psi(S_1) \subset (0, +\infty].$$

**Proof.** Clearly the second statement in (7) is trivial. In order to prove the first statement concerning $\tilde{\Omega}$ we note that

$$\sup (\Psi(\tilde{\Omega}_0)) < 0.$$

In fact, arguing indirectly we find otherwise a sequence $(q_n) \subset bd(\Lambda_\varepsilon)$ such that

$$-\| \partial q_n \|_L^2 = \Psi(-\partial q_n) \to 0.$$
Hence $\partial q_n$ is $(u f p c)$ and therefore $(q_n)$ is precompact. Eventually taking a subsequence we may assume $q_n \to q_0$ in $\Lambda$ and $\|\partial q_0\| = 0$. Hence $q_0 \in \Lambda_0 = p(M)$ giving the contradiction
$$\varepsilon = d_\Lambda(q_n, \Lambda_0) \to d_\Lambda(q_0, \Lambda_0) = 0.$$ Applying Proposition 2 to $\Psi$ (consider $\Psi = \Psi_\infty$ with $h = 0$) the following compactness property holds (Palais-Smale-condition).
$$\|\Psi'(x_n)\|_L \to 0 \quad \text{and} \quad \Psi'(x_n) \to d$$ for some sequence $(x_n)$ imply that $(x_n)$ is precompact.

Since $\Psi$ has only the critical level 0 we find a constant $r > 0$ such that
$$\|\Psi'(x)\|_L \geq r \quad \text{for all} \quad x \in \Lambda \Lambda \quad \text{such that} \quad \Psi(x) \in [-c, \sup \Psi(\Omega_0)].$$ Hence there exists a $t_0 > 0$ such that (7) holds. \qed

Now let $h$ be the Hamiltonian with compact support from Chapter IV. Since $h$ has compact support we have for a suitable constant $c > 0$
$$|\gamma_n(x)| \leq \frac{1}{3} c \quad \text{for all} \quad x \in \Lambda \Lambda$$ and all $n \in \mathbb{N}$. This implies with the $t_0 > 0$ given in Lemma 18 corresponding to $c$
$$\inf \Psi_n(S_1) > \sup \Psi_n(\tilde{\Omega}_t)$$ for all $t \geq t_0$. For the following let
$$S_2 = \Omega_{t_0}, \tilde{S}_2 = \tilde{\Omega}_{t_0}.$$

In the following section we shall study the problem $(S_2 *_{n t}) \cap S_1$.

VI.2. Fixed points of fibre-preserving maps and index maps.

Fix $n \in \mathbb{N}$. We have to calculate
$$\text{cat}((S_2 *_{n t} s) \cap S_1)$$ for all $s \in \mathbb{R}^+$ and to show that $c(M)$ is a lower bound for the expression on (1). This will imply that $i_n(S) \geq c(M)$. Now consider the intersection problem
$$(S_2 *_{n t} s) \cap S_1 = \phi.$$ Using the fact that $s \to \Psi_n(x *_{n t} s)$ is decreasing and the estimate (11) in IV.1 we infer
$$(S_2 *_{n t} s) \cap S_1 = \phi \quad \forall s \in \mathbb{R}^+.$$ Using the maps $E_0$, $\mu_0$, $E_n$ and $\mu_n$ introduced in V.2, we define a conti-
nuous family of fibre-preserving map \((\Lambda_c \times \mathbb{R}^+ \text{ considered as bundle over } \Lambda_c)\).

\begin{equation}
\Xi_n : \Lambda_c \times \mathbb{R}^+ \rightarrow \Lambda
\end{equation}

by

\[
\Xi_n(q, t) = \begin{bmatrix}
\partial q - \frac{1}{2} \mu_0(-\partial q, 0, t)K_0(-\partial q, t) & \text{for } t \in [0, t_0] \\
\partial q - \frac{1}{2} \mu_0(-\partial q, 0, t_0)K_0(-\partial q, t_0) \\
- \frac{1}{2} \mu_0(-\partial q, 0, t_0)\mu_n(-\partial q \circ t_0, 0, t - t_0)K_n(-\partial q \circ t_0, t - t_0) & \text{for } t > t_0.
\end{bmatrix}
\]

Clearly \(\Xi_n\) is continuous by the properties of \(E_0, E_n, \mu_0, \mu_n, K_0\) and \(K_n\). Moreover

**Lemma 19.** — For all \(t \in \mathbb{R}^+\) we have

\begin{equation}
\Xi_n(q, t) = 0 \quad \text{if} \quad q \in bd(\Lambda_c).
\end{equation}

Moreover the following statements are equivalent

(i) \(x \ast_n s \in S_1\) where \(x \in S_2\)

and

(ii) \(x = (-\partial q) \circ t_0\) for some \(q \in \Lambda_c\) and \(\Xi_n(q, t_0 + s) = 0\).

**Proof.** — We infer if \(x \in S_2\) and \(x \ast_n s \in S_1\) that

\[x = (-\partial q) \circ t_0\] for some \(q \in \Lambda_c\)

and

\begin{equation}
E_n((-\partial q) \circ t_0, s) = 0.
\end{equation}

(6) is equivalent to \(0 = \mu_n((-\partial q) \circ t_0, 0, s)E_n((-\partial q) \circ t_0, s)\).

Hence (6) is equivalent to

\[
0 = E_n((-\partial q) \circ t_0, 0) + \mu_n((-\partial q) \circ t_0, 0, s)\mu_n((-\partial q) \circ t_0, s).
\]

That is

\[
0 = E_0((-\partial q), t_0) + \mu_n((-\partial q) \circ t_0, 0, s)\mu_n((-\partial q) \circ t_0, s).
\]

Multiplying this with \(\mu_0(-\partial q, 0, t_0)\) gives an equivalent equation to which the representation result Proposition 3 can be applied. In order to get (7) we use the identity \(\mu_0(-\partial q, 0, t_0)^{-1} = \mu_0(-\partial q, t_0, 0)\)

\begin{equation}
0 = -2\partial q + \mu_0(-\partial q, 0, t_0)K_0(-\partial q, t_0) \\
+ \mu_0(-\partial q, 0 , t_0)\mu_n((-\partial q) \circ t_0, 0, s)K_n((-\partial q) \circ t_0, s).
\end{equation}

However (7) is equivalent to

\[ 0 = \Xi_n(q, t_0 + s) . \]

Hence we have proved the equivalence (i) \(\Rightarrow\) (ii). (5) follows from this equivalence together with (3). \(\square\)

For the following define \(\sigma_n : \Lambda_\varepsilon \times \mathbb{R}^+ \to \Lambda A\)

by

\[ \Xi_n(q, t) = \partial q + \sigma_n(q, t) . \]

**Lemma 20.** For all \(n \in \mathbb{N}\) there exists a monotone increasing map \(\kappa_n : \mathbb{R}^+ \to \mathbb{R}^+\) such that

\[ ||Z(\tau)\sigma_n(q, t) - \sigma_n(q, t)|| \leq \kappa_n(t)|\tau|^\delta\]

for all \(t \in \mathbb{R}^+, \tau \in [-1, 1]\) and \(q \in \Lambda_\varepsilon\).

**Proof.** We have for \(t \in [0, t_0]\)

\[ \sigma(q, t) = -\frac{1}{2} \mu_0 (-\partial q, 0, t) K_0 (-\partial q, t) . \]

Now observe that \(||\partial q|| \leq c\) for a suitable \(c\) for all \(q \in \Lambda_\varepsilon\) by the construction of \(\Lambda_\varepsilon\) (Recall \(\Lambda_\varepsilon \subset U = \Phi(B_s^0))\).

Now combining Proposition 3 (4) and Proposition 1 we conclude that (9) holds for all \(t \in [0, t_0]\) for a suitable \(\kappa_n\). Similarly we can estimate \(\sigma\) for \(t > t_0\). The details of the proof are shorter but similar to that of Proposition 3 and are left to the reader. \(\square\)

Recall the definition of the bundles \(\pi_M^0\), the sets \(\Lambda_\varepsilon\) and \(bd(\Lambda_\varepsilon)\) and the contents of Lemma 1. We define an open subset \(V\) of \(B_s^0\) by

\[ V = \{ x \in B_s^0 | \Phi(x) \in \Lambda_\varepsilon \setminus bd(\Lambda_\varepsilon) \} . \]

Moreover define

\[ bd(V) = \{ x \in B_s^0 | \Phi(x) \in bd(\Lambda_\varepsilon) \} . \]

Note that in fact \(bd(V)\) is the boundary of \(V\) in \(B_s^0\). We consider \(V, \overline{V} = V \cup bd(V)\) and \(bd(V)\) as bundles over \(M\). Define \(U = \Phi(B_s^0)\). Denote by \(\alpha_1\) the diffeomorphism defined by

\[ B_s^0 \oplus B^0 \to B_s^0 \oplus \Lambda A |_{\Lambda_0} : (x, y) \to (x, \delta y) . \]

Moreover define the map \(\alpha_2\) by

\[ B_s^0 \oplus \Lambda A |_{\Lambda_0} \to \Lambda A |_U : (x, z) \to \Phi_{s_M^{\infty}}(x, z) \]

where the \(\Phi_m\) are the maps already used in III. 1. Here we identify course again \(B\) and \(TA |_{\Lambda_0}\). Define \(\alpha : B_s^0 \oplus B^0 \to \Lambda A |_U\) to be the composition of \(\alpha_1\) and \(\alpha_2\)

\[ \alpha = \alpha_2 \alpha_1 . \]

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It turns out that \( \alpha \) is a diffeomorphism. In fact, it is enough to show that \( \alpha_2 \) is a diffeomorphism. Clearly \( \alpha_2 \) is smooth. The inverse is the smooth map \( \tilde{\alpha}_2 \) defined by
\[
\tilde{\alpha}_2(z) = (\Phi^{-1}(\pi(z)), \Phi_{\pi_M^0}(\Phi^{-1}(\pi(z))), \ldots)^{-1} z.
\]

Define a map \( \tilde{\Xi}_n \) by
\[
\tilde{\Xi}_n : (\nabla, bd(V)) \times \mathbb{R}^+ \to (B^0, \tilde{B}^0)
\]
where \( \tilde{B}^0 \) denotes the complement of the zero section in \( B^0 \), by
\[
\tilde{\Xi}_n(x, s) = pr_2 \circ \alpha^{-1} \circ \Xi_n(\Phi(x), s).
\]

Here \( pr_2 : B_0^0 \oplus B^0 \to B^0 \) denotes the projections onto the second factor. Clearly \( \tilde{\Xi}_n \) is continuous. Moreover it preserves for fixed \( s \in \mathbb{R}^+ \) the fibre over \( M \). In fact, if \( \pi_M^0(x) = m \) we find
\[
\pi(\Xi_n(\Phi(x), s)) = \Phi(x).
\]
Hence
\[
\pi_M^0 \circ \alpha^{-1} \circ \Xi_n(\Phi(x), s) = \pi_M^0 \circ \Phi^{-1} \circ \Phi(x) = \pi_M^0(x) = m.
\]

Further, in order to see that \( bd(V) \times \mathbb{R}^+ \) is mapped into \( \tilde{B}^0 \) we observe that \( \tilde{\Xi}_n(x, s) = 0 \) is equivalent to \( \Xi_n(\Phi(x), s) = 0 \). This implies \( \Phi(x) \notin bd(A_\epsilon) \). Hence \( x \notin bd(V) \). In the following we shall relate the cohomological properties of the zero set of \( \Xi_n(s + t_0) \) for some fixed \( s \in \mathbb{R}^+ \) to the cohomological properties of the set \( (S_2 \ast_\epsilon s) \cap S_1 \). We need

**Lemma 21.** — The map \( \gamma : LA|_{\Lambda_\epsilon} \to B^0 \) defined by
\[
\gamma(z) = pr_2 \circ \alpha^{-1}(z)
\]
maps \((ufp c)\) sets into precompact sets.

**Proof.** — Let \( D \subset LA|_{\Lambda_\epsilon} \) be \((ufp c)\) and \((z_n) \subset \gamma(D) \). We have to show that \((z_n)\) possesses a convergent subsequence. Let \((x_n) \subset D\) such that \( z_n = \gamma(x_n) \). Since \( \delta : B^0 \to LA|_{\Lambda_\epsilon} \) establishes a diffeomorphism it is enough to show that \((y_n)\), \( y_n = \delta z_n \), possesses a convergent subsequence. Let \( q_n = \pi(x_n) \) and \( m_n = \pi_M^0(z_n) \). Eventually taking a subsequence we may assume
\[
m_n \to m \quad \text{in} \quad M
\]
\[
q_n \to q \quad \text{in} \quad C, \quad \text{where} \quad q \in \Lambda.
\]

We have
\[
\alpha(\Phi^{-1}(\pi(x_n)), z_n) = x_n.
\]

Then
\[
\Phi_{m_n}(\Phi^{-1}(q_n), y_n) = x_n.
\]

Clearly \( \Phi^{-1}(q_n) \to \Phi^{-1}(q) \) uniformly for the metric \( d_{TM} \), i.e.
\[
d_{TM}(\Phi^{-1}(q_n)(t), \Phi^{-1}(q)(t)) \to 0
\]

uniformly in \( t \in [0, 1] \). This implies that \( \Phi^{-1}(q) \in (B^0)_m \). For \( n \in \mathbb{N} \) we define a smooth path \( a_n : [0, 1] \rightarrow \Lambda \) by

\[
a_n(s) = \Phi((1 - s)\Phi^{-1}(q_n)) = \Phi((1 - s)\Phi^{-1}(q_n), \Phi^{-1}(q_n)).
\]

We have \( a'_n(s) = -\Phi_{m*}(1 - s)\Phi^{-1}(q_n), \Phi^{-1}(q_n)) \). Since the parallel transport along a curve is only depending on the curve but not on its parameterisation we can apply Proposition 1 to find that the images \( x_n \) by the parallel transport along \( a_n \) of \( x_n \) build a \( (u fp c) \) sequence. Since \( (\pi(x_n)) \subset \Lambda_0 = M \) (we identify \( \Lambda_0 \) and \( M \), \( B \) and \( T\Lambda \mid_M \)) we infer by an easy application of Proposition 1 by the compactness of \( M \) that \( (x_n) \) is precompact. (If \( (y_n) \subset \Lambda \) is a sequence such that \( (\pi(y_n)) \) converges in \( \Lambda \) and \( (y_n) \) is \( (u fp c) \), then \( (y_n) \) is precompact; this is an easy exercise using Proposition 1). Hence eventually taking a subsequence we may assume

\[
\tilde{x}_n \rightarrow \tilde{x} \text{ in } \Lambda
\]

where \( \tilde{x} \in \Lambda \mid_M \). Let \( \tilde{m} = \pi(\tilde{x}) \).

We have

\[
\pi(\tilde{x}_n) = \Phi(0, \Phi^{-1}(q_n)) = \pi_0(\Phi^{-1}(q_n)) = m_n.
\]

Hence

\[
\tilde{m} = \lim \pi(\tilde{x}_n) = \lim m_n = m.
\]

We introduce local co-ordinates based at \( m \in M \) by (for \( n \) large enough)

\[
q_n = \exp_m(\xi_n) \quad \xi_n \in T_m\Lambda
\]

\[
x_n = \Phi_m(\xi_n, \eta_n) \quad \eta_n \in L_m\Lambda
\]

\[
\tilde{x}_n = \Phi_m(\tilde{n}_n, \tilde{\eta}_n) \quad \tilde{\eta}_n \in L_m\Lambda
\]

\[
m_n = \exp_m(b_n)
\]

\[
a_n(s) = \exp_m(\tilde{a}_n(s)).
\]

Denoting by \( c_n \) the solution of

\[
c_n(s) = -\Gamma_m(\tilde{a}_n(s))(\tilde{a}_n'(s), c_n(s))
\]

with

\[
c_n(0) = \eta_n
\]

we have

\[
c_n(1) = \tilde{\eta}_n.
\]

By the special form of the \( a_n \) it is clear that \( \tilde{a}_n(s) \) and \( \tilde{a}_n'(s) \) converge uniformly for \( s \in [0, 1] \) to \( \tilde{a}(s) \) and \( \tilde{a}'(s) \) respectively given by \( a(s) = \exp_m(\tilde{a}(s)) \) and

\[
a(s) = \Phi((1 - s)\Phi^{-1}(q)) \in (B^0)_m \). Since \( \tilde{\eta}_n \rightarrow \tilde{\eta} \) in \( L_m\Lambda \) where \( \tilde{x} = \Phi_m(0, \tilde{\eta}) \) we infer that \( \eta_n \) converges to some \( \eta \) in \( L_m\Lambda \) where \( \eta = c(0), \tilde{\eta} = c(1) \) and \( \tilde{c}'(s) = -\Gamma_m(a(s))(a'(s), c(s)) \). Now

\[
\Phi_m(\Phi^{-1}(q_n), y_n) = \Phi_m(\xi_n, \eta_n).
\]

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This gives
\[(\Phi^{-1}(q_n), y_n) = \Phi_{m_n}^{-1}\Phi_m(\xi_n, \eta_n).\]

Therefore
\[y_n = pr_2 \circ \Phi_{m_n}^{-1}\Phi_m(\xi_n, \eta_n).\]

Since \(\xi_n \to \xi\) uniformly where \(\xi\) represents \(q\) and \(\eta_n \to n\) in \(L_m\).

\[m_n \to m\] we infer \(y_n \to y\) in \(L^1|_M\). \(\square\)

**Lemma 22.** — Define \(\tilde{\sigma}_n : \bar{V} \times \mathbb{R}^+ \to \mathcal{B}^0\) by
\[
(12) \quad \tilde{\sigma}_n(x, t) = \tilde{\xi}_n(x, t) - x.
\]

Then \(\tilde{\sigma}_n\) maps sets of the form \(\bar{V} \times [0, s]\) into precompact sets.

**Proof.** — By the definition of \(\tilde{\xi}_n\) we have
\[
\tilde{\sigma}_n(x, s) = \tilde{\xi}_n(x, s) - x
\]
\[
= pr_2 \circ \alpha^{-1}((\partial\Phi(x) + \sigma_n(\Phi(x), s)) - x
\]
\[
= pr_2 \circ \alpha^{-1}(\Phi_{m_n}(\Phi, \delta x) + \sigma_n(\Phi(x), s)) - x
\]
\[
= pr_2 \circ \alpha^{-1} \circ \alpha(x, x) - x + pr_2 \circ \alpha^{-1} \circ \sigma_n(\Phi(x), s)
\]
\[
= pr_2 \circ \alpha^{-1} \circ \sigma_n(\Phi(x), s)
\]
\[
= \gamma \circ \sigma_n(\Phi(x), s).
\]

By Lemma 21 \(\sigma_n\) maps bounded sets in \(\bar{V} \times \mathbb{R}^+\) into \((ufpc)\) sets. By Lemma 22 \(\gamma\) maps \((ufpc)\) sets into precompact sets. \(\square\)

Denote by \(TM \oplus i TM \to M\) the complexification of the tangent bundle. For \(j \in \mathbb{Z}\) we define smooth maps
\[e_j : [0, 1] \to \mathbb{C}\]
by
\[
e_j(t) = \begin{cases} t - \frac{1}{2} & \text{if } j = 0 \\ e^{2\pi i j t} & \text{if } j \neq 0. \end{cases}
\]

It is well-known that the family \((e'_j)\) (' denotes the derivative) is a total subset of \(L^2(0, 1; \mathbb{C})\). Since all \(e_j\) have mean value 0 the span of the \(e_j\) will be a subspace of the \(H^1\)-functions with mean value zero. Since the derivative-map induces a linear isomorphism we infer that \((e_j)\) is a total subset in \(\left\{ x \in H^1([0, 1], \mathbb{C}) \big| \int_0^1 x = 0 \right\}\). For \(k \in \mathbb{N}\) we introduce smooth sub-bundles \(\pi_k^0 : \mathcal{B}^0_k \to M\) of \(\mathcal{B}^0 \to M\) by
\[\mathcal{B}^0_k = \left\{ x \in \mathcal{B}^0 \mid x = \sum_{j=-k}^k a_je_j, \text{ where } a_j \in TM \oplus i TM \text{ and } \bar{a}_j = a_{-j}\right. \]
\(\overline{\text{(a}_j\text{ the complex conjugate)}}\) \].
Clearly the maps in $B^0_k$ are real valued. A local trivialisation, say centred at $m \in M$, is given by

$$V_m \times (B^0_k)_m \to B^0_k \mid_{\exp(V_m)}$$

$$(a, x) \to \Phi_m(a, x),$$

where $V_m \subset T_mM$ is open such that the restriction of the exponential map gives a diffeomorphism onto $\exp(V_m)$. Note that

$$\Phi_m(a, \Sigma^k_{j=1} a_j e_j(t)) = \Sigma^k_{j=1} (D \exp_m(a) a_j)(e_j(t)).$$

Denote by $P_k : B^0 \to B^0_k$ the fibre-preserving map which projects on each fibre in $B^0$ orthogonally onto the corresponding fibre in $B^0_k$. Here, of course, « orthogonally » refers to the $T\Lambda$-inner product. (Recall that we identified $B^0 \subset B \cong \Lambda|_{\Theta}$. It is easy to see that $P_k$ is smooth and that

$$P_k x \to x$$

in $T\Lambda$ as $k \to +\infty$.

**Lemma 23.** Given any $s_0 \in \mathbb{R}^+$ and an open neighbourhood $W$ of the set

$$\{ x \in \overline{V} \mid \tilde{E}_n(x, s_0) = 0 \}$$

(which is allowed to be empty) there exists a $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ the following holds

(12) $$\{ x \in B^0_k \cap \overline{V} \mid P_k \tilde{E}(x, s_0) = 0 \} \subset W$$

and

(13) $$\{ x \in B^0_k \cap \overline{V} \mid \text{There exists } s \in [0, s_0] \text{ with } P_k \tilde{E}_n(x, s) = 0 \}$$

is contained in $V_k = V \cap B^0_k$.

**Proof.** We carry out the proof of (13). The proof of (12) is similar and left to the reader. Arguing indirectly we find sequences $k_j \to +\infty$, $x_j \in bd(V) \cap B^0_{k_j}$ and $s_j \in [0, s_0]$ such that

$$x_j + P_{k_j} \tilde{E}_n(x_j, s_j) = 0.$$

By Lemma 21 the sequence $(x_j)$ must be precompact. Hence we may assume eventually taking a subsequence

$$s_j \to s, \quad x_j \to x$$

where $s \in [0, s_0]$ and $x \in bd(V)$, and moreover

$$\tilde{E}_n(x, s) = 0.$$

However, we know by Lemma 19 that this is impossible. This proves (13).

**Lemma 24.** Given any open neighbourhood $X$ of $\{ x \in V \mid \tilde{E}_n(x, t_0) = 0 \}$ in $V$ the projection map $\pi_x := \pi^0_M | X : X \to M$ induces an injective map

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\[ \pi^*_k : H(M) \to H(X) \] in cohomology. \( t_0 \) is the number introduced in Lemma 18.

**Proof.** We use an approach similar to that used by Dold in the definition of the fixed point transfer, [8]. We find \( k \in \mathbb{N} \) such that

\[ F(t) := \{ x \in V \cap B_0 \mid \tilde{P}_k \tilde{\omega}_m(x, t) = 0 \} \] (it could be of course empty)

is compact for all \( t \in [0, t_0] \) and \( F(t_0) \subset X \). (This can be done by Lemma 22). We have the following commutative diagram \( (X_k = X \cap B_0) \).

\[ \begin{array}{ccc}
X & \xrightarrow{j} & X_k \\
\downarrow{\pi_X} & & \downarrow{\pi_k} \\
M & & M
\end{array} \]

where \( j \) is the inclusion and \( \pi_X \) and \( \pi_k \) denote the projections induced from \( \pi^*_M \). If we can show that the map \( \pi^*_k \) induced in cohomology is injective, it follows from \( \pi^*_k = j^* \pi^*_k \), that \( \pi^*_k \) is injective. Hence we have reduced the infinite-dimensional problem to a finite-dimensional problem. The finite-dimensional vector-bundle \( B_0 \to M \) is an ENRM in the sense of Dold [8]. That is there are fibre-preserving maps over \( M \):

\[ \begin{array}{ccc}
B_0 & \xrightarrow{i} & B_0 \\
\downarrow{\tau} & & \downarrow{\tau} \\
M & & M
\end{array} \]

such that \( ri = \text{id} \) and \( \mathcal{O} \) is an open subset of \( \mathbb{R}^m \times M \) for some \( m \in \mathbb{N} \). Here, of course, \( \mathbb{R}^m \times M \) is considered as the trivial bundle over \( M \) : \( (x, m) \to m \). Moreover \( \tau \) is the induced projection. Define

\[ \tilde{V}_k = r^{-1}(V_k), \]

where \( V_k = V \cap B_0 \) and maps \( \tilde{f} \) by

\[ \tilde{f} : \tilde{V}_k \to \mathbb{R}^m \times M : \tilde{f}(a) = i(-P_k \tilde{\omega}_n(r(a), t)). \]

Moreover let

\[ \tilde{F}(t) = \{ a \in \tilde{V}_k \mid \tilde{f}(a) = a \}. \]

If \( a \in \tilde{F}(t) \) we infer

\[ a = i(-P_k \tilde{\omega}_n(r(a), t)). \]

Hence

\[ r(a) = ri(-P_k \tilde{\omega}_n(r(a), t)) = -P_k \tilde{\omega}_n(r(a), t) \]

or equivalently

\[ P_k \tilde{\omega}_n(r(a), t) = 0. \]

Therefore

\[ r(\tilde{F}(t)) \subset F(t). \]
If \( x \in F(t) \) we have similarly \( x = -P_k \tilde{\sigma}_n(x, t) \) which implies
\[
i(x) = i(-P_k \tilde{\sigma}_n(x, t)) = \tilde{f}_i(i(x)).
\]

Therefore
\[
i(F(t)) = \tilde{F}(t)
\]
and in fact \( i \) and \( r \) induce homeomorphisms
\[
F(t) \rightarrow \tilde{F}(t), \quad \tilde{F}(t) \rightarrow F(t)
\]
which are inverse to each other. This implies the compactness of \( \tilde{F}(t) \)
for \( t \in [0, t_0] \) and that \( \tilde{X}_k = r^{-1}(X_k) \) is a neighbourhood of \( \tilde{F}(t_0) \). We have
the commutative diagram
\[
\begin{array}{ccc}
\tilde{X}_k & \rightarrow & X_k \\
\tau | \tilde{X}_k & \downarrow & \pi_k \\
& & M
\end{array}
\]

If we can show that \((\tau | \tilde{X}_k)^* \) is injective we infer from \((\tau | \tilde{X}_k)^* = r^* \pi_k^* \)
that \( \pi_k^* \) is injective. So, in order to complete the proof we show that \((\tau | \tilde{X}_k)^* \)
is injective. To simplify notation we shall write \( \tau = \tau | \tilde{X}_k \). Define
\[
\tilde{F} = \cup_{t \in [0, t_0]} \tilde{F}(t).
\]

Consider the following diagram. (We denote all inclusion maps by the
letter \( j \), \( pr_1: \mathbb{R}^m \times M \rightarrow \mathbb{R}^m \) denotes the projection onto the first factor.
Moreover maps denoted by « exc » are inclusion maps inducing in coho-
mology the natural excision isomorphism.)
\[
\begin{array}{ccc}
(R^m, R^m \setminus \{0\}) \times \tilde{X}_k & \xrightarrow{(pr_1(j - \tilde{f}_0), id)} & (\tilde{X}_k, \tilde{X}_k \setminus \tilde{F}(t_0)) \\
\downarrow{id \times \tau} & & \downarrow{\text{exc}} \\
(R^m, R^m \setminus \{0\}) \times M & \xleftarrow{(pr_1(j - \tilde{f}_0), \tau)} & (\tilde{V}_k, \tilde{V}_k \setminus \tilde{F}(t_0)).
\end{array}
\]

By the homotopy-invariance of the cohomology the bottom map induces
in cohomology
\[
d^* = (pr_1(j - \tilde{f}_0), \tau)^*.
\]

Hence we obtain the diagram
\[
\begin{array}{ccc}
H((R^m, R^m \setminus \{0\}) \times \tilde{X}_k) & \xrightarrow{(pr_1(j - \tilde{f}_0), id)^*} & H(\tilde{X}_k, \tilde{X}_k \setminus \tilde{F}(t_0)) \\
\downarrow{(id \times \tau)^*} & & \downarrow{j^*} \\
H((R^m, R^m \setminus \{0\}) \times M) & \xrightarrow{d^*} & H(\tilde{V}_k, \tilde{V}_k \setminus \tilde{F}).
\end{array}
\]

If we can show the bottom map is injective the map \((id \times \tau)^* \) must be
injective.
Clearly $d : (\tilde{V}_k, \tilde{V}_k \setminus \tilde{F}) \to (\mathbb{R}^m, \mathbb{R}^m \setminus \{0\}) \times M$ has the form
\[
d(a, m) = (a - pr_1 \circ i(0_{r(a,m)}), m)
\]
where $0_x$ denotes the zero element in $(B^0_x)$. Denote by $W$ a compact neighbourhood of $\tilde{F}$ such that $W \subset \tilde{V}_k$. Then the inclusion map $(W, W \setminus \tilde{F}) \to (\tilde{V}_k, \tilde{V}_k \setminus \tilde{F})$ will induce an isomorphism in cohomology. For $B_\rho$ being the open ball of radius $\rho$ around 0 in $\mathbb{R}^m$ we find for $\rho > 0$ large enough a continuous extension $e: \mathbb{R}^m \times M \to B_\rho$ of the map $W \to B_\rho : (x, m) \to pr_1 \circ i(0_{r(x,m)})$ (of course using Tietze’s Theorem).

Define a map $d_1 :$
\[
(\mathbb{R}^m \times M, (\mathbb{R}^m \setminus B_\rho) \times M) \to (\mathbb{R}^m \times M, (\mathbb{R}^m \setminus \{0\}) \times M)
\]
by $d_1(x, m) = (x - e(x, m), m)$. Note that $d_1$ is homotopic to the inclusion map. In order to show that $d^*$ is injective, consider the following diagrams where we assume that $B_\rho$ defined above satisfies in addition $B_\rho \times M \supset \tilde{F}$.

Note that in cohomology $d_1^* = j^*$ where $j :$ \[
(\mathbb{R}^m \times M, \mathbb{R}^m \setminus B_\rho) \times M) \to (\mathbb{R}^m \times M, (\mathbb{R}^m \setminus \{0\}) \times M)
\]
denotes the inclusion map. By the homotopy invariance and the long exact sequence $j^*$ will be an isomorphism. Hence we obtain for some homomorphism $\alpha$ the following diagram in cohomology.

\[
\begin{array}{ccc}
H((\mathbb{R}^m, \mathbb{R}^m \setminus \{0\}) \times M) & \xrightarrow{id} & H((\tilde{V}_k, \tilde{V}_k \setminus \tilde{F})) \\
\downarrow{id} & & \downarrow{id} \\
H((\mathbb{R}^m, \mathbb{R}^m \setminus \{0\}) \times M) & \xrightarrow{\alpha} & H(W, W \setminus \tilde{F}).
\end{array}
\]

Hence $(d | W)^*$ is injective which implies the injectivity of $d^*$. Summing up we have shown that the map
\[
H((\mathbb{R}^m, \mathbb{R}^m \setminus \{0\}) \times M) \xrightarrow{(id \times \iota)^*} H((\mathbb{R}^m, \mathbb{R}^m \setminus \{0\}) \times \tilde{X}_k)
\]
is injective. Denote by \( \sigma : H^i(\Gamma) \to H^{i+1}(\mathbb{R}, \mathbb{R}\setminus\{0\} \times \Gamma) \) the natural suspension isomorphism. Clearly the map \((id \times \tau)^*\) is the map induced under \(m\)-fold suspension. Therefore we finally obtain

\[
\begin{array}{ccc}
H^{i+m}(\mathbb{R}^m, \mathbb{R}^m\setminus\{0\}) \times M & \xrightarrow{(id \times \tau)^*} & H^{i+m}(\mathbb{R}^m, \mathbb{R}^m\setminus\{0\}) \times \tilde{X}_k \\
\xRightarrow{\sigma^m} & & \xRightarrow{\sigma^m} \\
H^i(M) & \xrightarrow{\tau^*} & H^i(\tilde{X}_k)
\end{array}
\]

Hence the map \(\tau^*\) is injective. That completes the proof of the Lemma.

\[
\square
\]

**Proposition 4.** — For all \(n \in \mathbb{N}\) we have

\[
i_n(S_1) \geq c(M).
\]

**Proof.** — We prove (14) by arguing indirectly. Assume \(i_n(S_2) < c(M)\).

We find \(s_0 \in \mathbb{R}^*\) such that

\[
\text{cat } ((S_2 \ast_n s_0) \cap S_1) = i_n(S_2).
\]

Define

\[
D = (S_2 \ast_n s_0) \cap S_1.
\]

By Lemma 19 we have with

\[
E = \{ x \in \overline{V} \mid \Xi_n(x, t_0 + s_0) = 0 \}
\]

the following relation between \(D\) and \(E\)

\[
D = ((-\partial \Phi(E) \circ t_0) \ast_n s_0).
\]

By Lemma 21 the set \(E\) is compact. Hence \(D\) is compact. Clearly

\[
(D \ast_n (-s_0)) \circ (-t_0) = -\partial \Phi(E)
\]

and

\[
\text{cat } (\partial \Phi(E)) = \text{cat } (-\partial \Phi(E)) = \text{cat } (D) < c(M).
\]

We find an open neighbourhood \(W \subset LA\) of \(\partial \Phi(E)\) such that

\[
\text{cat } (cl(W)) = \text{cat } (\partial \Phi(E))
\]

\[
\pi(cl(W)) \subset \Phi(V).
\]

We shall show that the inclusion map

\[
i : W \to LA
\]

cannot induce an injective map \(i^* : H(LA) \to H(W)\) in cohomology. Assume, arguing indirectly, it does. We find \(U_1, \ldots, U_k\) open in \(LA\) such that

\[
\bigcup_{i=1}^k U_i = W
\]

\(U_i\) is contractible in \(LA\).
From the long exact sequence for the pair $(\Lambda, U_i)$ we infer in dimension $j > 0$ that the induced map

$$H^j(\Lambda, U_i) \to H^j(\Lambda)$$

is bijective. Hence any cohomology class in $H^j(\Lambda)$, $j \geq 1$, can be represented by some cohomology class in $H^j(\Lambda, U_i)$. Consider the maps

$$M \to \Lambda \to M$$

$$m \to 0_m, \quad x \to \pi(x)(0)$$

which constitutes a factorisation of the identity map of $M$ through $\Lambda$. Hence the second map induces an injective map in cohomology

$$H(M) \to H(\Lambda).$$

Therefore

$$c(\Lambda) \geq c(M).$$

Let $v_i \in H^{n(i)}(\Lambda); \quad n(i) \geq 1; \quad i = 1, \ldots, c(M) - 1$, such that

$$v := v_1 \cup \ldots \cup \mu_{c(M) - 1} \neq 0.$$ 

Take representatives $\mu_1, \ldots, \mu_{c(M) - 1}$ such that $\mu_i \in H^{n(i)}(\Lambda, U_i)$. Then

$$\mu := \mu_1 \cup \ldots \cup \mu_{c(M) - 1} \neq 0.$$ 

On the other hand by standard properties of the cup-product we have

$$\mu \in H^{2n(i)}(\Lambda, W).$$

Since the inclusion $W \to \Lambda$ induces an injective map in cohomology we infer from the long exact sequence for the pair $(\Lambda, W)$ that the map $H(\Lambda, W) \to H(\Lambda)$ is the zero map. This implies, since $\mu \to v$ that $v = 0$ which establishes a contradiction. Up to now we have shown that $i_*(S_2) < c(M)$ leads to the conclusion that there exists a certain neighbourhood $W$ of $\partial \Phi(E)$ such that the inclusion map $W \to \Lambda$ cannot induce an injective map in cohomology. Define a map $\chi$ by

$$\chi : \Phi^{-1}(\pi(W)) \to M : \chi(x) = \pi_M^W(x).$$

Consider moreover the factorisation of the map

$$\tilde{\chi} : \Phi^{-1}(\pi(W)) \to M : \tilde{\chi}(x) = \Phi(x)(0)$$

by ($i$ denotes the inclusion, $\beta(x) = \partial \Phi(x)$).

$$\Phi^{-1}(\pi(W)) \xrightarrow{\beta} W \xrightarrow{i} \Lambda \xrightarrow{\phi} M$$

where $\beta(q) = q(0)$. Let us show that $\chi$ and $\tilde{\chi}$ are homotopic if $\varepsilon > 0$ is small enough (which we may assume, recall $\tilde{V} = \Phi^{-1}(\Lambda_\varepsilon)$). Clearly if $\varepsilon > 0$ is small enough we have

$$d_M(\chi(x), \tilde{\chi}(x)) < \delta_\varepsilon$$

for all $x \in \Phi^{-1}(\pi(W))$ where $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$. We can embed $\hat{t} : M \to \mathbb{R}^m$.
for some \( m \in \mathbb{N} \) and take a tubular neighbourhood of \( M \) in \( \mathbb{R}^m \). Let \( \gamma \) denote the projection of the tubular neighbourhood onto \( M \). Then the map \( [0, 1] \times \Phi^{-1}(\pi(W)) \to M \) defined by \( (t, x) \to \gamma(t\tilde{h}(\tilde{x}(x)) + (1 - t)\tilde{h}(\tilde{x}(x))) \) gives a homotopy. This implies for the induced maps in cohomology

\[
\chi^* = \tilde{\chi}^* = \beta^* i^* \pi^* \alpha^* .
\]

If we can show that \( \pi^* \alpha^* \) is an isomorphism we find since \( i^* \) is not injective that \( \chi^* \) is not injective. Consider \( \alpha : \Lambda \to M \) and define

\[
\tilde{\alpha} : [0, 1] \times \Lambda \to \Lambda : \tilde{\alpha}(s, q)(t) = q(st) .
\]

Then \( \tilde{\alpha} \) is continuous and a deformation retraction onto \( \Lambda_0 \), which is homeomorphic to \( M \). Therefore \( \alpha^* \) is an isomorphism. Next consider \( \tau : \Lambda \to M \) and define

\[
\tilde{\tau} : [0, 1] \times \Lambda \to \Lambda : \tilde{\tau}(s, x) = sx .
\]

Denote by \( \Sigma_0 \) the zero section of \( \Lambda \). Hence the inclusion map of \( \Sigma_0 \to \Lambda \) induces an isomorphism in cohomology. Since \( \Sigma_0 \) and \( \Lambda \) are naturally isomorphic we find that \( \pi^* \) is an isomorphism.

Therefore

\[
\chi : \Phi^{-1}(\pi(W)) \to M
\]
does not induce an injective map in cohomology. This, however, contradicts Lemma 24 and Proposition 4 is proved.

\[
\square
\]

\section{VII. PROOF OF THEOREM 2}

We give now the proof of Theorem 2. Clearly it will essentially consist of collecting the results we have already proved.

In IV we have constructed a \( \mathbb{N} \)-family \( (\Psi_n) \) for \( \Psi_\infty \in C^1(\Lambda, \mathbb{R}) \), where

\[
\Psi_\infty(x) = (\partial \pi x, x) - \alpha_\infty(x) .
\]

In VI. 1, 2 we have constructed the sets

\[
S_1 = \{ \partial q \mid q \in \Lambda \}
\]

and

\[
S_2 = \{ (-\partial q) \circ t_0 \mid q \in \Lambda \}
\]

and proved that

\[
i_{\delta}(S_2) \geq c(M)
\]

for all \( n \in \mathbb{N} \). By Theorem 3 the functional \( \Psi_\infty \) has at least \( c(M) \) critical points. If \( x \) is a critical point of \( \Psi_\infty \) we have \( (q = \pi x) \)

\[
\partial q = \tilde{\mathbf{K}}(x)
\]
and
\[(x, \delta y) = ((T\pi)H(x), y) \quad \text{for all} \quad y \in T_q\Lambda.\]

This implies \(x \in T_q\Lambda\). Integrating by parts yields
\[-\langle \delta x, y \rangle + \langle x(1), y(1) \rangle - \langle x(0), y(0) \rangle = ((T\pi)H(x), y)\]
for all \(y \in T_q\Lambda\). Therefore
\[x(0) = 0_{q(0)} \quad \text{and} \quad x(1) = 0_{q(1)}\]
and
\[-\delta x = (T\pi)H(x).\]

Denote by \(\sigma : TM \to T^*M\) the natural isomorphism for Riemannian manifolds
\[\sigma(a) = \langle a, . \rangle.\]

Then the pull back \(\tilde{\omega}\) of \(\omega\) via \(\sigma\) is given by
\[\tilde{\omega}(a, b) = \langle T\pi a, Kb \rangle - \langle Ka, T\pi b \rangle\]
(see [16], 3.1.3). Hence we have for all \(y \in T_q\Lambda\)
\[\int_0^1 \omega(x, y)dt = \int_0^1 dh^*_\sigma(\sigma(x))(T\sigma)ydt\]
(Here we have used the definition of \(h\)). Define \(\gamma : [0, 1] \to T^*M\) by \(\gamma(t) = \sigma(x(t))\). We infer
\[\int_0^1 \omega(\dot{\gamma}, (T\sigma)y)dt = \int_0^1 dh^*_\sigma(\gamma)(T\sigma)ydt.\]

That is
\[\dot{\gamma} = X_\gamma(\gamma).\]

Since \(x(0) = 0, x(1) = 0\) we infer \(\gamma(0), \gamma(1) \in \Sigma\). Therefore we have found \(c(M)\) different solutions for \(\dot{\gamma} = X_\gamma(\gamma), \gamma(0), \gamma(1) \in \Sigma.\)

\[\Box\]

ACKNOWLEDGEMENT

I thank Professor E. Zehnder, Bochum, for drawing my attention to the problem of Lagrangian embeddings and their intersections, and for some remarks concerning an earlier version of this paper. Moreover I would like to thank my former colleagues Professor J. Toland and Dr. F. Burstall at the University of Bath for helpful discussions.
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