ANNALES DE L'I. H. P., SECTION C

DENNIS M. DETURCK
NORIHITO KOISO

Uniqueness and non-existence of metrics with prescribed Ricci curvature

Annales de l'I. H. P., section C, tome 1, nº 5 (1984), p. 351-359
<http://www.numdam.org/item?id=AIHPC_1984__1_5_351_0>

Ann. Inst. Henri Poincaré,
Vol. 1, n° 5, 1984, p. 351-359. *Analyse non linéaire*

Uniqueness and non-existence of metrics with prescribed Ricci curvature

by

Dennis M. DeTURCK (*)
University of Pennsylvania

and

Norihito KOISO
Osaka University

ABSTRACT. — We investigate whether the Ricci tensor uniquely determines the Riemannian structure, and we give conditions that a doubly covariant tensor has to satisfy in order to be the Ricci tensor for some Riemannian structure.

RÉSUMÉ. — Nous cherchons si le tenseur de Ricci détermine la structure riemannienne, et nous donnons des conditions que doit satisfaire un tenseur deux fois covariant pour être le tenseur de Ricci d'une certaine structure riemannienne.

INTRODUCTION

Ever since Gauss proved his *Theorema Egregium*, geometers have tried to determine to what extent the geometry of a Riemannian manifold is determined by its curvature. In particular, since the Ricci tensor of a metric is the same size as the metric tensor, one expects reasonable exis-

(*) Partially supported by NSF Grant # 83-01095.

tence and uniqueness properties for the partial differential equation $\mathrm{Ric}(g) = \mathrm{R}$, where Ric is the differential operator that associates to a metric its (doubly covariant) Ricci tensor (see [D] for some basic notions and a satisfactory theory in two dimensions).

In this paper, we are concerned with the uniqueness aspect of the equation $\mathrm{Ric}(g) = \mathrm{R}$. Our main result shows that certain positive definite tensors on certain manifolds do indeed uniquely determine their metrics. Our results are extensions of a result of Hamilton [H], who showed that the standard metric on the sphere S^n is uniquely determined by its Ricci tensor. As a bonus, we find (on any manifold) conditions on R that guarantee that there is *no* Riemannian metric g with $\mathrm{Ric}(g) = \mathrm{R}$.

A few words about notation and conventions: We work on a *compact* manifold M of dimension at least 3. We use standard tensor analysis notation, and the summation convention usually applies.

The first author would like to thank Richard Hamilton, Eugenio Calabi and Wolfgang Ziller for useful discussions. The second author thanks Tadashi Nagano for informing him of Hamilton's work. Finally, both authors would like to thank Jerry Kazdan for bringing the work of each to the other's attention.

1. UNIQUENESS DEFINED

In order to discuss the extent to which the Ricci tensor can uniquely determine its metric, we must first decide what we mean by « uniquely determine ». Of course, it is well-known that $\mathrm{Ric}(g) = \mathrm{Ric}(cg)$ for any constant $c > 0$, so the most one can hope for is uniqueness up to homothety. It turns out, however, that even this is too much to ask, since one can scale the factor metrics of a product metric and preserve the Ricci tensor, i. e.,

$$\mathrm{Ric}(g_1 \oplus g_2) = \mathrm{Ric}(c_1 g_1 \oplus c_2 g_2)$$

for any $c_1, c_2 > 0$.

On the other hand, if two metrics g and $\bar{g}$ induce the same Levi-Civita connection, i. e., if $\Gamma^i_{jk} - \overline{\Gamma}^i_{jk}$ is identically zero, then the definition of the Ricci tensor as

$$\mathrm{Ric}(g)_{ij} = \frac{\partial \Gamma^\alpha_{ij}}{\partial x^\alpha} - \frac{\partial \Gamma^\alpha_{i\alpha}}{\partial x^j} + \Gamma^\alpha_{ij}\Gamma^\beta_{\alpha\beta} - \Gamma^\alpha_{i\beta}\Gamma^\beta_{j\alpha}$$

makes it manifestly clear that $\mathrm{Ric}(g) = \mathrm{Ric}(\bar{g})$. The circumstances under which two metrics induce the same Levi-Civita connection are well understood. The decomposition theorem of deRham [KN1] tells us that if (M, g) is a complete, simply connected Riemannian manifold, then (M, g) is isometric to the product

$$(\mathrm{M}_0, g_0) \times (\mathrm{M}_1, g_1) \times \ldots \times (\mathrm{M}_k, g_k)$$

where M_0 is a Euclidean space (possibly of dimension zero) and $M_1, \ldots, M_k$ are all simply connected, complete, irreducible Riemannian manifolds, and that this decomposition is unique up to ordering. Since the splitting is defined via holonomy, it is completely determined by the connection. Thus, if g and $\overline{g}$ induce the same connection, then

$$(M, g) \cong (M_0, g_0) \times (M_1, g_1) \times \ldots \times (M_k, g_k)$$

and

$$(M, \overline{g}) \cong (M_0, \overline{g}_0) \times (M_1, \overline{g}_1) \times \ldots \times (M_k, \overline{g}_k)$$

where g_i induces the same connection as $\overline{g}_i$ on M_i for $i = 0, \ldots, k$. But then $\overline{g}_i$ is parallel with respect to g_i, and irreducibility implies that $\overline{g}_i = c_i g_i$ for $i = 1, \ldots, k$ (and g_0 is isometric to $\overline{g}_0$ so we may and shall assume that $\overline{g}_0 = g_0$). So for simply connected M, we are back in the situation of product metrics described above.

If $\pi_1(M) \neq 0$, then we simply pull g and $\overline{g}$ back to the universal cover of M. This proves:

PROPOSITION 1.1. — *Let* M *be a complete manifold with respect to the metrics g and $\overline{g}$, and let $\phi : \tilde{M} \to M$ be the universal cover of* M. *If g and $\overline{g}$ induce the same Levi-Civita connection on* M, *then $\phi^*(g)$ and $\phi^*(\overline{g})$ can differ only by homotheties on the irreducible factors of $\tilde{M}$.*

The upshot of all this is that, without special assumptions on M, the strongest statement we can make is that the Ricci curvature uniquely determines the Levi-Civita connection of its metric — and it is precisely this statement that we shall prove for the Ricci curvatures we consider.

2. THE BASIC ESTIMATE

All of the results in this paper are based on the following lemma, which appears in [H]. Since the proof is so short, we include it for completeness, and in order to fix some notation.

To begin, we assume that the manifold M is equipped with two Riemannian metrics g and $\overline{g}$, and we continue to indicate to which metric various curvature tensors, Christoffel symbols, Laplacians, etc. belong by the presence or absence of a bar.

LEMMA 2.1. — *Set* $T^i_{jk} = \overline{\Gamma}^i_{jk} - \Gamma^i_{jk}$. *If* $g^{jk}T^i_{jk} = 0$, *then*

$$\frac{1}{2}\Delta(g^{ik}\overline{g}_{ik}) = g^{ik}g^{jl}(\overline{g}_{ij}R_{kl} - \overline{R}_{ijkl}) + g^{ik}g^{jl}\overline{g}_{pq}T^p_{jk}T^q_{il}\,.$$

REMARK 2.2. — The condition $g^{jk}T^i_{jk} = 0$ is geometrically natural in the sense that this equation expresses the fact that the identity mapping from (M, g) to $(M, \bar{g})$ is a harmonic mapping, and $g^{ik}\bar{g}_{ik}$ is precisely the harmonic mapping energy density [EL].

Proof of Lemma 2.1. — First, recall that if ∇ is the covariant derivative of g's Levi-Civita connection, then

$$\text{(2.3)} \qquad T^p_{jk} = \frac{1}{2}\bar{g}^{pq}(\nabla_j\bar{g}_{kq} + \nabla_k\bar{g}_{jq} - \nabla_q\bar{g}_{jk}),$$

$$\text{(2.4)} \qquad \nabla_l\bar{g}_{ip} = \bar{g}_{pq}T^q_{il} + \bar{g}_{iq}T^q_{pl}$$

and

$$\text{(2.5)} \qquad R^p_{jkl} - \bar{R}^p_{jkl} = \nabla_l T^p_{jk} - \nabla_k T^p_{jl} + T^p_{lq}T^q_{jk} - T^p_{kq}T^q_{jl}.$$

We claim that the formula of the lemma is the contraction of (2.5) with $g^{ik}g^{jl}\bar{g}_{ip}$. Upon this contraction, the left side of (2.5) is

$$\begin{aligned} g^{ik}g^{jl}\bar{g}_{ip}(R^p_{jkl} - \bar{R}^p_{jkl}) &= g^{ik}\bar{g}_{ip}R^p_k - g^{ik}g^{jl}\bar{R}_{ijkl} \\ &= g^{ik}g^{jl}(\bar{g}_{ij}R_{kl} - \bar{R}_{ijkl}) \end{aligned}$$

as claimed, while on the right, the second and fourth terms cancel because $g^{jl}T^p_{jl}$ is zero. The first term on the right side of (2.5) becomes (using (2.3))

$$\begin{aligned} g^{ik}g^{jl}\bar{g}_{ip}\nabla_l T^p_{jk} &= \frac{1}{2}g^{ik}g^{jl}\bar{g}_{ip}\nabla_l[\bar{g}^{pq}(\nabla_j\bar{g}_{kq} + \nabla_k\bar{g}_{jq} - \nabla_q\bar{g}_{jk})] \\ &= \frac{1}{2}\Delta(g^{ik}\bar{g}_{ik}) - g^{ik}g^{jl}(\nabla_l\bar{g}_{ip})T^p_{jk}. \end{aligned}$$

Now we use (2.4) and add in the third term; the right side of the contraction of (2.5) is:

$$\begin{aligned} \frac{1}{2}\Delta(g^{ik}\bar{g}_{ik}) - g^{ik}g^{jl}(\bar{g}_{pq}T^q_{il} + \bar{g}_{iq}T^q_{pl})T^p_{jk} &+ g^{ik}g^{jl}\bar{g}_{ip}T^p_{lq}T^q_{jk} \\ &= \frac{1}{2}\Delta(g^{ik}\bar{g}_{ik}) - g^{ik}g^{jl}T^q_{il}T^p_{jk}. \end{aligned}$$

All of the terms in the equality of the lemma have now been accounted for.

q. e. d.

We can use the lemma in connection with Ricci curvature in the following way: If g is a metric, and $\mathrm{Ric}(g) = \bar{g}$ is positive definite, then we can consider $\mathrm{Ric}(g)$ to be a Riemannian metric in its own right. From (2.3), one immediately recognizes that the condition $g^{jk}T^i_{jk} = 0$ is precisely the Bianchi identity for Ricci curvature (multiplied by the inverse of the Ricci tensor). In fact,

$$\bar{g}_{ip}g^{jk}T^p_{jk} = g^{jk}\left(\nabla_j\bar{g}_{ik} - \frac{1}{2}\nabla_i\bar{g}_{jk}\right) = 0.$$

Thus, *if* $\mathrm{Ric}(g)$ *is positive definite, then the identity mapping*

$$\mathrm{id} : (\mathrm{M}, g) \to (\mathrm{M}, \mathrm{Ric}(g))$$

is a harmonic mapping. We can restate Lemma 2.1 as follows:

COROLLARY 2.6. — *If* $\mathrm{Ric}(g) = \overline{g}$ *is positive definite, then*

$$\frac{1}{2}\Delta\,\mathrm{Scal}(g) = g^{ik}g^{jl}(\overline{g}_{ij}\overline{g}_{kl} - \overline{\mathrm{R}}_{ijkl}) + g^{ik}g^{jl}\overline{g}_{pq}\mathrm{T}^{p}_{jk}\mathrm{T}^{q}_{il}.$$

In particular, since the last term on the right is clearly nonnegative,

$$\frac{1}{2}\Delta\,\mathrm{Scal}(g) \geqq g^{ik}g^{jl}(\overline{g}_{ij}\overline{g}_{kl} - \overline{\mathrm{R}}_{ijkl})$$

with equality holding only if $\mathrm{T}^{i}_{jk} = 0$.
Here, Scal is the scalar curvature operator, and the last term on the right is simply a hybrid norm squared of T.

3. THE MAIN RESULTS

To state our main results on uniqueness and nonexistence, we need a geometric notion that will reflect the sign of the right-hand side of the inequality in Corollary 2.6. The appropriate notion turns out to be the eigenvalues of the curvature operator $\overset{\circ}{\mathrm{R}}$ (see [BK]).

DEFINITION 3.1. — Given a metric g, its sectional curvature tensor R_{ijkl} acts on the space $\mathrm{S}^2\mathrm{T}^*$ of symmetric tensors as follows: If $h_{ij} \in \mathrm{S}^2\mathrm{T}^*$, then *define*

$$(\overset{\circ}{\mathrm{R}}(h))_{ik} = \mathrm{R}_{ijkl}g^{jp}g^{lq}h_{pq} = \mathrm{R}_{i}{}^{p}{}_{k}{}^{q}h_{pq}.$$

For each point $x \in \mathrm{M}$, $\overset{\circ}{\mathrm{R}}$ is a symmetric linear map of $\mathrm{S}^2\mathrm{T}^*_x$ to itself, and as such has eigenvalues. Note also that the identity map of $\mathrm{S}^2\mathrm{T}^*$ is derived from the tensor $g_{ij}g_{kl}$ in the same way as $\overset{\circ}{\mathrm{R}}$ is from R_{ijkl}. We can now state the main theorem and its proof, after which we give some corollaries and examples that should clarify the situation.

THEOREM 3.2. — *Let* M *be a compact Riemannian manifold, let* $\overline{g}$ *be a metric (positive definite tensor) on* M, *and let* $\Lambda(x)$ *be the largest eigenvalue of the operator* $\overset{\circ}{\overline{\mathrm{R}}}$ *at each point* $x \in \mathrm{M}$. *Suppose* $\Lambda(x) \leqslant 1$ *for all* x *in* M.

a) If there is a Riemannian metric g *on* M *such that* $\mathrm{Ric}(g) = \overline{g}$, *then* $\overline{g}$ *is Einstein (so* $\mathrm{Ric}(\overline{g}) = \overline{g}$*), and* g *and* $\overline{g}$ *have the same Levi-Civita connection.*

b) If in addition to the above hypotheses, $\Lambda(x) < 1$ *at some point* $x \in \mathrm{M}$, *then there is* no *Riemannian metric* g *such that* $\mathrm{Ric}(g) = \overline{g}$ *on all of* M.

Proof. — Both conclusions follow from the inequality of Corollary 2.6 and the fact that on a compact manifold, the inequality $\Delta u \geqq 0$ implies that u is a constant (and so $\Delta u = 0$). In particular we have

$$\begin{aligned}\frac{1}{2}\Delta \operatorname{Scal}(g) &\geqq g^{ik}g^{jl}(\bar{g}_{ij}\bar{g}_{kl} - \overline{\mathrm{R}}_{ijkl}) = g^{ik}(\overline{\mathrm{Id}}(g^{-1}) - \mathring{\overline{\mathrm{R}}}(g^{-1}))_{ik} \\ &\geqq (1 - \Lambda(x))g^{ik}(\overline{\mathrm{Id}}(g^{-1}))_{ik} \\ &\geqq 0.\end{aligned}$$

Thus, if g exists, we must have $\Lambda(x) \equiv 1$, so $\Delta \operatorname{Scal}(g) = 0$. But then the equality of Corollary 2.6 shows that $\mathrm{T}^i_{jk} = 0$ so g and $\bar{g}$ have the same Levi-Civita connection. q. e. d.

Our first corollary shows that the conclusion (a) of the theorem is not vacuous.

COROLLARY 3.3 (*Uniqueness*). — *Let* $(\mathrm{M}, \bar{g})$ *be a compact Einstein manifold with* $\operatorname{Ric}(\bar{g}) = \bar{g}$ *and with non-negative sectional curvature. If* g *is another Riemannian metric on* M *with* $\operatorname{Ric}(g) = \bar{g}$, *then* g *has the same Levi-Civita connection as* $\bar{g}$ (*Here,* $n = \dim \mathrm{M}$).

COROLLARY 3.4 (*Non-existence*). — *Let* $(\mathrm{M}, \bar{g})$ *be a compact Einstein manifold with* $\operatorname{Ric}(\bar{g}) = \varepsilon\bar{g}$ *for some* $\varepsilon < 1$ *and with all sectional curvatures strictly greater than* $(\varepsilon - 1)/n$. *Then there is* no *Riemannian metric* g *such that* $\operatorname{Ric}(g) = \bar{g}$.

Proof. — We prove these corollaries simultaneously by showing that all the eigenvalues of $\mathring{\overline{\mathrm{R}}}$ are $\leqslant 1$ (and strictly less for Corollary 3.4). So, let $\bar{g}$ be Einstein with $\operatorname{Ric}(\bar{g}) = \varepsilon g$ ($\varepsilon = 1$ for Corollary 3.3). We see immediately that $\bar{g}$ is an eigenvector of $\mathring{\overline{\mathrm{R}}}$ with eigenvalue ε. To find other eigenvectors h of $\mathring{\overline{\mathrm{R}}}$, we can assume that they are $\bar{g}$-traceless. Choose coordinates around the point x so that at x, $\bar{g}_{ij} = \delta_{ij}$, and so that h is diagonal at x, i. e., $h_{ij} = \lambda_i\delta_{ij}$ $\left(\text{with } \sum_i \lambda_i = 0, \text{ so that the maximum } \lambda_i \text{ is positive}\right)$. Assume $\mathring{\overline{\mathrm{R}}}(h) = \alpha h$, and that $\lambda_{\mathrm{M}} \geqq \lambda_i$ for all i. Then

$$\alpha = \frac{(\mathring{\overline{\mathrm{R}}}(h))_{\mathrm{MM}}}{\lambda_{\mathrm{M}}} = \frac{\sum_j \lambda_j \overline{\mathrm{R}}_{\mathrm{M}j\mathrm{M}j}}{\lambda_{\mathrm{M}}} = \frac{\sum_j \lambda_j(\overline{\mathrm{R}}_{\mathrm{M}j\mathrm{M}j} - \min \overline{\mathrm{R}})}{\lambda_{\mathrm{M}}}$$

where $\min \overline{\mathrm{R}}$ is the minimum of the sectional curvatures at x. Note that

$$\overline{\mathrm{R}}_{\mathrm{M}j\mathrm{M}j} - \min \overline{\mathrm{R}} \geqq 0$$

since $\overline{R}_{MjMj}$ is itself the sectional curvature of the two-plane spanned by the orthonormal vectors $\frac{\partial}{\partial x^M}$ and $\frac{\partial}{\partial x^j}$. Thus

$$\alpha \leqslant \frac{\sum_j \lambda_M(\overline{R}_{MjMj} - \min \overline{R})}{\lambda_M} = \varepsilon - n(\min \overline{R}).$$

So all of the eigenvalues of $\overset{\circ}{\overline{R}}$ are bounded above by $\max \{ \varepsilon, \varepsilon - n(\min \overline{R}) \}$. For Corollary 3.3, $\varepsilon = 1$ and $\min \overline{R} \geqq 0$, so the eigenvalues are all $\leqslant 1$, and for Corollary 3.4, $\varepsilon < 1$ and $\min \overline{R} > \frac{\varepsilon - 1}{n}$, so the eigenvalues are < 1. Now apply the theorem. q. e. d.

REMARK 3.5. — It is clear that the hypotheses of Corollary 3.3 are satisfied by the Ricci tensors of the canonical metrics on the Riemannian symmetric spaces of compact type (see [KN2, p. 258]). In particular, the corollary is true for the compact rank-one symmetric spaces. Thus Corollary 3.3 is a generalization of the corresponding result of Hamilton for the spheres S^n.

Two more corollaries will clarify the situation for non-existence.

COROLLARY 3.6. — *Let* $(M, \overline{g})$ *be a compact Riemannian manifold with all sectional curvatures less than* $\frac{1}{n-1}$. *Then there is no Riemannian metric* g *such that* $\mathrm{Ric}(g) = \overline{g}$.

Proof. — As before, work at a point x where $\overline{g}_{ij} = \delta_{ij}$, and suppose that $\overset{\circ}{\overline{R}}(h) = \alpha h$, where h is a diagonal matrix $h_{ij} = \lambda_i \delta_{ij}$ at x. We compute the inner product of h with $\overset{\circ}{\overline{R}}(h) - \overline{\mathrm{Id}}(h)$ and show that it is negative (the inner product is taken with respect to $\overline{g}$). This will show that $\alpha < 1$.

$$\begin{aligned}\langle h, \overset{\circ}{\overline{R}}(h) - \overline{\mathrm{Id}}(h) \rangle &= \sum_{i \neq j} \lambda_i \lambda_j \overline{R}_{ijij} - \sum_i \lambda_i^2 \\ &< \frac{1}{n-1} \sum_{i \neq j} \lambda_i \lambda_j - \sum_i \lambda_i^2 \\ &= \frac{1}{n-1} \Big(\sum_{i,j} \lambda_i \lambda_j - n \sum_i \lambda_i^2 \Big) \\ &\leqslant 0 .\end{aligned}$$

q. e. d.

COROLLARY 3.7. — *For any metric* $\overline{g}$ *on the compact Riemannian mani-*

fold M, *there is a constant* $c_0(\bar{g})$ *such that for any real number* $c > c_0(\bar{g})$, *there is no Riemannian metric* g *for which* $\mathrm{Ric}(g) = c\bar{g}$. *In particular, for an Einstein metric* $\bar{g}$ *with* $\mathrm{Ric}(\bar{g}) = \bar{g}$ *and nonnegative sectional curvature we have* $c_0(\bar{g}) = 1$.

Proof. — The first assertion follows from the fact that the eigenvalues of $\overset{\circ}{\bar{R}}$ (and the sectional curvatures) are multiplied by c^{-1} when the metric $\bar{g}$ is multiplied by c. Then we invoke Theorem 3.2(*b*) (or Corollary 3.6). The assertion about Einstein metrics follows immediately from Corollary 3.4. q. e. d.

4. CONCLUDING REMARKS

We have shown that certain Einstein metrics are determined « uniquely » (in the sense of section 1) by their Ricci tensors. It is natural to ask which metrics are so determined, although the answer is not all metrics. Standing as counterexamples are the 19-dimensional family of non-cohomologous Ricci-flat metrics on the K3-surface guaranteed by Yau's solution of the Calabi conjecture (see [Be]) and some other explicit examples of non-cohomologous Kähler metrics with the same Ricci tensor constructed by Calabi (see [C1] [C2]). We note here that, if one is willing to look at Lorentz metrics, then counterexamples can be constructed on much less exotic manifolds.

EXAMPLE 4.1. — Let $\bar{g}$ be the standard « round » metric on S^3 with $\mathrm{Ric}(\bar{g}) = \bar{g}$. Then there is a one-parameter family $\{ g_t \}$ of non-isometric Lorentz metrics on S^3 with $\mathrm{Ric}(g_t) = 4\bar{g}$ for all t.

Proof. — We follow Milnor [M] or Hamilton [H] and consider S^3 as the Lie group SU(2), and look for left-invariant metrics on it. A metric is thus determined by its value at one point, so we let $\bar{g}_{ij} = \delta_{ij}$ at our point, and look for $g_t = \lambda_i\delta_{ij}$. Then $\mathrm{Ric}(g_t)$ is also diagonal and, if we determine μ_i by

$$\mu_1 + \mu_2 = 2\lambda_3 \qquad \mu_1 + \mu_3 = 2\lambda_2 \qquad \mu_2 + \mu_3 = 2\lambda_1 ,$$

then $\mathrm{Ric}(g_t)_{ij} = \sigma_i\delta_{ij}$ where

$$\sigma_1 = \frac{\mu_2\mu_3}{\lambda_2\lambda_3} \qquad \sigma_2 = \frac{\mu_1\mu_3}{\lambda_1\lambda_3} \qquad \sigma_3 = \frac{\mu_1\mu_2}{\lambda_1\lambda_2}.$$

We seek solutions for which $\sigma_1 = \sigma_2 = \sigma_3$, and we fix $\lambda_1 = 1$ to avoid the scaling redundancy. There are two cases: $\lambda_1 = \lambda_2 = \lambda_3 = 1$ (of course), and

$$\lambda_1 = 1 \qquad \lambda_2 = t \qquad \lambda_3 = -(1 + t) .$$

For the latter case, at least one of λ_2 or λ_3 is always negative, and one easily computes that $\mathrm{Ric}(g_t) = 4\delta_{ij}$. q. e. d.

REFERENCES

[Be] A. BESSE, *Géométrie riemannien en dimension* 4, Cedic Fernand Nathan, Paris, 1981.

[BK] J. BOURGUIGNON and H. KARCHER, « Curvature operators: pinching estimates and geometric examples », *Ann. Scient. Ec. Norm. Sup.*, 4e série, t. **11**, 1978, p. 71-92.

[C1] E. CALABI, « Extremal Kähler metrics », in *Seminar on Differential Geometry*, S. T. Yau, ed., Annals of Math Study 102, Princeton U. Press, 1982, p. 259-290.

[C2] E. CALABI, « Métriques Kählériennes et fibrés holomorphes », *Ann. Sci. Ec. Norm. Sup.*, 4e série, t. **12**, 1979, p. 269-294.

[D] D. DETURCK, « Metrics with prescribed Ricci curvature » in *Seminar on Differential Geometry*, S. T. Yau ed., Annals of Math Study 102, Princeton U. Press, 1982, p. 525-537.

[EL] J. EELLS and L. LEMAIRE, « A report on harmonic maps », *Bull. London Math. Soc.*, t. **10**, 1978, p. 1-68.

[H] R. HAMILTON, *The Ricci curvature equation*, preprint, MSRI Berkeley, 1983.

[KN1] S. KOBAYASHI and K. NOMIZU, *Foundations of Differential Geometry*, vol. 1, Wiley-Interscience, New York, 1963.

[KN2] S. KOBAYASHI and K. NOMIZU, *Foundations of Differential Geometry*, vol. 2, Wiley-Interscience, New York, 1969.

[M] J. MILNOR, « Curvatures of left-invariant metrics on Lie groups », *Adv. in Math.*, t. **21**, 1976, p. 293-329.

(Manuscrit reçu le 15 mars 1984)