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A rapid convergence method for a singular perturbation problem

by

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ABSTRACT. — The existence of spatially periodic solutions for a singular perturbation of elliptic type is established. A rapid convergence method is used to obtain the result.

RÉSUMÉ. — On démontre l'existence de solutions périodiques par rapport aux variables d'espace pour un problème de perturbation singulière de type elliptique. La démonstration repose sur une méthode de convergence rapide.

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INTRODUCTION

Consider the equation

$$(0.1) \qquad Lu \equiv - \sum_{i,j=1}^{n} (a_{ij}(x)u_{x_j})_{x_i} + u = \varepsilon f(x, u, Du, D^2u, D^3u)\,.$$

In (0.1), $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, L is uniformly elliptic with coefficients a_{ij}

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which are periodic in $x_1, \ldots, x_n$, and $\varepsilon \in \mathrm{R}$. The function f depends on u and its derivatives up to order three and is also periodic in $x_1, \ldots, x_n$ with the same periods as the coefficients a_{ij}. Our goal is to establish the existence of periodic solutions of (0.1) for small values of $|\varepsilon|$. This is a singular perturbation problem since the f term is of third order while L is merely of order two. We will show (0.1) possesses a one parameter family of periodic solutions depending continuously on ε for small $|\varepsilon|$ provided that the coefficients a_{ij} and f are sufficiently smooth. Surprisingly other than this differentiability requirement, no hypotheses are needed concerning the dependence of f on u and its derivatives.

We assume the functions f and the a_{ij} have the same period, say 2π, in each of $x_1, \ldots, x_n$. The analysis is unchanged if they have different periods $\mathrm{T}_1, \ldots, \mathrm{T}_n$ with respect to $x_1, \ldots, x_n$. For notational convenience we further set

$$\mathrm{F}(x, u) \equiv f(x, u, \mathrm{D}u, \mathrm{D}^2 u, \mathrm{D}^3 u).$$

Note that when $\varepsilon = 0$, (0.1) has a unique solution $u \equiv 0$. A natural way in which to attempt to solve (0.1) for small $|\varepsilon|$ is via the iteration scheme: $u_0 = 0$ and for $j \geqslant 0$,

$$\mathrm{L}u_{j+1} = \varepsilon \mathrm{F}(x, u_j). \tag{0.2}$$

For various choices of function spaces, L can be inverted with a gain of two derivatives. However since F depends on u and its derivatives up to order three, in passing from u_j to u_{j+1}, we have a net loss of one derivative. Thus if $f \in \mathrm{C}^m$, we can only iterate for a finite number of steps and even if $f \in \mathrm{C}^\infty$, convergence of this scheme is unlikely due to the above loss of derivatives phenomenon.

Methods have been developed by several authors to treat « loss of derivatives » and « small divisor » problems. See e. g. Nash [*1*], Moser [*2*], Schwartz [*3*], Sergeraert [*4*], Zehnder [*5*], Hörmander [*6*], and Hamilton [*7*]. We shall show how the approach of Moser can be applied to (0.1). The main difficulty in doing so is in finding approximate solutions of the corresponding linearized equation

$$\hat{\mathrm{L}}v \equiv \mathrm{L}v - \varepsilon \sum_{|\sigma| \leqslant 3} \mathrm{A}_\sigma(u)\mathrm{D}^\sigma v = g. \tag{0.3}$$

In (0.3), the usual multiindex notation is being employed, $\mathrm{A}_\sigma = \dfrac{\partial f}{\partial \mathrm{D}^\sigma u}$ for $|\sigma| \leqslant 3$, and the dependence of A_σ on x has been suppressed. Approximate solutions of (0.3) will be obtained as exact solutions of an elliptic regularization of (0.3):

$$(-1)^m \gamma \Delta^m v + \hat{\mathrm{L}}v = g \tag{0.4}$$

where Δ denotes the usual Laplacian.

In § 1, we will state Moser's result from [*2*] and show how it can be used to solve (0.1). With the exception of the technicalities associated with (0.3)-(0.4), this is not a difficult process. The technicalities of treating (0.3)-(0.4) are carried out in § 2. In § 3, a local uniqueness result will be obtained. Our approach to (0.1) relies in part on ideas from [*8*]. See also [*9*].

In [*10*], the written version of a talk delivered at the University of Alabama in Birmingham International Conference on Differential Equations, a one dimensional version of (0.1), was discussed. As an outgrowth of that lecture, Tosio Kato has found another approach to the problem using the stationary version of his theory of quasilinear evolution equations.

§ 1. MOSER'S THEOREM AND ITS APPLICATION TO (0.1)

Some functional analytic preliminaries are required before Moser's result can be stated. Let H^m denote the closure of the set of C^∞ functions on R^n which are 2π periodic in $x_1, \ldots, x_n$ with respect to

$$(1.1) \qquad \| u \|_m \equiv \left(\sum_{|\tau| \leqslant m} \int | D^\tau u |^2 dx \right)^{1/2} .$$

In (1.1) and elsewhere in this paper, integration is over the set

$$\{ x \in R^n \mid x_i \in [0, 2\pi], 1 \leqslant i \leqslant n \} .$$

Let $0 < \rho < r$ and $u_0 \in H^r$. Set

$$U = \{ u \in H^0 \mid \| u - u_0 \|_\rho < 1 \}$$

and $U_r = U \cap H^r$. Suppose $\mathscr{F} : U_r \to H^s$ where $s < r$. The equation $\mathscr{F}(u) = \phi$ *is said to have an approximate solution of order* $\lambda (> 0)$ in U_r if for all large K, there exists $u \equiv u_K \in U_r$ such that

$$\| \mathscr{F}(u) - \phi \|_0 < K^{-\lambda} \qquad \text{and} \qquad \| u \|_r < K .$$

For $u \in U_r$, let $\mathscr{F}'(u)$ denote the Frechet derivative of $\mathscr{F}$ at u. The equation $\mathscr{F}'(u)v = g$ is said to have an *approximate solution of order* $\mu (> 0)$ if there exists a constant $c > 0$ and a function $\psi(M)$ such that whenever $u \in U_r$, $g \in H^s$, and

$$(1.2) \qquad \| \mathscr{F}(u) - \phi \|_0 < K^{-\lambda} \qquad \text{and} \qquad \| u \|_r < K .$$

then for all large Q, there is a $v = v_Q \in H^s$ satisfying

$$(1.3) \qquad \| \mathscr{F}'(u)v - g \|_0 \leqslant \psi(M) K Q^{-\mu} ,$$

$$(1.4) \qquad \| v \|_r \leqslant \psi(M) K Q ,$$

and

$$(1.5) \qquad \| \mathcal{F}'(u)v \|_0 \geqslant c \| v \|_0 .$$

For $u \in U_r$ and $v \in H^r$, let

$$(1.6) \qquad \mathcal{Q}(u, v) \equiv \mathcal{F}(u + v) - \mathcal{F}(u) - \mathcal{F}'(u)v .$$

THEOREM 1.7 (*Moser* [2]). — Let $\mathcal{F} : H^r \to H^s$ and suppose there are constants c, ρ, λ, μ, β, and M and a function $\psi(M)$ such that

1° $$\mathcal{F} \in C^1(U_r, H^0)$$

2° If $u \in U_r$, then $\| \mathcal{F}(u) - \phi_0 \|_0 < M$ and $\| \mathcal{F}(u) - \phi_0 \|_s < \infty$.

3° For all large K, $\| \mathcal{F}(u) \|_s \leqslant MK$ whenever $u \in U_r$ and $\| u \|_r < K$.

4° The equation $\mathcal{F}'(u)v = g$ admits approximate solutions of order μ.

5° $\| \mathcal{Q}(u, v) \|_0 \leqslant M \| v \|_0^{2-\beta} \| v \|_r^{\beta}$ for all $u \in U_r$, $v \in H^r$

6° (*i*) $$\beta \in (0, 1)$$

ii) $$\rho/r < \frac{\lambda}{\lambda + 2}$$

iii) $$0 < \lambda + 1 < \frac{1}{2}(\mu + 1)$$

iv) $$0 < \beta < \frac{\lambda}{\lambda + 1} \frac{\mu}{\mu + 1}\left(1 - 2\frac{\lambda + 1}{\mu + 1}\right).$$

Then there exists a constant K_0 (depending on M, c, β, μ, λ) > 0 such that if

$$(1.8) \quad i) \qquad \| \phi - \phi_0 \|_0 < K_0^{-\lambda}$$

$$ii) \qquad \| u_0 \|_r < K_0$$

and $$iii) \qquad \| \phi \|_s \leqslant MK_0$$

hold, the equation $\mathcal{F}(u) = \phi$ possesses a sequence of approximate solutions of order λ in U_r. Moreover the sequence is a Cauchy sequence in H^ρ with $u_m \to u_\infty \in U$ and $\mathcal{F}(u_\infty) = \phi$.

REMARK 1.9. — Moser states the result somewhat less formally in [2]. The proof of Theorem 1.7 shows that if $\mathcal{F}$ depends continuously on a parameter ε, then so does u_∞.

We will demonstrate how Theorem 1.7 yields a solution of (0.1). Before doing so it is convenient to make a technical modification of f. When $\varepsilon = 0$, $u = 0$ is the unique solution of (0.1). Therefore we expect a small solution in $\| \cdot \|_{C^3}$ for small ε so the behavior of f only when e. g. $\| u \|_{C^3} < \frac{1}{2}$ should be of importance. Therefore we can multiply $f(x, \xi)$ (where $\xi \in R^{1+n+n^2+n^3}$)

by a smooth function $\chi(\xi)$ with $\chi(\xi) = 1$ if $|\xi_i| < \frac{1}{2}$ for all i and $\chi(\xi) = 0$ if any $|\xi_i| \geqslant 1$. Thus we can and will assume $f(x, \xi)$ has compact support with respect to ξ. Of course it must be shown later that $\| u(\varepsilon) \|_{C^3} < \frac{1}{2}$ for the solution we find.

To apply Theorem 1.7 to (0.1), set $r = k + 3$ and $s = k$ where k is free for the moment. We will determine lower bounds on k later when 5° is verified. Choose ρ to be the smallest integer that exeeds $4 + \frac{n}{2}$. The Sobolev inequality then implies $u \in C^4$ whenever $u \in U$. Define

$$\mathscr{F}(u) \equiv Lu - \varepsilon F(x, u). \tag{1.10}$$

Further set $u_0 = 0$, $\phi_0 = \mathscr{F}(0) = -\varepsilon F(x, 0)$, and $\phi = 0$. Our choice of ρ shows there is a constant $R > 0$ such that

$$\| u \|_{C^4} \leqslant R \tag{1.11}$$

for all $u \in U$. Moreover $\mathscr{F} \in C^1(C^3 \cap H^0, C^0)$ and *a fortiori* $\mathscr{F} \in C^1(U_r, H^0)$ so 1° of Theorem 1.7 holds.

The following « composition of functions » inequality from [2] is useful in verifying 2° and 3° of Theorem 1.7 for (0.1).

PROPOSITION 1.12. — Suppose $G(x, \xi) \in C^m(R^n \times R^{1+n+n^2+n^3}, R)$ and G is 2π periodic in $x_1, \ldots, x_n$. If $u \in H^{m+3} \cap C^3$ with $\| u \|_{C^3} \leqslant R$, then $G(x, u, Du, D^2u, D^3u) \in H^m$. Moreover there is a constant $\bar{c} = \bar{c}(m, R)$ such that

$$\| G(x, u, Du, D^2u, D^3u) \|_m \leqslant \bar{c}(m, R) (\| u \|_{m+3} + 1).$$

With the aid of Proposition 1.12 and our choice of ρ, 2° of Theorem 1.7 follows trivially. For 3°, by (1.10), (1.11), and Proposition 1.12, we have

$$\begin{aligned} \| \mathscr{F}(u) \|_k &\leqslant \alpha_1 \| u \|_{k+2} + |\varepsilon| \; \| F(x, u) \|_k \\ &\leqslant \alpha_1 \| u \|_{k+2} + |\varepsilon| \bar{c}(k, R) (\| u \|_{k+3} + 1) \leqslant MK \end{aligned} \tag{1.13}$$

provided that $|\varepsilon| \leqslant 1 \leqslant K$ and $\alpha_1 + 2\bar{c}(k, R) \leqslant M$. In (1.13), α_1 depends $\mathscr{A} \equiv \max_{1 \leqslant i,j \leqslant n} \| a_{ij} \|_{C^{k+1}}$.

To verify 4°, some notational preliminaries are needed. Let

$$A_\tau(u) \equiv \frac{\partial F}{\partial \xi_\tau}(x, u)$$

where ξ_τ corresponds to the $D^\tau u$ argument of F. Define

$$A(u)v \equiv \sum_{|\tau| \leqslant 3} A_\tau(u) D^\tau v.$$

Set

$$\| A(u) \|_j \equiv \sum_{|\tau| \leqslant 3} \| A_\tau(u) \|_j .$$

In § 2, we will prove

PROPOSITION 1.14. — If $\gamma > 0$, a_{ij}, $f \in C^{k+1}$, $u \in U_r$, and $g \in H^k$, then there is an $\varepsilon_k > 0$ such that for $| \varepsilon | < \varepsilon_k$, the equation

$$\text{(1.15)} \qquad Lv \equiv (-1)^m \gamma \Delta^m v + \mathscr{F}'(u)v = g$$

possesses a unique solution $v \in H^{2m+k}$. Moreover there is a $\overline{K}(M)$ such that if u, g satisfy (1.2), $K \geqslant \overline{K}$, and $\gamma \leqslant 1$, then

$$\text{(1.16)} \qquad \gamma \| v \|_{2m+k-1} + \| v \|_{k+2} \leqslant b_k(\| g \|_k + \varepsilon \| A(u) \|_k)$$

where b_k depends on k, the ellipticity constant of L, and $\mathscr{A}$.

Proposition 1.14 implies (1.3)-(1.4). Indeed by (1.16), (1.2) and Proposition 1.2,

$$\text{(1.17)} \qquad \| v \|_{k+2} \leqslant b_k(MK + | \varepsilon | \bar{c}(k, R)(K + 1)) \leqslant 2b_k MK$$

for $\| \varepsilon | \leqslant 1 \leqslant K$ and $2\bar{c}(k, R) \leqslant M$. Also by (1.16) and (1.16),

$$\text{(1.18)} \qquad \| v \|_{2m+k-1} \leqslant \gamma^{-1} 2b_k MK .$$

A standard interpolation inequality—see e. g. [*2*]—asserts if $0 < p < q$,

$$\text{(1.19)} \qquad \| w \|_p \leqslant \hat{c} \| w \|_0^{1-p/q} \| w \|_q^{p/q}$$

for all $w \in H^q$ where $\hat{c}$ is a constant depending only on p and q. Let $w = D^\tau v$ where $| \tau | = k + 2$. By (1.19),

$$\text{(1.20)} \qquad \| w \|_1 \leqslant \alpha_2 \| w \|_0^{\frac{2m-4}{2m-3}} \| w \|_{2m-3}^{\frac{1}{2m-3}} .$$

Hence combining (1.17), (1.18), and (1.20) yields

$$\text{(1.21)} \qquad \| v \|_{k+3} \leqslant 2\alpha_3 \gamma^{-\frac{1}{2m-3}} b_k MK .$$

Set $Q \equiv \gamma^{-\frac{1}{2m-3}}$ so (1.21) becomes

$$\text{(1.22)} \qquad \| v \|_{k+3} \leqslant 2\alpha_3 b_k MKQ \leqslant \psi(M)RQ$$

where

$$\text{(1.23)} \qquad \psi(M) \equiv 2(1 + \alpha_3)(1 + \alpha_4) b_k M$$

and the constant α_4 is defined in (1.24). Thus (1.4) holds.

Next note that from (1.15) we get

$$\text{(1.24)} \qquad \| \mathscr{F}'(u)v - g \|_0 = \gamma \| D^m v \|_0 \leqslant Q^{-(2m-3)} \alpha_4 \| v \|_{2m} .$$

Choose m so that $2m = k + 2$ if k is even and $2m = k + 3$ if k is odd.

In the first case, (1.17) and (1.24) show

(1.25) $$\| \mathscr{F}'(u)v - g \|_0 \leqslant (Q^{-(k-1)}\alpha_4)2b_k MK\,.$$

In the second case, (1.22) and (1.24) imply

(1.26) $$\| \mathscr{F}'(u)v - g \|_0 \leqslant (Q^{-k}\alpha_4)2\alpha_3 b_k MQK\,.$$

Hence in either case we have

$$\| \mathscr{F}'(u)v - g \|_0 \leqslant \psi(M)KQ^{-\mu}$$

where $\mu = k - 1$. Thus (1.3) is satisfied.

At this point 4° of Theorem 1.7 has been verified except for (1.5). In § 2 we shall show that (1.5) holds with c depending on the ellipticity constant of L provided that $|\varepsilon|$ is sufficiently small.

Next let $u \in U_r$ and $v \in H^{k+3}$. Then u, $v \in C^3$ and by Taylor's Theorem we have

(1.27) $$\mathscr{Q}(u, v) \equiv \frac{\varepsilon}{2} \sum_{|\sigma|,|\tau| \leqslant 3} \frac{\partial^2 F}{\partial \xi_0 \partial \xi_\tau} D^\sigma v D^\tau v$$

where ξ_τ again corresponds to the $D^\tau u$ argument of F. In (1.27), F is evaluated at $(x, u(x) + \theta(x)v(x))$ where $\theta(x) \in (0, 1)$ via Taylor's Theorem. By earlier remarks about truncating f, there is a constant α_5 such that

(1.28) $$| \mathscr{Q}(u, v) | \leqslant |\varepsilon| \alpha_5 \sum_{|\tau| \leqslant 3} | D^\tau v |^2 .$$

Consequently

(1.29) $$\| \mathscr{Q}(u, v) \|_0 \leqslant |\varepsilon| \alpha_6 \| v \|_{C^3} \| v \|_3 .$$

Applying (1.19) gives

(1.30) $$\| v \|_3 \leqslant \alpha_7 \| v \|_0^{\frac{k}{k+3}} \| v \|_{k+3}^{\frac{3}{k+3}} .$$

The Gagliardo-Nirenberg inequality [*11*] further implies

(1.31) $$\| v \|_{C^3} \leqslant \alpha_8 \| v \|_0^{\frac{k-\frac{n}{2}}{k+3}} \| v \|_{k+3}^{\frac{3+\frac{n}{2}}{k+3}} .$$

Thus for $|\varepsilon| \leqslant 1$ and $M \geqslant \alpha_\Omega(\alpha_\Lambda + \alpha_U)$, 5° of Theorem 1.7 obtains with $\beta = \left(6 + \frac{n}{2}\right)/(k + 3)$.

We turn now to the verification of 5°, determining k in the process. If $k > 3 + \frac{n}{2}$, (*i*) holds and (*iii*) is satisfied via setting $\lambda = \frac{k}{4} - 1$. (Recall $\mu = k - 1$.) To get (*ii*), we need

(1.32) $$\frac{\rho}{k + 3} < \frac{k - 4}{k + 4}.$$

Since $\rho \leqslant s + \frac{n}{2}$, it is easy to check that (1.32) holds for e. g. $k > 12 + n$. Lastly (*iv*) requires that

$$\frac{6 + \frac{n}{2}}{k + 3} < \frac{1}{2} \frac{k - 4}{k} \frac{k - 1}{k} \tag{1.33}$$

and $k \geqslant 28+2n$ is sufficient for (1.33). Thus if $k \geqslant 28+2n$ and $|\varepsilon| \leqslant \varepsilon_{28+2n}$, all of the hypothesis of Theorem 1.7 are satisfied and there is a $K_0(M, c, \beta, \lambda, \mu) > 1$ such that if (*i*)-(*iii*) of (1.8) holds, (0.1) has a solution. But by our choices of ϕ_0, ϕ, and u_0, (*ii*) and (*iii*) are trivially true and (*i*) also obtains if $|\varepsilon|$ is so small that

$$|\varepsilon| \, \| f(x, 0) \|_0 < K_0^{-\lambda}. \tag{1.34}$$

With this further restriction on $|\varepsilon|$, by Theorem 1.7 and Remark 1.9, (0.1) with the modified f possesses a curve of solutions $u(x;\varepsilon) \in C^3$ with $u(x;0)=0$ and u continuous in ε. Therefore for small $|\varepsilon|$, $\| u(x;\varepsilon) \|_{C^3} < \frac{1}{2}$ and (0.1) is satisfied with the original f. Thus we have shown:

THEOREM 1.35. — If f and the coefficients of L are sufficiently smooth there is an $\varepsilon^* > 0$ such that for all $|\varepsilon| < \varepsilon^*$, (0.1) has a solution $u(x;\varepsilon)$ which is C^3 in x and continuous in ε with $u(x;0)=0$.

REMARK 1.36. — Our above estimates show the conclusions of theorem 1.35 hold if f and the coefficients of L lie in C^{28+2n}. However, this is a rather crude lower bound for k.

§ 2. THE MODIFIED PROBLEM

The goal of this section is to find approximate solutions of $\mathscr{F}'(u)v = g$ in the sense of (1.2)-(1.5). This will be accomplished via Propositions 2.1, 2.18, and 2.36 below. The inequality (1.5) happens to be valid for all $v \in H^{k+3}$. To make this precise a few notational preliminaries are needed. Set

$$\tilde{A}(u)v \equiv \sum_{|\tau| \leqslant 2} A_\tau(u) D^\tau v \quad \text{and} \quad A_3(u)v \equiv \sum_{|\tau| = 3} A_\tau(u)v$$

so

$$A(u)v = \tilde{A}(u)v + A_3(u)v$$

Set

$$\| \tilde{A}(u) \|_{C^1} = \sum_{|\tau| \leqslant 2} \| A_\tau(u) \|_{C^1}, \qquad \| A_3(u) \|_{C^1} = \sum_{|\tau| = 3} \| A_\tau(u) \|_{C^1}$$

and

$$\| A(u) \|_{C^1} = \| \tilde{A}(u) \|_{C^1} + \| A_3(u) \|_{C^1} .$$

The H^0 inner product will be denoted by $(. , .)$. Finally note that

$$\mathscr{F}'(u)v = Lv - \varepsilon A(u)v .$$

PROPOSITION 2.1. — There are constants ε_1 and c depending on the ellipticity constant of L and on $\sum_{i,j=1}^{n} \| a_{ij} \|_{C^1}$ such that if $| \varepsilon | \leqslant \varepsilon_1$, $u \in U_r$, and $v \in H^3$,

$$(2.2) \qquad \| \mathscr{F}'(u)v \|_0 \geqslant c \| v \|_2 .$$

Proof. — To establish (2.2), we will estimate (a) $(\mathscr{F}'(u)v, v)$ and $(b)$$(\mathscr{F}'(u)\, v, -\Delta v)$. The first quantity is easy to treat:

$$(2.3) \quad \| \mathscr{F}'(u)v \|_0 \, \| v \|_0 \geqslant (\mathscr{F}'(u)v, v) \geqslant (Lv, v) - \\ - | \varepsilon | \, \| \tilde{A}(u) \|_{C^1} \| v \|_2 \, \| v \|_0 - | \varepsilon \, (A_3(u)\, v, v) | .$$

Since L is uniformly elliptic, there is an $\overline{\omega} > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geqslant \overline{\omega} \, | \xi |^2$$

for all x, $\xi \in R^n$. Therefore

$$(Lv, v) \geqslant \omega \| v \|_1^2$$

where $\omega = \min(1, \overline{\omega})$. Expanding the last term in (2.3) gives

$$(2.4) \qquad (A_3(u)v, v) = \int \sum_{|\tau| = 3} A_\tau(u)(D^\tau v) v dx .$$

Writing $D^\tau v = v_{x_i x_j x_m}$, a typical term in (2.4) can be integrated by parts:

$$(2.5) \qquad \int A_\tau(u) v_{x_i x_j x_m} v dx = - \int [(A_\tau(u))_{x_i} v_{x_j x_m} v \\ + A_\tau(u) v_{x_j x_m} v_{x_i}] dx .$$

Thus (2.4)-(2.5) and crude estimates yield

$$(2.6) \qquad | (A_3(u)v, v) | \leqslant \| A_3(u) \|_{C^1} \, \| v \|_1 \, \| v \|_2 .$$

Combining (2.3) and (2.6) then gives

$$\| \mathscr{F}'(u)v \|_0 \geqslant \omega \| v \|_1 - | \varepsilon | \, \| A(u) \|_{C^1} \| v \|_2 . \tag{2.7}$$

The estimate for (*b*) requires more care. As in (2.3) we have

$$\begin{aligned} \| \mathscr{F}'(u)v \|_0 \sum_{|\sigma|=2} \| D^\sigma v \|_0 &\geqslant (\mathscr{F}'(u)v, - \Delta v) = \\ &= (Lv, - \Delta v) + \varepsilon(\tilde{A}(u)v, \Delta v) + \varepsilon(A_3(u)v, \Delta v) . \end{aligned} \tag{2.8}$$

The terms on the right hand side of (2.8) will be estimated separately. First

$$(Lv, - \Delta v) \geqslant \omega \sum_{i=1}^{n} \| v_{x_i} \|_1^2 - \sum_{i,j=1}^{n} \| a_{ij} \|_{C^1} \| v \|_1 \sum_{|\sigma|=2} \| D^\sigma v \|_0 . \tag{2.9}$$

Next

$$| (\tilde{A}(u)v, \Delta v) | \leqslant \| \tilde{A}(u) \|_{C^1} \| v \|_2 \sum_{|\sigma|=2} \| D^\sigma v \|_0 . \tag{2.10}$$

A typical term in $(A_3(u)v, \Delta v)$ is

$$I = \int A_\tau v_{x_i x_j x_m} v_{x_p x_p} dx . \tag{2.11}$$

This must be handled carefully. Integrating by parts,

$$\begin{aligned} I = \int [- (A_\tau(u))_{x_i} v_{x_j x_m} v_{x_p x_p} &+ (A_\tau(u))_{x_p} v_{x_j x_m} v_{x_i x_p} \\ &+ A_\tau(u) v_{x_j x_m x_p} v_{x_i x_p}] dx . \end{aligned} \tag{2.12}$$

Interchanging the roles of i and j and adding the resulting expression to (2.12) yields

$$\begin{aligned} 2I = \int \{ - & [(A_\tau(u))_{x_i} v_{x_j x_m} + (A_\tau(u))_{x_j} v_{x_i x_m}] v_{x_p x_p} \\ &+ (A_\tau(u))_{x_p} [v_{x_j x_m} v_{x_i x_p} + v_{x_i x_m} v_{x_j x_p}] \\ &+ A_\tau(u) (v_{x_j x_p} v_{x_i x_p})_{x_m} \} \, dx . \end{aligned} \tag{2.13}$$

Thus one final integration by parts shows

$$| I | \leqslant \frac{5}{2} \| A_\tau(u) \|_{C^1} \| v \|_2 \sum_{|\sigma|=2} \| D^\sigma v \|_0 . \tag{2.14}$$

Consequently

$$| (A_3(u)v, \Delta v) | \leqslant \frac{5n}{2} \| A_3(u) \|_{C^1} \| v \|_2 \sum_{|\sigma|=2} \| D^\sigma v \|_0 \tag{2.15}$$

and combining (2.15) with (2.8)-(2.10) shows

$$(2.16)\quad \| \mathcal{F}'(u)v \|_0 \geqslant \omega \sum_{i=1}^{n} \| v_{x_0} \|_1 - \\ - \sum_{i,j=1}^{n} \| a_{ij} |_{C^1} \| v \|_1 - | \varepsilon | \beta_1 \| A(u) \|_{C^1} \| v \|_2 .$$

Adding β_2 times (2.16) to (2.7) yields

$$(2.17)\quad (1+\beta_2) \| \mathcal{F}'(u)v \|_0 \geqslant (\omega - \beta_2 \sum_{i,j=1}^{n} \| a_{ij} \|_{C^1}) \| v \|_1 + \\ + \omega\beta_2 \sum_{i=1}^{n} \| v_{x_i} \|_1 - (1+\beta_1\beta_2) | \varepsilon | \| A(u) \|_{C^1} \| v \|_2 .$$

Choosing

$$\beta_2 = \omega \left(2 \sum_{i,j=1}^{n} \| a_{ij} \|_{C^1} \right)^{-1} ; \quad \varepsilon_1 \| A(u) \|_{C^1} (1+\beta_1\beta_2) = \frac{1}{2} \min \left(\frac{\omega}{2}, w\beta_2 \right)$$

and $| \varepsilon | \leqslant \varepsilon_1$ gives (2.2).

It remains to prove Proposition 1.14. Its existence and uniqueness assertions follow from the next result and the estimates follow from Proposition 2.36 below.

PROPOSITION 2.18. — Suppose $f \in C^{k+1}$, $u \in U_r$, $g \in H^k$, $m > 1$, and $| \varepsilon | \leqslant \varepsilon_1$. Then there exists a unique $v \in H^{2m+k}$ satisfying (1.15) .

Proof. — First we will establish the existence and uniqueness of a weak solution of (1.15). Regularity will then follow easily from elliptic theory.

For $\zeta \in H^{2m}$, let $\Lambda\zeta \equiv \zeta - \beta_2 \Delta\zeta$. The estimates of (2.3)-(2.17) show for $| \varepsilon | \leqslant \varepsilon_1$,

$$(2.19)\qquad (\mathcal{L}\zeta, \Lambda\zeta) \geqslant \gamma \sum_{|\tau|=m}^{m+1} \| D^\tau \zeta \|_0^2 + c \| \zeta \|_2^2 .$$

Let H^{-s} denote the negative norm dual of H^s with respect to H^0. (Recall

$$(2.20)\qquad \| \zeta \|_{-s} = \sup_{0 \neq w \in H^s} \frac{(\zeta, w)}{\| w \|_s}$$

see e. g. Lax [11].) Let $\phi \in C^\infty \cap H^0$. Using e. g. Fourier series, it is easy to see that there is a unique $w \in C^\infty \cap H^0$ such that $\Lambda w = \phi$. Let $\psi = \mathcal{L}^*\phi$

where $\mathscr{L}^*$ denotes the formal adjoint of $\mathscr{L}$. Then by (2.19) and (2.20),

$$(2.21)\qquad \| w \|_{m+1} \| \psi \|_{-(m+1)} \geqslant (w, \psi) = (w, \mathscr{L}^*\phi) = (\mathscr{L}w, \phi) \\ = (\mathscr{L}w, \Lambda w) \geqslant \gamma_1 \| w \|_{m+1}^2$$

where γ_1 depends on γ and c. Moreover

$$(2.22)\qquad \| \phi \|_{-2} = \sup_{0 \neq z \in \mathrm{H}^2} \frac{(\phi, z)}{\| z \|_2} = \sup_{0 \neq z \in \mathrm{H}^2} \frac{(\Lambda w, z)}{\| z \|_2} \leqslant \gamma_2 \| w \|_2 \leqslant \gamma_2 \| w \|_{m+1}.$$

Consequently by (2.21)-(2.22),

$$(2.23)\qquad \| \phi \|_{-2} \leqslant \gamma_3 \| \psi \|_{-(m+1)}$$

for all $\phi \in \mathrm{C}^\infty \cap \mathrm{H}^0$.

Now for fixed $g \in \mathrm{H}^2$, define a linear functional on $\mathrm{C}^\infty \cap \mathrm{H}^0$ via

$$(2.24)\qquad l(\phi) \equiv (\phi, g).$$

Setting $\psi = \mathscr{L}^*\phi$, (2.24) can be used to define a new linear functional:

$$(2.25)\qquad l^*(\psi) \equiv l(\phi)$$

for $\psi \in \mathscr{L}^*(\mathrm{C}^\infty \cap \mathrm{H}^0)$. By (2.23)-(2.25),

$$(2.26)\qquad | l^*(\psi) | \leqslant \gamma_3 \| g \|_2 \| \psi \|_{-(m+1)}.$$

Thus l^* is continuous on $\mathscr{L}^*(\mathrm{C}^\infty \cap \mathrm{H}^0) \subset \mathrm{H}^{-(m+1)}$. Therefore by the Hahn-Banach Theorem it can be continuously extended to all of $\mathrm{H}^{-(m+1)}$ with preservation of norm. It then follows from a lemma of Lax [*11*] that there exists $v \in \mathrm{H}^{m+1}$ satisfying

$$(2.27)\qquad l^*(\psi) = (\psi, v)$$

for all $\psi \in \mathrm{H}^{-(m+1)}$ and

$$(2.28)\qquad \| v \|_{m+1} \leqslant \gamma_3 \| g \|_2.$$

In particular for $\psi = \mathscr{L}^*\phi$ with $\phi \in \mathrm{C}^\infty \cap \mathrm{H}^0$, by (2.24)-(2.25), (2.27),

$$(2.29)\qquad l(\phi) = (\phi, g) = l^*(\psi) = (\mathscr{L}^*\phi, v).$$

Hence v is a weak solution of (1.15). The uniqueness of v follows from (2.28).

It remains to establish the regularity of v. The following lemma is helpful for that purpose as well as in the sequel.

LEMMA 2.30. — If $\phi, \psi \in \mathrm{H}^r \cap \mathrm{C}$ and $|\sigma| = r$, then $\phi\psi \in \mathrm{H}^r$ and

$$(2.31)\qquad \| \mathrm{D}^\sigma(\phi\psi) \|_0 \leqslant c_r(\| \phi \|_{\mathrm{L}^\infty} \| \psi \|_r + \| \psi \|_{\mathrm{L}^\infty} \| \phi \|_r).$$

If further $\phi \in \mathrm{C}^1$, then

$$(2.32)\qquad \| \mathrm{D}^\sigma(\phi\psi) - \phi \mathrm{D}^\sigma \psi \|_0 \leqslant c_r(\| \phi \|_{\mathrm{C}^1} \| \psi \|_{r-1} + \| \psi \|_{\mathrm{L}^\infty} \| \phi \|_r)$$

where c_r depends only on r.

Proof. — We argue in a similar fashion to related results in [*2*] or [*8*]. By the Hölder inequality

$$(2.33)\qquad \int |\mathrm{D}^{\sigma}(\phi\psi)|^2 dx = \int\Big(\sum_{\tau+\theta=\sigma} \mathrm{D}^{\tau}\phi\mathrm{D}^{\theta}\psi\Big)^2 dx \leqslant$$

$$\leqslant \text{const} \sum_{\tau+\theta=\sigma} \int |\mathrm{D}^{\tau}\phi|^2 \,|\mathrm{D}^{\theta}\psi|^2 dx \leqslant \sum_{|\tau|+|\theta|=r} \|(\mathrm{D}^{\tau}\phi)^2\|_{\mathrm{L}^{\frac{r}{|\tau|}}} \|(\mathrm{D}^{\theta}\psi)^2\|_{\mathrm{L}^{\frac{r}{|\theta|}}}.$$

By the Gagliardo-Nirenberg inequality [*11*], if $a \in \mathrm{H}^r \cap \mathrm{L}^{\infty}$ and $0 \leqslant |\nu| \leqslant r$,

$$(2.34)\qquad \|\mathrm{D}^{\nu}a\|_{\mathrm{L}^{\frac{|\nu|}{2r}}} \leqslant \tilde{c}\, \|a\|_{\mathrm{L}^{\infty}}^{1-\frac{|\nu|}{r}} \|a\|_r^{\frac{|\nu|}{r}}.$$

Employing (2.34) in (2.33) and using Young's inequality then gives (2.31). Inequality (2.32) is proved in a similar fashion.

Completion of proof of Proposition 2.18. — Set

$$\tilde{\mathrm{L}}\phi \equiv (-1)^m\gamma\Delta^m\phi + \mathrm{L}\phi.$$

Standard elliptic results [*12*] [*13*] imply if $h \in \mathrm{H}^s$ there is a unique $w \in \mathrm{H}^{2m+s}$ such that $\tilde{\mathrm{L}}w = h$. Suppose $f \in \mathrm{C}^{k+1}$, $u \in \mathrm{U}_r$, and $v \in \mathrm{H}^{m+1}$. By Proposition 1.12, the coefficients of $\mathrm{A}(u)$ belong to H^k. Hence Lemma 2.30 shows $\mathrm{A}(u)v \in \mathrm{H}^t$ where $t = \min(k, m+1)$. (For our application to Theorem 1.35, $m \in \left[\frac{k}{2}+1, \frac{k}{2}+\frac{3}{2}\right]$ in which case $t = m+1$.) Then by our above remarks about $\tilde{\mathrm{L}}$, there is a unique $w \in \mathrm{H}^{2m+s}$ such that

$$(2.35)\qquad \tilde{\mathrm{L}}w = g + \varepsilon\mathrm{A}(u)v.$$

A fortiori w is a weak solution of (2.35). But we already have obtained v as a unique weak solution. Hence $v = w \in \mathrm{H}^{2m+s}$. In particular if $g \in \mathrm{H}^k$, $v \in \mathrm{H}^{2m+t}$. A standard bootstrap argument shows $v \in \mathrm{H}^{2m+k}$. The proof of Proposition 2.18 is complete.

The estimate (1.16) requires a more delicate analysis.

PROPOSITION 2.36. — Under the hypotheses of Proposition 2.18, there are constants ε_k, $\overline{b}_k$ depending on k, ω, and $\mathscr{A}$ such that for $|\varepsilon| \leqslant \varepsilon_k$, the solution v of (1.15) satisfies

$$(2.37)\qquad \min(\gamma, 1)\|v\|_{2m+k-1} + \|v\|_{k+2} \leqslant \overline{b}_n(\|g\|_k + |\varepsilon|\,\|\mathrm{A}(u)\|_k\|v\|_{\mathrm{C}^3})$$

If further u and g satisfy (1.2) with $\lambda = \frac{k}{4} - 1$ and $\gamma \leqslant 1$, then there exists a $\overline{\mathrm{K}} = \overline{\mathrm{K}}(\mathrm{M})$ and $\overline{\varepsilon}$ such that for $\mathrm{K} \geqslant \overline{\mathrm{K}}$ and $|\varepsilon| \leqslant \overline{\varepsilon}$,

$$(2.38)\qquad \|v\|_{\mathrm{C}^3} \leqslant 1.$$

Proof. — By (2.19) we have

$$\| g \|_0 \geqslant c \| v \|_2 . \tag{2.39}$$

Suppose we have shown

$$\| v \|_q \leqslant c_q(\| g \|_{q-2} + | \varepsilon | \; \| v \|_{C^3} \|A(u) \|_{q-2}) . \tag{2.40}$$

By (2.39), (2.40) holds for $q=2$. We will then establish (2.40) for $q+1$. Consider

$$(\mathscr{L} v, \Delta^q v) = (g, \Delta^q v) . \tag{2.41}$$

On the one hand,

$$(g, \Delta^q v) \leqslant \| g \|_{q-1} \sum_{|\sigma| = q+1} \| D^\sigma v \|_0 . \tag{2.42}$$

On the other hand,

$$(\mathscr{L} v, \Delta^q v) \geqslant (Lv, \Delta^q v) - \varepsilon(\tilde{A}(u)v, \Delta^q v) - \varepsilon(A_3(u)v, v) \equiv I_1 - \varepsilon(I_2 + I_3) . \tag{2.43}$$

Integration by parts and crude estimates show

$$I_1 \geqslant \omega \sum_{|\sigma| = q}^{q+1} \| D^\sigma v \|_0^2 - \bar{\alpha}_q \sum_{i,j=1}^{n} \| a_{ij} \|_{C^q} \| v \|_q \sum_{|\sigma| = q+1} \| D^\sigma v \|_0 \tag{2.44}$$

where $\bar{\alpha}_q$ depends only on q. (A more careful estimate could be made using Lemma 2.30.)

To estimate I_2 and I_3, we will make use of Lemma 2.30. A typical term in I_2 has the form

$$(D^\sigma(\tilde{A}(u)v), D^\sigma v_{x_p x_p}) \tag{2.45}$$

where $| \sigma | = q - 1$. Therefore (2.31) implies

$$\begin{aligned} | I_2 | &\leqslant \| \tilde{A}(u)v \|_{q-1} \sum_{|\sigma| = q+1} \| D^\sigma v \|_0 \leqslant \\ &\leqslant \hat{\alpha}_q(\| \tilde{A}(u) \|_{L^\infty} \| v \|_{q+1} + \| v \|_{C^2} \| \tilde{A}(u) \|_{q-1}) \sum_{|\sigma| = q+1} \| D^\sigma v \|_0 . \end{aligned} \tag{2.46}$$

A typical term in I_3 has the form

$$\begin{aligned} \int D^\sigma(A_\tau(u) v_{x_i x_j x_m}) D^\sigma v_{x_p x_p} dx &\equiv \int A_\tau(u) w_{x_i x_j x_m} w_{x_p x_p} dx + (\mathscr{R}, w_{x_p x_p}) \\ &\equiv I_4 + I_5 \end{aligned} \tag{2.47}$$

where $w = D^\sigma v$. Comparing I_4 to (2.11), we have

$$| I_4 | \leqslant \frac{5}{2} \| A_\tau(u) \|_{C^1} \| v \|_{q+1} \sum_{|\sigma| = q+1} \| D^\sigma v \|_0 . \tag{2.48}$$

Next

$$(2.49) \qquad | I_5 | \leqslant \| \mathscr{R} \|_0 \sum_{|\sigma| = q+1} \| D^\sigma v \|_0$$

and by (2.32),

$$(2.50) \qquad \| \mathscr{R} \|_0 \leqslant \tilde{\alpha}_q(\| A_\tau(u) \|_{C^1} \| v \|_{q+1} + \| v \|_{C^3} \| A_\tau(u) \|_{q-1} .$$

Now combining (2.41)-(2.50) yields

$$(2.51) \quad \| g \|_{q-1} \geqslant \omega \sum_{|\sigma| = q+1} \| D^\sigma v \|_0 - \alpha_q^* \Big[\sum_{i,j=1}^{n} \| a_{ij} \|_{C^q} \| v \|_q + | \varepsilon |(\| A(u) \|_{C^1} \| v \|_{q+1} + \| v \|_{C^3} \| A(u) \|_{q-1}] .$$

Multiplying (2.51) by α_q where $\alpha_q \alpha_q^* \sum_{i,j=1}^{n} \| a_{ij} \|_{C^q} < \frac{1}{2}$, adding it to (2.40), and choosing $| \varepsilon | \leqslant \varepsilon_{q-2}$ where $\varepsilon_{q-2} \alpha_q \alpha_q^* \| A(u) \|_{C^1} \leqslant \frac{1}{2} \min(\omega, 1)$ yields (2.40) with q replaced by $q + 1$. In particular we have (2.40) for $q = k + 2$ if $| \varepsilon | \leqslant \varepsilon_k$. By (1.15) and (2.31),

$$(2.52) \quad \gamma \| \Delta^m v \|_{k-1} = \| Lv - \varepsilon A(u) v - g \|_{k-1}$$
$$\leqslant \beta_q [(\| v \|_{k+1} + \| g \|_{k-1} + | \varepsilon | (\| A(u) \|_{k-1} \| v \|_{C^3} + \| A(u) \|_{L^\infty} \| v \|_{k+2})] .$$

Using e. g. Fourier series, it is easily seen that

$$(2.53) \qquad \| \Delta^m v \|_{k-1} + \| v \|_0 \geqslant \tilde{c} \| v \|_{2m+k-1} .$$

Hence combining (2.40), (2.52), and (2.53) gives (2.37).

Lastly suppose u and g satisfy (1.2) with $\lambda = \frac{k}{4} - 1$. Set $q = \rho - 1$ in (2.40). Recalling (1.11), by Proposition 1.12 and the Sobolev inequality we have

$$(2.54) \quad \bar{\gamma} \| v \|_{C^3} \leqslant \| \mathscr{v} \|_{\rho - 1} \leqslant c_{\rho-1}(\| g \|_{\rho-3} + | \varepsilon | \; \| v \|_{C^3} \| A(u) \|_{\rho-3})$$
$$\leqslant c_{\rho-1}(\| g \|_{\rho-3} + | \varepsilon | \; \| v \|_{C^3} \bar{c}(\rho - 3, R)(\| u \|_\rho + 1))$$

(with $\| u \|_\rho < 1$). By (1.2) and (1.19)),

$$(2.55) \qquad \| g \|_{\rho-3} \leqslant \hat{c} \| g \|_0^{1 - \frac{\rho-3}{k}} \| g \|_k^{\frac{\rho-3}{k}} \leqslant \hat{c} M^{\frac{\rho-3}{k}} K^\delta$$

with $\delta = 1 + 4^{-1}(\rho - 3 - k)$. The restrictions imposed on ρ and

$$k \left(\rho \leqslant 5 + \frac{n}{2}, k \geqslant 28 + 2n \right)$$

show $\delta < 0$. By choosing $\bar{\varepsilon} = (4c_{\rho-1}\bar{c}(\rho - 3, R))^{-1}$ and $| \varepsilon | \leqslant \bar{\varepsilon}$, we find

$$(2.56) \qquad \bar{\gamma}/2 \| v \|_{C^3} \leqslant \hat{c} M^{\frac{\rho-3}{k}} K^\delta$$

and further choosing $K \geqslant \overline{K}$ where $2\overline{\gamma}^{-1}\hat{c}M^{\frac{\rho-3}{k}}\overline{K} < 1$ gives (2.38). The proof is complete.

Now finally Proposition 2.18 and 2.37 imply Proposition 1.14 and complete the proof of Theorem 1.35.

§ 3. UNIQUENESS

In this section we will prove that $u(x; \varepsilon)$, the solution of (0, 1) obtained in § 1-2, is the only small solution of (0, 1).

THEOREM 3.1. — Suppose $u_1, u_2 \in C^4 \cap H^0$ and satisfy (0.1) for the same value of ε. If $\| u_i \|_{C^4} \leqslant R$, $i = 1, 2$, and $|\varepsilon| \leqslant \varepsilon_1$, then $u_1 = u_2$.

Proof. — Let $v = u_1 - u_2$. Then

$$(3.2) \quad \mathcal{F}(u_1) - \mathcal{F}(u_2) = 0 = Lv - \varepsilon(F(x, u_1) - F(x, u_2))$$
$$= Lv - \varepsilon \int_0^1 \frac{d}{d\theta} F(x, u_2 + \theta(u_1 - u_2))d\theta$$
$$= Lv - \varepsilon \int_0^1 A(u_2 + \theta v)v d\theta .$$

Forming

$$(3.3) \quad (\mathcal{F}(u_1) - \mathcal{F}(u_2), v - \beta_2 \Delta v)$$

with β_2 as in the proof of Proposition 2.1 and arguing as in that proof shows

$$(3.4) \quad 0 = \| \mathcal{F}(u_1) - \mathcal{F}(u_2) \|_0 \geqslant c \| v \|_2$$

for $|\varepsilon| \leqslant \varepsilon_1$. Hence $v = 0$ and $u_1 = u_2$.

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