Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2015, Vol. 51, No. 2, 648–671
DOI: 10.1214/14-AIHP600

ANNALES
DE L'INSTITUT
HENRI
POINCARÉ
PROBABILITÉS
ET STATISTIQUES
www.imstat.org/aihp

Fourier coefficients of invariant random fields on homogeneous spaces of compact Lie groups

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Received 11 December 2012; revised 13 January 2014; accepted 16 January 2014

Abstract. Let T be a random field invariant under the action of a compact group G. In the line of previous work we investigate properties of the Fourier coefficients as orthogonality and Gaussianity. In particular we give conditions ensuring that independence of the random Fourier coefficients implies Gaussianity. As a consequence, in general, it is not possible to simulate a non-Gaussian invariant random field through its Fourier expansion using independent coefficients.

Résumé. Soit T un champ aléatoire invariant par rapport à l'action d'un groupe compact G. On étudie les propriétés de ses coefficients de Fourier telles que l'orthogonalité et la gaussianité. En particulier on établit des conditions qui garantissent que l'indépendance de ces coefficients entraîne qu'ils sont gaussiens. Une conséquence remarquable est que, en général, il n'est pas possible de générer par simulation un champ aléatoire non gaussien invariant à l'aide de son développement par des coefficients indépendants.

MSC: Primary 60B15; secondary 60E05; 43A30

Keywords: Invariant random fields; Fourier expansions; Characterization of Gaussian random fields

1. Introduction

Recently much interest has been attracted to the investigation of properties of random fields on the sphere $\mathbb{S}^2$ that are invariant (in distribution) with respect to the action of the rotation group $SO(3)$, highlighting a certain number of interesting features (see [1,9] e.g.). This interest is motivated mainly by the modeling and the investigation of cosmological data.

For instance, in [1] it was proved that assumptions of independence of the Fourier coefficients of the development in spherical harmonics

$$T = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m} \tag{1.1}$$

of a real random field T, in addition to invariance, imply Gaussianity. More precisely it was proved that if the field is invariant and the coefficients $a_{\ell m}$, $\ell = 1, 2, \dots, 0 \le m \le \ell$ are independent, then the field is necessarily Gaussian (since the field is real, the other coefficients are constrained by the condition $a_{\ell,-m} = (-1)^m \overline{a_{\ell m}}$). This result implies, in particular, the relevant consequence that a non-Gaussian invariant random field on $\mathbb{S}^2$ *cannot* be simulated using independent coefficients.

It is then a natural question whether this property also holds for invariant random fields on more general structures. A result in this direction was obtained in [2] where it was proved that a similar statement holds in general for an invariant random field on the homogeneous space of a compact Lie group, provided the development is made with respect to a suitable Fourier basis satisfying a particular condition.

The main object of this paper is to pursue this line of investigation in the direction of determining new examples where assumptions of independence for the coefficients imply Gaussianity for an invariant random field and to better understand this phenomenon.

In particular we give a new condition, equivalent to the one that was introduced in [2] and therefore ensuring the property above, which is satisfied for every self-conjugated Fourier basis of the sphere $\mathbb{S}^2$ (and not just the spherical harmonics) and also for an important class of self-conjugated bases of the sphere $\mathbb{S}^3$. A relevant consequence is that it is not possible to simulate a non-Gaussian invariant real random field on $\mathbb{S}^2$ using independent coefficients with respect to *any* self-conjugated bases of the irreducible G-modules of $L^2(\mathbb{S}^2)$.

Besides this characterization of Gaussianity, we discuss other properties of the Fourier coefficients of an invariant random field such as orthogonality (it is well known that the $a_{\ell m}$'s of the development (1.1) on $\mathbb{S}^2$ for an invariant random field with finite variance are orthogonal, see [9], p. 140) and invariance of their distribution with respect to rotations of the complex plane.

The plan of the paper is as follows. In Section 2 we recall the main properties of the Fourier development of a random field on the homogeneous space $\mathscr{X}$ of a compact group and give necessary and sufficient conditions for the invariance of the field in terms of its development. In Section 3 we investigate properties of its coefficients as orthogonality, among other things. It turns out that, unlike the case of $\mathbb{S}^2$, they are not orthogonal in general and precisions are made concerning this phenomenon. In Section 4 we recall (from [2]) results giving the characterization of Gaussianity which is our main concern. As mentioned above, these results, in many cases of interest, hold under the assumption that the Fourier basis that is chosen for the Fourier development enjoys a certain property (Assumption 4.3) with respect to the action of the group. This section also contains a converse result, giving conditions on Gaussian coefficients in order to produce an invariant random field.

The remainder of the paper is devoted to the investigation of the validity of Assumption 4.3. In Section 5 we give a new equivalent condition which is the main tool for the investigation of the two main examples ($\mathbb{S}^2$ and $\mathbb{S}^3$) which are the objects of Section 6 and Section 7. We are also able to prove that independence of the Fourier coefficients entails Gaussianity in some situations of interest in which it is known that Assumption 4.3 does not hold (Theorem 6.4), in particular covering the case of the basis of the spherical harmonics $(Y_{\ell m})_m$ for $\ell = 1$ (which completes the proof of the result of [1]).

Finally Section 8 points out some open questions.

2. A.s. square integrable random fields and Fourier developments

Throughout this paper we denote by G a compact group, by K a closed subgroup and by $\mathscr{X}$ the homogeneous space G/K. We denote $(g,x) \mapsto gx$, $g \in G$ the action of G and by $\pi : G \to G/K$ the canonical projection. We denote respectively by $\mathrm{d}g$ and $\mathrm{d}x$ the normalized Haar measure on G and the unique G-invariant probability measure on $\mathscr{X}$. We write $L^2(\mathscr{X})$ for $L^2(\mathscr{X}, \mathrm{d}x)$ and $L^2(G)$ for $L^2(G, \mathrm{d}g)$. The spaces L^2 are spaces of *complex valued* square integrable functions.

Let us denote by $\widehat{G}$ the set of equivalence classes of irreducible representations of G. For every $\sigma \in \widehat{G}$, let H_σ a Hilbert unitary G-module of class σ fixed from now on. As soon as an orthonormal basis $h_1, \dots, h_m$, $m = \dim\sigma$, of H_σ is chosen, one can define the matrix coefficients

$$D^\sigma_{ij}(g) = \langle \sigma(g) h_j, h_i \rangle.$$

The Peter–Weyl theorem (see [12] or [3] e.g.) states that the normalized matrix elements $\sqrt{\dim\sigma}\, D^\sigma_{ij}$, $\sigma \in \widehat{G}$, $1 \le i, j \le \dim\sigma$, form an orthonormal complete basis of $L^2(G)$, so that, if we set, for $f \in L^2(G)$,

$$\widehat{f}(\sigma)_{ij} = \sqrt{\dim\sigma} \int_G f(g) D^\sigma_{ij}\bigl(g^{-1}\bigr)\,\mathrm{d}g = \sqrt{\dim\sigma} \int_G f(g) \overline{D^\sigma_{ji}(g)}\,\mathrm{d}g = \sqrt{\dim\sigma} \langle f, D^\sigma_{ji} \rangle_{L^2(G)}$$

we have the Fourier development

$$f(g)=\sum_{\sigma\in\widehat{G}}\sqrt{\dim\sigma}\sum_{i,j=1}^{\dim\sigma}\widehat{f}(\sigma)_{ij}D^{\sigma}_{ji}(g)=\sum_{\sigma\in\widehat{G}}\dim\sigma\sum_{i,j=1}^{\dim\sigma}\langle f,D^{\sigma}_{ji}\rangle_{L^2(G)}D^{\sigma}_{ji}(g). \tag{2.1}$$

The development (2.1) is actually independent of the choice of orthonormal bases of the G-modules H_σ: if for $f\in L^2(G)$ we set

$$\widehat{f}(\sigma):=\sqrt{\dim\sigma}\int_G f(g)\sigma\bigl(g^{-1}\bigr)\,\mathrm{d}g\in\operatorname{End}(H_\sigma)$$

then (2.1) can be written

$$f(g)=\sum_{\sigma\in\widehat{G}}\sqrt{\dim\sigma}\,\operatorname{tr}\bigl(\widehat{f}(\sigma)\sigma(g)\bigr). \tag{2.2}$$

Let us denote by L the left action of G on $L^2(G)$, so that for all $g,h\in G$ and all $f\in L^2(G)$, $L_gf(h)=f(g^{-1}h)$. It is immediate that the functions $(D^\sigma(g)_{ij})_{1\le i\le\dim\sigma}$, appearing in the columns of D^σ, span a subspace of $L^2(G)$ that is invariant and irreducible with respect to this left action. To be precise the action of G on this subspace is not in general equivalent to σ, but to its dual.

From (2.2) a similar development can be derived for $L^2(\mathscr{X})$, $\mathscr{X}=G/K$. Actually remark that, if e is the identity of G and $x_0=\pi(e)\in\mathscr{X}$, the relation $\widetilde{f}(g)=f(gx_0)$ uniquely identifies functions in $L^2(\mathscr{X})$ as functions in $L^2(G)$ that are right invariant under the action of K which is the isotropy group of x_0. For such functions f we have, for every $k\in K$,

$$\widehat{f}(\sigma)=\int_G f(g)\sigma\bigl(g^{-1}\bigr)\,\mathrm{d}g=\int_G f(gk)\sigma\bigl(g^{-1}\bigr)\,\mathrm{d}g=\int_G f(t)\sigma\bigl(kt^{-1}\bigr)\,\mathrm{d}t=\sigma(k)\int_G f(t)\sigma\bigl(t^{-1}\bigr)\,\mathrm{d}t=\sigma(k)\widehat{f}(\sigma)$$

which implies that $\widehat{f}(\sigma)$ is H^K_σ-valued, H^K_σ denoting the subspace of H_σ of vectors that are invariant under the action of K, i.e. $\widehat{f}(\sigma)\in\operatorname{Hom}(H_\sigma,H^K_\sigma)$. Hence, for a choice of an orthonormal basis $h_1,\dots,h_m$ of H_σ such that $h_1,\dots,h_k$ span H^K_σ, the matrix $\widehat{f}(\sigma)$ will have all zeros in the rows from the $(k+1)$th to the mth. For $f\in L^2(\mathscr{X})$ we shall consider, for simplicity, that $\widehat{f}(\sigma)$ is a $\dim\sigma\times\dim\sigma$ matrix with zeros on every row but for the first $\dim H^K_\sigma$ ones, corresponding to the first elements of the basis, that are supposed to be K-invariant. Remark that H^K_σ might be reduced to $\{0\}$.

As a consequence of the aforementioned Peter–Weyl theorem we have the decomposition, that we shall need later,

$$L^2(\mathscr{X})=\bigoplus_{\sigma\in\widehat{G}}\bigoplus_{1\le j\le\dim(H^K_\sigma)}V^\sigma_j, \tag{2.3}$$

where the V^σ_j are irreducible sub-G-modules of $L^2(\mathscr{X})$. For instance one can choose V^σ_j to be the span of the column $(D^\sigma_{ij})_{0\le i\le\dim\sigma}$. Even if it is not going to be relevant in the rest of the paper recall that, as mentioned above, the action of G is not, in general, equivalent to σ.

We consider on $\mathscr{X}$ a real or complex random field $(T(x))_{x\in\mathscr{X}}$. This means that there exists a probability space $(\Omega,\mathscr{F},\mathrm{P})$ on which the r.v.'s $T(x)$ are defined and we shall always assume joint measurability, i.e. $(x,\omega)\mapsto T(x,\omega)$ is $\mathscr{B}(\mathscr{X})\otimes\mathscr{F}$ measurable, $\mathscr{B}(\mathscr{X})$ denoting the Borel σ-field of $\mathscr{X}$.

T is said to be *a.s. continuous* if the map $x\mapsto T(x)$ is continuous a.s. It is said to be *a.s. square integrable* if

$$\int_{\mathscr{X}}\bigl|T(x)\bigr|^2\,\mathrm{d}x<+\infty,\quad\text{a.s.} \tag{2.4}$$

Remark that a.s. square integrability does not imply existence of moments of the r.v.'s $T(x)$. If T is a a.s. square integrable random field on G then the function $x\mapsto T(x,\omega)$ belongs to $L^2(\mathscr{X})$ a.s. and one can define "pathwise"

$$\widehat{T}(\sigma)=\sqrt{\dim\sigma}\int_G T(g)\sigma\bigl(g^{-1}\bigr)\,\mathrm{d}g \tag{2.5}$$

which is now a $\mathrm{End}(H_\sigma)$-valued r.v. Similarly we have the analog of (2.2), i.e.

$$T(h) = \sum_{\sigma\in\widehat{G}} \sqrt{\dim\sigma}\,\mathrm{tr}\big(\widehat{T}(\sigma)\sigma(h)\big) \tag{2.6}$$

or

$$T(h) = \sum_{\sigma\in\widehat{G}} \sqrt{\dim\sigma} \sum_{i,j=1}^{\dim\sigma} \widehat{T}(\sigma)_{ij} D^\sigma_{ji}(h) = \sum_{\sigma\in\widehat{G}} \dim\sigma \sum_{i,j=1}^{\dim\sigma} \big\langle T, D^\sigma_{ji}\big\rangle_{L^2(G)} D^\sigma_{ji}(h) \tag{2.7}$$

the series converging a.s. in $L^2(G)$.

For a random field T we define the rotated random field T^g as $T^g(x) = T(gx)$.

Definition 2.1. *A a.s. square integrable random field T on $\mathscr{X}$ is said to be G-*invariant *if, as a $L^2(\mathscr{X})$-valued random variable, it has the same distribution as the rotated random field T^g for every $g \in G$, in the sense that the joint laws of*

$$\big(T(x_1), \dots, T(x_m)\big) \quad \textit{and} \quad \big(T(gx_1), \dots, T(gx_m)\big) \tag{2.8}$$

coincide for every $g \in G$ and $x_1, \dots, x_m \in \mathscr{X}$.

More generally a family $(T_i)_{i\in\mathscr{I}}$ of random fields on $\mathscr{X}$ is said to be invariant if and only if for every choice of $g \in G$, $i_1, \dots, i_m \in \mathscr{I}$ and $x_1, \dots, x_m \in \mathscr{X}$, the joint laws of

$$\big(T_{i_1}(x_1), \dots, T_{i_m}(x_m)\big) \quad \textit{and} \quad \big(T_{i_1}(gx_1), \dots, T_{i_m}(gx_m)\big) \tag{2.9}$$

coincide.

The following will have some importance later. We thank D. Marinucci and G. Peccati for informing us of the existence of this result.

Proposition 2.2. *Let T be a a.s. square-integrable invariant random field on $\mathscr{X}$ and define, for $f \in L^2(\mathscr{X})$,*

$$T(f) := \int_{\mathscr{X}} T(x)\overline{f(x)}\,\mathrm{d}x. \tag{2.10}$$

Then, for every $g \in G$ and every $f_1, \dots, f_m \in L^2(G)$, the two random variables

$$\big(T(f_1), \dots, T(f_m)\big) \quad \textit{and} \quad \big(T^g(f_1), \dots, T^g(f_m)\big)$$

have the same distribution.

Proof. See [10]. □

Proposition 2.3. *Let T be a a.s. square integrable random field on G. Then T is invariant if and only if, for every $g \in G$, the two families of r.v.'s*

$$\big(\widehat{T}(\sigma)\big)_{\sigma\in\widehat{G}} \quad \textit{and} \quad \big(\widehat{T}(\sigma)\sigma(g)\big)_{\sigma\in\widehat{G}}$$

are equi-distributed.

Proof. Let us assume T invariant and let $\sigma \in \widehat{G}$. Then for every $v, w \in H_\sigma$ the function $g \mapsto \langle\sigma(g^{-1})v, w\rangle$ is bounded and therefore in $L^2(G)$. Therefore, thanks to Proposition 2.2 and denoting by $\sim$ equality in law, we have for every

$g \in G$,

$$\begin{aligned}\langle \widehat{T}(\sigma)v, w\rangle &= \sqrt{\dim\sigma}\int_G T(h)\langle\sigma(h^{-1})v, w\rangle\,\mathrm{d}h \sim \sqrt{\dim\sigma}\int_G T(gh)\langle\sigma(h^{-1})v, w\rangle\,\mathrm{d}h\\ &= \sqrt{\dim\sigma}\int_G T(t)\langle\sigma(t^{-1}g)v, w\rangle\,\mathrm{d}t = \sqrt{\dim\sigma}\int_G T(t)\langle\sigma(t^{-1})\sigma(g)v, w\rangle\,\mathrm{d}t = \langle\widehat{T}(\sigma)\sigma(g)v, w\rangle.\end{aligned}$$

This being true for every $v, w \in H_\sigma$, we have that, as $\mathrm{End}(H_\sigma)$-valued r.v.'s, $\widehat{T}(\sigma)$ and $\widehat{T}(\sigma)\sigma(g)$ have the same distribution. Quite similarly, only in a just more complicated way to write,

$$\bigl(\widehat{T}(\sigma_1), \dots, \widehat{T}(\sigma_n)\bigr) \quad\text{and}\quad \bigl(\widehat{T}(\sigma_1)\sigma_1(g), \dots, \widehat{T}(\sigma_n)\sigma_n(g)\bigr)$$

have the same distribution as $\mathrm{End}(H^{\sigma_1}) \oplus \cdots \oplus \mathrm{End}(H^{\sigma_n})$-valued r.v., thus proving the only if part of the statement. The converse follows easily from development (2.6). □

Let $f \in L^2(\mathscr{X})$ and $V \subset L^2(\mathscr{X})$ an irreducible G-module. We can then consider the orthogonal projection $P_V f$ of f on V. Similarly, for a a.s. square integrable random field T on $\mathscr{X}$, let us denote by T_V its orthogonal projection on V. Remark that by definition (the functions of V are necessarily continuous) T_V is always a continuous random field.

Let us denote by $D_{ij}(g)$ the matrix elements of the left regular action of G on V with respect to a fixed orthonormal basis $(v_i)_i$ of V and let us consider the random coefficients of the development of T with respect to this basis

$$a_i = \int_{\mathscr{X}} T(x)\overline{v_i(x)}\,\mathrm{d}x. \tag{2.11}$$

We denote by a the complex vector with components a_i, $i = 1, \dots, d$. Then the coefficients of the rotated random field T^g are

$$\begin{aligned}a_i^g &= \int_{\mathscr{X}} T(gx)\overline{v_i(x)}\,\mathrm{d}x = \int_{\mathscr{X}} T(x)\overline{v_i(g^{-1}x)}\,\mathrm{d}x\\ &= \sum_{k=1}^d \overline{D_{ki}(g)}\int_{\mathscr{X}} T(x)\overline{v_k(x)}\,\mathrm{d}x = \sum_{k=1}^d D_{ik}(g^{-1})a_k\end{aligned} \tag{2.12}$$

that is

$$a^g = D(g^{-1})a. \tag{2.13}$$

As

$$T_V(x) = \sum_{k=1}^d a_k v_k(x),$$

it is immediate that T_V is invariant if and only if the random vectors a and $D(g)a$ have the same distribution.

With respect to the Peter–Weyl decomposition (2.3) we have

$$T = \sum_{\sigma\in\widehat{G}}\sum_{i=1}^{\dim H_\sigma^K} T_{V_i^\sigma}.$$

Using the fact that the projections are G-equivariant (i.e. commute with the action of G) it is easy to prove the following, not really unexpected, statement (anyway see Proposition 3 of [11] for a proof).

Proposition 2.4. *T is invariant if and only if the family $(T_{V_i^\sigma})_{\sigma\in\widehat{G}, 1\le i\le \dim H_\sigma^K}$ of random fields is invariant.*

We shall therefore concentrate our attention mainly on the projected random fields T_V. When dealing with a real random field it is natural to require that the basis $v_1, \dots, v_d$ of the G-module V, with respect to which the coefficients are computed "respects" the real and imaginary parts and, in particular, if $V = \overline{V}$, that this basis is stable under conjugation. As explained in [2], Section 2 and the Appendix, it is actually possible to decompose $L^2(\mathscr{X})$ into a direct sum of irreducible G-modules in the form

$$L^2(\mathscr{X}) = \bigoplus_{i \in \mathscr{I}^0} V_i \oplus \bigoplus_{i \in \mathscr{I}^+} (V_i \oplus \overline{V_i}), \tag{2.14}$$

where the direct sums are orthogonal and

$$i \in \mathscr{I}^0 \quad \Leftrightarrow \quad V_i = \overline{V_i}, \qquad i \in \mathscr{I}^+ \quad \Leftrightarrow \quad V_i \perp \overline{V_i}.$$

We can therefore choose an orthonormal basis $(v_{ik})_{ik}$ of $L^2(\mathscr{X})$ such that ($d_i = \dim V_i$):

- for $i \in \mathscr{I}^0$, $(v_{ik})_{1 \le k \le d_i}$ is an orthonormal basis of V_i stable under conjugation;
- for $i \in \mathscr{I}^+$, $(v_{ik})_{1 \le k \le d_i}$ is an orthonormal basis of V_i and $(\overline{v_{ik}})_{1 \le k \le d_i}$ is an orthonormal basis of $\overline{V_i}$.

It is immediate that if T is a real random field and $i \in \mathscr{I}^0$ then T_{V_i} is also a real random field. On the other hand, if $i \in \mathscr{I}^+$ then T_{V_i} and $T_{\overline{V_i}}$ may not be real (actually they cannot be real unless they vanish), whereas $T_{V_i} + T_{\overline{V_i}}$ will be real.

Remark 2.5. *Representations of a compact Lie group G are classically classified as of real, complex or quaternionic type (see for instance* [3], *p.* 93). *In order to be self-contained let us recall that a* conjugation *J of a G-module V is an antilinear ($J(\alpha v) = \overline{\alpha} J(v)$) equivariant map $J : V \to V$.*

A G-module V is said to be real *if there exists a conjugation $J : V \to V$ such that $J^2 = 1$ and* quaternionic *if there exists a conjugation $J : V \to V$ such that $J^2 = -1$. It is* complex *if it is neither real nor quaternionic.*

The important thing is that an irreducible G-module is of one and only one of these types and that equivalent G-modules are necessarily of the same type. If an irreducible G-module $V \subset L^2(\mathscr{X})$ is such that $\overline{V} = V$, the usual conjugation $J : v \to \overline{v}$ is a real conjugation, so that V must be of real type. In particular, if a representation is of quaternionic or complex type, it cannot contain in its isotypical space a G-module that is self-conjugated, so that in the decomposition (2.14) *it cannot be of type $\mathscr{I}^0$.*

The irreducible representations of even dimension of $SU(2)$ are quaternionic and the corresponding G-modules of this group cannot, therefore, be self-conjugated.

3. Properties of the coefficients

In this section we give results concerning two properties that are enjoyed by the coefficients $\widehat{T}(\sigma)_{ij}$, $\sigma \in \widehat{G}$, $1 \le i, j \le \dim \sigma$, of the Fourier development of an invariant random field on $\mathscr{X}$.

A random field T is said to have finite variance if

$$\mathrm{E}\left(\int_{\mathscr{X}} \left| T(x) \right|^2 \mathrm{d}x \right) < +\infty. \tag{3.1}$$

Remark 3.1. *If* (3.1) *holds, then the map $x \mapsto T(x)$ necessarily belongs to $L^2(\mathscr{X})$ a.s., so that T is also a.s. square integrable. Moreover, by the Cauchy–Schwarz inequality, if T has finite variance, the random variables $T(f)$, $f \in L^2(\mathscr{X})$, defined in* (2.10), *have finite variance. In particular the Fourier coefficients of T, with respect to any Fourier basis, also have finite variance.*

It is well known (see [1], [9], p. 126) that in the case $\mathscr{X} = \mathbb{S}^2$, $G = SO(3)$, if T is invariant and has finite variance, its Fourier coefficients with respect to the basis formed by the spherical harmonics (see for instance [9], p. 64, for definitions) are pairwise orthogonal. Our first concern in this section is to investigate this question in the case of a more general basis and for a general homogeneous space of a compact Lie group.

Remark that, if T is invariant and has finite variance, then for every σ that is not the trivial representation we have

$$\mathrm{E}\big[\widehat{T}(\sigma)\big]=\mathrm{E}\big[\widehat{T}(\sigma)\sigma(g)\big]=\mathrm{E}\big[\widehat{T}(\sigma)\big]\int_G \sigma(g)\,\mathrm{d}g=0. \tag{3.2}$$

Theorem 3.2. *Let T be a finite variance invariant random field on $\mathscr{X}$ and $\sigma_1,\sigma_2\in\widehat{G}$.*

(a) *If σ_1 and σ_2 are not equivalent, then, for every orthonormal bases of H^{σ_1} and H^{σ_2} the r.v.'s $\widehat{T}(\sigma_1)_{ij}$ and $\widehat{T}(\sigma_2)_{k\ell}$ are orthogonal*, $1\le i,j\le\dim\sigma_1$, $1\le k,\ell\le\dim\sigma_2$.
(b) *If $\sigma_1=\sigma_2=\sigma$ let $\Gamma(\sigma)=\mathrm{E}[\widehat{T}(\sigma)\widehat{T}(\sigma)^*]$. Then $\operatorname{Cov}(\widehat{T}(\sigma)_{ij},\widehat{T}(\sigma)_{k\ell})=\delta_{j\ell}\Gamma(\sigma)_{ik}$. In particular coefficients belonging to different columns are orthogonal and the covariance between entries in different rows of a same column does not depend on the column.*

Proof. Let us denote by $D^{\sigma}_{ij}(g)$ the matrix elements of the action of G on H_σ with respect to a given orthonormal basis. Recall that $\widehat{T}(\sigma)_{ij}=\sqrt{\dim\sigma}\langle T,D^{\sigma}_{ji}\rangle_{L^2(G)}$ so that, thanks to Remark 3.1, the r.v.'s $\widehat{T}(\sigma)_{ij}$'s have themselves finite variance.

(a) By Proposition 2.3 $(\widehat{T}(\sigma_1),\widehat{T}(\sigma_2))$ has the same joint distribution as $(\widehat{T}(\sigma_1)\sigma_1(g),\widehat{T}(\sigma_2)\sigma_2(g))$ for every $g\in G$, which implies that, as matrices, $(\widehat{T}(\sigma_1),\widehat{T}(\sigma_2))$ and $(\widehat{T}(\sigma_1)D^{\sigma_1}(g),\widehat{T}(\sigma_2)D^{\sigma_2}(g))$ have the same joint distribution. Therefore we have

$$\begin{aligned}\mathrm{E}\big[\widehat{T}(\sigma_1)_{ij}\overline{\widehat{T}(\sigma_2)_{k\ell}}\big]&=\mathrm{E}\big[\big(\widehat{T}(\sigma_1)D^{\sigma_1}(g)\big)_{ij}\overline{\big(\widehat{T}(\sigma_2)D^{\sigma_2}(g)\big)_{k\ell}}\big]\\&=\sum_{r=1}^{\dim\sigma_1}\sum_{m=1}^{\dim\sigma_2}D^{\sigma_1}_{rj}(g)\overline{D^{\sigma_2}_{m\ell}(g)}\mathrm{E}\big[\widehat{T}(\sigma_1)_{ir}\overline{\widehat{T}(\sigma_2)_{km}}\big].\end{aligned}$$

This being true for every $g\in G$, it is also true if we take the integral of the right-hand side over G in dg. As the functions $D^{\sigma_1}_{rj}$ and $D^{\sigma_2}_{m\ell}$ are orthogonal for every choice of the indices, the representations σ_1 and σ_2 being not equivalent, we find

$$\mathrm{E}\big[\widehat{T}(\sigma_1)_{ij}\overline{\widehat{T}(\sigma_2)_{k\ell}}\big]=0.$$

(b) If $\sigma_1=\sigma_2=\sigma$, the previous computation gives

$$\begin{aligned}\mathrm{E}\big[\widehat{T}(\sigma)_{ij}\overline{\widehat{T}(\sigma)_{k\ell}}\big]&=\sum_{r,m=1}^{\dim\sigma}\mathrm{E}\big[\widehat{T}(\sigma)_{ir}\overline{\widehat{T}(\sigma)_{km}}\big]\int_G D^{\sigma}_{rj}(g)\overline{D^{\sigma}_{m\ell}(g)}\,\mathrm{d}g\\&=\sum_{r,m=1}^{\dim\sigma}\mathrm{E}\big[\widehat{T}(\sigma)_{ir}\overline{\widehat{T}(\sigma)_{km}}\big]\delta_{rm}\delta_{j\ell}=\delta_{j\ell}\sum_{r=1}^{\dim\sigma}\mathrm{E}\big[\widehat{T}(\sigma)_{ir}\overline{\widehat{T}(\sigma)_{kr}}\big]=\delta_{j\ell}\Gamma(\sigma)_{ik}.\end{aligned}$$

□

Theorem 3.2 states that the entries of $\widehat{T}(\sigma)$ are not pairwise orthogonal unless the matrix Γ is diagonal. This fact has actually already been remarked by other authors (see [8], Theorem 2, e.g.) and Example 3.3 below provides an instance of this phenomenon.

Of course there are situations in which orthogonality is still guaranteed: when the dimension of H^K_σ is 1 at most (i.e. in every irreducible G-module the dimension of the space H^K_σ of the K-invariant vectors in one at most) as is the case for $G=SO(d)$, $K=SO(d-1)$, $G/K=\mathbb{S}^{d-1}$. In this case actually the matrix $\widehat{T}(\sigma)$ has just one row that does not vanish and $\Gamma(\sigma)$ is all zeros, but one entry in the diagonal.

Let us recall first a well known definition.

Let $Z=Z_1+\mathrm{i}Z_2$ a complex r.v. Z is said to be *Gaussian complex valued* if (Z_1,Z_2) is jointly Gaussian. Z is said to be *complex Gaussian* if, in addition, Z_1 and Z_2 are independent and have the same variance. If Z is centered this is equivalent to the requirement that their distribution is invariant with respect to rotations of the complex plane. We shall use the following properties.

- A centered Gaussian complex valued r.v. Z is complex Gaussian if and only if $\mathrm{E}[Z^2]=0$.

- Two centered complex valued Gaussian r.v.'s Z_1, Z_2 are independent if and only if $\mathrm{E}[Z_1\overline{Z_2}] = \mathrm{E}[Z_1 Z_2] = 0$.

Example 3.3. *Let $\sigma \in \widehat{G}$ and $V \subset L^2(G)$ an irreducible G-module of dimension $d \ge 2$ of class σ and denote by the matrix $D^\sigma(g)$ the action of G on V with respect to a fixed basis. Let $Z_1, \dots, Z_d$ be independent centered complex Gaussian r.v.'s such that $\mathrm{E}[|Z_j|^2] = 1$ for every j. Let $B = (b_{ij})_{ij}$ be the random matrix defined as $b_{ij} = \alpha_i Z_j$, $\alpha_i \in \mathbb{C}$. Then the random field*

$$T(g) = \sqrt{d}\,\mathrm{tr}\big(B D^\sigma(g)\big)$$

is invariant and, as it is immediate that $\widehat{T}(\sigma) = B$, its coefficients $\widehat{T}(\sigma)_{ij}$ are not pairwise orthogonal. Let us check invariance. Let $C = \widehat{T}(\sigma) D^\sigma(g)$, then $c_{ij} = \alpha_i \sum_{k=1}^d Z_k D^\sigma_{kj}(g) = \alpha_i W_j$ where

$$W_j = \sum_{k=1}^d Z_k D^\sigma_{kj}(g).$$

In view of Proposition 2.3 we must therefore just prove that the W_j's are complex Gaussian, independent and that $\mathrm{E}[|W_j|^2] = 1$. First it is immediate that they are Gaussian complex valued. We have also

$$\begin{aligned}\mathrm{E}[W_j\overline{W_k}] &= \mathrm{E}\Bigg[\sum_{h,r=1}^d Z_h\overline{Z_r} D^\sigma_{hj}(g)\overline{D^\sigma_{rk}(g)}\Bigg] = \sum_{h,r=1}^d \delta_{hr} D^\sigma_{hj}(g)\overline{D^\sigma_{rk}(g)} \\ &= \sum_{r=1}^d D^\sigma_{rj}(g)\overline{D^\sigma_{rk}(g)} = \sum_{r=1}^d D^\sigma_{kr}\big(g^{-1}\big) D^\sigma_{rj}(g) = \delta_{kj}.\end{aligned} \tag{3.3}$$

Similarly, as $\mathrm{E}[Z_h Z_r] = 0$ for every $1 \le h, r \le d$ (recall that $\mathrm{E}[Z^2] = 0$ for a centered complex Gaussian r.v. Z),

$$\mathrm{E}[W_j W_k] = \mathrm{E}\Bigg[\sum_{h,r=1}^d Z_h Z_r D^\sigma_{hj}(g) D^\sigma_{r\ell}(g)\Bigg] = 0. \tag{3.4}$$

(3.3) for $k = j$ gives $\mathrm{E}[|W_j|^2] = 1$, whereas (3.3) and (3.4) together imply that W_j and W_k, $k \ne j$, are independent. Finally (3.4) for $k = j$ gives $\mathrm{E}[W_j^2] = 0$ for every j so that the W_j's are complex Gaussian, which completes the proof.

Arguments similar to the proof of Theorem 3.2(c) allow to prove the following.

Corollary 3.4. *Let T an invariant random field with finite variance on $\mathscr{X}$ and let $V \subset L^2(\mathscr{X})$ an irreducible G-module different from the constants. Then the coefficients $(a_k)_k$ of the development of the projection T_V of T on V with respect to any orthonormal basis of V are centered, orthogonal, and have a common variance c.*

Proof. As pointed out in Remark 3.1 the coefficients a_k's have themselves finite variance and, thanks to (2.12) and V being different from the constants, they are also centered. From (2.13) we have, for every $g \in G$,

$$\mathrm{E}[a_k\overline{a_\ell}] = \mathrm{E}\big[\big(D(g)a\big)_k\overline{\big(D(g)a\big)_\ell}\big] = \sum_{j,r=1}^d D_{kr}(g)\overline{D_{\ell j}(g)}\mathrm{E}[a_r\overline{a_j}].$$

Integrating in $\mathrm{d}g$ and using the orthonormality properties of the matrix elements $D_{ij}(g)$ we find

$$\mathrm{E}[a_k\overline{a_\ell}] = \frac{1}{\dim V}\sum_{j,r=1}^{\dim V} \delta_{k\ell}\delta_{rj}\mathrm{E}[a_r\overline{a_j}] = \frac{1}{\dim V}\delta_{k\ell}\sum_{j=1}^{\dim V}\mathrm{E}\big[|a_j|^2\big].$$

For $k \neq \ell$ this gives immediately the orthogonality, whereas for $k = \ell$ we have

$$\mathrm{E}\big[|a_k|^2\big] = \frac{1}{\dim V} \sum_{j=1}^{\dim V} \mathrm{E}\big[|a_j|^2\big]$$

so that the a_k's have the same variance. □

Another feature appearing in the case $G = SO(3)$, $\mathscr{X} = \mathbb{S}^2$ is that if we consider the development of an invariant random field with respect to the classical basis of the spherical harmonics (see for instance [9], p. 64, for definitions) as in (1.1), then the coefficients $a_{\ell m}$ have each a distribution that is invariant with respect to rotations of the complex plane *if* $m \neq 0$. See for instance [9], p. 142, on this point. The following discussion aims at seeing what can be said for a general homogeneous space $\mathscr{X}$ concerning this property.

Remark 3.5. *Let G be a compact connected Lie group, $V \subset L^2(\mathscr{X})$ an irreducible G-module of dimension $d > 1$ and $\mathbb{T} \subset G$ a maximal torus. Let*

$$V = \bigoplus_{k=1}^{d} U_k \tag{3.5}$$

be a decomposition of V into orthogonal irreducible components of the action of $\mathbb{T}$ on V. As $\mathbb{T}$ is Abelian, $\dim(U_k) = 1$ for every $k = 1, \dots, d$. Let $u_k \in U_k$ be a unit vector. Then $L_t u_k = u_k(t^{-1}\cdot) = \chi_k(t) u_k$ for $t \in \mathbb{T}$, where χ_k denotes the character of the representation of $\mathbb{T}$ on U_k. If we consider the Fourier development

$$T = \sum_{k=1}^{d} a_k u_k$$

of an invariant random field T with respect to the orthonormal basis $(u_1, \dots, u_d)$ then, as for $t \in \mathbb{T}$

$$T(tx) = \sum_{k=1}^{d} a_k u_k(tx) = \sum_{k=1}^{d} a_k \overline{\chi_k(t)} u_k(x),$$

and $T(tx)$ and $T(x)$ have the same distribution, necessarily for every k such that the action of $\mathbb{T}$ on U_k is not trivial (that is $\chi_k(t) \not\equiv 1$) the coefficient a_k must be invariant in distribution with respect to rotations of the complex plane (and therefore, if it is Gaussian, it must be complex Gaussian).

Remark also that the action of $\mathbb{T}$ on V cannot be trivial, that is $\chi_k \not\equiv 1$ for some k necessarily. Actually, as all maximal tori are conjugated (that is if $\mathbb{T}'$ is another maximal torus then $\mathbb{T}' = g\mathbb{T}g^{-1}$ for some $g \in G$) then the action of all maximal tori on V would be trivial which is impossible as the union of all maximal tori is the group itself so that this would imply that the action of G itself is trivial, whereas we assumed V to be irreducible and with dimension $d > 1$.

The property, mentioned above, of the random coefficients with respect to the basis of the spherical harmonics in the case of the $\mathbb{S}^2$, appears now as a particular case.

4. Invariant random fields with independent Fourier coefficients

In this section we see results that state that independence assumptions on the Fourier coefficients imply Gaussianity of the coefficients and of the corresponding random field.

Theorems 4.1 and 4.4 below are already known (see [2]) and we reproduce them only to be self-contained, our main concern being the investigation of the validity of Assumption 4.3, which is a necessary condition in many situations of interest.

Theorem 4.1. *Assume that G is a compact connected Lie group. Let T be a a.s. square integrable G-invariant random field on the homogeneous space $\mathscr{X}$ of G. Let V be an irreducible G-module of $L^2(\mathscr{X})$ with dimension $d > 1$ and let us assume that coefficients $(a_k)_k$ of the development*

$$T_V = \sum_{k=1}^{d} a_k v_k$$

with respect to an orthonormal basis $(v_k)_k$ of V are independent. Then they are necessarily Gaussian and the random field T_V is Gaussian itself.

Remark that in the statement of Theorem 4.1, as in Theorem 4.4 below, we make no assumption concerning the integrability or the existence of finite moments of the r.v.'s $T(x)$ and/or a_k. But, of course, under the assumptions of the theorem it follows that necessarily the r.v.'s $T_V(x)$ and a_k has finite moments of every order.

The proof of Theorem 4.1 relies on the following Skitovich–Darmois theorem, actually proved in this version by Ghurye and Olkin [6] (see also [7]).

Theorem 4.2. *Let $X_1, \dots, X_r$ be mutually independent random vectors with values in $\mathbb{R}^n$. If, for some real nonsingular $n \times n$ matrices A_j, B_j, $j = 1, \dots, r$, there are two linear statistics*

$$L_1 = \sum_{j=1}^{r} A_j X_j, \qquad L_2 = \sum_{j=1}^{r} B_j X_j$$

that are independent, then the vectors $X_1, \dots, X_r$ are Gaussian.

Proof of Theorem 4.1. Let us denote again by $D(g)$ the representative matrix of the left action of $g \in G$ on V with respect to the orthonormal basis $(v_k)_k$ and by a the vector of the coefficients a_k. Thanks to (2.13) we have

$$a \overset{\text{distr}}{=} a^g = D\bigl(g^{-1}\bigr)a.$$

Let $1 \le k_1 < k_2 \le \dim V$. Then the joint distribution of a_{k_1} and a_{k_2} is the same as the joint distribution of

$$\sum_{j=1}^{d} D_{k_1 j}\bigl(g^{-1}\bigr)a_j \quad \text{and} \quad \sum_{j=1}^{d} D_{k_2 j}\bigl(g^{-1}\bigr)a_j$$

which are therefore themselves independent. Thus we have found two linear statistics of the r.v.'s a_k that are independent. Therefore, it will follow from the Skitovich–Darmois theorem (Theorem 4.2) that the joint distribution of the a_k's is Gaussian complex-valued as soon as we will have proved that there exists at least one element $g \in G$ such that the real linear transformations

$$\mathbb{C} \ni z \mapsto D_{k_1 j}(g)z \quad \text{and} \quad \mathbb{C} \ni z \mapsto D_{k_2 j}(g)z, \qquad j = 1, \dots, d$$

are nondegenerate. This follows from analyticity properties of the coefficients, as explained at the end of the proof of Proposition 4.8 of [2] (in this part of the proof the connectedness of G is required). □

Remark that the result above does not hold for 1-dimensional G-modules. As shown in [2], Example 3.7, it is possible to construct a non-Gaussian invariant random field on the torus having all its coefficients independent.

Theorem 4.1 is not really satisfactory because its assumptions are not satisfied in the case of *real* random fields, for which the coefficients are necessarily constrained by the fact that the imaginary parts must cancel and therefore cannot, in general, be independent. Theorem 4.1 has however its own interest because it contains the essence of the arguments that we use in the sequel and because it holds without any assumption concerning the orthonormal basis $(v_k)_k$ of V.

We consider now the case of a *real* G-invariant random field T.

The results are different according to the fact that the irreducible G-module V under consideration is of type $\mathscr{I}^+$ or $\mathscr{I}^0$, as classified at the end of Section 2.

In the case $V \in \mathscr{I}^+$ the fact that we deal with a real random field does not impose constraints on the coefficients of T_V and $T_{\overline{V}}$ but, of course, in order to obtain a real random field one must impose that in the sum $T_V + T_{\overline{V}}$ the imaginary parts cancel. In this situation T_V and $T_{\overline{V}}$ will be both complex, in general, but their sum will give rise to a real random field.

In this case Theorem 4.1 states that, if $\dim V > 1$ and the coefficients $(a_k)_k$ are independent, then T_V and $T_{\overline{V}}$ are both Gaussian and their sum is a Gaussian real random field.

On the other hand if $V \in \mathscr{I}^0$ the coefficients must satisfy some constraints in order to ensure that the random field is real. It is natural then to consider the setting of an orthonormal basis that is self-conjugated.

It is actually appealing to consider a basis that is formed by real functions. For such a basis, say $(v_k)_k$, and under the assumption that the coefficients $a_k = \int_{\mathbb{S}^2} T(x) v_k(x)\,\mathrm{d}x$ are independent, Theorem 4.1 applies so that if the random field T is invariant then the a_k's are jointly Gaussian. Such a statement turns out however to be much weaker than the one we are going to state now, as explained below in Remark 4.6.

We shall consider the case of a basis $(v_k)_{-\ell \le k \le \ell}$ of V such that

$$v_{-k} = \overline{v_k}. \tag{4.1}$$

This means that we assume that, if the dimension of V is odd, the basis contains only one element v_0 which is a real function. If the dimension of V is even we shall still write $(v_k)_{-\ell \le k \le \ell}$ in order to simplify the notations (there is however no v_0 function). In the following arguments we shall consider the case where $\dim V$ is odd, the case $\dim V$ even being quite similar.

For a basis satisfying (4.1) the fact that T is real imposes to the coefficients the requirement $a_{-k} = \overline{a_k}$. This is the usual setting in the case $\mathscr{X} = \mathbb{S}^2$, where, to be precise, usually one considers the basis of the spherical harmonics for which it holds $v_{-k} = (-1)^k \overline{v_k}$ so that the condition above becomes $a_{-k} = (-1)^k \overline{a_k}$, a slight difference that does not change things. The argument in the case $V = \overline{V}$ can be implemented along the same lines as in Theorem 4.1: let us assume that the coefficients $(a_k)_{k \ge 0}$, are independent, then, if $m_1 \ge 0$, $m_2 \ge 0$, $m_1 \ne m_2$, and we denote as above by $D_{m,m'}(g)$ the matrix elements of the action of G on V, the two complex r.v.'s

$$\widetilde{a}_{m_1} = \sum_{m=-\ell}^{\ell} D_{m_1,m}\bigl(g^{-1}\bigr) a_m \quad \text{and} \quad \widetilde{a}_{m_2} = \sum_{m=-\ell}^{\ell} D_{m_2,m}\bigl(g^{-1}\bigr) a_m \tag{4.2}$$

have the same joint distribution as a_{m_1} and a_{m_2} and are therefore independent. The Skitovich–Darmois theorem cannot be applied as before, as a_m and $a_{-m} = \overline{a_m}$ are certainly not independent. But (4.2) can be written

$$\widetilde{a}_{m_1} = D_{m_1,0}\bigl(g^{-1}\bigr) a_0 + \sum_{m=1}^{\ell} \bigl(D_{m_1,m}\bigl(g^{-1}\bigr) a_m + D_{m_1,-m}\bigl(g^{-1}\bigr) \overline{a_m}\bigr),$$

$$\widetilde{a}_{m_2} = D_{m_2,0}\bigl(g^{-1}\bigr) a_0 + \sum_{m=1}^{\ell} \bigl(D_{m_2,m}\bigl(g^{-1}\bigr) a_m + D_{m_2,-m}\bigl(g^{-1}\bigr) \overline{a_m}\bigr)$$

so that $\widetilde{a}_{m_1}$, $\widetilde{a}_{m_2}$ are (real) linear functions of the independent r.v.'s $a_0, \dots, a_\ell$. Therefore we can again apply Theorem 4.2 as soon as we prove that an element $g \in G$ can be chosen so that the real linear applications

$$z \mapsto D_{m_i,m}\bigl(g^{-1}\bigr) z + D_{m_i,-m}\bigl(g^{-1}\bigr) \overline{z}, \quad m = 1, \dots, \ell, i = 1, 2 \tag{4.3}$$

are non singular. It is immediate that this is equivalent to require that

$$\bigl|D_{m_i,m}\bigl(g^{-1}\bigr)\bigr| \ne \bigl|D_{m_i,-m}\bigl(g^{-1}\bigr)\bigr|, \quad m = 1, \dots, \ell, i = 1, 2. \tag{4.4}$$

We are therefore led to state our main result under the assumption that (4.4) is fulfilled.

Assumption 4.3 (The mixing condition). *Let $V \subset L^2(\mathscr{X})$ a self-conjugated irreducible G-module and let $(v_i)_{-\ell\le i\le\ell}$ a self-conjugated orthonormal basis of V. Let us denote by $D(g)$ the representative matrix of the action of G on V. We say that the self-conjugated basis $(v_i)_{-\ell\le i\le\ell}$ is* mixing *if there exist $g\in G$ and $0\le m_1<m_2\le\ell$ such that*

$$\left|D_{m_i,m}(g)\right| \ne \left|D_{m_i,-m}(g)\right| \tag{4.5}$$

for every $0<m\le\ell$, $i=1,2$.

We have therefore proved the following.

Theorem 4.4. *Let G be a compact connected Lie group and T a a.s. square integrable* real *G-invariant random field on the homogeneous space $\mathscr{X}$ of G. Let $V\subset L^2(\mathscr{X})$ be an irreducible G-module such that $\overline{V}=V$. Let $(v_k)_{-\ell\le k\le\ell}$ be a self-conjugated mixing* (*see Assumption 4.3*) *basis of V. Consider the* real *random field*

$$T_V(x)=\sum_k a_k v_k(x), \tag{4.6}$$

where the r.v.'s a_k, $k\ge 0$ are independent. Then if T_V is G-invariant the r.v.'s $(a_k)_k$ are jointly Gaussian and therefore also T_V is Gaussian.

Remark 4.5. *It is relevant to point out that in Theorems 4.1 and 4.4 we do not assume independence of the real and imaginary parts of the coefficients. Actually under this additional assumption the statement becomes almost trivial* (*and much weaker*) *in many situations, as often the invariance of the random field implies that the coefficients* (*some of them at least, see Remark 3.5*) *have a distribution that is invariant with respect to rotations of the complex plane. And it is well known that this assumption together with independence of the components implies a joint Gaussian distribution, with no need of Assumption 4.3* (*immediate consequence of the Bernstein–Kac theorem as recalled in Proposition 6.3 below*).

This point is important with respect to one of the practical consequences of these results, which is the simulation of invariant random fields. Actually a natural and computationally efficient procedure in order to simulate a random field on $\mathscr{X}$ is by sampling its Fourier coefficients. For the case $\mathscr{X}=\mathbb{S}^2$, for instance, Theorem 4.4 together with the fact that the basis of the spherical harmonics is mixing ([9], *p.* 145) *entails that if the coefficients $a_{\ell m}$'s, $m\ge 0$, of the corresponding development are independent, then, in order to obtain an invariant random field, they must be Gaussian and the resulting random field will be Gaussian itself. Different choices of their distribution will lead to a random field which cannot be invariant. In particular the choice of independent r.v.'s $a_{\ell m}$'s, $m\ge 1$ with a complex Cauchy distribution, for example, cannot produce an invariant random field*, even if the real and imaginary parts of $a_{\ell m}$ are not independent.

We shall improve this statement in the following sections. As a consequence of Theorem 6.2 below it is not possible to simulate a non-Gaussian random field on $\mathbb{S}^2$ using independent coefficients, with respect to any *self-conjugated basis.*

Remark 4.6. *As remarked above, if the G-module V is self-conjugated as in Theorem 4.4, it is natural to consider an orthonormal basis on V that is formed by real functions. For instance in the case $\mathscr{X}=\mathbb{S}^2$, denoting as usual by $(Y_{\ell,m})_{\ell,m}$ the Fourier basis of the spherical harmonics* (*see again* [9], *p.* 64) *one might consider the orthonormal basis given by $v_{\ell 0}=Y_{\ell 0}$ and*

$$v_{\ell m}=\frac{1}{\sqrt{2}}\big(Y_{\ell m}+(-1)^m Y_{\ell,-m}\big), \qquad v_{\ell,-m}=\frac{1}{\mathrm{i}\sqrt{2}}\big(Y_{\ell m}-(-1)^m Y_{\ell,-m}\big), \quad m\ge 0.$$

The functions $v_{\ell m}$ are real and, if we denote $a_{\ell m}$ the coefficients of the real random field T with respect to the basis of the spherical harmonics, then the coefficients with respect to the basis $(v_{\ell m})_m$ would be $b_{\ell 0}=a_{\ell 0}$ and

$$b_{\ell m}=\sqrt{2}\,\mathrm{Re}\,a_{\ell m}, \qquad b_{\ell,-m}=\sqrt{2}\,\mathrm{Im}\,a_{\ell,m}, \quad m\ge 1.$$

They are of course real r.v.'s. A repetition of the arguments of Theorem 4.1 now gives immediately that invariance of the random field and independence of the coefficients $(b_{\ell m})_m$ imply joint Gaussianity of the coefficients $(b_{\ell m})_m$ without bothering with Assumption 4.3. This would however be a much weaker result, as independence of the $(b_{\ell m})_m$'s would imply independence of the real and imaginary parts of the $(a_{\ell m})_m$'s, which is not required in Theorem 4.4 as pointed out above in Remark 4.5.

Let V an irreducible self-conjugated G-module, T_V a real random field as in (4.6) and $(v_k)_{-\ell\le k\le \ell}$ a self-conjugated basis as above. Then by Theorem 4.4, under Assumption 4.3, if the coefficients a_k, $k\ge 0$, with respect to the given basis are independent they are Gaussian. Moreover, by Corollary 3.4, as they must have the same variance and be orthogonal, there exists $c\ge 0$ such that for $k\ne 0$ $\mathrm{E}[(\Re a_k)^2]=\mathrm{E}[(\Im a_k)^2]=\frac{c}{2}$ (this is a consequence of the orthogonality of a_k and $a_{-k}=\overline{a_k}$) and $\mathrm{E}[a_0^2]=c$ (if the basis contains a real function v_0). Conversely, is a real random field T_V with these properties invariant? This is the object of the next statement.

Theorem 4.7. *Let G be a compact Lie group and $V\subset L^2(\mathscr{X})$ a self-conjugated irreducible G-module and $(v_k)_{-\ell\le k\le \ell}$ a self-conjugated orthonormal basis of V (possibly $k\ne 0$ if $\dim H$ is even). Let T be a real a.s. square integrable random field on $\mathscr{X}$ and let $(a_k)_{-\ell\le k\le\ell}$ be its random coefficients with respect to the basis above. Then if the real and imaginary parts of the r.v.'s a_k, $k\ge 0$ (resp. $k>0$ if $\dim V$ is even) are centered, independent and Gaussian and, for $k\ne 0$, there exists $c\ge 0$ such that $\mathrm{E}[(\Re a_k)^2]=\mathrm{E}[(\Im a_k)^2]=\frac{c}{2}$ and $\mathrm{E}[a_0^2]=c$, then the random field*

$$T_V=\sum_{k=-\ell}^{\ell} a_k v_k$$

is invariant.

Proof. We make the proof under the assumption that $\dim V$ is odd, the case of an even dimension being quite similar. Therefore in the basis $(v_k)_{-\ell\le k\le\ell}$ we have $v_{-k}=\overline{v_k}$ and v_0 is a real function. Let A be the matrix of the transformation $\mathbb{C}^{2\ell+1}\to\mathbb{C}^{2\ell+1}$

$$\begin{pmatrix} z_\ell\\ \vdots\\ z_1\\ z_0\\ z_{-1}\\ \vdots\\ z_{-\ell}\end{pmatrix}\mapsto\begin{pmatrix}\frac{1}{\sqrt2}(z_\ell+z_{-\ell})\\ \vdots\\ \frac{1}{\sqrt2}(z_1+z_{-1})\\ z_0\\ \frac{1}{\mathrm{i}\sqrt2}(z_1-z_{-1})\\ \vdots\\ \frac{1}{\mathrm{i}\sqrt2}(z_\ell-z_{-\ell})\end{pmatrix}. \tag{4.7}$$

Lemma 4.8 below proves that the matrix $\widetilde D(g)=AD(g)A^{-1}$ is real orthogonal. Let $a_k=X_k+\mathrm{i}Y_k$, $k>0$, and $a_0=Z$. The real r.v.'s Z, X_k, Y_k, $k=1,\dots,\ell$, are independent and the matrix A maps the vector $a=(a_\ell,\dots,a_{-\ell})^t$ into

$$\widetilde a=\begin{pmatrix}\sqrt2 X_\ell\\ \vdots\\ \sqrt2 X_1\\ Z\\ \sqrt2 Y_1\\ \vdots\\ \sqrt2 Y_\ell\end{pmatrix}.$$

As the distribution of $\widetilde a$ is Gaussian with all its coordinates centered and independent with a common variance, $\widetilde a$ is invariant in distribution under the action of every orthogonal matrix and therefore under the action of $\widetilde D(g)$ for every

$g \in G$. Therefore the random vector a is invariant in distribution under the action of $D(g)$ for every $g \in G$, which implies the invariance of T_V. □

Lemma 4.8. $\widetilde{D}(g) = AD(g)A^{-1}$ *is a real orthogonal matrix for every* $g \in G$.

Proof. It is immediate that the rows of A are pairwise orthogonal and unitary. Therefore A is a unitary matrix as well as $\widetilde{D}(g)$. Let us prove that $\widetilde{D}(g)$ maps $\mathbb{R}^{2\ell+1}$ into $\mathbb{R}^{2\ell+1}$, which will end the proof. Let $\Xi = (\xi_\ell, \dots, \xi_1, \zeta, \eta_1, \dots, \eta_\ell)^t \in \mathbb{R}^{2\ell+1}$ and $z_k = \xi_k + \mathrm{i}\eta_k$, $k \geq 1$, $z_0 = \zeta$. Set $z = (z_\ell, \dots, z_1, z_0, \overline{z_1}, \dots, \overline{z_\ell})^t$, then $z = A^{-1}\Xi$ and z has the form

$$z = \begin{pmatrix} \frac{1}{\sqrt{2}}(\xi_\ell + \mathrm{i}\eta_\ell) \\ \vdots \\ \frac{1}{\sqrt{2}}(\xi_1 + \mathrm{i}\eta_1) \\ \zeta \\ \frac{1}{\sqrt{2}}(\xi_1 - \mathrm{i}\eta_1) \\ \vdots \\ \frac{1}{\sqrt{2}}(\xi_\ell - \mathrm{i}\eta_\ell) \end{pmatrix}. \tag{4.8}$$

Now the function f defined as

$$f(z) = \sum_{k>0} z_k v_k + \zeta v_0 + \sum_{k>0} z_{-k} v_{-k}$$

is a real function. Remark that the matrix A changes any vector of the form (4.8) into a vector of $\mathbb{R}^{2\ell+1}$. As $D(g)z$ is the vector of the coefficients of the function $L_g f$, which is still a real function, its coefficients are again of the form (4.8), so that $AD(g)A^{-1}\Xi \in \mathbb{R}^{2\ell+1}$. □

5. On the validity of the main assumption

In this section we investigate the validity of Assumption 4.3. Throughout this section we assume that G is a compact Lie group.

Let us remark first that, for a given self-conjugated G-module V of $L^2(\mathscr{X})$, Assumption 4.3, as far as we know, might be true for some orthonormal bases of V and not for other ones. So far it is known to be true for the basis formed by the spherical harmonics when $\mathscr{X} = \mathbb{S}^2$ (see [9], p. 144, for a proof), if $\dim V > 3$. Actually, as explained below, if $\dim V \leq 3$ Assumption 4.3 cannot hold. We shall investigate the implication between independence of the coefficients and Gaussianity for the 3-dimensional irreducible G-module of $L^2(\mathbb{S}^2)$ in Theorem 6.4.

Remark that, as $D_{mk}(g) = \langle L_g v_k, v_m\rangle$, condition (4.5) is equivalent to

$$\left|\langle L_g v_m, v_{m_i}\rangle\right| \neq \left|\langle L_g v_{-m}, v_{m_i}\rangle\right| \quad \text{for some } g \in G \text{ and every } 0 < m \leq \ell, i = 1, 2. \tag{5.1}$$

The main results of this section are Proposition 5.3 and Corollary 5.4 where we state a condition equivalent to Assumption 4.3 carrying a more geometric meaning. This will be the key tool in the next section, where we prove that every self-conjugated orthonormal basis of an irreducible G-module of $L^2(\mathbb{S}^2)$ with $\dim(V) > 3$ is mixing. In Section 7 we check the validity of Assumption 4.3 for the sphere $\mathscr{X} = \mathbb{S}^3$ under the action of $G = SO(4)$, at least for a class of self-conjugated orthonormal bases.

Let us first state some remarks.

Remark 5.1. (a) *As remarked above, mixing (Assumption 4.3), might hold for some orthonormal self-conjugated basis and not for other ones of a given irreducible G-module* $V \subset L^2(\mathscr{X})$. *However if it holds for a self-conjugated* *orthonormal basis* $(v_k)_{-\ell\le k\le\ell}$, *then it also holds for every other basis* $(w_k)_{-\ell\le k\le\ell}$ *of the form* $w_k = L_{g_0}v_k$ *for some* $g_0 \in G$. *Actually if* $\widetilde{D}(g)$ *denotes the matrix of the action of* G *on* V *with respect to the basis* $(w_k)_{-\ell\le k\le\ell}$, *that is*

$$w_k(g^{-1}x) = \sum_{i=-d}^{d} \widetilde{D}_{ik}(g) w_i(x),$$

then $\widetilde{D}(g) = D(g_0^{-1}gg_0)$ *so that Assumption* 4.3 *holds also for* $(w_k)_{-\ell\le k\le\ell}$.

(b) *A given self-conjugated orthonormal basis* $(v_k)_{-\ell\le k\le\ell}$ *can of course have some of its elements that are real functions, actually one at least if the dimension is odd. Remark however that Assumption* 4.3 *cannot be true if* v_{m_i} *is a real function of* $L^2(\mathscr{X})$. *Actually, as the left regular action commutes with conjugation,*

$$D_{m_i,-m}(g) = \langle L_g v_{-m}, v_{m_i}\rangle = \langle \overline{L_g v_m}, v_{m_i}\rangle = \overline{\langle L_g v_m, v_{m_i}\rangle} = \overline{D_{m_i,m}(g)}.$$

Therefore $|D_{m_i,-m}(g)| = |D_{m_i,m}(g)|$ *for every* $g \in G$. *This implies that Assumption* 4.3 *cannot be satisfied if* $\overline{V} = V$ *and* $\dim V = 2$ *or* $\dim V = 3$. *Actually in the first case there is only one* $m_i \ge 0$, *whereas, if* $\dim V = 3$, *the values* $m_i = 0, 1$ *are possible, but* v_0 *must be a real function and* (4.5) *cannot be satisfied for* v_0. *The case* $\dim V = 3$ *is of interest because it appears in the Peter–Weyl decomposition of* $L^2(\mathbb{S}^2)$.

In the next section we prove however that also for this G*-module, if the random field is invariant and the coefficients* a_0, a_1 *are independent, then they are necessarily Gaussian. In this proof we do not use Skitovitch–Darmois theorem so that Assumption* 4.3 *is not required.*

This raises the question whether one might prove Theorem 4.4 *using a different characterization of the Gaussian distribution than the one provided by the Skitovich–Darmois Theorem* 4.2. *This might lead to an argument in which Assumption* 4.3 *is not needed (see Section* 8 *for a more precise discussion on open questions).*

Let H be a irreducible unitary G-module of real type (recall Remark 2.5) and let $J : H \to H$ a conjugation such that $J^2 = 1$ and let us denote by $\langle\cdot,\cdot\rangle$ the corresponding G-invariant scalar product on H. Let $(h_k)_{-\ell\le k\le\ell}$ be a orthonormal basis of H which is self-conjugated with respect to J, i.e. such that $h_{-k} = J(h_k)$. We shall say that such a basis is J*-mixing* if (4.5) holds, now denoting by $D(g)$ the matrix of the action of G on H with respect to this basis. Of course J-mixing coincides with mixing if $H \subset L^2(\mathscr{X})$ with the left regular action and J is the usual conjugation $Jv = \overline{v}$.

Lemma 5.2. *Let* H *be a irreducible unitary* G*-module of real (resp. quaternionic) type (recall Remark* 2.5*) and let* $J : H \to H$ *a conjugation such that* $J^2 = 1$ *(resp.* $J^2 = -1$*), then, for every* $v, w \in H$,

$$\langle Jv, Jw\rangle = \overline{\langle Jv, Jw\rangle}.$$

Proof. It is immediate that

$$\langle v, w\rangle' = \overline{\langle Jv, Jw\rangle}$$

is also a G-invariant scalar product on H, hence, by Schur lemma, there exists a real number $\lambda > 0$ such that $\langle v, w\rangle' = \lambda\langle v, w\rangle$. The relation $J^2 = 1$ (resp. $J^2 = -1$) easily implies $\lambda = 1$. □

We are going to express Assumption 4.3 in terms of the action of G on the wedge product $\bigwedge^2 H$. Recall that $\bigwedge^2 H$ is endowed with the usual G-invariant scalar product

$$\langle v_1 \wedge w_1, v_2 \wedge w_2\rangle_2 := \langle v_1, v_2\rangle\langle w_1, w_2\rangle - \langle v_1, w_2\rangle\langle w_1, v_2\rangle.$$

We denote by $g(v \wedge w) = gu \wedge gw$ the action of G on $\bigwedge^2 H$.

Proposition 5.3. *Let* $g \in G$, *then* $|D_{m_i,m}(g)| = |D_{m_i,-m}(g)|$ *if and only if*

$$\big\langle g(h_{m_i} \wedge h_{-m_i}), h_m \wedge h_{-m}\big\rangle_2 = 0.$$

Proof. We have

$$\begin{aligned}\big\langle g(h_s\wedge h_{-s}),h_m\wedge h_{-m}\big\rangle_2&=\langle gh_s\wedge gh_{-s},h_m\wedge h_{-m}\rangle_2\\&=\langle gh_s,h_m\rangle\langle gh_{-s},h_{-m}\rangle-\langle gh_s,h_{-m}\rangle\langle gh_{-s},h_m\rangle\\&=\langle gh_s,h_m\rangle\langle gJh_s,Jh_m\rangle-\langle gh_s,h_{-m}\rangle\langle gJh_s,Jh_{-m}\rangle\\&=\big|\langle gh_s,h_m\rangle\big|^2-\big|\langle gh_{-s},h_m\rangle\big|^2=\big|D_{m,s}(g)\big|^2-\big|D_{m,-s}(g)\big|^2,\end{aligned}$$

where we used the fact that $\langle Jv,Jw\rangle=\overline{\langle Jv,Jw\rangle}$ thanks to Lemma 5.2. □

Assumption 4.3 can therefore be rephrased in terms of orthogonality of the G-orbits of the vectors $h_m\wedge h_{-m}$ in $\bigwedge^2 H$.

To be precise, let us denote, for every $1\le m\le\ell$, by W_m the subspace of $\bigwedge^2 H$ generated by the G-orbit of $h_m\wedge h_{-m}$; let S the set of the pairs (i,j), $1\le i,j\le\ell$, such $W_i\subset W_j^\perp$. Let $\widetilde S$ the set of the indices $1\le i\le\ell$ such that $(i,j)\in S$ for some $1\le j\le\ell$.

Corollary 5.4. *Assumption 4.3 holds if and only if the complement set $\widetilde S^c$ contains at least two indices. In particular Assumption 4.3 is verified if $\ell\ge 2$ and S is empty.*

Proof. Let (i,j), $1\le i,j\le\ell$ and let

$$F_{i,j}=\big\{g\in G;\big|D_{i,j}(g)\big|\ne\big|D_{i,-j}(g)\big|\big\}.$$

If $F_{i,j}\ne\varnothing$ then it is a dense open set of G. Assumption 4.3 holds if and only if for every $1\le m_1<m_2\le\ell$ we have that $F_{m_1,m}\ne\varnothing$ for every $1\le m\le\ell$ and $F_{m_2,m'}\ne\varnothing$ for every $1\le m'\le\ell$. Now it is sufficient to observe that, by Proposition 5.3, $F_{i,j}=\varnothing$ if and only if $(i,j)\in S$. □

In the remainder of this section we introduce a family of orthonormal bases of a G-module that arises naturally (the spherical harmonics are of this type) and for which the investigation of the validity of Assumption 4.3 might be simpler.

Let H be an irreducible G-module, $\mathbb T$ a maximal torus of G and let us go back to the setting of Remark 3.5 and consider the decomposition (3.5). It is possible to assemble an orthonormal basis of H by picking a unitary vector u_k in each of the U_k's. We say that such a basis is *associated to the torus* $\mathbb T$.

If among the U_k's there is only one subspace at most that is associated to a given character of $\mathbb T$, then the decomposition (3.5) is unique and an associated orthonormal basis is also unique, up to multiplication of its elements by unitary complex numbers. In this case (i.e. if among the U_k's there is only one subspace at most that is associated to a given character of $\mathbb T$) we say that H is $\mathbb T$-*simple*.

We shall see in the next sections that all irreducible sub-G-modules of $L^2(\mathbb S^2)$ and $L^2(\mathbb S^3)$ are $\mathbb T$ simple with respect to the maximal tori of $G=SO(3)$ or $G=SO(4)$ respectively.

Let us now suppose that H is a real G-module and denote by J a real conjugation. If $u\in U_k$ and $t\in\mathbb T$ we have, denoting $u\mapsto gu$ the action of G,

$$tu=\chi_k(t)u$$

for some character χ_k of $\mathbb T$, so that

$$tJu=Jtu=\overline{\chi_k(t)}Ju=\chi_{-k}(t)Ju.\tag{5.2}$$

Therefore it is easy to see that an orthonormal basis of H associated to $\mathbb T$ can be chosen in such a way that it is self-conjugated with respect to J. We shall denote by $(h_k)_{-\ell\le k\le\ell}$ such an orthonormal basis associated to $\mathbb T$, where the index k ranges among the corresponding characters of $\mathbb T$ appearing in the decomposition (3.5). Then it is clear that if the relation

$$\big|\langle gh_k,h_{m_i}\rangle\big|\ne\big|\langle gh_{-k},h_{m_i}\rangle\big|,\quad\text{for some } g\in G\text{ and for every }k\ne 0\tag{5.3}$$

holds for the basis $(h_k)_{-\ell\le k\le\ell}$, then it holds also for every other basis that is associated to $\mathbb{T}$, as two such bases only differ by multiplication by a unitary complex number.

It is also clear that if $H \subset L^2(\mathscr{X})$ with the usual conjugation J and (5.3) holds, then also Assumption 4.3 holds for the basis $(h_k)_{-\ell\le k\le\ell}$.

It is immediate that if H is $\mathbb{T}$-simple then it is also $\widetilde{\mathbb{T}}$-simple for any other maximal torus $\widetilde{\mathbb{T}}$. Actually, $\mathbb{T}$ and $\widetilde{\mathbb{T}}$ being conjugated, if $\widetilde{\mathbb{T}} = g^{-1}\mathbb{T}g$, a basis $(h_k)_{-\ell\le k\le\ell}$ is associated to $\mathbb{T}$ if and only if $(gh_k)_{-\ell\le k\le\ell}$ is associated to $\widetilde{\mathbb{T}}$. Thanks to Remark 5.1(a), if (5.3) is satisfied for a basis associated to $\mathbb{T}$, then it is also satisfied by all bases associated to $\widetilde{\mathbb{T}}$.

The following result states that if an irreducible G-module H is $\mathbb{T}$-simple and satisfies (5.3), then the same is true for every irreducible G-module that is equivalent to H.

Proposition 5.5. *Let $V \subset L^2(\mathscr{X})$ an irreducible G-module with $\overline{V} = V$ and H a $\mathbb{T}$-simple G-module equivalent to V. Then also V is $\mathbb{T}$-simple. Moreover if (5.3) is satisfied by the orthonormal bases of H associated to $\mathbb{T}$, then the same is true for V and every self-conjugated basis of V associated to a maximal torus is mixing.*

The proof is straightforward.

6. The sphere $\mathbb{S}^2$ and related examples

In this section we prove first that, for every irreducible G-module, $G = SO(3)$, of dimension > 3 of $L^2(\mathbb{S}^2)$, every self-conjugated orthonormal basis is mixing. This extends previous results: see [9], p. 144, where this is proved for the basis of the spherical harmonics. We also give a proof of the fact that the statement of Theorem 4.4 is true for every self-conjugated orthonormal basis of the irreducible d-dimensional $SO(d)$-module of $L^2(\mathbb{S}^{d-1})$. This covers in particular the case of the 3-dimensional $SO(3)$-module of $L^2(\mathbb{S}^2)$ a situation in which we know that Assumption 4.3 is not satisfied (Remark 5.1(b)).

Let us recall that in the Peter–Weyl decomposition of $L^2(\mathbb{S}^d)$, $d \ge 3$, all the irreducible modules for the action of $SO(d+1)$ are self-conjugated (see [5], pp. 196–197), so that, when dealing with a real random field, in order to apply Theorem 4.4 the validity of Assumption 4.3 must be checked.

It is well-known that $SO(3) = SU(2)/\{\mathrm{id}, -\mathrm{id}\}$ so that the irreducible representations of $SO(3)$ are the representations of $SU(2)$ which are trivial on $\{\mathrm{id}, -\mathrm{id}\}$ (see again [3] or [5]). The group $G = SU(2)$ acts on the modules H_ℓ formed by the homogeneous polynomials in 2 complex variables z_1, z_2 of degree ℓ in the following way: if $p \in H_\ell$, then, if $z = (z_1, z_2)$,

$$gp(z_1, z_2) = p(az_1 - \overline{b}z_2, bz_1 + \overline{a}z_2) = p(zg), \tag{6.1}$$

where

$$g = \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix}, \quad a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \tag{6.2}$$

denotes a generic element of $G = SU(2)$. The $SU(2)$-modules H_ℓ are irreducible and every irreducible $SU(2)$-module is equivalent to H_ℓ for some $\ell = 0, 1, \dots$. The action of $-\mathrm{id}$ in these representations is trivial if and only if ℓ is even, so that every irreducible representation of $SO(3)$ is equivalent to H_ℓ for some ℓ even.

Lemma 6.1. *Let P, Q be homogeneous polynomials of degree $\ell \ge 1$ in the two complex variables z_1, z_2. Let*

$$D(P, Q) := \det\begin{pmatrix} \frac{\partial P}{\partial z_1} & \frac{\partial P}{\partial z_2} \\ \frac{\partial Q}{\partial z_1} & \frac{\partial Q}{\partial z_2} \end{pmatrix}. \tag{6.3}$$

Then $D(P, Q)$ is a homogeneous polynomial of degree $2\ell - 2$ which vanishes if and only if $P = 0$ or if $Q = \lambda P$ for some $\lambda \in \mathbb{C}$.

We give the proof of Lemma 6.1 after the following main result.

Theorem 6.2. *Let $V \subset L^2(G)$, $G = SO(3)$, an irreducible self-conjugated G-module of dimension $2m+1$. Then, if $m > 1$, every self-conjugated orthonormal basis of V is mixing.*

Proof. The proof relies on the characterization of Corollary 5.4. Let $\ell = 2m$ and H_ℓ as above. Then (6.3) defines a map $(P, Q) \mapsto D(P, Q)$ from $H_\ell \otimes H_\ell$ to $H_{2\ell-2}$ which is obviously bilinear and antisymmetric. It is also equivariant with respect to the action of $SU(2)$ (acting both on $H_\ell \otimes H_\ell$ and $H_{2\ell-2}$). Actually, denoting $\partial_z P = (\frac{\partial P}{\partial z_1}, \frac{\partial P}{\partial z_2})$,

$$\partial_z (gP)(z) = \partial_z P\big(g^t \cdot\big)(z) = (\partial_z P)\big(g^t z\big) g^t$$

and one concludes easily, as $\det g^t = 1$. $L(P \wedge Q) = D(P, Q)$ therefore defines a linear equivariant map $\bigwedge^2 H_\ell \to H_{2\ell-2}$, such that $L(P \wedge Q) = 0$ if and only if $P \wedge Q = 0$ (thanks to Lemma 6.1).

Let us prove that every orthonormal basis self-conjugated with respect to some conjugation $\widetilde{J} : H_\ell \to H_\ell$ (i.e. such that $\widetilde{J}(f_{-r}) = f_r$) is $\widetilde{J}$-mixing. Let $(f_{-m}, \dots, f_m)$ such a $\widetilde{J}$-self-conjugated orthonormal basis. If it were not $\widetilde{J}$-mixing, then by Corollary 5.4 there would exist $r, s > 0$ and two mutually orthogonal invariant subspaces U_1, U_2 of $\bigwedge^2 H_\ell$ such that $f_r \wedge f_{-r} \in U_1$, $f_s \wedge f_{-s} \in U_2$ (recall that we assume $m > 1$). As f_r and f_{-r} are orthogonal, $L(f_r \wedge f_{-r}) \neq 0$, so that L does not vanish on U_1 and by Schur lemma U_1 must contain a G-submodule equivalent to $H_{2\ell-2}$. By the same argument also U_2 must contain a G-submodule equivalent to $H_{2\ell-2}$, which is not possible, as, by the Clebsch–Gordan decomposition (see [9], Section 3.5, or (7.1) below) the representation $H_{2\ell-2}$ appears only once in $H_\ell \otimes H_\ell$ and, a fortiori, in $\bigwedge^2 H_\ell$.

Therefore, by Corollary 5.4, the basis $(f_{-m}, \dots, f_m)$ is $\widetilde{J}$-mixing.

Now let $(v_{-m}, \dots, v_m)$ an orthonormal self-conjugated (in the sense of ordinary conjugation, noted J) basis of V. The actions of $G = SO(3)$ on V and H_{2m} are equivalent and therefore there exists a map $A : V \to H_{2m}$ that intertwines the two actions, that is such that $AL_g v = gAv$ for every $g \in G$, $v \in V$. Up to multiplication by a constant we can assume that A preserves the scalar product. If we note $\widetilde{J} f = AJA^{-1} f$, $\widetilde{J}$ defines a conjugation on H_{2m} with respect to which $f_r = Av_r$ is a self-conjugated orthonormal basis. By the first part of the proof we know that such a basis is $\widetilde{J}$-mixing. Therefore there exists $g \in SO(3)$ such that

$$\big|\langle L_g v_r, v_s\rangle_V\big| = \big|\langle g f_r, f_s\rangle_{H_{2m}}\big| \neq \big|\langle g f_r, f_{-s}\rangle_{H_{2m}}\big| = \big|\langle L_g v_r, v_{-s}\rangle_V\big|$$

and $(v_{-m}, \dots, v_m)$ is mixing itself. □

Proof of Lemma 6.1. Assume $P \not\equiv 0$. If $D(P, Q) = 0$ and $\partial_z P \not\equiv 0$, then there exists a function $\lambda : \mathbb{C}^2 \to \mathbb{C}$ such that, for every $z \in \mathbb{C}$,

$$\partial_z Q = \lambda(z) \partial_z P. \tag{6.4}$$

Recall Euler formula for homogeneous functions of exponent ℓ:

$$\frac{\partial P}{\partial z_1} z_1 + \frac{\partial P}{\partial z_2} z_2 = \ell P$$

and similarly for Q, so that from (6.4) we get $Q = \lambda P$. On the open set $\mathbb{C}^2 \setminus \Gamma$, where Γ is the set of zeros of P, we have $\lambda = \frac{Q}{P}$, so that, on $\mathbb{C}^2 \setminus \Gamma$,

$$\partial_z Q = \frac{Q}{P} \partial_z P.$$

But from $\lambda = \frac{Q}{P}$ we have also, for $j = 1, 2$,

$$\frac{\partial \lambda}{\partial z_j} = \frac{1}{P^2}\left(P \frac{\partial Q}{\partial z_j} - Q \frac{\partial P}{\partial z_j}\right) = \frac{1}{P^2}\left(P\left(\frac{Q}{P}\frac{\partial P}{\partial z_j}\right) - Q \frac{\partial P}{\partial z_j}\right) = 0.$$

As λ is analytic on $\mathbb{C}^2 \setminus \Gamma$, this implies that $Q = \mathit{const}\, P$ on a nonempty open set of $\mathbb{C}^2$ and therefore $Q = \mathit{const}\, P$ everywhere. □

We address now the question of the 3-dimensional irreducible G-module of $L^2(\mathbb{S}^2)$ to which the previous result does not apply and for which Assumption 4.3 is not satisfied (see Remark 5.1(b)). Actually we prove a more general statement. The key argument is the following classical characterization of the normal distribution.

Proposition 6.3. *Let $X = (X_1, \dots, X_m)$ a $\mathbb{R}^m$-valued r.v. such that*:

(a) *the distribution of X is invariant with respect to the action of $SO(m)$*;
(b) *there exist i, j, $1 \le i, j \le m$ such that X_i and X_j are independent*.

Then X is Gaussian.

Proof. We can assume for simplicity that X_1, X_2 are independent. Let

$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & \cdots & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & \cdots & & \\ 0 & & \cdots & 0 & 1 \end{pmatrix}.$$

Then $A \in SO(d)$ and, by assumption, X and AX have the same distribution. In particular (X_1, X_2) and $(\frac{1}{\sqrt{2}}(X_1 - X_2), \frac{1}{\sqrt{2}}(X_1 + X_2))$ have the same distribution, so that $\frac{1}{\sqrt{2}}(X_1 - X_2)$ and $\frac{1}{\sqrt{2}}(X_1 + X_2)$ are independent. By the classical Bernstein–Kac characterization of Gaussian measures, X_1 and X_2 are therefore Gaussian (see [4], pp. 74 and 85 for a simple proof). In order to prove joint Gaussianity of X, just remark that rotational invariance implies that the characteristic function of X is of the form

$$\phi_X(\theta) = \psi\bigl(|\theta|\bigr), \quad \theta \in \mathbb{R}^d$$

for some function $\psi : \mathbb{R} \to \mathbb{R}$. But, as X_1 is Gaussian, by choosing $\theta = (\theta_1, 0, \dots, 0)$ we have

$$\psi\bigl(|\theta|\bigr) = \phi_X(\theta) = \mathrm{e}^{-\sigma^2\theta_1^2/2} = \mathrm{e}^{-\sigma^2|\theta|^2/2},$$

where $\sigma^2 = \mathrm{Var}(X_1)$, which allows to conclude. □

Recall that in the Peter–Weyl decomposition of $L^2(\mathbb{S}^{d-1})$ the smallest irreducible G-module besides the constants, V_d say, has always dimension d exactly (see [5], p. 197, e.g.).

Theorem 6.4. *Let $(v_{d/2}, \dots, v_{-1}, v_1, \dots, v_{-d/2})$ for d even (resp. $(v_{(d-1)/2}, \dots, v_0, \dots, v_{-(d-1)/2})$ for odd d) be a self-conjugated orthonormal basis of the $SO(d)$-module $V_d \subset L^2(\mathbb{S}^{d-1})$. Let T a real invariant random field on $L^2(\mathbb{S}^{d-1})$ such that its coefficients $(a_{d/2}, \dots, a_1)$ (resp. $(a_{(d-1)/2}, \dots, a_0)$) with respect to this basis are independent. Then they are Gaussian*.

Proof. Let us make the proof for d odd, $d = 2m + 1$. By assumption we can write $a_k = X_k + \mathrm{i}Y_k$, $a_{-k} = X_k - \mathrm{i}Y_k$ for $1 \le k \le d$, $a_0 = Z$, where the r.v.'s Z, $(X_1, Y_1), \dots, (X_m, Y_m)$ are independent.

Let $a = (a_{-m}, \dots, a_0, \dots, a_m)^t$ and denote by $D(g)$, $g \in G$, the matrices of the left regular action of $G = SO(d)$ on the G-module V_d with respect to the given orthonormal basis. By assumption the random vectors a and $D(g)a$

have the same distribution for every $g \in G$. Let now A be the matrix $\mathbb{C}^d \to \mathbb{C}^d$ defined as in (4.7) (with ℓ replaced by m) so that

$$Aa = \begin{pmatrix} \sqrt{2}X_m \\ \vdots \\ \sqrt{2}X_1 \\ Z \\ \sqrt{2}Y_1 \\ \vdots \\ \sqrt{2}Y_m \end{pmatrix} := \widetilde{a}.$$

Thanks to Lemma 4.8 the matrices $\widetilde{D}(g) = AD(g)A^{-1}$ are orthogonal. Moreover $g \mapsto \widetilde{D}(g)$ is an irreducible representation of G of dimension d, the representations D and $\widetilde{D}$ being equivalent. As there is only one d-dimensional irreducible representation of $SO(d)$ (up to equivalence), $g \mapsto A^{-1}D(g)A$ is equivalent to the natural action of $SO(d)$ on $\mathbb{C}^d$ and therefore $1 = \det(D(g)) = \det \widetilde{D}(g)$ so that $\widetilde{D}(g) \in SO(d)$ and the image of $\widetilde{D}$ is $SO(d)$ itself, as the map $g \mapsto \widetilde{D}(g)$ is injective.

Therefore invariance of the distribution of a with respect to the matrices $D(g)$ entails invariance of the distribution of $\widetilde{a}$ with respect to $SO(d)$. As the r.v.'s X_1 and Z, for instance, are independent, Proposition 6.3 implies that $\widetilde{a}$ is Gaussian, and therefore also a. □

The previous theorem ensures that, if T is an invariant random field on $\mathbb{S}^2$ and V is an irreducible G-module of $L^2(\mathbb{S}^2)$ of dimension 3, independence of the coefficients a_1 and a_0 with respect to any self-conjugated orthonormal basis of V entails Gaussianity of T_V, even if Assumption 4.3 is not true for such V.

Remark 6.5. *In the case where $V_d \subset L^2(\mathbb{S}^{d-1})$ is the d-dimensional irreducible $SO(d)$-module and if $d \geq 4$, the conclusion of Theorem 6.4 follows from Theorem 4.4 since, as a consequence of Corollary 5.4, every self-conjugated orthonormal basis is mixing. In fact for $d \geq 5$ the module $\bigwedge^2 V_d$ is irreducible whereas, for $d = 4$, $\bigwedge^2 V_d$ has two irreducible non-isomorphic components which are the eigenspaces of the Hodge $*$ operator. Therefore a non-zero real vector of the form $\mathrm{i}v \wedge \overline{v}$ cannot be contained in either eigenspace (see [3], pp. 272–274).*

***Example 6.6** ($SO(3)$ **and** $SU(2)$).* *In the same line of arguments it is easy to check that, for a real invariant random field, independence of the coefficients entails Gaussianity in the cases $\mathscr{X} = G = SO(3)$ and $\mathscr{X} = G = SU(2)$.*

Actually if $\mathscr{X} = G = SO(3)$ this is partially known when considering the basis given by the normalized columns (or rows) of the Wigner matrices: in every isotypical submodule one of the columns is generated by the spherical harmonics for which it is known that, for $\ell > 1$, Assumption 4.3 holds so that Theorem 4.4 applies. As for the other columns, they are not self-conjugated but conjugated pairwise, so that one can apply Theorem 4.1.

However Theorems 6.2 and 6.4 ensure that, even considering a different decomposition of the isotypical spaces, it is not possible to simulate an invariant non-Gaussian random field using independent coefficients.

This is true also for $\mathscr{X} = G = SU(2)$ as in the Peter–Weyl decomposition, in addition to those already considered for $G = SO(3)$, other representations appear that are quaternionic, so that the corresponding isotypical modules cannot contain self-conjugated irreducible modules (recall Remark 2.5).

7. The sphere $\mathbb{S}^3$

In this section we prove that for every irreducible G-module, $G = SO(4)$, of $L^2(\mathbb{S}^3)$ every basis adapted to a maximal torus is mixing.

We shall need some known facts about the group $SO(4)$ and its representations. $G = SO(4)$ is isomorphic to $SU(2) \times SU(2)/\{(\mathrm{id}, \mathrm{id}), (-\mathrm{id}, -\mathrm{id})\}$. Therefore its irreducible representations are of the form $H_\ell \otimes H_k$, H_ℓ, H_k being the irreducible modules of $SU(2)$ introduced at the beginning of Section 6, with the condition that the action of

$(-\mathrm{id}, -\mathrm{id})$ is trivial. As these modules are formed by the homogeneous polynomials of degree ℓ and k respectively in the complex variables z_1, z_2, one has

$$(-\mathrm{id}, -\mathrm{id})(p \otimes q) = (-1)^{\ell+k} p \otimes q$$

and therefore the irreducible modules of $SO(4)$ are of the form $H_\ell \otimes H_k$ with $\ell + k$ even. In order to determine the Peter–Weyl decomposition of $L^2(\mathbb{S}^3)$, $\mathbb{S}^3 = SO(4)/SO(3)$, one must recall that, in the isomorphism $G \simeq SU(2) \times SU(2)/\{(\mathrm{id}, \mathrm{id}), (-\mathrm{id}, -\mathrm{id})\}$, $SO(3)$ is mapped into the diagonal and therefore the action of $SO(3)$ on $H_\ell \otimes H_k$ is $g(p \otimes q) = gp \otimes gq$ of $SU(2)$. By the Clebsch–Gordan formula for $SU(2)$, the action of $SU(2)$ on the tensor product $H_\ell \otimes H_k$ can be decomposed as

$$H_\ell \otimes H_k = \bigoplus_{j=0}^{d_q} H_{\ell+k-2j}, \quad d_q = \min(\ell, k) \tag{7.1}$$

and therefore the trivial representation appears in this decomposition if and only if

$$\ell + k \text{ is even}, \qquad \frac{\ell + k}{2} \le \ell, \qquad \frac{\ell + k}{2} \le k$$

that is if and only if $\ell = k$. We have therefore found that the representations of $SO(4)$ appearing in the Peter–Weyl decomposition of $L^2(\mathbb{S}^3)$ are exactly those that are equivalent to $H_\ell \otimes H_\ell$. Remark that the smallest dimension of these, besides the case $\ell = 1$ of the constants, is 4, so that we do not have to bother with the problem of dimension 3 appearing for the sphere $\mathbb{S}^2$, as discussed in Remark 5.1(b).

On the $SU(2)$-module H_ℓ introduced in Section 5 let us consider the polynomials $p_s(z_1, z_2) = z_1^s z_2^{\ell-s}$, $s = 0, \dots, \ell$ which form an orthogonal basis with respect to the scalar product

$$\langle p_s, p_r \rangle = \frac{s!(\ell - s)!}{\ell!} \delta_{s,r} = \frac{1}{\binom{l}{s}} \delta_{s,r} \tag{7.2}$$

which turns out to be $SU(2)$-invariant. Therefore the polynomials $e_s = c_s p_s$, $s = 0, \dots, \ell$ with $c_s = \sqrt{\binom{\ell}{s}}$ form an orthonormal basis of the unitary $SU(2)$-module H_ℓ.

A maximal torus of $SU(2)$ is the subgroup of the elements

$$t_\theta = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\theta} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\theta} \end{pmatrix}$$

whose action on the polynomials $p_s(z_1, z_2) = z_1^s z_2^{\ell-s}$ is

$$t_\theta p_s = \mathrm{e}^{\mathrm{i}(2s-\ell)\theta} p_s.$$

Thus, with respect to the invariant scalar product (7.2), the elements $e_s = c_s p_s$ with $c_s = \sqrt{\binom{\ell}{s}}$ form an orthonormal basis of H_ℓ that is adapted to the maximal torus $\mathbb{T}$. In particular H_ℓ is $\mathbb{T}$-simple.

The following computation is our key argument. We have

$$ge_s = c_s(az_1 - \overline{b}z_2)^s(bz_1 + \overline{a}z_2)^{\ell-s} = c_s \sum_{r=0}^{\ell} \underbrace{\sum_{\substack{h+k=r \\ 0\le h\le s \\ 0\le k\le \ell-s}} \binom{s}{h}\binom{\ell-s}{k} a^h \overline{a}^{\ell-s-k} b^k (-\overline{b})^{s-h}}_{:=H_{r,s}} z_1^r z_2^{2m-r}$$

$$= c_s \sum_{r=0}^{\ell} H_{r,s} z_1^r z_2^{\ell-r} = c_s \sum_{\ell=-m}^{m} \frac{1}{c_r} H_{r,s} e_r$$

and therefore

$$\langle ge_s, e_j\rangle = \frac{c_s}{c_j} H_{j,s} = \frac{c_s}{c_j} \sum_{\substack{h+k=j\\ 0\le h\le s\\ 0\le k\le \ell-s}} \binom{s}{h}\binom{\ell-s}{k} a^h \overline{a}^{\ell-s-k} b^k \overline{b}^{s-h} (-1)^{s-h}.$$

Taking into account the condition $h+k=j$ this can be written

$$\begin{aligned}\langle ge_s, e_j\rangle &= \frac{c_s}{c_j} H_{j,s} = \frac{c_s}{c_j} \sum_{\substack{0\le h\le s\\ 0\le j-h\le \ell-s}} \binom{s}{h}\binom{\ell-s}{j-h} a^h \overline{a}^{\ell-s-j+h} b^{j-h} \overline{b}^{s-h} (-1)^{s-h}\\ &= \frac{c_s}{c_j}\overline{a}^{\ell-s-j}\overline{b}^{s-j} \sum_{\substack{0\le h\le s\\ 0\le j-h\le \ell-s}} \binom{s}{h}\binom{\ell-s}{j-h} |a|^{2h}|b|^{2(j-h)}(-1)^{s-h}\end{aligned}$$

and therefore

$$\begin{aligned}&\left|\langle ge_s, e_j\rangle\right|^2\\ &\quad= \frac{c_s^2}{c_j^2}|a|^{2(\ell-s-j)}|b|^{2(s-j)}\Bigg(\sum_{\substack{0\le h\le s\\ 0\le j-h\le \ell-s}} \binom{s}{h}\binom{\ell-s}{j-h} |a|^{2h}|b|^{2(j-h)}(-1)^{s-h}\Bigg)^2\\ &\quad= \frac{c_s^2}{c_j^2}\Bigg(\sum_{\substack{0\le h\le s\\ 0\le j-h\le \ell-s}} \binom{s}{h}\binom{\ell-s}{j-h} |a|^{\ell-s-j+2h}|b|^{s+j-2h}(-1)^{s-h}\Bigg)^2 := \mathscr{P}^{\ell}_{s,j}\big(|a|, |b|\big)\end{aligned} \tag{7.3}$$

which is a homogeneous polynomial of degree 2ℓ in the variables $|a|$, $|b|$. Let us point out that in the sum inside the square defining $\mathscr{P}^{\ell}_{s,j}$ the range of h is

$$\max(0, -\ell+s+j) \le h \le \min(s, j). \tag{7.4}$$

Let us assume first ℓ even, $\ell = 2m$. Let

$$f_k = e_{m+k}.$$

$(f_k)_{-m\le k\le m}$ is also an orthonormal basis with respect to the $SU(2)$-invariant scalar product (7.2) and adapted to $\mathbb{T}$. The maximal torus of $SU(2)\times SU(2)$ is $\mathbb{T}\times\mathbb{T}$. As

$$(t_{\theta_1}, t_{\theta_2}) f_{k_1}\otimes f_{k_2} = \underbrace{\mathrm{e}^{\mathrm{i}(2k_1\theta_1+2k_2\theta_2)}}_{\chi_{k_1,k_2}(t_{\theta_1},t_{\theta_2})} f_{k_1}\otimes f_{k_2}, \quad -m\le k_1, k_2\le m,$$

$H_\ell\otimes H_\ell$ is simple with respect to $\mathbb{T}\times\mathbb{T}$ and the basis $(f_{k_1}\otimes f_{k_2})_{k_1,k_2}$ is adapted to the maximal torus above. Moreover if $f_{k_1}\otimes f_{k_2}$ is the eigenvector of the character χ_{k_1,k_2} of $\mathbb{T}\times\mathbb{T}$, then $f_{-k_1}\otimes f_{-k_2}$ is an eigenvector of $\overline{\chi}_{k_1,k_2}$. We proceed now to check condition (5.3) in view of taking advantage of Proposition 5.5. We must show that for some $m_1>0$, $m_2>0$

$$\begin{aligned}&\left|\big\langle (g_1, g_2)(f_{m_1}\otimes f_{m_2}), f_{r_1}\otimes f_{r_2}\big\rangle\right|^2\\ &\quad\ne \left|\big\langle (g_1, g_2)(f_{m_1}\otimes f_{m_2}), f_{-r_1}\otimes f_{-r_2}\big\rangle\right|^2 \quad \text{for every } -m\le r_1, r_2\le m\end{aligned} \tag{7.5}$$

for some $g_1, g_2\in SU(2)$. We have

$$\left|\big\langle (g_1, g_2)(f_{m_1}\otimes f_{m_2}), f_{r_1}\otimes f_{r_2}\big\rangle\right|^2 = \left|\langle g_1 f_{m_1}, f_{r_1}\rangle\right|^2\left|\langle g_2 f_{m_2}, f_{r_2}\rangle\right|^2$$

and taking into account (7.3)

$$\left|\langle g f_k, f_r\rangle\right|^2 = \left|\langle g e_{k+m}, e_{r+m}\rangle\right|^2 = P^\ell_{k+m,r+m}\big(|a|, |b|\big)$$

so that, denoting by a_1, b_1 and a_2, b_2 the coordinates of g_1 and g_2 in the representation (6.1),

$$\left|\big\langle (g_1, g_2)(f_{m_1} \otimes f_{m_2}), f_{r_1} \otimes f_{r_2}\big\rangle\right|^2 = \mathscr{P}^{2m}_{m+m_1,m+r_1}\big(|a_1|, |b_1|\big)\mathscr{P}^{2m}_{m+m_2,m+r_2}\big(|a_2|, |b_2|\big) \tag{7.6}$$

and

$$\left|\big\langle (g_1, g_2)(f_{m_1} \otimes f_{m_2}), f_{-r_1} \otimes f_{-r_2}\big\rangle\right|^2 = \mathscr{P}^{2m}_{m+m_1,m-r_1}\big(|a_1|, |b_1|\big)\mathscr{P}^{2m}_{m+m_2,m-r_2}\big(|a_2|, |b_2|\big). \tag{7.7}$$

In order to conclude we must prove that for some values of a_1, b_1, a_2, b_2 with $|a_1|^2 + |b_1|^2 = |a_2|^2 + |b_2|^2 = 1$ the right-hand sides in (7.6) and (7.7) are different. For every $r \neq 0$ if the two polynomials $\mathscr{P}^\ell_{m+m_i,m-r}$ and $\mathscr{P}^\ell_{m+m_i,m+r}$, both homogeneous of degree $4m$, coincide on the circle $|a|^2 + |b|^2 = 1$, they would coincide on the whole of $\mathbb{R}^2$. In order to see that this cannot happen we look at the monomial that exhibits the highest exponent in $|a|$ and see that the degrees are different. Recalling (7.4), we must show that the two values

$$h_1 = \min(m + m_i, m + r) \quad \text{and} \quad h_2 = \min(m + m_i, m - r)$$

are different. This is done by checking directly all possibilities: as $m_i > 0$, then

	h_1	h_2
$0 < r \le m_i$	$m + m_i$	$m - r$
$m_i < r \le m$	$m + r$	$m - r$
$-m_i \le r < 0$	$m + r$	$m + m_i$
$-m \le r < -m_i$	$m - r$	$m + r$

Therefore, unless $r = 0$ of course, $h_1 \neq h_2$ in all possible occurrences. Therefore the two polynomials at the right-hand side of (7.6) and (7.7) are different if one at least between r_1 and r_2 is different from 0.

Along the same lines goes the proof for ℓ odd. Thanks to Proposition 5.5, we have

Theorem 7.1. *Let $V \subset L^2(\mathbb{S}^3)$ a irreducible G-module of dimension > 1. Then every self-conjugated basis of V associated to a maximal torus is mixing.*

8. Some open questions

This paper gives some precisions about properties of the Fourier coefficients of an invariant random field and clarifies some important points in the direction of characterizing the random fields on a homogeneous space whose coefficients in their Fourier development are independent (or at least that can be simulated through the generation of independent r.v.'s) on the track of [1] and [2].

However it also points out some natural questions that remain open to conjecture. We make here a tentative list.

(1) In order to prove Gaussianity of such random fields we used the Skitovitch–Darmois theorem whose application in turn requires, in many cases of interest, to ascertain that Assumption 4.3 is verified. But we have also remarked that Gaussianity still holds in situations where Assumption 4.3 is not true (see Remark 5.1(b)). So one might think of taking advantage of a characterization of Gaussianity different form the one that is provided by the Skitovitch–Darmois theorem, and thus be ridden of Assumption 4.3.

(2) It is nevertheless of interest also to investigate the validity of Assumption 4.3. Is it always true (at least for self-conjugated G-modules of dimension > 3)? We do not know of counterexamples so far.

(3) In a less ambitious perspective, is Assumption 4.3 true for the groups $SO(d)$ and for the spheres $\mathbb{S}^{d-1}$, $d \geq 5$? For which classes of orthonormal bases? Intuition should point towards the positive: as these structures contain $SO(3)$ and $SO(4)$ for which the result is proved. Remark that one possible way of attacking this problem is through an extension of Proposition 6.3: does its statement remain true if the assumption of invariance with respect to the group $SO(d)$ is replaced by invariance with respect to a subgroup of $SO(d)$ that acts irreducibly on $\mathbb{R}^d$?

Acknowledgements

The authors wish to thank the Referee for many illuminating remarks.

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