Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2014, Vol. 50, No. 4, 1161–1164
DOI: 10.1214/13-AIHP563

ANNALES DE L'INSTITUT HENRI POINCARÉ
PROBABILITÉS ET STATISTIQUES
www.imstat.org/aihp

A stationary random graph of no growth rate

Ádám Timár[1]

Bolyai Institute, University of Szeged, Aradi v. tere 1, 6720 Szeged, Hungary. E-mail: madaramit@gmail.com

Received 22 August 2012; revised 19 March 2013; accepted 16 April 2013

Abstract. We present a random automorphism-invariant subgraph of a Cayley graph such that with probability 1 its exponential growth rate does not exist.

Résumé. Nous construisons un sous-graphe aléatoire invariant par automorphismes d'un groupe de Cayley qui n'a presque sûrement pas un taux de croissance exponentiel bien défini.

MSC: 60C05; 05C80

Keywords: Unimodular graph; Stationary random graph; Growth rate

At the Banff workshop "Graphs, groups and stochastics" in 2011, Vadim Kaimanovich asked the following question:

Question 1. *For an arbitrary (Γ, o) unimodular random rooted graph, does the exponential growth rate*

$$\lim_{n\to\infty} \log\big|B_\Gamma(o,n)\big|/n$$

exist almost surely? Here $B_\Gamma(o,n)$ denotes the ball of radius n around o in Γ.

We give a negative answer to the question.

Theorem 2. *There is a transitive unimodular graph G with a random invariant spanning subgraph Γ such that for any point $o \in G$ and almost every Γ*

$$\liminf_{n\to\infty} \log\big|B_\Gamma(o,n)\big|/n = 0 \quad \textit{and}$$

$$\limsup_{n\to\infty} \log\big|B_\Gamma(o,n)\big|/n = c$$

with $c > 0$. Moreover, there exists such a Γ that is a spanning tree with one end.

Unimodular random graphs are random rooted graphs with a certain stationarity property (see the next paragraph). Unimodular random graphs provide a common framework to examples such as automorphism-invariant random subgraphs of transitive unimodular graphs (e.g., Cayley graphs), augmented Galton–Watson trees, Borel equivalence

[1]Research was supported by Sinergia grant CRSI22-130435 of the Swiss National Science foundation and by MTA Rényi "Lendület" Groups and Graphs Research Group. This research was partly realized in the frames of TÁMOP 4.2.4. A/1-11-1-2012-0001 "National Excellence Program," subsidized by the European Union and co-financed by the European Social Fund.

relations… . Many such examples arise as the local weak limit (or Benjamini–Schramm limit) of a sequence of finite graphs; a main open question by Aldous and Lyons is whether all of them arise this way, [1].

Suppose that the random rooted graph (G, o) has every degree bounded by d almost surely, and define a new graph (G', o), by adding loops to vertices of G in a way that makes it d-regular. Choose an edge uniformly of all edges in G' incident to o, and let x be o if the chosen edge is a loop, otherwise let x be the endpoint of the edge different from o. If the resulting distribution on the triples (G', o, x) is the same as the distribution of the triples (G', x, o), then we call (G, o) unimodular. The bound on the degrees can be removed, and several alternative definitions exist, see [1].

The exponential growth rate of an infinite graph is one of the important invariants that are defined as a limit. For other invariants, such as the speed of the random walk, the defining limit is known to exist almost surely for every unimodular random graph, using some subadditive inequality, [1]. Hence it may be surprising that a similar argument does not work for the existence of growth, as it does for every Cayley graph.

Proof of Theorem 2.

Construction: Let Δ be the triangular lattice, $L = \mathbb{Z}_2 \wr \mathbb{Z}$ be the Cayley graph of the lamplighter group, where we think about each vertex of L as a pair (ξ, k), where $\xi \in \{0,1\}^{\mathbb{Z}}$ of finite support is the status of lamps and $k \in \mathbb{Z}$ is the position of the marker, and the generators defining edges are $(\mathbf{1}_0, 0), (\mathbf{0}, 1), (\mathbf{0}, -1)$ (hence a vertex is adjacent to vertices that result from switching the lamp at the location of the marker, or from making the marker step to the right or to the left). Define $G = \Delta \times L \times K_{28}$, where K_{28} is the complete graph on vertex set $\{1, 2, \ldots, 28\}$. Here the product $H_1 \times H_2$ of graphs H_1, H_2 is defined on vertex set $V(H_1) \times V(H_2)$ with (x_1, x_2) adjacent to (y_1, y_2) if and only if $x_i = y_i$ or x_i is adjacent to y_i in H_i, for $i = 1, 2$.

Consider critical percolation on the vertices of Δ, that is, delete each vertex with probability $1/2$ independently from each other, and call the deleted vertices *closed*, the remaining ones *open*. Let $\mathcal{C}_0$ and $\mathcal{C}_1$ be the set of components (*clusters*) induced by closed and open vertices respectively; $\mathcal{C} := \mathcal{C}_0 \cup \mathcal{C}_1$. From percolation theory it is known that every component in $\mathcal{C}$ is finite. Furthermore, any component C in $\mathcal{C}_0$ is separated from infinity by a unique component C' in $\mathcal{C}_1$ to which it is adjacent to (meaning that there is a unique $C' \in \mathcal{C}_1$ such that C is contained in a finite component of $\Delta \setminus C'$ and there is an edge between C and C'), and conversely, any component C in $\mathcal{C}_1$ is separated from infinity by a unique component C' in $\mathcal{C}_0$. Hence there is a naturally defined oriented tree T on $\mathcal{C}$ as vertex set: let there be an edge pointing from x to y ($x \to y$), if the cluster y separates x from infinity and they are adjacent (in particular, if one of them is closed then the other has to be open). Call the set of leaves in the tree $\mathcal{L}_1$, and recursively, define $\mathcal{L}_i$ as the set of vertices x in T with the property that the longest path oriented towards x has length i (measured in the number of vertices in it).

Fix sequence $c_1 \ll c_2 \ll \cdots$ to be defined later. Define a random variable X_i to be uniformly chosen in $\{1, \ldots, c_i\}$ $(i = 1, 2, \ldots)$. Consider the equivalence relation $\mathcal{R}_i$ on $\mathbb{Z}$ where x and y are equivalent if for $\alpha_i := \sum_{j=1}^{i} X_j \prod_{k=1}^{j-1} c_k$ we have $[(x - \alpha_i)/c_1 \cdots c_i] = [(y - \alpha_i)/c_1 \cdots c_i]$ (where $[\cdot]$ denotes the floor function). That is, the sequence of $\mathbb{Z}$-invariant partitions defined by the $(\mathcal{R}_i)$ is coarser and coarser, and each $\mathcal{R}_i$ consists of classes of $c_1 \cdots c_i$ consecutive integers. This gives rise to a sequence $\mathcal{P}_i$ of invariant coarser and coarser partitions of L, where each class of $\mathcal{P}_i$ have size $c_1 \cdots c_i 2^{c_1 \cdots c_i}$: let points (ξ_1, k_1) and (ξ_2, k_2) be in the same class if k_1 and k_2 are in the same class of $\mathcal{R}_i$ and ξ_1 and ξ_2 only differ on this class.

Let $\mathcal{S}_i$ be the set of subgraphs of G induced by the finite sets of the form $\delta \times \sigma \times K_{28}$, where $\delta \in \mathcal{L}_i$ and $\sigma \in \mathcal{P}_i$. So each element of $\mathcal{S}_i$ has the form of the product of a percolation cluster of Δ in $\mathcal{L}_i$, a class of the partition of $\mathcal{P}_i$ and K_{28}. Let $\mathcal{S}_i^0$ be the set of those elements in $\mathcal{S}_i$ where the δ above is closed, and let $\mathcal{S}_i^1$ be the set of those where it is open. Hence $\mathcal{S}_i = \mathcal{S}_i^0 \cup \mathcal{S}_i^1$; call the elements of $\bigcup_i \mathcal{S}_i$ *cans*. Cans in some $\mathcal{S}_i^0$ will be called *type* 0, those in some $\mathcal{S}_i^1$ are *type* 1.

First, we will define the edges of Γ that go between two distinct cans, then those that go inside one. Suppose that $\delta \times \sigma \times K_{28}$ and $\delta' \times \sigma' \times K_{28}$ are in $\bigcup \mathcal{S}_i$, and such that $\sigma \subset \sigma'$ and $\delta \to \delta'$ in T. Then choose a random edge of G between $\delta \times \sigma \times K_{28}$ and $\delta' \times \sigma' \times K_{28}$ and add it to Γ. Denote the endpoint of this edge in $\delta \times \sigma \times K_{28}$ by $v(\delta, \sigma)$. Do it for every such pair independently. This way we have defined a tree on the set $\bigcup_i \mathcal{S}_i$ of cans.

What is left is to define edges within cans. For cans of type 1, let every edge of G induced in the can be in Γ. The c_i will be chosen so that with high probability the diameter of this induced graph in the can in logarithmic in its size. If we wanted Γ to be a tree, we can choose uniformly a spanning tree of the can C at this point that is geodesic with respect to some uniformly chosen point x, i.e. the distance of any y from x in C is the same as in this tree. The diameter of the resulting spanning tree is still logarithmic in the size with high probability (as a consequence of the triangle inequality), and this is the only thing we will use later. For cans $\delta \times \sigma \times K_{28}$ of type 0, choose uniformly a Hamiltonian path starting from $v(\delta, \sigma)$ that is contained in some Hamiltonian cycle induced by $\delta \times \sigma \times K_{28}$ in G, independently from all other choices. To see that such a Hamiltonian path always exists, we need to show that a Hamiltonian cycle exists. Choose any spanning tree in $\delta \times \sigma$, take a closed depth-first walk $v_1, \dots, v_m$ (with $v_1 = v_m$) that visits every vertex of $\delta \times \sigma$ at least once and at most degree $(\Delta \times L) = 28$ times, and then replace every copy of a vertex v in $v_1, \dots, v_m$ by one or more of the vertices $(v, 1), (v, 2), \dots, (v, 28) \in \delta \times \sigma \times K_{28}$ in such a way that each of these occur exactly once in the resulting new cycle. Note that every Hamiltonian cycle O in $\delta \times \sigma \times K_{28}$ gives rise to exactly two Hamiltonian paths starting from $v(\delta, \sigma)$: delete one of the edges in O adjacent to $v(\delta, \sigma)$. This also shows that for any $x \in V(\delta \times \sigma \times K_{28})$, the distance between x and $v(\delta, \sigma)$ in one of these paths is at least $|\delta \times \sigma \times K_{28}|/2$.

This finishes the construction of Γ. It is easy to check that the resulting Γ is ergodic.

Verification: We sketch here why the upper and lower growth rates are different, before giving a rigorous proof in the rest. There are radii R when the ball $B_R(o) := B_\Gamma(o, R)$ of radius R in Γ just exits a can C of type 0, in the direction of the can that separates it from infinity (meaning that $B_R(o)$ intersects only finite components of $\Gamma \setminus C$, but B_{R+1} intersects the infinite component as well). Then with high probability some constant proportion of $B_R(o)$ is within C, so it has to contain some constant proportion of the Hamiltonian path in C, which will imply that its radius is also close to the volume of C (this gives the claim about the lower growth rate). On the other hand, there are radii R when $B_R(o)$ just exits a can C of type 1, in which case a significant proportion of the volume of $B_R(o)$ is contained in C with high probability. Since the growth of the ball in C is exponential, the radius increase between entering and exiting C is at most the logarithm of the size of $|C|$. By the fast increase of the c_i we can conclude that R is also about logarithmic in $|B_R(o)|$ (for this, it is enough to assume that the sum of the sizes of the smaller cans that are intersected by a minimal o–C path is of order $\log c_i$ with high probability). This gives the statement about the upper growth.

Now we work out the above argument in details. For a cluster $\delta \in \mathcal{C}$ let $\operatorname{int}\delta$ be the complement of the infinite component of $\Delta \setminus \delta$ (that is, the union of all clusters that δ separates from infinity, including δ). Denote by π_Δ the projection from G to Δ and π_H be the projection to $L \times K_{28} =: H$. From now on, condition on the event $E_0 := \{o$ is in a can $C \in \mathcal{S}_1\}$. Let $C = C_1, C_2, \dots$ be consecutive cans that an infinite simple path from o in Γ visits (in particular, $\pi_\Delta(C_1), \pi_\Delta(C_2), \dots$ determines a simple path in T oriented from $\pi_\Delta(C_1)$ to infinity). Note that by definition if $i < i'$ then $C_{i'}$ separates C_i from infinity in Γ. We will now specify how the c_i's have to be chosen. Let $c_1 = 1$. For $i > 1$, let c_i be a number such that with probability at least $1 - 2^{-i}$ we have

$$\log c_i \geq 28 c_1 \cdots c_{i-1} 2^{c_1 \cdots c_{i-1}} \left|\operatorname{int}\pi_\Delta(C_i)\right|. \tag{3}$$

Let E_i be the event that (3) holds.

Let e be the edge connecting C_i to C_{i+1} in Γ, and let R be such that the ball $B_o(R)$ of radius R around o in Γ contains exactly one endpoint of e (the one in C_i). Then this ball is contained in the finite component of $\Gamma \setminus e$. The vertex set of the finite component of $\Gamma \setminus e$ arises as a union of the vertices in cans C', namely, it is $V(\bigcup\{C' \colon \pi_H(C') \subset \pi_H(C_i), \pi_\Delta(C') \subset \operatorname{int}\pi_\Delta(C_i)\})$. In particular,

$$\left|B_o(R)\right| \leq \left|\bigcup\{C' \colon \pi_H(C') \subset \pi_H(C_i), \pi_\Delta(C') \subset \operatorname{int}\pi_\Delta(C_i)\}\right| = \left|\operatorname{int}\pi_\Delta(C_i)\right|\left|\pi_H(C_i)\right|. \tag{4}$$

Now, in the case that C_i is of type 0, the restriction $C_i|_\Gamma$ is a path P. By the choice of P and our remark on the Hamiltonian paths, with probability $\geq 1/2$ the ball $B_o(R)$ contains all of the points in P, hence $|P|/2$ gives a lower bound for R. We conclude that conditioned on E_0 and E_i, with probability at least $1/2$ there is an R such that $B_R(o)$ has radius

$$R \geq |C_i|/2 = \left|\pi_\Delta(C_i)\right|\left|\pi_H(C_i)\right|/2$$

and has volume

$$\left|B_o(R)\right| \leq \left|\operatorname{int}\pi_\Delta(C_i)\right|\left|\pi_H(C_i)\right| \leq \left(\operatorname{diam}\left(\operatorname{int}\pi_\Delta(C_i)\right) + 1\right)^2 \left|\pi_H(C_i)\right| \leq \left|\pi_\Delta(C_i)\right|^2 \left|\pi_H(C_i)\right|,$$

where $\operatorname{diam}(\operatorname{int}\pi_\Delta(C_i))$ denotes the diameter of $\operatorname{int}\pi_\Delta(C_i)$, and the second inequality follows from the quadratic growth of Δ. We obtain that $\log|B_o(R)|/R$ gets arbitrarily close to 0 with probability at least $1/2$ (and hence, by ergodicity, with probability 1). Obviously this remains true if we do not condition on $o \in C \in \mathcal{S}_1$.

Suppose finally, that C_i is of type 1. Then the diameter of $\Gamma|_{C_i}$ is logarithmic to its size, so conditioned on E_i we have for some constant a that $R \le |C_1| + \cdots + |C_{i-1}| + a\log|C_i| \le 28c_1\cdots c_{i-1}2^{c_1\cdots c_{i-1}}|\operatorname{int}\pi_\Delta(C_i)| + a\log|C_i| \le 2a\log|C_i|$ with probability at least $1/2$, using (3). On the other hand with probability at least $1/2$ the volume of $B_o(R)\cap C_i$ is at least $|C_i|/2$, since the choice of e is uniform. We obtain that with some positive constant c

$$\log\big|B_o(R)\big|/R > c,$$

which holds for infinitely many R almost surely (by an argument similar to the end of last paragraph). □

Acknowledgements

I thank Lewis Bowen, Russ Lyons and Gábor Pete for valuable discussions, and an anonymous referee for suggestions for improvement.

Reference

[1] D. Aldous and R. Lyons. Processes on unimodular random networks. *Electron. Commun. Probab.* **12** (2007) 1454–1508. MR2354165