Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2013, Vol. 49, No. 1, 95–118
DOI: 10.1214/11-AIHP447

ANNALES DE L'INSTITUT HENRI POINCARÉ
PROBABILITÉS ET STATISTIQUES
www.imstat.org/aihp

Poincaré inequalities and hitting times[1]

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Received 23 December 2010; revised 15 June 2011; accepted 4 July 2011

Abstract. Equivalence of the spectral gap, exponential integrability of hitting times and Lyapunov conditions is well known. We give here the correspondence (with quantitative results) for reversible diffusion processes. As a consequence, we generalize results of Bobkov in the one dimensional case on the value of the Poincaré constant for log-concave measures to superlinear potentials. Finally, we study various functional inequalities under different hitting times integrability conditions (polynomial, ...). In particular, in the one dimensional case, ultracontractivity is equivalent to a bounded Lyapunov condition.

Résumé. L'équivalence entre le trou spectral, l'intégrabilité exponentielle des temps de retour et des conditions de Lyapunov est bien connue pour les chaînes de Markov. Nous donnons ici cette même équivalence (quantitative) pour des diffusions réversibles. Une des conséquences est la généralisation de résultats de Bobkov dans le cas unidimensionnel sur la valeur de la constante de l'inégalité de Poincaré des mesures log-concaves à des potentiels super linéaires. En conclusion, nous étudions diverses inégalités fonctionnelles sous difffentes conditions d'intégrabilité des temps de retour (polynomiale, ...). En particulier, en dimension 1, nous montrons l'équivalence entre ultracontractivité et condition de Lyapunov bornée.

MSC: 26D10; 39B62; 47D07; 60G10; 60J60

Keywords: Poincaré inequalities; Lyapunov functions; Hitting times; Log-concave measures; Poincaré–Sobolev inequalities

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1. Introduction

During the recent years a lot of progress has been made in the understanding of functional inequalities and their links with the long time behavior of stochastic processes. Very recently, starting with [3], the interplay between functional inequalities and the Lyapunov functions used in the "Meyn–Tweedie" theory [20,26] has emerged (see [2,12,16] and the recent survey [15]).

In the present paper we shall go a step further by showing the equivalence between the (usual) Poincaré inequality, the existence of a Lyapunov function and the exponential integrability of the hitting times of open bounded subsets.

As we shall recall below, this equivalence is well known in the Markov chains setting, a key tool being the renewal theory. We shall discuss here the diffusion process setting. In order to avoid technical intricacies, we mainly look at "very regular" cases, i.e. hypoelliptic processes.

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[1] Supported by the Agence Nationale de la Recherche project EVOL.

Note that the question of the existence of exponential moments for hitting times when a Poincaré inequality holds was addressed in [8] almost thirty years ago. We will precise explicit values for the constants, and add Lyapunov functions to the picture.

The one dimensional situation was recently discussed in [24], but as it is well known, monotonicity arguments make things easier in the one dimensional situation.

The main theorem is derived in Section 2. Actually most of the statements in the theorem are known or part of the "folklore" but in fact not clearly (or simply) written and most importantly often not quantitative! The most important new point is the fact that a Poincaré inequality implies the existence of exponential moments for hitting times not only starting from the invariant measures bu also pointwise. However the proofs we shall give (even for already known statements) are simple (in comparison with the existing ones) and constructive. Thus, they allow us to give quantitative estimates or bounds. This is done in Section 3. In Section 4 we look at the one dimensional setting. We show that Boltzmann–Gibbs measures with a super-linear potential at infinity satisfy a Poincaré inequality and recover (up to the universal constant) the control of the Poincaré constant for log-concave Probability measures obtained by Bobkov [5]. In the final Section 5 we shall discuss various statements concerning integrability of return times: a first surprising feature is that boundedness of exponential moments is in fact equivalent to ultraboundedness in dimension one. Such result is false in larger dimension. We then consider polynomial moments of hitting times, instead of exponential ones, in connection with weak Poincaré inequalities. This section is reminiscent of the work of Mathieu [25]. We shall however prove here a direct link between the existence of polynomial moments and weak Poincaré inequalities which are new, providing thus the first (non-optimal) reciprocical implication, in the sense that we can prove that some weak Poincaré inequality implies existence of polynomial moments.

2. Poincaré inequality and hitting times

2.1. *The main result*

Let us first recall the known situation for Markov chains. For simplicity assume that the state space E is countable, and that Q is a Markov transition kernel on E which is irreducible and aperiodic. Denote by $(X_n)_{n\in\mathbb{N}}$ the associated Markov chain. For $a \in E$ we denote by T_a the hitting time of $\{a\}$ i.e. $T_a = \inf\{n \geq 0; X_n = a\}$. Then

Theorem 2.1. *Under the previous assumptions, the following statements are equivalent*

(1) *there exist $a \in E$ and $\rho > 1$ such that for all $x \in E$, $\mathbb{E}_x(\rho^{T_a}) < +\infty$,*
(2) *there exist an invariant probability measure π and $0 < \theta < 1$ such that for all $x \in E$ one can find $C(x)$ with*

$$\left\| Q^n(x,\cdot) - \pi(\cdot)\right\|_{\mathrm{TV}} \leq C(x)\theta^n,$$

where $\|\nu - \mu\|_{\mathrm{TV}}$ denotes the total variation distance between μ and ν,
(3) *there exists a Lyapunov function, i.e. a function $W : E \to \mathbb{R}$, such that $W \geq 1$, $(Q - \mathrm{Id})W := LW \leq \alpha W + b\mathbb{1}_a$ for some $0 < \alpha < 1$ and some $b \geq 0$.*

In addition if the (unique) invariant measure is symmetric, these statements are equivalent to the following two additional ones

(4) *there exists a constant C_P such that the Poincaré inequality*

$$\mathrm{Var}_\pi(f) \leq C_P\big\langle\big(\mathrm{Id} - Q^2\big)f, f\big\rangle$$

holds for all $f \in l^2(\pi)$ ($\langle\cdot,\cdot\rangle$ being the scalar product in $l^2(\pi)$),
(5) *there exists some $0 < \lambda < 1$ such that $\mathrm{Var}_\pi(Q^n f) \leq \mathrm{Var}_\pi(f)\lambda^{2n}$.*

The equivalence between (1) and (3) is an exercise, while (3) implies (2) can be nicely shown as remarked by M. Hairer and J. C. Mattingly [22] even in a stronger form. The converse (2) implies (1) is more intricate, and usual proofs call upon Kendall's renewal theorem and an argument of analytic continuation (see e.g. S. Meyn and

R. Tweedie's monograph [26]). In particular we can give explicit expressions for the constants for all implications, except this one (i.e. if (2) holds, we only know that (1) holds for some non-explicit ρ).

The equivalence between (4) and (5) is well known, while (5) clearly implies (2). Finally, (3) implies that (2) holds for Q hence for Q^2 changing θ. Hence (3) holds for Q^2, and this implies that the Poincaré inequality (4) holds according to an argument due to Mu-Fa Chen ([18], pp. 221–235).

The aim of this section is to extend this result to some continuous time diffusion processes on $\mathbb{R}^d$ (or a finite dimensional Riemannian manifold). We also want to get bounds for all the constants, as precisely as possible. Actually, as we said in the Introduction, an accurate study of the literature provides (in possibly more general situations) almost all the results we shall state. One possible way is to use some skeleton chain and Theorem 2.1 (with some loss for the constants). Our approach will be more direct and elementary.

For simplicity we shall consider $\mathbb{R}^n$ valued diffusion processes $(X_t)_{t>0}$ with generator

$$L = \sum_{i,j} a_{ij} \partial^2_{ij} + \sum_i b_i \partial_i,$$

where $a = \sigma^* \sigma$, σ_{ij} and b_i being smooth enough (C^∞ for instance). We introduce the "carré du champ" operator

$$\Gamma(f,g) = \frac{1}{2}\big(L(fg) - fLg - gLf\big) = \langle \sigma \nabla f, \sigma \nabla g \rangle.$$

In addition we assume that $\mu(\mathrm{d}x) = \mathrm{e}^{-V(x)}\,\mathrm{d}x$ is a symmetric probability measure for the process, where the potential V is also assumed to be smooth. Thus L generates a μ-symmetric semi-group P_t and the $\mathbb{L}^2$ ergodic theorem (in the symmetric case) tells us that for all $f \in \mathbb{L}^2(\mu)$,

$$\lim_{t \to +\infty} \left\| P_t f - \int f \,\mathrm{d}\mu \right\|_{\mathbb{L}^2(\mu)} = 0.$$

If U is an open subset of $\mathbb{R}^d$ we define

$$T_U = \inf\{t > 0; X_t \in U\}.$$

Consider the following statements:

(H1) There exists a Lyapunov function W, i.e. there exist a smooth function $W : \mathbb{R}^n \to \mathbb{R}$, s.t. $W \geq 1$, a constant $\lambda > 0$ and an open connected bounded subset U such that

$$LW \leq -\lambda W \quad \text{on } (\bar{U})^c.$$

(H2) There exist an open connected bounded subset U and a constant $\theta > 0$ such that for all x,

$$\mathbb{E}_x\big(\mathrm{e}^{\theta T_U}\big) < +\infty,$$

and $x \mapsto \mathbb{E}_x(\mathrm{e}^{\theta T_U})$ is locally bounded.

(H2μ) There exist an open connected bounded subset U and a constant $\theta > 0$ such that,

$$\mathbb{E}_\mu\big(\mathrm{e}^{\theta T_U}\big) < +\infty.$$

(H3) There exist constants $\beta > 0$ and $C > 0$ and a function $W \geq 1$ belonging to $\mathbb{L}^1(\mu)$ such that for all x

$$\big\| P_t(x, \cdot) - \mu \big\|_{\mathrm{TV}} \leq C W(x) \mathrm{e}^{-\beta t}.$$

(H4) μ satisfies a Poincaré inequality, i.e. there exists a constant C_P such that for all smooth f,

$$\mathrm{Var}_\mu(f) \leq C_P \int \Gamma(f,f)\,\mathrm{d}\mu.$$

(H5) There exist constants $\eta > 0$ and $C > 0$ such that for all bounded f,

$$\mathrm{Var}_\mu(P_t f) \le C \mathrm{e}^{-\eta t}\,\mathrm{Osc}^2(f),$$

where $\mathrm{Osc}(f)$ denotes the oscillation of f.

(H6) There exists a constant C_S such that for all $f \in \mathbb{L}^2(\mu)$,

$$\mathrm{Var}_\mu(P_t f) \le \mathrm{e}^{-C_S t}\,\mathrm{Var}_\mu(f).$$

Finally we also introduce the following definition

Definition 2.2. *We shall say that L is strongly hypoelliptic if it can be written in Hörmander form $L = \sum_j X_j^2 + Y$ where the X_j's and Y are smooth vector fields such that the Lie algebra generated by the X_j's is full at each $x \in \mathbb{R}^n$ (i.e. spans the tangent space at each x). Note that in this situation $\Gamma(f, f) = \sum_j |X_j f|^2$.*

We shall say that L is uniformly strongly hypoelliptic if all the X_j's are bounded with bounded derivatives (of any order) and there exist $N \in \mathbb{N}$, $\alpha > 0$ such that for all $\xi \in \mathbb{R}^n$,

$$\sum_{Z \in L_N(x)} \langle Z(x), \xi\rangle^2 \ge \alpha |\xi|^2,$$

where $L_N(x)$ denotes the set of Lie brackets of length smaller or equal to N computed at x.

We may state now our main

Theorem 2.3. *The following relations hold true (recall that μ is symmetric)*

(1) (H1) $\Rightarrow$ (H3) $\Rightarrow$ (H4) $\Leftrightarrow$ (H5) $\Leftrightarrow$ (H6),
(2) (H1) $\Rightarrow$ (H2) *and* (H2μ),
(3) *if L is uniformly strongly hypoelliptic then* (H4) $\Rightarrow$ (H2) *and* (H2μ), *and* (H2) *or* (H2μ) $\Rightarrow$ (H1).

Hence if L is uniformly strongly hypoelliptic all statements (H1) *up to* (H6) *are equivalent.*

As we said, except for the third item, all these statements are mainly known. Let us make a few remarks on the hypotheses.

Remark 2.4 (Hypo-ellipticity). *The diffusion with a gradient drift $L = \Delta - \nabla V \cdot \nabla$ is of course hypo-elliptic. We will see later precise computations for the constants in this case, under the additional assumption:*

$$LV + \frac{1}{2}\Gamma(V, V) \le C_m < \infty. \tag{2.5}$$

Of course hypoellipticity is too strong. In general what is required for a Lyapunov function in (H1) *is to belong to the domain (or the extended domain) of the generator L. In particular if the exponential moment in* (H2) *belongs to $\mathbb{L}^2(\mu)$, the general theory of Dirichlet forms implies that it belongs to the domain of L and satisfies* (H1). *The situation is more intricate when it is only locally square integrable. In this case one has to adapt the arguments in Remark* 2.11. *Though it is certainly of big interest to study more general situations, we prefer to avoid some technicalities, and stay at a very regular level.*

Remark 2.6 (Symmetry). *Actually several implications are still true without the symmetry assumption. However symmetry is required for* (H5) $\Rightarrow$ (H6) *(counter-examples are known in the non-symmetric situation, see e.g.* [3], *Section* 6, *with the kinetic Fokker–Planck equation). It is also required for our proof of* (H1) $\Rightarrow$ (H4), *but it is not for the one in* [19,20].

Symmetry is used in the proof of (H4) $\Rightarrow$ (H2), *but it is not required for the first partial result i.e. the existence of the exponential moment for* μ *almost all* x (*which holds in much more general cases according to the framework of* [13]). *This result appears in the paper by Carmona and Klein* [8] *where the exponential integrability of hitting times is shown under exponential rate of convergence in the ergodic theorem* (*hence Poincaré*) *and we are able to give a precise bound for the exponent* (*answering the question in Remark* 2 *of* [8]).

Note also that the implications (H1) *to* (H5) *holds also, with additional assumptions* (*local Poincaré inequality and* (*slight*) *conditions on the constants involved in* (H1)) *using Lyapunov–Poincaré inequalities as in* [3].

Let us finally remark that Röckner–Wang [27] *proves* (H5) *to* (H6) *without symmetry but assuming that* L *is normal* (*i.e.* $LL^* = L^*L$).

Remark 2.7. *Of course, provided* W *is everywhere defined and smooth,* (H1) *can be rewritten: there exists a Lyapunov function* W, *i.e. there exist a smooth function* $W \geq 1$, *a constant* $\lambda > 0$ *and an open connected bounded subset* U *such that*

$$LW \leq -\lambda W + b\mathbb{1}_U$$

with $b = \sup_U(LW + \lambda W)$. *This formulation is the one used in* [2] *yielding another bound for the Poincaré constant, namely*

$$C_P \leq \frac{1}{\lambda}\big(1 + bC_P(U)\big). \tag{2.8}$$

The bound we will get below (*Eq.* (2.16)) *is not immediately comparable with this one.*

In particular if (H2) *holds in our strong hypoelliptic framework,* $x \mapsto \mathbb{E}_x(\mathrm{e}^{\theta T_U})$ *is smooth* (*provided the boundary* ∂U *is non-characteristic*) *on* $\bar{U}^c$ (*see again* [10]) *hence can be smoothly extended to the whole* R^n *according to Seeley's theorem. But an explicit bound for* b *is difficult to obtain.*

Remark 2.9. *An important ingredient in our proofs is that the measure* μ *satsified a Poincaré inequality restricted on* U. *Indeed, as we suppose the potential* V *to be smooth, one has then that it is locally bounded, and as in all assumptions* (H1), (H2) *or* (H3), *the set* U *is supposed to be connected and bounded, we have that* μ *satisfies a local Poincaré inequality on* U *in particular via easy perturbation arguments from the one satisfied by the Lebesgue measure on* U *when* L *is for example* $L = \Delta - \nabla V \cdot \nabla$ (*in a quantitative way*). *Note that this local Poincaré inequality is also verified by general strongly hypoelliptic generator* L, *see* [3].

Remark 2.10. *Let us emphasize that in the simple symmetric case* $L = \Delta - \nabla V \cdot \nabla$ *on* $\mathbb{R}^n$, *condition* (H1) *is quite easy to verify: for example, suppose one of the following*

$$x \cdot \nabla V \geq c > 0 \quad \text{or} \quad \exists 0 < a < 1, \quad (1-a)|\nabla V|^2 - \Delta V \geq c > 0$$

outside some large ball. Indeed use $W(x) = \mathrm{e}^{a|x|}$ *for the first condition and* $W(x) = \mathrm{e}^{aV(x)}$ *in the second.*

2.2. *Proof of the main theorem*

Let us begin by a small remark on (H1).

Remark 2.11 (Integrability of W***).*** *We did not impose any integrability condition for* W *in* (H1). *Actually if* W *satisfies* (H1), W *automatically belongs to* $\mathbb{L}^1(\mu)$.

Indeed choose some smooth, non-decreasing, concave function ψ *defined on* $\mathbb{R}^+$, *satisfying* $\psi(u) = u$ *if* $u \leq R$, $\psi(u) = R+1$ *if* $u \geq R+2$ *and with* $\psi'(u) \leq 1$ (*such a function exists*). *Then* $\psi(W)$ *is smooth and bounded. According to the chain rule*

$$L\big(\psi(W)\big) = \psi'(W)LW + \psi''(W)\Gamma(W,W) \leq -\lambda\psi'(W)W \quad \text{on } \bar{U}^c, \tag{2.12}$$

thanks to our assumptions. For R large enough, $W \le R$ on U, so that $\psi(W) = W$ on U. It follows

$$\begin{aligned}\lambda \int W \mathbb{1}_{W\le R}\, d\mu &\le \lambda \int \psi'(W) W \, d\mu \\ &= \int L\big(\psi(W)\big)\, d\mu + \lambda \int \psi'(W) W \, d\mu \quad \textit{since} \left(\int Lg \, d\mu = 0\right) \\ &\le \int_U \big(L\big(\psi(W)\big) + \lambda \psi'(W) W\big)\, d\mu \quad \textit{using } (2.12) \\ &\le \int_U (LW + \lambda W)\, d\mu = C(U),\end{aligned}$$

where $C(U)$ does not depend on R. We conclude by letting R go to ∞.

We now turn to the proof of the theorem.

(H4) $\Leftrightarrow$ (H6). This is well known and we have in addition $C_S = 2/C_P$.

(H6) $\Leftrightarrow$ (H5). (H6) clearly implies (H5). Since μ is symmetric the converse is proven in [27] using the spectral resolution. For the sake of completeness we shall give below a very elementary proof of this fact based on the following

Lemma 2.13. *$t \mapsto \log \|P_t f\|_{\mathbb{L}^2(\mu)}$ is convex.*

Indeed if $n(t) = \|P_t f\|^2_{\mathbb{L}^2(\mu)}$, the sign of the second derivative of $\log n$ is the one of $n''n - (n')^2$. But

$$n'(t) = 2 \int P_t f \, L P_t f \, d\mu$$

and

$$n''(t) = 2 \int (L P_t f)^2 \, d\mu + 2 \int P_t f \, L P_t L f \, d\mu = 4 \int (L P_t f)^2 \, d\mu,$$

so that the lemma is just a consequence of Cauchy–Schwarz inequality.

This convexity is a key argument in the proof of the following

Lemma 2.14. *Let $\mathcal{C}$ be a dense subset of $\mathbb{L}^2(\mu)$. Suppose that there exists $\beta > 0$, and, for any $f \in \mathcal{C}$, a constant c_f such that*:

$$\forall t, \quad \operatorname{Var}_\mu(P_t f) \le c_f e^{-\beta t}.$$

Then

$$\forall f \in \mathbb{L}^2(\mu), \forall t, \quad \operatorname{Var}_\mu(P_t f) \le e^{-\beta t} \operatorname{Var}_\mu(f).$$

Our claim (H5) implies (H6) immediately follows with $\eta = C_S$. In order to prove Lemma 2.14, assuming that $\int f \, d\mu = 0$ which is not a restriction, it is enough to look at

$$t \mapsto \log \|P_t f\|_{\mathbb{L}^2(\mu)} + (\beta t/2),$$

which is convex, according to Lemma 2.13, and bounded since $\operatorname{Var}_\mu(P_t f) \le c_f e^{-\beta t}$. But a bounded convex function on $\mathbb{R}^+$ is necessarily non-increasing. Hence

$$\|P_t f\|_{\mathbb{L}^2(\mu)} \le e^{-\beta t/2} \|P_0 f\|_{\mathbb{L}^2(\mu)}$$

for all $f \in \mathcal{C}$, the result follows using the density of $\mathcal{C}$.

(H3) ⇒ (H5). This is shown in [3], Theorem 2.1, and we may choose the constant C in (H5) equal to $8C\int W\,\mathrm{d}\mu$ where C is the constant in (H3), and $\eta = \beta$.

(H1) ⇒ (H3). This is the key result in [20] (also see [19]), unfortunately with an essentially non-explicit control of the constants.

Combining all these results we get the first statement of the theorem, in particular we already know that (H1) implies (H4).

A direct and short proof of (H1) ⇒ (H4) is given in [2] for $L = \Delta - \nabla V \cdot \nabla$ which is the natural symmetric operator associated with μ. Let us give a slightly modified proof, yielding a better control on the constants and extending it to more general operators.

The key is the following (f being smooth)

$$\int \frac{-LW}{W} f^2\,\mathrm{d}\mu \le \int \Gamma(f,f)\,\mathrm{d}\mu, \tag{2.15}$$

which is a consequence of

$$\begin{aligned}\int \frac{-LW}{W} f^2\,\mathrm{d}\mu &= \int \Gamma\left(\frac{f^2}{W}, W\right)\mathrm{d}\mu\\ &= 2\int \frac{f}{W}\Gamma(f,W)\,\mathrm{d}\mu - \int \frac{f^2}{W^2}\Gamma(W,W)\,\mathrm{d}\mu\\ &= -\int \left|\frac{f}{W}\sigma\nabla W - \sigma\nabla f\right|^2\mathrm{d}\mu + \int \Gamma(f,f)\,\mathrm{d}\mu.\end{aligned}$$

Next for $r > 0$ introduce $U_r = \{x; d(x,U) < r\}$ for the (Euclidean or Riemannian) distance d. Let $0 \le \chi \le 1$ be a C^∞ function such that $\chi = 1$ on U and $\chi = 0$ on U_r^c. Then

$$\begin{aligned}\int f^2\,\mathrm{d}\mu &= \int \big(f(1-\chi) + f\chi\big)^2\,\mathrm{d}\mu\\ &\le 2\int f^2(1-\chi)^2\,\mathrm{d}\mu + 2\int f^2\chi^2\,\mathrm{d}\mu\\ &\le \frac{2}{\lambda}\int \frac{-LW}{W} f^2(1-\chi)^2\,\mathrm{d}\mu + 2\int_{U_r} f^2\,\mathrm{d}\mu\\ &\le \frac{2}{\lambda}\int \Gamma\big(f(1-\chi), f(1-\chi)\big)\,\mathrm{d}\mu + 2\int_{U_r} f^2\,\mathrm{d}\mu\end{aligned}$$

by (2.15). Since $\Gamma(fg,fg) \le 2(f^2\Gamma(g,g) + g^2\Gamma(f,f))$, we get:

$$\begin{aligned}\int f^2\,\mathrm{d}\mu &\le \frac{4}{\lambda}\int \Gamma(f,f)\,\mathrm{d}\mu + \frac{4}{\lambda}\int f^2\Gamma(\chi,\chi)\,\mathrm{d}\mu + 2\int_{U_r} f^2\,\mathrm{d}\mu\\ &\le \frac{4}{\lambda}\int \Gamma(f,f)\,\mathrm{d}\mu + \left(\frac{4\|\Gamma(\chi,\chi)\|_\infty}{\lambda} + 2\right)\int_{U_r} f^2\,\mathrm{d}\mu.\end{aligned}$$

Now, if μ satisfies a Poincaré inequality in restriction to U_r, i.e.

$$\int_{U_r} f^2\,\mathrm{d}\mu \le C_P(U,r)\int_{U_r} \Gamma(f,f)\,\mathrm{d}\mu \quad \text{if } \int_{U_r} f\,\mathrm{d}\mu = 0,$$

we may apply the previous inequality with $g = f - \frac{1}{\mu(U_r)}\int_{U_r} f\,\mathrm{d}\mu$, yielding, since $\sigma\nabla f = \sigma\nabla g$,

$$\mathrm{Var}_\mu(f) \le \int g^2\,\mathrm{d}\mu \le \left(\frac{4}{\lambda} + \left(\frac{4\|\Gamma(\chi,\chi)\|_\infty}{\lambda} + 2\right)C_P(U,r)\right)\int \Gamma(f,f)\,\mathrm{d}\mu, \tag{2.16}$$

i.e. the Poincaré inequality (H4). Note that we may always replace U by a larger Euclidean ball, i.e. we may assume that U is an Euclidean ball. According to the discussion in [3], pp. 744–745, if L is strongly hypoelliptic, μ satisfies the Poincaré inequality in restriction to any Euclidean ball, so that we have shown that (H1) $\Rightarrow$ (H4) in this case.

We now turn to the part of the results involving the stochastic process.

(H1) $\Rightarrow$ (H2). This is a simple application of Ito's formula applied to $(t,x) \mapsto \mathrm{e}^{at} W(x)$ (notice that (H1) implies that the diffusion process is non-explosive or conservative). Indeed let $x \in U^c$, and $a \leq \lambda$. Define T_{UR} as the first hitting time of $U \cup \{|y| > R\}$. For $R > |x|$ we thus have

$$\begin{aligned}
\mathbb{E}_x\big(\mathrm{e}^{a(t\wedge T_{UR})}\big) &\leq \mathbb{E}_x\big(\mathrm{e}^{a(t\wedge T_{UR})} W(X_{t\wedge T_{UR}})\big) \\
&\leq W(x) + \mathbb{E}_x\left(\int_0^{t\wedge T_{UR}} (aW + LW)(X_s)\mathrm{e}^{as}\,\mathrm{d}s\right) \\
&\leq W(x) + \mathbb{E}_x\left(\int_0^{t\wedge T_{UR}} (a-\lambda)W(X_s)\mathrm{e}^{as}\,\mathrm{d}s\right) \\
&\leq W(x),
\end{aligned}$$

so that letting first R then t go to infinity we obtain (H2) for $\theta = \lambda$, thanks to Lebesgue's monotone convergence theorem.

The same proof shows that (H2μ) holds since we know that $W \in \mathbb{L}^1(\mu)$.

Conversely, assume (H2) and the strong hypoellipticity of L. Again we may assume that U is an Euclidean ball so that for any $R > 0$, the boundary of the Euclidean shell $U_R - U$ is non-characteristic for L. We may then use the results in e.g. [10], Theorem 5.14 (local boundedness in (H2) ensures that hypothesis (HC) in [10] is satisfied), showing that

$$x \mapsto W_R(x) = \mathbb{E}_x\big(\mathrm{e}^{\theta(T_U \wedge T_{U_R^c})}\big)$$

is smooth and solves the Dirichlet problem

$$LW_R + \theta W_R = 0 \quad \text{in } U_R - U, \qquad W_R = 1 \quad \text{on } \partial(U_R - U).$$

Using (H2) again it then follows that

$$x \mapsto W(x) = \mathbb{E}_x\big(\mathrm{e}^{\theta T_U}\big)$$

is well defined, solves the Dirichlet problem with $R = +\infty$ in the sense of Schwartz distributions, hence is smooth thanks to hypoellipticity. W is then a Lyapunov function in (H1). If (H2μ) is satisfied, then an argument below will show that (H2) is satisfied.

To conclude the proof of the theorem it remains to show that the Poincaré inequality (H4) implies (H2). Let U be an open bounded set. The idea is that, if T_U is large, the process stays for a long time in U^c, and spends no time at all in U. However, the ergodic properties given by the Poincaré inequality tell us that, for large times, the fraction of the time spent in U should be proportional to $\mu(U)$; therefore T_U cannot be too large.

To be more precise,

$$\{T_U > t\} \subseteq \left\{\frac{1}{t}\int_0^t \mathbb{1}_U(X_s)\,\mathrm{d}s = 0\right\}. \tag{2.17}$$

Hence

$$\begin{aligned}
\mathbb{P}_\nu(T_U > t) &\leq \mathbb{P}_\nu\left(\frac{1}{t}\int_0^t \mathbb{1}_U(X_s)\,\mathrm{d}s = 0\right) \\
&\leq \mathbb{P}_\nu\left(-\frac{1}{t}\int_0^t \mathbb{1}_U(X_s)\,\mathrm{d}s + \mu(U) \geq \mu(U)\right) \\
&\leq \left\|\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right\|_{\mathbb{L}^2(\mu)} \cdot \exp\left(-\frac{t\mu(U)}{8C_P(1-\mu(U))}\right),
\end{aligned} \tag{2.18}$$

provided $\mu(U) \leq 1/2$. The latter is a consequence of Proposition 1.4 and Remark 1.6 in [13].

From there, we get exponential moments, using the elementary lemma:

Lemma 2.19. *For any positive random variable*,

$$\mathbb{E}\big[\mathrm{e}^{\theta T}\big] = 1 + \int_0^\infty \theta \mathrm{e}^{\theta t} \mathbb{P}[T > t]\,\mathrm{d}t.$$

If for some s_0, θ_U and for $t > s_0$, $\mathbb{P}[T > t] \leq C \exp(-(t - s_0)\theta_U)$, then

$$\forall \theta < \theta_U, \quad \mathbb{E}\big[\mathrm{e}^{\theta T}\big] \leq \mathrm{e}^{\theta s_0}\left(1 + C\frac{\theta}{\theta_U - \theta}\right).$$

For $s_0 = 0$, $\theta_U = \mu(U)/8C_P$, and $\nu = \mu$, using (2.18) and this lemma, we get $\mathbb{E}_\mu(\mathrm{e}^{\theta T_U}) < +\infty$, for any $\theta < \theta_U$. This entails that $\mathbb{E}_x[\mathrm{e}^{\theta T_U}]$ is itself finite, for μ-almost any x.

If we assume the uniform strong hypoellipticity the marginal law at time t of $\mathbb{P}_x$ has an everywhere positive smooth density $r(t, x, \cdot)$ w.r.t. μ, and symmetry combined with the Chapman–Kolmogorov relation yield

$$\int r^2(t, x, y)\mu(\mathrm{d}y) = r(2t, x, x) < \infty,$$

showing that the $\mathbb{P}_x$ law of X_1 has a density $r(1, x, \cdot) \in \mathbb{L}^2(\mu)$. We may thus apply the previous result with $\nu = r(1, x, \cdot)\mu$.

Notice that this argument also shows that T_U has an exponential moment of order $\theta/2$ for $\mathbb{P}_x$ as soon as it has an exponential moment of order θ for $\mathbb{P}_\mu$, i.e. (H2μ) implies (H2).

Remark 2.20. *The proof shows that* (H4) *implies* (H2), *i.e. the hitting times have finite exponential moments*, *but do not give explicit bounds on the value of these moments* (*depending on x*). *Such explicit bounds will be given in the next section.*

3. Some consequences

We rephrase here the implication (H4) $\Rightarrow$ (H2) of the main theorem, and add explicit computations of the constants, and the dependence on x of the moments, in special cases.

Proposition 3.1. *Assume that the Poincaré inequality holds with constant C_P.*

Then for all open set U with $\mu(U) \leq 1/2$, $\mathbb{E}_x(\mathrm{e}^{\theta T_U}) < +\infty$ for

$$\theta < \mu(U)/8C_P\big(1 - \mu(U)\big) := \theta(U).$$

If $\mu(U) \geq 1/2$ we may take $\theta(U) = \mu^2(U)/2C_P$.

If the boundedness assumption (2.5) *holds*, *there exists C such that*:

$$\forall x, \forall \theta < \theta_U, \quad \mathbb{E}_x\big[\mathrm{e}^{\theta T_U}\big] \leq C\left(1 + \mathrm{e}^{V(x)/2}\frac{\theta}{\theta_U - \theta}\right). \tag{3.2}$$

If, *in addition*, *we are in the elliptic case $L = \Delta - \nabla V \cdot \nabla$*, (3.2) *holds with C replaced by $\mathrm{e}^{\theta s_0}$*, *where $s_0 = \frac{1}{2\pi}\mathrm{e}^{2C_m/n}$.*

Proof. The first statement has already been proved.

If we assume the additional boundedness hypothesis (Eq. (2.5)), we can use stochastic calculus to get good bounds: the idea is that the density of the law of X_t with respect to μ is computable, and its L^2 norm can be bounded.

First of all recall that $L = \sum X_j^2 + Y$. Since μ is symmetric

$$Y = \sum_j \operatorname{div} X_j X_j - \sum_j X_j V X_j.$$

If we denote by $\mathbb{Q}_x$ the law of the diffusion process starting from x with generator

$$L' = \sum_j X_j^2 + \sum_j \operatorname{div} X_j X_j$$

we have a Girsanov type representation

$$\begin{aligned} G_t := \frac{\mathrm{d}\mathbb{P}_x}{\mathrm{d}\mathbb{Q}_x}\bigg|_{\mathcal{F}_t} &= \exp\left(-\frac{1}{2}\int_0^t \langle X_j V(\omega_s), \mathrm{d}\omega_s\rangle - \frac{1}{4}\int_0^t \Gamma(V,V)(\omega_s)\,\mathrm{d}s\right) \\ &= \exp\left(\frac{1}{2}V(\omega_0) - \frac{1}{2}V(\omega_t) - \frac{1}{2}\int_0^t \left(\frac{1}{2}\Gamma(V,V) - L'V\right)(\omega_s)\,\mathrm{d}s\right), \end{aligned}$$

the latter (Feynman–Kac representation) being obtained by integrating by parts the stochastic integral. We can now follow an argument we already used in previous works. We write the details for the sake of completeness.

Thanks to the uniform strong hypoellypticity we know that the marginal law at time t of $\mathbb{Q}_x$ has an everywhere positive smooth density $q(t,x,\cdot)$ w.r.t. Lebesgue measure satisfying for some M (see e.g. [9], Theorem 1.5)

$$\left|q(t,x,y)\right| \le C(1 \wedge t)^{-M} \quad \text{for all } x, y \in \mathbb{R}^n.$$

Hence

$$\begin{aligned} \mathbb{E}_x\left[f(X_t)\right] &= \mathbb{E}^{\mathbb{Q}_x}\left[f(\omega_t)\mathbb{E}^{\mathbb{Q}_x}[G|\omega_t]\right] \\ &= \int f(y)\mathbb{E}^{\mathbb{Q}_x}[G|\omega_t = y]q(t,x,y)\,\mathrm{d}y \\ &= \int f(y)\mathbb{E}^{\mathbb{Q}_x}[G|\omega_t = y]q(t,x,y)\mathrm{e}^{V(y)}\mu(\mathrm{d}y). \end{aligned}$$

In other words, the law of X_t has a density with respect to μ given by

$$r(t,x,y) = \mathbb{E}^{\mathbb{Q}_x}[G|\omega_t = y]q(t,x,y)\mathrm{e}^{V(y)}.$$

Hence

$$\begin{aligned} \int_0^{+\infty} r^2(t,x,y)\mu(\mathrm{d}y) &= \int \left(\mathbb{E}^{\mathbb{Q}_x}[G|\omega_t = y]q(t,x,y)\mathrm{e}^{V(y)}\right)^2 \mathrm{e}^{-V(y)}\,\mathrm{d}y \\ &= \mathbb{E}^{\mathbb{Q}_x}\left[q(t,x,\omega_t)\mathrm{e}^{V(\omega_t)}\left(\mathbb{E}^{\mathbb{Q}_x}[G|\omega_t]\right)^2\right] \\ &\le \mathbb{E}^{\mathbb{Q}_x}\left[q(t,x,\omega_t)\mathrm{e}^{V(\omega_t)}\mathbb{E}^{\mathbb{Q}_x}\left[G^2|\omega_t\right]\right] \\ &\le \mathrm{e}^{V(x)}\mathbb{E}^{\mathbb{Q}_x}\left[q(t,x,\omega_t)\mathrm{e}^{-\int_0^t((1/2)\Gamma(V,V)-L'V)(\omega_s)\,\mathrm{d}s}\right] \\ &\le C(1 \wedge t)^{-M}\mathrm{e}^{V(x)}\mathrm{e}^{C_m t}. \end{aligned}$$

Hence the law at time 1 of $X.$ has a density belonging to $\mathbb{L}^2(\mu)$. Using the result in [13] we have recalled and the Markov property we thus have for $t > 1$

$$\mathbb{P}_x(T_U > t) \le D\mathrm{e}^{V(x)/2}\mathrm{e}^{-(t-1)\mu(U)/(8C_P(1-\mu(U)))}$$

hence the result by Lemma 2.19.

Finally, if $L = \Delta - \nabla V \cdot \nabla$, we can be even more precise.
Indeed $q(t,x,y) \leq (2\pi t)^{-n/2}$ so that for $t > s > 0$, using (2.18) we obtain

$$\mathbb{P}_x(T_U > t) \leq (2\pi s)^{-n/4} \mathrm{e}^{C_m s/2} \mathrm{e}^{V(x)/2} \mathrm{e}^{-(t-s)\theta(U)}.$$

Choosing $s_0 = \frac{1}{2\pi} \mathrm{e}^{2C_m/n}$ we get for $t > s_0$,

$$\mathbb{P}_x(T_U > t) \leq \mathrm{e}^{V(x)/2} \mathrm{e}^{-(t-s_0)\theta(U)},$$

so that for $\theta < \theta(U)$, using Lemma 2.19, we get

$$\mathbb{E}_x\left(\mathrm{e}^{\theta T_U}\right) \leq \mathrm{e}^{\theta s_0}\left(1 + \frac{\theta}{\theta(U) - \theta} \mathrm{e}^{V(x)/2}\right).$$

If μ satisfies a Poincaré inequality with constant C_P, so does $\mu^{\otimes k}$ for any $k \in \mathbb{N}^*$. It thus follows as before that for $x = (x_1, \dots, x_k)$ and $\theta < \theta(U)$,

$$\mathbb{P}_x(T_U > t) \leq (2\pi s)^{-nk/4} \mathrm{e}^{k C_m s/2} \mathrm{e}^{\sum_i V(x_i)/2} \mathrm{e}^{-(t-s)\theta(U)},$$

so that for the same s_0,

$$\mathbb{E}_x\left(\mathrm{e}^{\theta T_U}\right) \leq \mathrm{e}^{\theta s_0}\left(1 + \frac{e^{\sum_i V(x_i)/2}}{\theta(U) - \theta}\right). \qquad \square$$

Remark 3.3. *In the same way, when $L = \Delta - \nabla V \cdot \nabla$, one can improve upon the constant if we assume in addition that*

$$U \text{ has a smooth boundary and } \frac{\partial W}{\partial n} \leq 0 \text{ on } \partial U \text{ where } n \text{ denotes the inward normal to the boundary.} \tag{3.4}$$

Indeed in this case we can directly integrate by parts in U^c using the Green Rieman formula. This yields

$$\begin{aligned}
\int_{U^c} f^2 \, \mathrm{d}\mu &\leq \frac{1}{\lambda} \int_{U^c} f^2 \frac{-LW}{W} \mathrm{e}^{-V} \, \mathrm{d}x \\
&\leq \frac{1}{\lambda} \int_{U^c} \frac{f^2}{W}(-\Delta W + \nabla V \cdot \nabla W) \mathrm{e}^{-V} \, \mathrm{d}x \\
&\leq \frac{1}{\lambda} \int_{U^c} \left(\nabla\left(\frac{f^2 \mathrm{e}^{-V}}{W}\right) \cdot \nabla W \mathrm{e}^{V} + \frac{f^2}{W} \nabla V \cdot \nabla W\right) \mathrm{e}^{-V} \, \mathrm{d}x \\
&\quad + \frac{1}{\lambda} \int_{\partial U} \left(\frac{f^2 \mathrm{e}^{-V}}{W}\right) \frac{\partial W}{\partial n} \, \mathrm{d}m_{\partial U} \\
&\leq \frac{1}{\lambda} \int_{U^c} \nabla\left(\frac{f^2}{W}\right) \cdot \nabla W \mathrm{e}^{-V} \, \mathrm{d}x \\
&\leq \frac{1}{\lambda} \int_{U^c} \left(|\nabla f|^2 - \left|\nabla f - \frac{f}{W} \nabla W\right|^2\right) \mathrm{d}\mu \\
&\leq \frac{1}{\lambda} \int_{U^c} |\nabla f|^2 \, \mathrm{d}\mu.
\end{aligned}$$

Therefore we obtain in this case

$$C_P \leq \frac{1}{\lambda} + C_P(U), \tag{3.5}$$

which is of course much better than any other bound we gave.

Remark 3.6. *Since we are collecting some quantitative bounds here, let us recall what happens when we replace the mean by something else (but well suited) in the Poincaré inequality. Recall that if $m_\mu(f)$ denotes a μ median of f, one has*

$$\operatorname{Var}_\mu(f) \le \mathbb{E}_\mu\big[\big(f - m_\mu(f)\big)^2\big] \le 2\operatorname{Var}_\mu(f), \tag{3.7}$$

so that one may replace the variance by the squared distance to any median in Poincaré inequality up to some universal constants. Similar inequalities can be shown if we relace the median by any quantile.

We may also replace the mean of f by local means. Indeed let U be a subset such that $0 < \mu(U) < 1$. If $\int_U f\,\mathrm{d}\mu = 0$ we may write

$$\begin{aligned}
\int f^2\,\mathrm{d}\mu &\le C_P \int |\nabla f|^2\,\mathrm{d}\mu + \left(\int f\,\mathrm{d}\mu\right)^2 \\
&\le C_P \int |\nabla f|^2\,\mathrm{d}\mu + \left(\int_{U^c} f\,\mathrm{d}\mu\right)^2 \\
&\le C_P \int |\nabla f|^2\,\mathrm{d}\mu + \big(1-\mu(U)\big)\left(\int f^2\,\mathrm{d}\mu\right)
\end{aligned}$$

so that

$$\int f^2\,\mathrm{d}\mu \le \frac{C_P}{\mu(U)} \int |\nabla f|^2\,\mathrm{d}\mu.$$

4. Probability measures on the line

In this section we shall look at the case $n = 1$, $\mu(\mathrm{d}x) = Z^{-1}\mathrm{e}^{-V(x)}\,\mathrm{d}x$ where Z is a normalization constant. Since the Poincaré constant is unchanged by translating the measure we may also assume that $\int x\,\mathrm{d}\mu = 0$.

General bounds for the Poincaré constant are well known using Hardy–Muckenhoupt weighted inequalities (see e.g. [1]). Another approach was recently proposed in [24] where bounds for both the Poincaré constant and the exponential moment for hitting times are obtained, through the rate function and speed measure. Notice that the results of Section 2 seem to be less precise in the one-dimensional situation but cover all possible dimensions.

4.1. *Super-linear and log-concave one dimensional distributions*

Our interest here is to describe the Poincaré constant for particular μ including the log-concave situation. The log-concave situation indeed deserved a lot of interest due to the belief that, in the multidimensional isotropic case (namely the covariance matrix is the identity), it is close to the independent one. It is therefore particularly relevant to get bounds on functional inequalities in terms of the variance. For log-concave measures μ on the line Bobkov [5] proved that

$$\operatorname{Var}_\mu(x) \le C_P(\mu) \le 12\operatorname{Var}_\mu(x), \tag{4.1}$$

where x denotes the identity function. One can also look at another approach in [21].

In our previous work [2] we have shown how to use the Lyapunov function method to recover the general result of Bobkov saying that any log-concave probability measure (in any dimension) satisfies a Poincaré inequality. Here we shall be more precise for the one dimensional case and we shall recover a bad version of Bobkov's result (4.1), i.e. with a worse constant larger than 12 but for more general measures. We start with some definitions.

Definition 4.2. *Let $\mu(\mathrm{d}x) = Z^{-1}\mathrm{e}^{-V(x)}\,\mathrm{d}x$ be a probability measure on the line. We assume that there exists $V_{\min} > -\infty$ such that $V_{\min} \le V \le +\infty$ and that there exists $a \in \mathbb{R}$ such that $V(a) = V_{\min}$.*

For $\beta > 0$ we denote by $R_+(\beta)$ any positive number such that $V(a+u) - V(a) \le \beta$ for all $0 \le u \le R_+(\beta)$ and similarly $R_-(\beta)$ on the left-hand side of a. Finally $R(\beta) = R_+(\beta) \vee R_-(\beta)$.

We shall say that V is β-superlinear if for $t \geq R_+(\beta)$ (resp. $t \geq R_-(\beta)$) one has

$$V(a+t) - V(a) \geq \frac{c_\beta}{R(\beta)} t - h_\beta \quad \left(\textit{resp. } V(a-t) - V(a) \geq \frac{c_\beta}{R(\beta)} t - h_\beta\right)$$

for some non-negative constant c_β and some h_β.

Remark 4.3. *Let $\mu(\mathrm{d}x) = Z^{-1}\mathrm{e}^{-V(x)}\,\mathrm{d}x$ be a probability measure on the line, with V of class C^1. We assume that* $\min V = 0 = V(a)$ *and that there exist $\beta > 0$ and $\theta > 0$ such that* $\operatorname{sign}(x-a)V'(x) \geq \theta$ *outside some subset N_β of the level set $\{V \leq \beta\}$. Remark that this type of condition already appears in* [3] *as a sufficient condition for Poincaré inequality as it implies some Lyapunov condition. Constants there were however far from efficient concerning isotropic normalization.*

Since $\operatorname{sign}(x-a)V'(x) \geq \theta$ *outside N_β it is easily seen that N_β is necessarily a closed interval. We thus choose $R_+(\beta)$ and $R_-(\beta)$ such that $N_\beta = [a - R_-(\beta), a + R_+(\beta)]$. We may assume that $R_+(\beta) \geq R_-(\beta)$.*

For $x \geq a + R_+(\beta)$, our assumptions furnish

$$\begin{aligned}
V(x) &\geq V(x) - V\big(a + R_+(\beta)\big) \\
&\geq \theta\big(x - a - R_+(\beta)\big) \\
&\geq \frac{c_\beta}{R_+(\beta)}(x-a) - h_\beta,
\end{aligned}$$

where $c_\beta = \theta R_+(\beta) = h_\beta$.

For $x \leq a - R_-(\beta)$ we have the same result of course, still with $c_\beta = \theta R_+(\beta)$ a priori with $h = \theta R_-(\beta)$ which is smaller than h_β, so that the result still holds with h_β.

Hence V is β-superlinear. Actually it is β'-superlinear for any $\beta' \geq \beta$.

Our definition looks thus unnecessarily intricate. However, we shall see that it is well appropriate for the isotropic normalization.

The next lemma allows us to compare the variance and the β level set values,

Lemma 4.4. *Assume that V is β-superlinear and that $\int x\,\mathrm{d}\mu = 0$, then*

$$R_+^2(\beta) \vee R_-^2(\beta) \leq 12\,\mathrm{Var}_\mu(x)\mathrm{e}^\beta\left(1 + \frac{2\mathrm{e}^{h_\beta}}{c_\beta}\right)$$

and

$$R_+^2(\beta) \vee R_-^2(\beta) \geq \frac{1}{2}\,\mathrm{Var}_\mu(x)\mathrm{e}^{-\beta}\left(\frac{1}{3} + \frac{\mathrm{e}^{h_\beta}}{c_\beta}\left(1 + \frac{2}{c_\beta} + \frac{2}{c_\beta^2}\right)\right)^{-1}.$$

Proof. We may and will assume that $V(a) = 0$ (just modify Z). We fix once for all β and thus skip the dependence in β for notational convenience. Let $R = R_+ + R_-$ and denote by σ^2 the variance of μ.

Since V is β-superlinear we have

$$\begin{aligned}
R(\beta)\mathrm{e}^{-\beta} &\leq \int_{a-R_-}^{a+R_+} \mathrm{e}^{-\beta}\,\mathrm{d}x \leq \int_{a-R^-}^{a+R_+} \mathrm{e}^{-V(x)}\,\mathrm{d}x \leq Z \\
&\leq \int_{a-R_-}^{a+R_+} \mathrm{d}x + \mathrm{e}^h\left(\int_{a+R_+}^{+\infty} \mathrm{e}^{-(c/R(\beta))(x-a)}\,\mathrm{d}x + \int_{-\infty}^{a-R_-} \mathrm{e}^{-(c/R(\beta))(a-x)}\,\mathrm{d}x\right) \\
&\leq R(\beta)\left(1 + 2\frac{\mathrm{e}^h}{c}\right),
\end{aligned}$$

i.e.

$$R(\beta)\mathrm{e}^{-\beta} \le Z \le R(\beta)\left(1+\frac{2\mathrm{e}^h}{c}\right).$$

By symmetry we may also assume that $R_+ \ge R_-$ so that it is enough to get an upper bound for R_+. But

$$\frac{\mathrm{e}^{-\beta}}{3}\big((a+R_+)^3 - a^3\big) = \int_a^{a+R_+} \mathrm{e}^{-\beta} x^2\,\mathrm{d}x \le \int_a^{a+R_+} x^2 \mathrm{e}^{-V(x)}\,\mathrm{d}x \le Z\sigma^2.$$

Using $R(\beta) = R_+$ we thus obtain

$$R_+^2 + 3aR_+ + 3a^2 \le 3\sigma^2 \mathrm{e}^\beta\left(1+\frac{2\mathrm{e}^h}{c}\right). \tag{4.5}$$

If $a > 0$ we immediately obtain $R_+^2 \le 3\sigma^2(1+\frac{2\mathrm{e}^h}{c})$. If $a \le 0$ the minimal value of the left hand side in (4.5) (considered as a polynomial in a) is obtained for $a = -R_+/2$ and is equal to $R_+^2/4$ so that we obtain in all cases

$$R_+^2 \le 12\sigma^2 \mathrm{e}^\beta\left(1+\frac{2\mathrm{e}^h}{c}\right). \tag{4.6}$$

In the same way we see that if $a > 0$ then $a^2 \le 2\sigma^2\mathrm{e}^\beta(1+\frac{2\mathrm{e}^h}{c})$. If $a \le 0$ the minimal value of the left-hand side in (4.5) (considered as a polynomial in R_+) is obtained for $R_+ = -\frac{3}{2}a$ and is equal to $3a^2/4$ so that we obtain in all cases

$$a^2 \le 4\sigma^2 \mathrm{e}^\beta\left(1+\frac{2\mathrm{e}^h}{c}\right). \tag{4.7}$$

We turn to the second bound. Again we assume that $R_+ \ge R_-$. Recall that $\sigma^2 \le \int (x-a)^2\,\mathrm{d}\mu$. Similarly to the first bound we can thus write

$$\begin{aligned}
Z\sigma^2 &\le \int (x-a)^2 \mathrm{e}^{-V(x)}\,\mathrm{d}x \\
&\le \int_{a-R_-}^{a+R_+} (x-a)^2\,\mathrm{d}x + \mathrm{e}^h\left(\int_{a+R_+}^{+\infty} (x-a)^2 \mathrm{e}^{-(c/R(\beta))(x-a)}\,\mathrm{d}x + \int_{-\infty}^{a-R_-} (x-a)^2 \mathrm{e}^{-(c/R(\beta))(a-x)}\,\mathrm{d}x\right) \\
&\le \frac{1}{3}\big(R_+^3 + R_-^3\big) + \mathrm{e}^h \frac{R_+^3 + R_-^3}{c}\left(1+\frac{2}{c}+\frac{2}{c^2}\right) \\
&\le 2R_+^3\left(\frac{1}{3} + \frac{\mathrm{e}^h}{c}\left(1+\frac{2}{c}+\frac{2}{c^2}\right)\right).
\end{aligned}$$

Using $Z \ge R_+\mathrm{e}^{-\beta}$ we thus obtain

$$R_+^2 \ge \frac{1}{2}\sigma^2\mathrm{e}^{-\beta}\left(\frac{1}{3} + \frac{\mathrm{e}^h}{c}\left(1+\frac{2}{c}+\frac{2}{c^2}\right)\right)^{-1}.$$

□

We turn to the study of the Poincaré constant.

Remark 4.8. *If $m = \int x\,\mathrm{d}\mu$ the measure $\mathrm{e}^{-V(x+m)}\,\mathrm{d}x$ is centered and share the same Poincaré constant as μ. Replacing a by $a+m$ we may and will assume that $m = 0$.*

Similarly if we consider the probability measure $\mu_u = u\mathrm{e}^{-V(ux)}\,\mathrm{d}x$, we have $u^2\,\mathrm{Var}_{\mu_u}(x) = \mathrm{Var}_\mu(x)$ and an easy change of variables shows that $u^2 C_P(\mu_u) = C_P(\mu)$. So we can assume that $\mathrm{Var}_\mu(x) = 1$.

If V is β-superlinear, it is easy to see that V_u (defined by $V_u(x) = V(ux)$) is still β-superlinear, with the same constants c_β and h_β, but replacing R_β by $R_u(\beta) = R_\beta/u$.

Proposition 4.9. *Let $\mu(\mathrm{d}x) = Z^{-1}\mathrm{e}^{-V(x)}\,\mathrm{d}x$ be a probability measure on the line, with V of class C^1. We assume that $\min V = 0 = V(a)$ and that there exist $\beta_0 > 0$ and $\theta > 0$ such that $\mathrm{sign}(x-a)V'(x) \geq \theta$ outside some subset N_{β_0} of the level set $\{V \leq \beta_0\}$.*

Then there exists a constant $C(\beta_0, \theta)$ such that the Poincaré constant $C_P(\mu)$ satisfies

$$C_P(\mu) \leq C(\beta_0, \theta)\,\mathrm{Var}_\mu(x).$$

Proof. As we already remarked we can assume that μ is centered, and V being of C^1 class, we have $Lg = g'' - V'g'$.

We know that V is β-superlinear for any $\beta \geq \beta_0$. We denote by $N_\beta = [a - R_-(\beta), a + R_+(\beta)]$. We shall modify μ introducing

$$\mu_\beta(\mathrm{d}x) = Z_\beta^{-1}\big(\mathrm{e}^{-V(x)}\mathbb{1}_{x\notin N_\beta} + \mathrm{e}^{-\beta}\mathbb{1}_{x\in N_\beta}\big)\,\mathrm{d}x. \tag{4.10}$$

Note that, according to Lemma 4.4

$$1 \leq \frac{Z}{Z_\beta} \leq \mathrm{e}^\beta\left(1 + \frac{2\mathrm{e}^{h_\beta}}{c_\beta}\right).$$

It follows that

$$\mathrm{e}^{-\beta} \leq \frac{\mathrm{d}\mu_\beta}{\mathrm{d}\mu} \leq \mathrm{e}^{2\beta}\left(1 + \frac{2\mathrm{e}^{h_\beta}}{c_\beta}\right). \tag{4.11}$$

Accordingly we know that

$$C_P(\mu) \leq \mathrm{e}^{3\beta}\left(1 + \frac{2\mathrm{e}^{h_\beta}}{c_\beta}\right)C_P(\mu_\beta). \tag{4.12}$$

It remains to find an estimate for the Poincaré constant of μ_β.

We have to face a small problem since the potential V_β of μ_β is no more of class C^1, but we still have a drift, i.e. $V'(x)\mathbb{1}_{x\notin N_\beta}$, and an easy approximation procedure allows us to extend Theorem 2.3 in this situation.

We denote by L_β the associated generator i.e. $L_\beta f(x) = f''(x) - V'(x)\mathbb{1}_{x\notin N_\beta} f'(x)$.

We shall now introduce a well chosen Lyapunov function. We define

$$u(x) = |x|\mathbb{1}_{|x|>1} + \left(\frac{3}{8} + \frac{3}{4}x^2 - \frac{1}{8}x^4\right)\mathbb{1}_{|x|\leq 1}.$$

It is easily seen that u is of class C^2.

Now for $a_\beta = a + \frac{R_+(\beta) - R_-(\beta)}{2}$ (which is the center of L_β), and $R = R_+(\beta) + R_-(\beta)$ we define

$$W_\beta(x) = \mathrm{e}^{\gamma u(x - a_\beta)}.$$

An easy calculation shows that

$$L_\beta W_\beta \leq \gamma(\gamma - \theta)W_\beta \quad \text{if } |x - a| \geq R.$$

Choosing $\gamma = \theta/2$, it follows that W_β is a Lyapunov function i.e. satisfies (H1) with

$$\lambda = \frac{1}{2}\theta^2 = \frac{1}{2}\frac{c_\beta^2}{R^2} = \frac{v(\beta)}{\mathrm{Var}_\mu(x)},$$

according to Lemma 4.4.

It is thus enough to apply (2.16) with some care. First we replace U by N_β, then U_r by $N_{2\beta}$. We may thus choose some χ such that $\Gamma(\chi, \chi)$ is of order $(R_{2\beta} - R_\beta)^{-2}$ i.e. such that $\Gamma(\chi, \chi)/\lambda$ only depends on β (and not explicitly on $\mathrm{Var}_\mu(x)$).

Since μ_β is uniform on N_β, it is known that its Poincaré constant (in restriction to N_β) is equal to R^2/π^2, and again thanks to Lemma 4.4 it is bounded independently of V by some constant that only depends on β and λ.

The proof is completed, and the reader easily sees why we did not give an explicit value for the constant $C(\beta, \lambda, \mathrm{Var}_\mu(x))$. □

Remark 4.13. *The previous proposition is not surprising. It tells us that once the exponential concentration* (*which is a consequence of the Poincaré inequality*) *rate at infinity is known*, *and the bound of the density is given* (*at finite horizon*), *the Poincaré constant has to be controlled up to the natural scaling in the variance. We have given a proof of this natural conjecture under a strong form of concentration result. This result entails in particular double-well potentials. Note that no bound on the second derivative is needed* (*except that the first derivative has to stay greater than* λ), *so that the previous result contains much more general situations than the log-concave situation. We may even build examples with a Bakry–Emery curvature equal to* $-\infty$.

We turn to the log-concave situation. Since our method covers more general situations, it is certainly not sharp. So it is an illusion to hope to recover the constant 12 in Bobkov's result. Hence we shall even not try to give an explicit constant.

Theorem 4.14. *There exists a universal constant C such that for all log-concave probability measure μ on the real line*,

$$C_P(\mu) \le C\,\mathrm{Var}_\mu(x).$$

Proof. According to Remark 4.8 the result will follow if we prove the existence of the universal constant C for any log-concave measure with $\mathrm{Var}_\mu(x) = 1$.

First we assume that $\mu(\mathrm{d}x) = Z^{-1}\mathrm{e}^{-V(x)}\,\mathrm{d}x$ is a log-concave probability measure on the line, with V a C^1 function. We assume that $\min V = 0 = V(a)$ and $\mathrm{Var}_\mu(x) = 1$.

Since V is convex it is easily seen that for any $\beta > 0$, N_β is necessarily a closed interval denoted again $[a - R_-(\beta), a + R_+(\beta)]$.

In particular if $x \ge a + R_+(2\beta)$, the convexity of V yields

$$\begin{aligned} V(x) &\ge V(x) - V\bigl(a + R_+(\beta)\bigr) \\ &\ge \frac{\beta}{R_+(2\beta) - R_+(\beta)}\bigl(x - a - R_+(\beta)\bigr) \\ &\ge \frac{\beta}{R_+(2\beta)}\bigl(x - a - R_+(\beta)\bigr) = \frac{c_{2\beta}^+}{R_+(2\beta)}(x-a) - h_{2\beta}, \end{aligned}$$

where $c_{2\beta}^+ = \beta$ and $0 \le h_{2\beta} \le \beta$.

For $x \le a - R_-(2\beta)$ we have a similar result replacing R_+ by R_- hence with $c_-(2\beta) = \beta$ again. Since R_+ and R_- are both smaller than (or equal to) $R(\beta)$, V is 2β-superlinear and Lemma 4.4 yields

$$R^2(2\beta) \le 12\mathrm{e}^\beta\left(1 + \frac{2\mathrm{e}^{2\beta}}{\beta}\right).$$

In addition for $x \ge a + R_+(2\beta)$ convexity yields

$$V'(x) \ge \frac{\beta}{R_+(2\beta)} \ge \frac{\beta}{R(2\beta)} \ge \frac{\beta^{3/2}\mathrm{e}^{-\beta/2}}{2\sqrt{3}(\beta + 2\mathrm{e}^\beta)^{1/2}},$$

and the same result is true for $x \le a - R_-(2\beta)$. Proposition 4.9 yields a bound for each β ($\beta = 1$ for example). We should optimize in β but as we said we shall never attain the optimal bound 12.

In the general case (V convex with values in $]-\infty, +\infty]$) we first approximate V by everywhere finite convex functions, and then approximate such a function by a smooth one convoluing it for instance with Gaussian kernels with small variance. □

4.2. *Hardy type inequalities*

In the spirit of Remark 3.3 we can state another particular result of Hardy type, which is known to hold (with a better constant) if b below is a median of μ

Theorem 4.15. *Let $\mathrm{d}\mu = \mathrm{e}^{-V(x)}\,\mathrm{d}x$ be a probability measure on the real line satisfying a Poincaré inequality with constant C_P. We assume that there exist a sequence V_n of C^1 functions such that e^{-V_n} converges to e^{-V} weakly in $\sigma(\mathbb{L}^1, \mathbb{L}^\infty)$. Then for all $b \in \mathbb{R}$ the following inequality holds for all bounded smooth f,*

$$\int \big(f(x) - f(b)\big)^2 \mu(\mathrm{d}x) \leq \frac{8C_P}{\mu(]-\infty, b]) \wedge \mu([b, +\infty[)} \int \big(f'\big)^2(x)\mu(\mathrm{d}x). \tag{4.16}$$

Proof. Assume first that V is of class C^1. If X_t is the diffusion process with generator $Lf = f'' - V'f'$ (which is conservative since Poincaré inequality, hence (H1) holds), Proposition 3.1 tells us that for any $\theta < \mu(]-\infty, b]/8C_P$ the hitting time of $]-\infty, b]$ has an exponential moment of order θ. Hence one can find a Lyapunov function satisfying $LW = -\theta W$ on $[b, +\infty[$, namely $W(x) = \mathbb{E}_x(\mathrm{e}^{\theta T_b})$. It follows that for a smooth f and $A \geq b$,

$$\begin{aligned}\int_b^A \big(f(x) - f(b)\big)^2 \mu(\mathrm{d}x) &= \frac{-1}{\theta} \int_b^A \frac{LW}{W}(x)\big(f(x) - f(b)\big)^2 \mu(\mathrm{d}x) \\ &\leq \frac{1}{\theta}\left(\int_b^A \big(f'(x)\big)^2 \mu(\mathrm{d}x) - \left(\big(f(A) - f(b)\big)^2 \frac{W'(A)}{W(A)} \mathrm{e}^{-V(A)}\right)\right)\end{aligned}$$

the latter being obtained as in Remark 3.3 using integration by parts, since $f(x) - f(b) = 0$ for $x = b$. But W is clearly non-decreasing in x so that the last term into braces is non-negative, yielding the bound we claimed on $[b, +\infty[$ by letting A go to $+\infty$. The same holds on the left hand side of b.

Hence the Hardy–Poincaré–Sobolev inequality (4.16) holds for any constant larger than

$$\frac{8C_P}{\mu(]-\infty, b]) \wedge \mu([b, +\infty[)}$$

hence with this value by taking the infimum, and then for a non-necessarily smooth V using an approximation procedure. $\square$

As it is clear in the previous proof we may replace the full $\mathbb{R}$ by any interval containing b without any change in the constant. Since the Variance of f in restriction to an interval minimizes the square distance to any value, we thus obtain as a corollary

Corollary 4.17. *Let $\mathrm{d}\mu = \mathrm{e}^{-V(x)}\,\mathrm{d}x$ a probability measure on the real line satisfying a Poincaré inequality with constant C_P. Then for all interval $(a, b) \subseteq \mathbb{R}$ the following inequality holds for $\mu_{(a,b)}$ the restriction of μ to (a, b) and for all bounded smooth f,*

$$\mathrm{Var}_{\mu_{(a,b)}}(f) \leq \frac{8C_P}{\sup_{u\in(a,b)}\{\mu(]-\infty, u]) \wedge \mu([u, +\infty[)\}} \int \big(f'\big)^2(x)\mu_{(a,b)}(\mathrm{d}x). \tag{4.18}$$

In particular if $a \leq m_\mu \leq b$ then $\mu_{(a,b)}$ satisfies a Poincaré inequality with a constant not bigger than $16C_P$.

This bound can certainly be attained and improved by looking carefully at Muckenhoupt type constants, at least when the median belongs to the interval.

4.3. $\mathbb{L}^1$ *inequalities*

It is well known that one obtains a stronger inequality replacing the $\mathbb{L}^2$ norm by a $\mathbb{L}^p$ norm for $1 \le p \le 2$ (see e.g. [7], Chapter 2). The $\mathbb{L}^1$ Poincaré inequality (sometimes called Cheeger inequality) is of particular interest since it yields controls for the isoperimetric constant (see e.g. [5,6]). Due to the standard

$$\frac{1}{2}\int \bigl|f-\mu(f)\bigr|\,\mathrm{d}\mu \le \int \bigl|f-m_\mu(f)\bigr|\,\mathrm{d}\mu \le \int \bigl|f-\mu(f)\bigr|\,\mathrm{d}\mu, \tag{4.19}$$

where $\mu(f)$ and $m_\mu(f)$ denote respectively the mean and a μ median of f, such an inequality can be written indifferently

$$\int \bigl|f-\mu(f)\bigr|\,\mathrm{d}\mu \le C_C \int |\nabla f|\,\mathrm{d}\mu \quad \text{or} \quad \int \bigl|f-m_\mu(f)\bigr|\,\mathrm{d}\mu \le C'_C \int |\nabla f|\,\mathrm{d}\mu. \tag{4.20}$$

Equation (4.20) is true for any log-concave distribution and actually C_C and C_P differ by an universal multiplicative constant (see [23]). For one dimensional log-concave distribution C_C is universally bounded (see [6]). In our previous paper [2] we have shown that the existence of a Lyapunov function W as in (H1) implies a Cheeger type inequality, provided $\nabla W/W$ is bounded.

We shall here derive such an inequality, with the correct normalization factor $\mu(|x-\mu(x)|)$ which immediately follows by a linear change of variables in (4.20).

Theorem 4.21. *Under the hypotheses of Proposition 4.9 there exists a constant $C(\beta,\lambda,\mu(|x-\mu(x)|))$ such that the Cheeger constant $C_C(\mu)$ satisfies*

$$C_C(\mu) \le C\bigl(\beta,\lambda,\mu\bigl(\bigl|x-\mu(x)\bigr|\bigr)\bigr).$$

In particular if μ is a log-concave probability measure on the line, there exists an universal constant C such that $C_C(\mu) \le C\mu(|x-\mu(x)|)$.

Proof. We follow the proof of Proposition 4.9 (see the notations therein) proving a Cheeger inequality for the measure μ_β. Recall that W_β satisfies $L_\beta W_\beta \le -\lambda^2/4W_\beta + b(R,\beta,\lambda)\mathbb{1}_{N_\beta}$.

The first thing to do is to show that $R_+ \vee R_-$ is controlled, from above and from below by a quantity depending only on β, λ and $\mu(|x-\mu(x)|)$ i.e. to prove the analogue of Lemma 4.4. Denote by s the quantity $\mu(|x-\mu(x)|)$. Then mimiking the proof of Lemma 4.4 we can prove

$$R_+ \vee R_- \le 2\left(1+\frac{2\mathrm{e}^h}{c}\right)s$$

and

$$R_+ \vee R_- \ge \frac{1}{2}s\mathrm{e}^{-\beta}C(h,c)$$

for some $C(h,c)>0$.

Now we may assume that $s=1$. The second thing to do is to recall the reasoning in [2] i.e. if f is smooth and $g=f-m$ for some constant m we have

$$\begin{aligned}\int |g|\mu_\beta(\mathrm{d}x) &\le \frac{4}{\lambda^2}\int |g|\left(-\frac{L_\beta W_\beta}{W_\beta}\right)\mu_\beta(\mathrm{d}x) + \frac{4b(\beta,\lambda)\mathrm{e}^\beta}{\lambda^2 Z_\beta}\int_{N_\beta}|g|\,\mathrm{d}x \\ &\le \frac{4}{\lambda^2}\int\left(\bigl|g'\bigr|\left(\frac{|W'_\beta|}{W_\beta}\right) - |g|\left(\frac{|W'_\beta|^2}{W_\beta^2}\right)\right)\mu_\beta(\mathrm{d}x) + \frac{4b(\beta,\lambda)\mathrm{e}^\beta}{\lambda^2 Z_\beta}\frac{R}{\pi}\int_{N_\beta}\bigl|g'\bigr|\,\mathrm{d}x\end{aligned}$$

if we choose $m=\int_{N_\beta} f(x)\,\mathrm{d}x$. The first term is obtained after integrating by parts, the second one is using the standard Cheeger inequality for the uniform measure on an interval.

Now remark that $|W'_\beta|/W_\beta$ is bounded by some constant depending only on β and λ. Finally we have obtained (if $\mu(|x-\mu(x)|)=1$),

$$\int \left|f-\mu_\beta(f)\right| \mathrm{d}\mu_\beta \le 2\int \left|f-m_{\mu_\beta}(f)\right| \mathrm{d}\mu_\beta \le 2\int |g|\,\mathrm{d}\mu_\beta \le K(\beta,\lambda)\int \left|f'\right| \mathrm{d}\mu_\beta,$$

hence the result for μ_β and then for μ as in Proposition 4.9.

The log-concave case is then similar to Theorem 4.14. □

As we already said the previous Theorem contains Proposition 4.9 thanks to Cheeger's inequality $C_P \le 4C_C^2$. Actually our proof yields so bad constants in both cases that it is really tedious to check when the previous relation gives a better bound than Proposition 4.9. We also insist on the proof of both properties using Lyapunov function. As we have seen, the proof of a $\mathbb{L}^1$ inequality requires the boundedness of W'/W. In particular if we choose for W the Laplace transform of hitting times $\mathbb{E}_x(\mathrm{e}^{\theta T_b})$, this latter property is not ensured. So we cannot obtain similar results as in Section 4.2.

5. ϕ moments and Poincaré like inequalities

Since the status of the existence of exponential moments for hitting times is now characterized through the results of Section 2, it is certainly interesting to look at more general ϕ moments. The first result is a direct consequence of (2.15):

Proposition 5.1. *Assume that L is uniformly strongly hypoelliptic. If U is an open connected set with $\mu(U)<1$, then*

$$\sup\left\{\lambda, \text{ such that } \mathbb{E}_\mu\left(\mathrm{e}^{\lambda T_U}\right) < +\infty\right\} < +\infty.$$

In particular if ϕ growths faster, at infinity, than any exponential $\mathbb{E}_\mu(\phi(T_U)) = +\infty$.

Proof. We already saw that, in the uniform strong hypoelliptic situation, $\mathbb{E}_x(\mathrm{e}^{\lambda T_U}) < +\infty$ for all x as soon as $\mathbb{E}_\mu(\mathrm{e}^{2\lambda T_U}) < +\infty$. According to the proof of Theorem 2.3, we thus know that there exists a Lyapunov function satisfying (H1). Equation (2.15) implies that

$$\int f^2\,\mathrm{d}\mu \le \frac{1}{\lambda}\int \Gamma(f,f)\,\mathrm{d}\mu$$

for all smooth f with support in $\bar{U}^c$. This cannot hold for all λ since $\mu(U)<1$, just looking at $\lambda\to+\infty$. □

Remark 5.2. *Actually the hypoellipticity assumption is not needed. Indeed, if $\mathbb{E}_\mu(\mathrm{e}^{2\lambda T_U}) < +\infty$, $W(x)=\mathbb{E}_x(\mathrm{e}^{\lambda T_U})$ (which is defined μ almost everywhere, or even better quasi-everywhere) belongs to the domain of L. One can then use the contents of Remark 2.4.*

This result is in accordance with the fact that one cannot improve on the exponential convergence in $\mathbb{L}^2$ (or in total variation distance) even for very strong repelling forces. In order to discriminate diffusions satisfying a Poincaré inequality, one has to introduce new inequalities (e.g. F-Sobolev inequalities or super-Poincaré inequalities) or contraction properties of the semi-group (see e.g. [4,16]). Another connected possibility is to look at exponential decay to equilibrium for weaker norms than $\mathbb{L}^p$ norms (see e.g. [14]). The certainly best known case is the one when a logarithmic Sobolev inequality holds or equivalently the semi-group is hypercontractive or equivalently exponential convergence holds in entropy (or in $\mathbb{L}\log\mathbb{L}$ Orlicz norm) (see e.g. [1] for an elementary introduction to all these notions).

The use of Lyapunov functions for studying such stronger inequalities is detailed in [16]. It should be very interesting to understand these phenomena in terms of hitting times. We strongly suspect that what is important is the behavior of $W(x)=\mathbb{E}_x(\mathrm{e}^{\lambda T_U})$ as x goes to infinity. For instance if W is bounded, we suspect that the semi-group is ultracontractive (or more properly ultrabounded). Some hints in this direction are contained in [11], Theorem 7.3, at least for diffusion processes on the real line. Let us state a result in this direction:

Proposition 5.3. *Assume that $L = \Delta - \nabla V \cdot \nabla$, where V is smooth, is defined on $\mathbb{R}$. $\mu(\mathrm{d}x) = \mathrm{e}^{-V(x)}\,\mathrm{d}x$ (supposed to be a probability measure) is then symmetric for L. Assume in addition that $|\nabla V|^2 - \Delta V \geq -C$ for some non-negative constant C.*

Then there is an equivalence between

(1) *the associated semi-group P_t is ultrabounded (i.e. P_t maps continuously $\mathbb{L}^1(\mu)$ in $\mathbb{L}^\infty(\mu)$ for all $t > 0$), and there exists an open interval U such that for all $x \in \mathbb{R}$, $P_x(T_U < +\infty) = 1$,*
(2) *there exists a bounded Lyapunov function W,*
(3) *there exists an open interval U and $\lambda > 0$ such that*

$$\sup_x \mathbb{E}_x\left(\mathrm{e}^{\lambda T_U}\right) < +\infty.$$

Proof. The equivalence between (2) and (3) follows from the proof of Theorem 2.3, since L is uniformly elliptic.

If (1) holds, it follows from the arguments in Appendix B of [17], that there exists an unique quasi limiting distribution for the process (starting from the right of U) killed when hitting any interior point of U. For all definitions connected with quasi-stationary measures and quasi-limiting distributions we refer to [11,17]. The same holds for the process coming from the left of U. According to [11], Theorem 7.3, this implies that the killed process "comes down from infinity" i.e. satisfies (3).

Conversely, [11], Theorem 7.3, tells us that (3) implies the condition (called (H5) therein)

$$\int_a^{+\infty} \mathrm{e}^{-V(y)} \int_a^y \mathrm{e}^{V(z)}\,\mathrm{d}z\,\mathrm{d}y < +\infty$$

for $a = \sup U$. Define, for $x > a$, $u(x) = \mu([x, +\infty[)$ and

$$F(z) = z\left(\int_a^{u^{-1}(1/z)} \mathrm{e}^{V(y)}\,\mathrm{d}y\right)^{-1}.$$

$z \mapsto F(z)/z$ is thus non-increasing and we have

$$u(x) F\left(\frac{1}{u(x)}\right) \int_a^x e^{V(y)}\,\mathrm{d}y = 1. \tag{5.4}$$

According to results in [4] (see Remark 3.3 in [14]), μ satisfies a $\tilde{F}$-Sobolev inequality for a slight modification of F. Condition (H5) of [11] recalled above implies that

$$\int^{+\infty} \frac{1}{uF(u)}\,\mathrm{d}u < +\infty.$$

The same holds with $\tilde{F}$ in place of F. According to a result of [28] explained p. 135 of [14], this implies that the semi-group is ultrabounded. □

5.1. *Weak Poincaré inequalities and polynomials moments*

In this section we shall look at the existence of ϕ-moments for functions ϕ growing slower at infinity than an exponential, and actually we shall mainly focus on power functions. In all the discussion below we shall assume, for simplicity, that L is uniformly strongly hypoelliptic and our symmetry assumption.

First of all recall that under our assumptions, defining for $q \in \mathbb{N}$,

$$v_q(x) = \mathbb{E}_x\left(T_U^q\right) \tag{5.5}$$

and provided v_q is well defined for all x, v_q is smooth and satisfies for $q \geq 1$

$$Lv_q(x) = -q v_{q-1}(x) \quad \text{for all } x \in U^c, \tag{5.6}$$

as a simple application of the Markov property. We thus have some "nested" Lyapunov functions.

Henceforth we assume that U is bounded (which is clearly not a restriction). Then, since $v_q(x) > 0$ when $d(x, U) \geq 1$, the Markov property together with the continuity of v_q and the compactness of $\{d(x, U) = 1\}$, show that there exists $\kappa > 0$ such that for all x with $d(x, U) \geq 1$, $v_q(x) \geq \kappa$ and $v_{q-1}(x) \geq \kappa$. Remark that equality (5.6) is still true for all x such that $d(x, U) \geq 1$. We then obtain the following consequences

Proposition 5.7. (1) *Weak Poincaré like inequalities: Assume that a local Poincaré inequality holds. Suppose that $v_q(x)$ is finite for all x. Then for all positive $s < 1$, there exists a positive function β such that for all bounded f*

$$\mathrm{Var}_\mu(f) \leq \beta(s) \int \Gamma(f, f)\,\mathrm{d}\mu + s\,\mathrm{Osc}(f)^2 \tag{5.8}$$

and $\beta(s) = C(\inf\{u; \mu(\frac{v_{q-1}}{1+v_q} < u) > s\})^{-1}$ for some explicit constant C.

(2) *Assume that $L = \Delta - \nabla V \cdot \nabla$, where V is smooth, is defined on $\mathbb{R}$. Then, if v_1 is bounded, the associated semi-group P_t is ultrabounded (hence for some $\lambda > 0$, $\sup_x \mathbb{E}_x(\mathrm{e}^{\lambda T_U}) < +\infty$).*

(3) *If there exists C such that $v_q(x) \leq C v_{q-1}(x)$ for all x with $d(x, U) \geq 1$, then μ satisfies a Poincaré inequality (and consequently T_U has some exponential moment for all $\mathbb{P}_x$).*

The first part of the theorem gives that in the reversible setting, finiteness of moments of return times implies a weak Poincaré like inequality, a result that we are not aware of in discrete times. It is however very difficult to get precise estimates of β as we need concentration properties of μ and sharp control of v_q and v_{q-1}. Using that $v_{q-1} \leq v_q^{(q-1)/q}$ we may get a lower bound for β using only v_q.

Note that the second part of the Proposition is not so surprising and corresponds to the similar discrete situation of birth and death processes on the half line (see Proposition 7.10 in [11]). The third part is only expressing that v_q is a Lyapunov function.

Proof of Proposition 5.7. The first part of the theorem, inspired by [12] and the proof of the main theorem, may be proved in two steps that we sketch here.

First, using the Lyapunov conditions (5.6) (and $L1 = 0$) and the same line of reasoning than (H1) implies (H4) in our main theorem, we get some weighted Poincaré inequality: for some constant C, we have

$$\inf_a \int \frac{v_{q-1}}{1+v_q}(f-a)^2\,\mathrm{d}\mu \leq C \int \Gamma(f, f).$$

Indeed, remark first that $v_{q-1} \leq 1 + v_q$ so that (5.6) may be rewritten for some constant $b > 0$

$$L(1+v_q) \leq -q\frac{v_{q-1}}{1+v_q}(1+v_q) + b1_U,$$

and one has for all a

$$\int \frac{v_{q-1}}{1+v_q}(f-a)^2\,\mathrm{d}\mu \leq \frac{1}{q}\int -\frac{L(1+v_q)}{1+v_q}(f-a)^2\,\mathrm{d}\mu + \frac{b}{q}\int_U (f-a)^2\,\mathrm{d}\mu.$$

Choose now $a = \int_U f\,\mathrm{d}\mu$ and use the local Poincaré inequality and inequality (2.15) to conclude.

Then, with $a_f = \mu(f\frac{v_{q-1}}{1+v_q})/\mu(\frac{v_{q-1}}{1+v_q})$, for all bounded f and for all $u > 0$

$$\begin{aligned}
\mathrm{Var}_\mu(f) &\leq \int (f-a_f)^2\,\mathrm{d}\mu \\
&= \int_{(v_{q-1})/(1+v_q)\geq u} (f-a_f)^2\,\mathrm{d}\mu + \int_{(v_{q-1})/(1+v_q)<u} (f-a_f)^2\,\mathrm{d}\mu \\
&\leq u^{-1}\inf_a \int \frac{v_{q-1}}{1+v_q}(f-a)^2\,\mathrm{d}\mu + \mu\left(\frac{v_{q-1}}{1+v_q} < u\right)\mathrm{Osc}(f)^2
\end{aligned}$$

which gives the result.

For the second part just remark that

$$Lv_1(x) \le -\frac{v_1(x)}{\sup v_1} \quad \text{for } x \in \bar{U}^c.$$

Hence v_1/κ is a bounded Lyapunov function satisfying (H1) (with $\bar{U}^c$ replaced by $\{d(x,U) > 1\}$) and we may apply Proposition 5.3.

For the third part we similarly have

$$Lv_q(x) \le -q\frac{v_{q-1}(x)}{v_q(x)}v_q(x) \quad \text{for } x \text{ such that } d(x,U) \ge 1.$$

Hence v_q/κ is a Lyapunov function satisfying (H1) (with $\bar{U}^c$ replaced by $\{d(x,U) > 1\}$), and we may apply Theorem 2.3. □

An immediate generalization of (3) in the previous proposition is the following assumption: there exists an increasing function φ growing to infinity and $R > 0$, such that

$$\varphi\big(v_q(x)\big) \le qv_{q-1}(x) \quad \text{for } |x| \ge R. \tag{5.9}$$

Indeed if (5.9) holds, we have

$$Lv_q(x) \le -\varphi\big(v_q(x)\big)$$

for $|x|$ large enough, and v_q is thus a φ-Lyapunov function in the terminology of [12] and [19] (see Definition 2.2 in the first reference).

Conversely, is it possible to get the existence of ϕ-moments starting from a functional inequality? The first answer to this question was given in [25] where some Nash type inequalities are shown to imply the existence of moments. The proof uses the fact that the Laplace transform of T_U, $h_t^U(x) = \mathbb{E}_x(\mathrm{e}^{-tT_U})$ satisfies $Lh - th = 0$ for all $t > 0$.

Using the results in Section 3 of [13] again we can derive similar (actually stronger) results. Recall that

$$\mathbb{P}_\mu(T_U > t) \le \mathbb{P}_\mu\left(-\frac{1}{t}\int_0^t \mathbb{1}_U(X_s)\,\mathrm{d}s + \mu(U) \ge \mu(U)\right).$$

According to Proposition 3.5 in [13] we thus have for t large enough,

$$\mathbb{P}_\mu(T_U > t) \le C(k)t^{-k}\big(\mu(U)\big)^{-2k}$$

provided the process is α-mixing with a mixing rate $\alpha(u) \le C(1+u)^{-k}$ for some integer $k \ge 1$.

The mixing rate is connected to the rate of convergence to equilibrium of the semi group, as explained in [13]. This rate of convergence can be bounded using either a Weak Poincaré Inequality (see [27]) or a φ-Lyapunov function (see [3,19]). Let us collect these results in the next (and final) theorem

Theorem 5.10. *Assume that L is uniformly strongly hypoelliptic. Let U be a bounded connected open set. Assume in addition one of the following conditions,*

(1) *μ satisfies a weak Poincaré inequality, i.e. there exists a non-increasing function β such that for all $s > 0$ and all bounded and smooth f,*

$$\mathrm{Var}_\mu(f) \le \beta(s)\int \Gamma(f,f)\,\mathrm{d}\mu + s\,\mathrm{Osc}^2(f),$$

where $\mathrm{Osc}(f)$ *denotes the oscillation of f; in which case the process is α-mixing with a mixing rate*

$$\alpha(t) \le \big(\inf\{s > 0; \beta(s)\log(1/s) \le t/2\}\big)^2,$$

(2) *there exists a φ-Lyapunov function W for some smooth increasing concave function φ with $\varphi' \to 0$ at infinity*; *in which case the process is α-mixing with a mixing rate*

$$\alpha(t) \le C\left(\int W \, \mathrm{d}\mu\right) \frac{1}{\varphi \circ H_\varphi^{-1}(t)},$$

where $H_\varphi(t) = \int_1^t (1/\varphi(s)) \, \mathrm{d}s$ *and we assume that* $\int W \, \mathrm{d}\mu < +\infty$.

If in addition $\alpha(t) \le C(1+t)^{-k}$ *for some positive integer* k, *then*

$$\mathbb{P}_\mu(T_U > t) \le C(k) t^{-k} \big(\mu(U)\big)^{-2k}$$

for some constant $C(k)$.

In particular for all $j < k$, $\mathbb{E}_\mu(T_U^j) < +\infty$. *The same holds for* μ *almost all* x, *and for all* x, *and* $j < k/2$, $\mathbb{E}_x(T_U^j) < +\infty$.

The interested reader will find in [3,12] in particular many examples (including the so called κ-concave measures) of measures satisfying one (or both) of the previous conditions. As a flavour of the examples presented in these references, one usually considers Cauchy type measures $\mathrm{d}\mu = Z^{-1}(1 + |x|)^{1+\alpha} \, \mathrm{d}x$ associated with the usual symmetric generator which satisfies a weak Poincaré type inequality with $\beta(s) = cs^{-2/\alpha}$ and for which $\alpha(t) \le c(1+t)^{-k}$ for all $k \le \alpha/2$.

Acknowledgments

The authors thank the referee, Associate Editor and Editor for their careful reviews and comments which helped to get a clearer presentation and to correct small mistakes.

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