Characterizations of processes with stationary and independent increments under $G$-expectation\(^1\)

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**Abstract.** Our purpose is to investigate properties for processes with stationary and independent increments under $G$-expectation. As applications, we prove the martingale characterization of $G$-Brownian motion and present a pathwise decomposition theorem for generalized $G$-Brownian motion.

**Résumé.** Notre but est d’étudier des propriétés de processus à accroissements stationnaires et indépendants sous une $G$-espérance. Comme application, nous démontrons la caractérisation de la martingale de $G$-mouvement Brownien et fournissons un théorème de décomposition trajectorielle pour le $G$-mouvement Brownien généralisé.

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1. Introduction

Recently, motivated by the modelling of dynamic risk measures, Shige Peng ([3–5]) introduced the notion of a $G$-expectation space. It is a generalization of probability spaces (with their associated linear expectation) to spaces endowed with a nonlinear expectation. As the counterpart of Wiener space in the linear case, the notion of $G$-Brownian motion was introduced under the nonlinear $G$-expectation.

Recall that if $\{A_t\}$ is a continuous process over a probability space $(\Omega, \mathcal{F}, P)$ with stationary, independent increments and finite variation, then there exists some constant $c$ such that $A_t = ct$. However, it is not the case in the $G$-expectation space $(\Omega_T, L^G_1(\Omega_T), \hat{E})$. A counterexample is $\{\langle B \rangle_t\}$, the quadratic variation process for the coordinate process $\{B_t\}$, which is a $G$-Brownian motion. We know that $\{\langle B \rangle_t\}$ is a continuous, increasing process with stationary and independent increments, but it is not deterministic.

The process $\{\langle B \rangle_t\}$ is very important in the theory of $G$-expectation, which shows, in many aspects, the difference between probability spaces and $G$-expectation spaces. For example, we know that for a probability space continuous local martingales with finite variation are trivial processes. However, [4] proved that in a $G$-expectation space all processes in form of $\int_0^t \eta_s \, d\langle B \rangle_s - \int_0^t 2G(\eta_s) \, ds$, $\eta \in M^G_L(0, T)$ (see Section 2 for the definitions of the function $G(\cdot)$ and the space $M^G_L(0, T)$), are nontrivial $G$-martingales with finite variation (in fact, they are even nonincreasing) and continuous paths. [4] also conjectured that any $G$-martingale with finite variation should have such representation. Up to

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now, some properties of the process \( \{ \langle B \rangle_t \} \) remain unknown. For example, we know that, if \( G(x) = \frac{1}{2} \text{sup}_{\sigma \leq \sigma \leq \pi} \sigma^2x \) generates the \( G \)-expectation, we have \( \sigma^2(t-s) \leq \langle B \rangle_t - \langle B \rangle_s \leq \pi(t-s) \) for all \( s < t \), but we do not know whether \( \{ \frac{d}{dt} \langle B \rangle_s \} \) belongs to \( M_G^1(0,T) \). This is a very important property since \( \{ \frac{d}{dt} \langle B \rangle_s \} \in M_G^1(0,T) \) would imply that the representation mentioned above of \( G \)-martingales with finite variation is not unique.

For the case of a probability space, a continuous local martingale \( \{ M_t \} \) is a standard Brownian motion if and only if the quadratic variation process \( \langle M \rangle_t = t \). However, it’s not the case for \( G \)-Brownian motion since its quadratic variation process is only an increasing process with stationary and independent increments. How can we give a characterization for \( G \)-Brownian motion?

In this article, we shall prove that if \( A_t = \int_0^t h_s \, ds \) (respectively \( A_t = \int_0^t h_s \, d\langle B \rangle_s \)) is a process with stationary, independent increments and \( h \in M_G^1(0,T) \) (respectively \( h \in M_G^{\beta,+}(0,T), \beta > 1 \)), then there exists some constant \( c \) such that \( h \equiv c \). As applications, we prove the following conclusions (Question 1 and 3 are put forward by Prof. Shige Peng in private communications):

1. \( \{ \frac{d}{dt} \langle B \rangle_s \} \notin M_G^1(0,T) \).

2. (Martingale characterization)

   A symmetric \( G \)-martingale \( \{ M_t \} \) is a \( G \)-Brownian motion if and only if its quadratic variation process \( \{ \langle M \rangle_t \} \) has stationary and independent increments;

   A symmetric \( G \)-martingale \( \{ M_t \} \) is a \( G \)-Brownian motion if and only if its quadratic variation process \( \langle M \rangle_t = c \langle B \rangle_t \), for some \( c \geq 0 \).

   The sufficiency of the second assertion is trivial, but not the necessity.

3. Let \( \{ X_t \} \) be a generalized \( G \)-Brownian motion with zero mean, then we have the following decomposition:

   \[ X_t = M_t + L_t, \]

   where \( \{ M_t \} \) is a (symmetric) \( G \)-Brownian motion, and \( \{ L_t \} \) is a nonpositive, nonincreasing \( G \)-martingale with stationary and independent increments.

This article is organized as follows: In Section 2 we recall some basic notions and results of \( G \)-expectation and the related space of random variables. In Section 3 we characterize processes with stationary and independent increments. In Section 4, as application, we prove the martingale characterization of \( G \)-Brownian motion and present a decomposition theorem for generalized \( G \)-Brownian motion. In Section 5 we present some properties for \( G \)-martingales with finite variation.

2. Preliminary

We recall some basic notions and results of \( G \)-expectation and the related space of random variables. More details of this section can be found in [3–8].

\textbf{Definition 2.1.} Let \( \Omega \) be a given set and let \( \mathcal{H} \) be a vector lattice of real valued functions defined on \( \Omega \) with \( c \in \mathcal{H} \) for all constants \( c \). \( \mathcal{H} \) is considered as the space of “random variables.” A sublinear expectation \( \hat{E} \) on \( \mathcal{H} \) is a functional \( \hat{E} : \mathcal{H} \rightarrow \mathbb{R} \) satisfying the following properties: For all \( X, Y \in \mathcal{H} \), we have

(a) Monotonicity: If \( X \geq Y \) then \( \hat{E}(X) \geq \hat{E}(Y) \).

(b) Constant preserving: \( \hat{E}(c) = c \).

(c) Sub-additivity: \( \hat{E}(X) - \hat{E}(Y) \leq \hat{E}(X - Y) \).

(d) Positive homogeneity: \( \hat{E}(\lambda X) = \lambda \hat{E}(X), \lambda \geq 0 \).

\((\Omega, \mathcal{H}, \hat{E})\) is called a sublinear expectation space.

\textbf{Definition 2.2.} Let \( X_1 \) and \( X_2 \) be two \( n \)-dimensional random vectors defined respectively in sublinear expectation spaces \( (\Omega_1, \mathcal{H}_1, \hat{E}_1) \) and \( (\Omega_2, \mathcal{H}_2, \hat{E}_2) \). They are called identically distributed, denoted by \( X_1 \sim X_2 \), if \( \hat{E}_1[\varphi(X_1)] = \hat{E}_2[\varphi(X_2)] \), for all \( \varphi \in C_{l,\text{Lip}}(R^n) \), where \( C_{l,\text{Lip}}(R^n) \) is the space of real continuous functions defined on \( R^n \) such that

\[ |\varphi(x) - \varphi(y)| \leq C(1 + |x|^k + |y|^k)|x - y|, \quad \text{for all} \ x, y \in R^n, \]
where \( k \) and \( C \) depend only on \( \varphi \).

**Definition 2.3.** In a sublinear expectation space \((\Omega, \mathcal{H}, \hat{E})\) a random vector \( Y = (Y_1, \ldots, Y_n), Y_i \in \mathcal{H}, \) is said to be independent of another random vector \( X = (X_1, \ldots, X_m), X_i \in \mathcal{H}, \) under \( \hat{E}(\cdot)\), denoted by \( Y \perp X \), if for every test function \( \varphi \in C_b,1_{\text{Lip}}(\mathbb{R}^m \times \mathbb{R}^n) \) we have \( \hat{E}[\varphi(X, Y)] = E[\hat{E}[\varphi(x, y)]_{x = Y}] \).

**Definition 2.4 (G-normal distribution).** A \( d \)-dimensional random vector \( X = (X_1, \ldots, X_d) \) in a sublinear expectation space \((\Omega, \mathcal{H}, \hat{E})\) is called \( G \)-normal distributed if for every \( a, b \in \mathbb{R}_+ \) we have
\[
aX + b\hat{X} \sim \sqrt{a^2 + b^2}X,
\]
where \( \hat{X} \) is an independent copy of \( X \). Here the letter \( G \) denotes the function
\[
G(A) := \frac{1}{2} \hat{E}[(AX, X)] : S_d \to \mathbb{R},
\]
where \( S_d \) denotes the collection of \( d \times d \) symmetric matrices.

The function \( G(\cdot) : S_d \to \mathbb{R} \) is a monotonic, sublinear mapping on \( S_d \) and \( G(A) = \frac{1}{2} \hat{E}[(AX, X)] \leq \frac{1}{2}|A|\hat{E}[|X|^2] =: \frac{1}{2}|A|\bar{\sigma}^2 \) implies that there exists a bounded, convex and closed subset \( \Gamma \subset S_d^+ \) such that
\[
G(A) = \frac{1}{2} \sup_{Y \in \Gamma} \text{Tr}(Y A).
\]
(2.1)

If there exists some \( \beta > 0 \) such that \( G(A) - G(B) \geq \beta \text{Tr}(A - B) \) for any \( A \geq B \), we call the \( G \)-normal distribution nondegenerate. This is the case we consider throughout this article.

**Definition 2.5.** (i) Let \( \Omega_T = C_0([0, T]; \mathbb{R}^d) \) be endowed with the supremum norm and \( \{B_t\} \) be the coordinate process. Set \( \mathcal{H}_T^0 := \{\varphi(B_{t_1}, \ldots, B_{t_n})| n \geq 1, t_1, \ldots, t_n \in [0, T], \varphi \in C_{1,\text{Lip}}(\mathbb{R}^{d \times n})\} \). \( G \)-expectation is a sublinear expectation defined by
\[
\hat{E}[X] = \hat{E}[\varphi(\sqrt{t_1 - t_0} \xi_1, \ldots, \sqrt{t_m - t_{m-1}} \xi_m)],
\]
for all \( X = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_m} - B_{t_{m-1}}) \), where \( \xi_1, \ldots, \xi_n \) are identically distributed \( d \)-dimensional \( G \)-normally distributed random vectors in a sublinear expectation space \((\hat{\Omega}, \hat{\mathcal{H}}, \hat{E})\) such that \( \xi_{i+1} \) is independent of \( (\xi_1, \ldots, \xi_i) \) for every \( i = 1, \ldots, m - 1 \). \((\Omega_T, \mathcal{H}_T^0, \hat{E})\) is called a \( G \)-expectation space.

(ii) Let us define the conditional \( G \)-expectation \( \hat{E}_t \) of \( \xi \in \mathcal{H}_T^0 \) knowing \( \mathcal{H}_t^0 \) for \( t \in [0, T] \). Without loss of generality we can assume that \( \xi \) has the representation \( \xi = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_m} - B_{t_{m-1}}) \) with \( t = t_i \) for some \( 1 \leq i \leq m \), and we put
\[
\hat{E}_t[\varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_m} - B_{t_{m-1}})]
= \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_i} - B_{t_{i-1}}),
\]
where
\[
\varphi(x_1, \ldots, x_i) = \hat{E}[\varphi(x_1, \ldots, x_i, B_{t_{i+1}} - B_{t_i}, \ldots, B_{t_m} - B_{t_{m-1}})].
\]

Define \( \|\xi\|_{1,G} = [\hat{E}(\|\xi\|^p)]^{1/p} \) for \( \xi \in \mathcal{H}_T^0 \) and \( p \geq 1 \). Then for all \( t \in [0, T] \), \( \hat{E}_t(\cdot) \) is a continuous mapping on \( \mathcal{H}_T^0 \) with respect to the norm \( \| \cdot \|_{1,G} \) and therefore can be extended continuously to the completion \( L^1_G(\Omega_T) \) of \( \mathcal{H}_T^0 \) under the norm \( \| \cdot \|_{1,G} \).

Let \( L^p(\Omega_T) := \{\varphi(B_{t_1}, \ldots, B_{t_n})| n \geq 1, t_1, \ldots, t_n \in [0, T], \varphi \in C_{b,1_{\text{Lip}}}(\mathbb{R}^{d \times n})\} \), where \( C_{b,1_{\text{Lip}}}(\mathbb{R}^{d \times n}) \) denotes the set of bounded Lipschitz functions on \( \mathbb{R}^{d \times n} \). [1] proved that the completions of \( C_b(\Omega_T), \mathcal{H}_T^0 \) and \( L^p(\Omega_T) \) under \( \| \cdot \|_{p,G} \) are the same; we denote them by \( L^p_G(\Omega_T) \).
**Definition 2.6.** (i) We say that \( \{X_t\} \) on \((\Omega_T, L^1_G(\Omega_T), \hat{E})\) is a process with independent increments if for any \(0 < t < T\) and \(s_0 \leq \cdots \leq s_m \leq t_0 \leq \cdots \leq t_n \leq T\),
\[
(X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}) \perp (X_{s_1} - X_{s_0}, \ldots, X_{s_m} - X_{s_{m-1}}).
\]

(ii) We say that \( \{X_t\} \) on \((\Omega_T, L^1_G(\Omega_T), \hat{E})\) with \(X_t \in L^1_G(\Omega_t)\) for every \(t \in [0, T]\) is a process with independent increments w.r.t. the filtration if for any \(0 < s < T\) and \(s_0 \leq \cdots \leq s_m \leq s \leq t_0 \leq \cdots \leq t_n \leq T\),
\[
(X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}) \perp (B_{s_1} - B_{s_0}, \ldots, B_{s_m} - B_{s_{m-1}}).
\]

**Remark 2.7.** (i) Let \(\xi \in L^1_G(\Omega_T)\). If there exists \(s \in [0, T]\) such that for any \(s_0 \leq \cdots \leq s_m \leq s\), \(\xi \perp (B_{s_1} - B_{s_0}, \ldots, B_{s_m} - B_{s_{m-1}})\), then we have \(\hat{E}_s(\xi) = \hat{E}(\xi)\). In fact, there is no loss of generality, we assume \(\hat{E}(\xi) = 1\) and \(C \geq \xi \geq \epsilon\) for some \(C, \epsilon > 0\). Set \(\eta = \hat{E}_s(\xi)\). For any \(n \in \mathbb{N}\), we have
\[
\hat{E}(\eta^{n+1}) = \hat{E}(\eta^n) = \cdots = \hat{E}(\eta) = 1.
\]
By this, we have
\[
\eta \leq 1, \quad \text{q.s.}
\]
On the other hand, we have
\[
\hat{E}\left((\eta - 1)^2\right) = \hat{E}[\eta(\eta - 2)] + 1 = \hat{E}[\eta(\xi - 2)] + 1.
\]
Since \(\xi - 2 \perp \eta\), we have
\[
\hat{E}\left((\eta - 1)^2\right) = \hat{E}(1 - \eta).
\]
By Theorem 2.12 below, there exists \(P \in \mathcal{P}\) such that
\[
E_P\left((\eta - 1)^2\right) = \hat{E}(1 - \eta^2).
\]
Noting that
\[
E_P(1 - \eta) \leq \hat{E}(1 - \eta) = \hat{E}\left((\eta - 1)^2\right) = E_P\left((\eta - 1)^2\right) \leq E_P(1 - \eta),
\]
we have
\[
E_P\left((\eta - 1)^2\right) = E_P(1 - \eta).
\]
By this, we have
\[
\eta^2 = \eta, \quad P\text{-a.s.}
\]
Since \(\eta \geq \epsilon\), we have \(\eta = 1, P\text{-a.s.}\) So we have
\[
\hat{E}\left((1 - \eta)^2\right) = E_P\left((1 - \eta)^2\right) = 0.
\]

(ii) Let \(\{X_t\} \) on \((\Omega_T, L^1_G(\Omega_T), \hat{E})\) be a process with stationary and independent increments and let \(c = \hat{E}(X_T)/T\). If \(\hat{E}(X_t) \to 0\) as \(t \downarrow 0\), then for any \(0 \leq s < t \leq T\), we have \(\hat{E}(X_t - X_s) = c(t - s)\).
Definition 2.8. Let \( \{X_t\} \) be a \( d \)-dimensional process defined on \((\Omega_T, L^1_G(\Omega_T), \hat{E})\) such that:

(i) \( X_0 = 0 \);
(ii) \( \{X_t\} \) is a process with stationary and independent increments w.r.t. the filtration;
(iii) \( \lim_{t \to 0} \hat{E}|X_t|^2|t|^{-1} = 0 \).

Then \( \{X_t\} \) is called a generalized \( G \)-Brownian motion.

If in addition \( \hat{E}(X_t) = \hat{E}(-X_t) = 0 \) for all \( t \in [0, T] \), \( \{X_t\} \) is called a (symmetric) \( G \)-Brownian motion.

Remark 2.9. (i) Clearly, the coordinate process \( \{B_t\} \) is a (symmetric) \( G \)-Brownian motion and its quadratic variation process \( \langle \{B_t\} \rangle \) is a process with stationary and independent increments (w.r.t. the filtration).

(ii) [4] gave a characterization for the generalized \( G \)-Brownian motion: Let \( \{X_t\} \) be a generalized \( G \)-Brownian motion. Then

\[
X_{t+s} - X_t \sim \sqrt{s} \xi + s \eta \quad \text{for} \ t, s \geq 0,
\]

where \((\xi, \eta)\) is \( G \)-distributed (see, e.g., [6] for the definition of \( G \)-distributed random vectors). In fact, this characterization presented a decomposition of generalized \( G \)-Brownian motion in the sense of distribution. In this article, we shall give a pathwise decomposition for the generalized \( G \)-Brownian motion.

Let \( H^0_G(0, T) \) be the collection of processes of the following form: for a given partition \( \{t_0, \ldots, t_N\} = \pi_T \) of \([0, T]\), \( N \geq 1 \),

\[
\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) 1_{[j,j+1)}(t),
\]

where \( \xi_i \in L^1_{\pi_T}(\Omega_T) \), \( i = 0, 1, 2, \ldots, N - 1 \). For every \( \eta \in H^0_G(0, T) \), let \( \|\eta\|_{H^0_G} = (\hat{E}(\int_0^T |\eta_s|^2 \, ds)^{1/2})^{1/p} \), \( \|\eta\|_{M^p_G} = (\hat{E}(\int_0^T |\eta_s|^p \, ds)^{1/p} \) and denote by \( H^0_G(0, T), M^p_G(0, T) \) the completions of \( H^0_G(0, T) \) under the norms \( \|\cdot\|_{H^0_G}, \|\cdot\|_{M^p_G} \) respectively.

Definition 2.10. For every \( \eta \in H^0_G(0, T) \) with the form

\[
\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) 1_{[j,j+1)}(t),
\]

we define

\[
I(\eta) = \int_0^T \eta(s) \, dB_s := \sum_{j=0}^{N-1} \xi_j(B_{t_{j+1}} - B_{t_j}).
\]

By B–D–G inequality (see Proposition 4.3 in [10] for this inequality under \( G \)-expectation), the mapping \( I : H^0_G(0, T) \to L^p_G(\Omega_T) \) is continuous under \( \|\cdot\|_{H^0_G} \) and thus can be continuously extended to \( H^0_G(0, T) \).

Definition 2.11. (i) A process \( \{M_t\} \) with values in \( L^1_G(\Omega_T) \) is called a \( G \)-martingale if \( \hat{E}_s(M_t) = M_s \) for any \( s \leq t \). If \( \{M_t\} \) and \( \{-M_t\} \) are both \( G \)-martingales, we call \( \{M_t\} \) a symmetric \( G \)-martingale.

(ii) A random variable \( \xi \in L^1_G(\Omega_T) \) is called symmetric if \( \hat{E}(\xi) + \hat{E}(-\xi) = 0 \).

A \( G \)-martingale \( \{M_t\} \) is symmetric if and only if \( M_T \) is symmetric.
Theorem 2.12 ([1,2]). There exists a tight subset \( \mathcal{P} \subset M_1(\Omega_T) \) such that

\[
\hat{E}(\xi) = \max_{P \in \mathcal{P}} E_P(\xi) \quad \text{for all } \xi \in \mathcal{H}_T^0.
\]

\( \mathcal{P} \) is called a set that represents \( \hat{E} \).

Remark 2.13. (i) Let \((\Omega^0, \mathcal{F}^0, P^0)\) be a probability space and \(\{W_t\}\) be a \(d\)-dimensional Brownian motion under \(P^0\). Let \(F^0 = \{\mathcal{F}^0_t\}\) be the augmented filtration generated by \(W\). [1] proved that

\[
\mathcal{P}_M := \left\{ P_h | P_h = P^0 \circ X^{-1}, X_t = \int_0^t h_s \, dW_s, h \in L^2_{F^0}(0, T); \Gamma^{1/2} \right\}
\]

is a set that represents \( \hat{E} \), where \( \Gamma^{1/2} := \{\gamma^{1/2} | \gamma \in \Gamma \} \) and \( \Gamma \) is the set in the representation of \( G(\cdot) \) in the formula (2.1).

(ii) For the 1-dimensional case, i.e., \(\Omega_T = C_0([0, T], R)\),

\[
L^2_{F^0} := L^2_{F^0}(0, T); \Gamma^{1/2} = \{h | h \text{ is adapted w.r.t. } F^0 \text{ and } \sigma \leq h_s \leq \sigma \},
\]

where \(\sigma^2 = \hat{E}(B_t^2)\) and \(\sigma^2 = -\hat{E}(-B_t^2)\).

\[
G(a) = 1/2\hat{E}[a^2] = 1/2[\sigma^2 a^+ - \sigma^2 a^-] \quad \text{for } a \in R.
\]

(iii) Set \(c(A) = \sup_{P \in \mathcal{P}_M} P(A), \) for \(A \in \mathcal{B}(\Omega_T)\). We say \(A \in \mathcal{B}(\Omega_T)\) is a polar set if \(c(A) = 0\). If an event happens except on a polar set, we say the event happens q.s.

3. Characterization of processes with stationary and independent increments

In what follows, we only consider the \(G\)-expectation space \((\Omega_T, L^1_G(\Omega_T), \hat{E})\) with \(\Omega_T = C_0([0, T], R)\) and \(\sigma^2 = \hat{E}(B_t^2) > -\hat{E}(-B_t^2) = \sigma^2 > 0\).

Lemma 3.1. For \(\xi \in M^1_G(0, T)\) and \(\varepsilon > 0\), let

\[
\xi^\varepsilon_t = \frac{1}{\varepsilon} \int_{(0-\varepsilon)^+}^t \xi_s \, ds
\]

and

\[
\xi^{\varepsilon,0}_t = \frac{1}{\varepsilon} \sum_{k=1}^{k_\varepsilon-1} \int_{(k-1)\varepsilon}^{k\varepsilon} \xi_s \, ds 1_{[k\varepsilon,(k+1)\varepsilon]}(t),
\]

where \(t \in [0, T], k_\varepsilon \varepsilon \leq T < (k_\varepsilon + 1)\varepsilon\). Then as \(\varepsilon \to 0\)

\[
\| \xi^\varepsilon - \xi \|_{M^1_G(0, T)} \to 0 \quad \text{and} \quad \| \xi^{\varepsilon,0} - \xi \|_{M^1_G(0, T)} \to 0.
\]

Proof. The proofs of the two cases are similar. Here we only prove the second case. Our proof starts with the observation that for any \(\xi, \xi' \in M^1_G(0, T)\)

\[
\| \xi^{\varepsilon,0} - \xi''^{\varepsilon,0} \|_{M^1_G(0, T)} \leq \| \xi - \xi' \|_{M^1_G(0, T)}. \tag{3.1}
\]

By the definition of the space \(M^1_G(0, T)\), we know that for every \(\xi \in M^1_G(0, T)\), there exists a sequence of processes \(\{\xi^n\}\) with

\[
\xi^n_t = \sum_{k=0}^{m-1} \xi^n_{t_k} 1_{[t_k,t_{k+1}]}(t)
\]
and \( \xi_{tn} \in L_{lip}(\Omega_{tn}) \) such that
\[
\| \zeta - \zeta^n \|_{M_1^0(0,T)} \to 0 \quad \text{as } n \to \infty. \tag{3.2}
\]
It is easily seen that for every \( n \)
\[
\| \xi^n : 0 - \zeta^n \|_{M_1^0(0,T)} \to 0 \quad \text{as } \varepsilon \to 0. \tag{3.3}
\]
Thus we get
\[
\| \xi : 0 - \zeta \|_{M_1^0(0,T)} \leq \| \xi^n : 0 - \zeta^n \|_{M_1^0(0,T)} + \| \zeta^n - \zeta \|_{M_1^0(0,T)} + \| \zeta^n - \zeta^n : 0 - \varepsilon, 0 \|_{M_1^0(0,T)}. 
\]
The second inequality follows from (3.1). Combining (3.2) and (3.3), first letting \( \varepsilon \to 0 \), then letting \( n \to \infty \), we have
\[
\| \xi : 0 - \zeta \|_{M_1^0(0,T)} \to 0 \quad \text{as } \varepsilon \to 0. \]

**Theorem 3.2.** Let \( A_T = \int_0^t h_s \, ds \) with \( h \in M_{G}(0, T) \) be a process with stationary and independent increments (w.r.t. the filtration). Then we have \( h \equiv c \) for some constant \( c \).

**Proof.** Let \( \tilde{c} := \hat{E}(A_T) / T \geq -\hat{E}(-A_T) / T =: \zeta \). For \( n \in \mathbb{N} \), set \( \varepsilon = T/(2n) \), and define \( h^{T/(2n), 0} \) as in Lemma 3.1. Then we have
\[
\| h - h^{T/(2n), 0} \|_{M_1^0(0,T)} 
\]
\[
= \hat{E} \left[ \sum_{k=0}^{2n-1} \int_{kT/(2n)}^{(k+1)T/(2n)} \left| h_s - h^{T/(2n), 0}_s \right| \, ds \right].
\]

Consequently, from the condition of independence of the increments and their stationarity, we have
\[
\| h - h^{T/(2n), 0} \|_{M_1^0(0,T)} 
\]
\[
\geq \hat{E} \left[ \sum_{k=1}^{n-1} \int_{2kT/(2n)}^{(2k+1)T/(2n)} \left( h_s - h^{T/(2n), 0}_s \right) \, ds \right].
\]

Then we get
\[
\| \xi : 0 - \zeta \|_{M_1^0(0,T)} \to 0 \quad \text{as } \varepsilon \to 0. \]

\[
\| \xi : 0 - \zeta \|_{M_1^0(0,T)} \to 0 \quad \text{as } \varepsilon \to 0. \]

\[
\| \xi : 0 - \zeta \|_{M_1^0(0,T)} \to 0 \quad \text{as } \varepsilon \to 0. \]

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\[
\| \xi : 0 - \zeta \|_{M_1^0(0,T)} \to 0 \quad \text{as } \varepsilon \to 0. \]
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So by Lemma 3.1, letting $n \to \infty$, we have $\hat{c} = c$. Furthermore, we note that $M_t := A_t - ct$ is a $G$-martingale. In fact, for $t > s$, we see

\[
\hat{E}_s(M_t) = \hat{E}_s(M_t - M_s) + M_s = \hat{E}(M_t - M_s) + M_s = M_s.
\]

The second equality is due to the independence of increments of $M$ w.r.t. the filtration.

So \{${M_t}$\} is a symmetric $G$-martingale with finite variation, from which we conclude that $M_t \equiv 0$, hence that $A_t = \overline{c}t$. \hfill \Box

**Corollary 3.3.** Assume $\sigma > \sigma > 0$. Then we have that \{${d\over d_s}(B)_s$\} $\notin M^1_G(0, T)$.

**Proof.** The proof is straightforward from Theorem 3.2. \hfill \Box

**Corollary 3.4.** There is no symmetric $G$-martingale \{${M_t}$\} which is a standard Brownian motion under $G$-expectation (i.e. $(M)_t = t$).

**Proof.** Let \{${M_t}$\} be a symmetric $G$-martingale. If \{${M_t}$\} is also a standard Brownian motion, by Theorem 4.8 in [10] or Corollary 5.2 in [11], there exists \{${h_s}$\} $\in M^2_G(0, T)$ such that

\[
M_t = \int_0^t h_s \, dB_s
\]

and

\[
\int_0^t h_s^2 \, dB_s = t.
\]

Thus we have \{${d\over d_s}(B)_s = h_s^2$\} $\in M^1_G(0, T)$, which contradicts the conclusion of Corollary 3.3. \hfill \Box

**Proposition 3.5.** Let $A_t = \int_0^t h_s \, ds$ with $h \in M^1_G(0, T)$ be a process with independent increments. Then $A_t$ is symmetric for every $t \in [0, T]$.

**Proof.** By arguments similar to those in the proof of Theorem 3.2, we have

\[
\| h - h^{T/(2n)} \|_{M^1_G(0, T)} \geq \hat{E}_{n-1} \sum_{k=0}^{n-1} \left[ (A_{(2k+1)T/2n} - A_{2kT/2n}) - (A_{2kT/2n} - A_{(2k-1)+T/2n}) \right] = \sum_{k=0}^{n-1} \left[ \hat{E}(A_{(2k+1)T/2n} - A_{2kT/2n}) + \hat{E}[-(A_{2kT/2n} - A_{(2k-1)+T/2n})] \right] .
\]

The right side of the first inequality is only the sum of the odd terms. Summing up the even terms only, we have

\[
\| h - h^{T/(2n)} \|_{M^1_G(0, T)} \geq \sum_{k=0}^{n-1} \left[ \hat{E}(A_{(2k+2)T/2n} - A_{(2k+1)T/2n}) + \hat{E}[-(A_{(2k+1)T/2n} - A_{2kT/2n})] \right] .
\]
Combining the above inequalities, we have
\[
2\|h - h^{T/(2n)}\|_{M^1_G(0,T)}^2 \\
\geq \sum_{k=0}^{2n-1} \left\{ \hat{E}\left[A_{kT/2n}A_{kT/2n} - A_{kT/2n}A_{kT/2n} \right] + \hat{E}\left[-A_{kT/2n}A_{kT/2n} \right] \right\} \\
\geq \hat{E}\sum_{k=0}^{2n-1} \left[A_{kT/2n}A_{kT/2n} - A_{kT/2n}A_{kT/2n} \right] + \hat{E}\sum_{k=0}^{2n-1} \left[-A_{kT/2n}A_{kT/2n} \right] \\
= \hat{E}(AT) + \hat{E}(-AT).
\]

Thus by Lemma 3.1, letting \( n \to \infty \), we have \( \hat{E}(AT) + \hat{E}(-AT) = 0 \), which means that \( AT \) is symmetric. \( \Box \)

For \( n \in N \), define \( \delta_n(s) \) in the following way:
\[
\delta_n(s) = \sum_{i=0}^{n-1} (-1)^i 1_{\left[ \frac{i}{n}, \frac{i+1}{n} \right]}(s) \quad \text{for all } s \in [0, T].
\]

In [12] we proved that \( \lim_{n \to \infty} \hat{E}(\int_0^T \delta_n(s)h_s ds) = 0 \) for \( h \in M^1_G(0,T) \).

Let \( F_t = \sigma\{B_s | s \leq t\} \) and \( \mathbb{F} = \{F_t\}_{t \in [0,T]} \).

In the following, we shall use some notations introduced in Remark 2.13.

For every \( P \in \mathcal{P}_M \) and \( t \in [0,T] \), set \( A_{t,P} := \{Q \in \mathcal{P}_M | Q(F_t) = P(F_t)\} \). Proposition 3.4 in [9] gave the following result: For \( t \in [0, T] \), assume \( \xi \in L^1_G(\Omega_T) \) and \( \eta \in L^1_G(\Omega_T) \). Then \( \eta = \hat{E}_t(\xi) \) if and only if for every \( P \in \mathcal{P}_M \)
\[
\eta = \text{ess sup}_P^{\mathcal{P}_M} E_Q(\xi | F_t), \quad P\text{-a.s.},
\]
where \( \text{ess sup}^{\mathcal{P}_M} \) denotes the essential supremum under \( P \).

**Theorem 3.6.** Let \( A_t = \int_0^t h_s dQ(s) \) be a process with stationary, independent increments (w.r.t. the filtration) and \( h \in M^1_G(0,T) \). If \( AT \in L^1_G(\Omega_T) \) for some \( \beta > 1 \), we have \( A_t = c(B)_t \) for some constant \( c \geq 0 \).

**Proof.** For the readability, we divide the proof into several steps:

**Step 1.** Set \( K_t := \int_0^t h_s ds \). We claim that \( K_T \) is symmetric.

**Step 1.1.** Let \( \mu = \hat{E}(AT)/T \) and \( \mu = \hat{E}(-AT)/T \). First, we shall prove that \( \frac{\mu}{\sigma^2} = \frac{\mu}{\sigma^2} \).

Actually, for any \( 0 \leq s < t \leq T \), we have
\[
\hat{E}_s\left(\int_s^t h_r dr \right) = \hat{E}_s\left(\int_s^t \theta^{-1}_r dA_r \right) \geq \frac{1}{\sigma^2} \hat{E}_s\left(\int_s^t dA_r \right) = \frac{\mu}{\sigma^2}(t-s) \quad \text{q.s.},
\]
where the inequality holds due to \( \theta_t := \frac{d(B)}{ds} \leq \sigma^2 \), q.s. Noting that \( \mu t - \mu \) is nonincreasing by Lemma 4.3 in Section 4 since it is a \( G \)-martingale with finite variation, we have, for every \( \eta \in L^2_{P_{\eta}} \), \( P_{\eta}\text{-a.s.} \),
\[
\hat{E}_s\left(\int_s^t h_r dr \right) = \text{ess sup}_{Q \in A_{t,P_{\eta}}}^{P_{\eta}} E_Q\left(\int_s^t h_r dr | F_s \right)
\]
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\[
\begin{align*}
= \operatorname{ess sup}_{Q \in \mathcal{A}_t, \mu} P_{\eta} P_{\eta} E_Q \left( \int_r^t \theta_r^{-1} \mathrm{d}A_r \bigg| \mathcal{F}_s \right) \\
\geq \mu \operatorname{ess sup}_{Q \in \mathcal{A}_t, \mu} P_{\eta} P_{\eta} E_Q \left( \int_r^t \theta_r^{-1} \mathrm{d}r \bigg| \mathcal{F}_s \right) \\
= \frac{\mu}{\sigma^2} (t - s).
\end{align*}
\]

So \( \hat{E}_s \left( \int_s^t h_r \mathrm{d}r \right) \geq \max \{ \frac{\mu}{\sigma^2}, \frac{\mu}{\sigma^2} \} (t - s) =: \lambda(t - s), \quad \text{q.s.} \)

On the other hand,

\[
\begin{align*}
\hat{E}_s \left( - \int_s^t h_r \mathrm{d}r \right) &= \hat{E}_s \left( \int_s^t -\theta_r^{-1} \mathrm{d}A_r \right) \\
&\geq \frac{1}{\sigma^2} \hat{E}_s \left( - \int_s^t \mathrm{d}A_r \right) = - \frac{\mu}{\sigma^2} (t - s), \quad \text{q.s.}
\end{align*}
\]

and for every \( \eta \in L^2_{F_0}, \ P_\eta \text{-a.s.}, \)

\[
\begin{align*}
\hat{E}_s \left( - \int_s^t h_r \mathrm{d}r \right) &= \operatorname{ess sup}_{Q \in \mathcal{A}_t, \mu} P_{\eta} P_{\eta} E_Q \left( - \int_s^t h_r \mathrm{d}r \bigg| \mathcal{F}_s \right) \\
&= \operatorname{ess sup}_{Q \in \mathcal{A}_t, \mu} P_{\eta} P_{\eta} E_Q \left( - \int_s^t \theta_r^{-1} \mathrm{d}A_r \bigg| \mathcal{F}_s \right) \\
&\geq \overline{\mu} \operatorname{ess sup}_{Q \in \mathcal{A}_t, \mu} P_{\eta} P_{\eta} E_Q \left( - \int_s^t \theta_r^{-1} \mathrm{d}r \bigg| \mathcal{F}_s \right) \\
&= - \frac{\overline{\mu}}{\sigma^2} (t - s)
\end{align*}
\]

since \( A_t - \overline{\mu} t \) is nonincreasing. So

\[
\hat{E}_s \left( - \int_s^t h_r \mathrm{d}r \right) \geq - \min \left\{ \frac{\overline{\mu}}{\sigma^2}, \frac{\mu}{\sigma^2} \right\} (t - s) =: - \lambda(t - s), \quad \text{q.s.}
\]

Noting that

\[
\begin{align*}
\hat{E} \left( \int_0^T \delta_{2n}(s) h_s \mathrm{d}s \right) &= \hat{E} \left[ \int_0^{(2n-1)T/(2n)} \delta_{2n}(s) h_s \mathrm{d}s + \hat{E}_{(2n-1)T/(2n)} \left( - \int_0^{T/(2n)} h_s \mathrm{d}s \right) \right] \\
&\geq \left( - \lambda \right) \frac{T}{2n} + \hat{E} \left[ \int_0^{(2n-2)T/(2n)} \delta_{2n}(s) h_s \mathrm{d}s + \hat{E}_{(2n-2)T/(2n)} \left( \int_0^{(2n-1)T/(2n)} h_s \mathrm{d}s \right) \right] \\
&\geq \frac{\lambda - \lambda}{2n} T + \hat{E} \left[ \int_0^{(2n-2)T/(2n)} \delta_{2n}(s) h_s \mathrm{d}s \right],
\end{align*}
\]

we have

\[
\hat{E} \left( \int_0^T \delta_{2n}(s) h_s \mathrm{d}s \right) \geq \frac{\lambda - \lambda}{2} T.
\]
So

\[
0 = \lim_{n \to \infty} \hat{E} \left( \int_0^T \delta_{2n}(s) h_s \, ds \right) = \frac{\lambda - \lambda}{2} T
\]

and \[ \frac{\mu}{\sigma^2} = \frac{\mu}{\sigma^2} = \lambda. \]

**Step 1.2.** For every \( \eta \in L^2_p \), \( E_{p_\eta}(K_T) = \lambda T \), which implies that \( K_T \) is symmetric.

**Step 1.2.1.** We now introduce some notations: For \( n \in \mathbb{N} \), set \( \eta_n = \bar{\eta}, \ \eta = \sigma, \ \eta^* = \sqrt{\frac{\sigma^2 + \eta^2}{2}} \) on \( [s, t] \) and \( \eta_n = \eta = \eta^* = \eta \) on \( [s, t]^c \). Actually, we have, \( P_\eta \)-a.s.,

\[
\mu(t - s) = \hat{E}_s\left( \int_s^t h_r \, d(B)_r \right) \geq E_{p_\eta}\left( \int_s^t h_r \, d(B)_r \bigg| \mathcal{F}_s \right) = \sigma^2 E_{p_\eta}\left( \int_s^t h_r \, d\mathcal{F}_s \right).
\]

So

\[
E_{p_\eta}\left( \int_s^t h_r \, d\mathcal{F}_s \right) \leq \lambda(t - s), \quad P_\eta \text{-a.s.} \tag{3.4}
\]

By similar arguments we have that

\[
E_{p_\eta}\left( \int_s^t h_r \, d\mathcal{F}_s \right) \geq \lambda(t - s), \quad P_\eta \text{-a.s.} \tag{3.5}
\]

Let’s compute the following conditional expectations:

\[
E_{p_\eta}\left( \int_s^t (h_r - \lambda) \delta_{2n}(r) \, dr \bigg| \mathcal{F}_s \right)
\]

\[
= E_{p_\eta}\left[ \sum_{i=0}^{n-1} \left( E_{p_\eta}\left( \int_{t_{2i+1}}^{t_{2i+2}} (h_r - \lambda) \, dr \bigg| \mathcal{F}_s \right) + E_{p_\eta}\left( \int_{t_{2i+1}}^{t_{2i+2}} (\lambda - h_r) \, dr \bigg| \mathcal{F}_s \right) \right) \right]
\]

where \( \delta_{2n}(r) = \sum_{i=0}^{n-1} (1_{[t_{2i+1}, t_{2i+2}]}(r) - 1_{[t_{2i+1}, t_{2i+2}]}(r)) \), \( t_j = s + \frac{j}{2n} (t - s), \ j = 0, \ldots, 2n; \)

\[
E_{p_\eta}\left( \int_s^t h_r \, d\mathcal{F}_s \right) = E_{p_\eta}\left[ \sum_{i=0}^{n-1} (A_i - B_i) \right].
\]

By (3.4) and (3.5) (noting that \( \eta \) and \( s, t \) are all arbitrary), we conclude that \( A_j, B_i \geq 0, \ P_{\eta_n} \text{-a.s.} \) So

\[
\left| E_{p_\eta}\left( \int_s^t (h_r - \lambda) \, dr \bigg| \mathcal{F}_s \right) \right| \leq E_{p_\eta}\left( \int_s^t (h_r - \lambda) \delta_{2n}(r) \, dr \bigg| \mathcal{F}_s \right), \quad P_\eta \text{-a.s.}
\]

Noting that

\[
E_{p_\eta}\left( \int_s^t (h_r - \lambda) \delta_{2n}(r) \, dr \bigg| \mathcal{F}_s \right) \leq \hat{E}_s\left[ \int_s^t (h_r - \lambda) \delta_{2n}(r) \, dr \right], \quad P_\eta \text{-a.s.}
\]
we have $E_{P,\eta} (\int_0^T (h_r - \lambda) \delta_{2n} (r) \, dr) \to 0$, q.s., as $n \to \infty$.

**Step 1.2.3.** For any $x \in L^1_T (\Omega_T)$, $E_{P,\eta} (\xi \mid \mathcal{F}_s) \to E_{P,\eta} (\xi \mid \mathcal{F}_s)$, $P_\eta$-a.s., as $n \to \infty$.

In fact, if $\xi = \varphi (B_{t_1} - B_{t_0}, \ldots, B_{t_m} - B_{t_{m-1}}) \in L_{lip} (\Omega_T)$, the conclusion is obvious. For general $x \in L^1_T (\Omega_T)$, there exists a sequence $\{m^n\} \subset L_{lip} (\Omega_T)$ such that $\hat{E} (\xi^m - \xi^m) = \hat{E} (\hat{E} (\xi^m - \xi^m)) \to 0$. So we can assume $\hat{E} (\xi^m - \xi) \to 0$ q.s.

Then, $P_\eta$-a.s., we have

$$
\left| E_{P,\eta} (\xi \mid \mathcal{F}_s) - E_{P,\eta} (\xi \mid \mathcal{F}_s) \right| \\
\leq \left| E_{P,\eta} (\xi \mid \mathcal{F}_s) - E_{P,\eta} (\xi \mid \mathcal{F}_s) \right| + \left| E_{P,\eta} (\xi \mid \mathcal{F}_s) - E_{P,\eta} (\xi \mid \mathcal{F}_s) \right| \\
+ \left| E_{P,\eta} (\xi \mid \mathcal{F}_s) - E_{P,\eta} (\xi \mid \mathcal{F}_s) \right| \\
\leq 2 \hat{E} (\xi^m - \xi) + \left| E_{P,\eta} (\xi^m \mid \mathcal{F}_s) - E_{P,\eta} (\xi^m \mid \mathcal{F}_s) \right|.
$$

First letting $n \to \infty$, then letting $m \to \infty$, we have $E_{P,\eta} (\xi \mid \mathcal{F}_s) \to E_{P,\eta} (\xi \mid \mathcal{F}_s)$, $P_\eta$-a.s. So combining Step 1.2.2 and Step 1.2.3, we have

$$
E_{P,\eta} \left( \int_s^T h_r \, dr \bigg| \mathcal{F}_s \right) = \lambda (t-s), \quad P_\eta \text{-a.s.} \quad (3.6)
$$

**Step 1.2.4.** For $0 \leq s \leq t \leq T$, $\eta \in L^2_{P\eta}$, $\sigma \in [\sigma, \overline{\sigma}]$, set $\eta^\sigma = \sigma$ on $]s, t]$ and $\eta^\sigma = \eta$ on $]s, t]$. We have

$$
E_{P,\eta} \left( \int_s^T h_r \, dr \bigg| \mathcal{F}_s \right) = \lambda (t-s), \quad P_\eta \text{-a.s.}
$$

In fact, Step 1.2.2–Step 1.2.3 proved the following fact: If $(3.4)$, $(3.5)$ hold for some $\sigma, \sigma' \in [\sigma, \overline{\sigma}]$, then $(3.6)$ holds for $\sqrt{\frac{\sigma^2 + \sigma'^2}{2}}$. So by repeating the Step 1.2.2–Step 1.2.3, we get the desired result.

**Step 1.2.5.** For any simple process $\eta \in L^2_{P\eta}$, $E_{P,\eta} (K_T) = \lambda T$.

Let $\eta_r = \sum_{i=0}^{n-1} \eta_{i+1} t_i, \ldots, t_{i+1}) \in L^2_{P\eta}$ with $\eta_{i+1} = \sum_{j=1}^{n_i} a_{j} (t_{i+1} \lambda_1 \lambda$ an $\mathcal{F}_i$ measurable simple function, where $\{t_0, \ldots, t_m\}$ is a given partition of $[0, T]$. Set $X_i = \int_0^1 t_r \, dW_r$. Let $F^x = \{F^x_t\}$ be the filtration generated by $X$.

Fix $0 \leq i < m$. Set $\eta^{x, i} = \eta_x [0, t_{i+1}] (s) + a_{j} (t_{i+1} \lambda \lambda$ and $X_{i}^{x, i} = \int_0^1 t_r \, dW_r$ for $\varepsilon > 0$ small enough. Let $F^{X,i} = \{F^{X,i}_t\}$ be the filtration generated by $X^{i, \varepsilon}$. Then

$$
E_{P} \left( \int_{t_i}^{t_{i+1}} h_r \, dr \bigg| F_{t_i} \right) = E_{P} \left( \int_{t_i}^{t_{i+1}} h_r \circ X \, dr \bigg| F_{t_i} \right) = E_{P} \left[ E_{P} \left( \int_{t_i}^{t_{i+1}} h_r \circ X \, dr \bigg| F^{X}_{t_i+\varepsilon} \right) \right].
$$

Since $A^i_j \in F^{X}_{t_i+\varepsilon}$ and $X_i = \sum_{j=0}^{n_i} X^{x, i, j} 1_{A^i_j}$ on $[0, t_{i+1}]$, we have

$$
E_{P} \left( \int_{t_i}^{t_{i+1}} h_r \circ X \, dr \bigg| F^{X}_{t_i+\varepsilon} \right) \\
= \sum_{j=1}^{n_i} E_{P} \left( 1_{A^i_j} \int_{t_i}^{t_{i+1}} h_r \circ X^{x, i, j} \, dr \bigg| F^{X}_{t_i+\varepsilon} \right) \\
= \sum_{j=1}^{n_i} 1_{A^i_j} E_{P} \left( \int_{t_i}^{t_{i+1}} h_r \circ X^{x, i, j} \, dr \bigg| F^{X}_{t_i+\varepsilon} \right).
$$
Noting that
\[ E_p^\alpha (\int_{t_i + \varepsilon}^{t_{i+1}} h_r \circ X^{i, \varepsilon} \, dr \bigg| \mathcal{F}_{t_i + \varepsilon}^{X^{i, \varepsilon}}) = E_{p_i}^\alpha (\int_{t_i + \varepsilon}^{t_{i+1}} h_r \, dr \bigg| \mathcal{F}_{t_i + \varepsilon}) \circ X^{i, \varepsilon} = \lambda(t_{i+1} - t_i - \varepsilon) \]  
by Step 1.2.4, we have \( E_{p_0} (\int_{t_i + \varepsilon}^{t_{i+1}} h_r \, dr) = \lambda(t_{i+1} - t_i) \) and \( E_{p_0} (K_T) = \lambda T \).

Step 2. \( h \equiv \lambda \).

Let \( M_t = \int_0^t h_r \, dB_r - \int_0^t 2G(h_s) \, ds \) and \( N_t = \int_0^t h_s \, dB_r - \bar{\mu}t \). As is mentioned in the Introduction, [4] proved that \( \{M_t\} \) is a \( G \)-martingale. Since \( \{\int_0^t h_r \, dB_r\} \) is a process with stationary and independent increments w.r.t. the filtration, we know that \( \{N_t\} \) is also a \( G \)-martingale. Let \( L_t = \hat{E}_t(\bar{\mu}T - \bar{\sigma}^2 K_T) \). Then \( \{L_t\} \) is a symmetric \( G \)-martingale since \( K_T \) is symmetric. By the symmetry of \( \{L_t\} \) we have
\[ M_t = \hat{E}_t(M_T) = \hat{E}_t(L_T + N_T) = L_t + N_t. \]

By the uniqueness of the \( G \)-martingale decomposition, we get \( L \equiv 0 \) and \( h \equiv \lambda \).

\[ \square \]

**Remark 3.7.** Clearly, \( h \in M^\beta_G(0, T) \) for some \( \beta > 1 \) implies \( A_T = \int_0^T h_r \, dB_r \in L^\beta_G(\Omega_T) \).

### 4. Characterization of the \( G \)-Brownian motion

A version of the martingale characterization for the \( G \)-Brownian motion was given in [13], where only symmetric \( G \)-martingales with Markovian property were considered. Here we shall present a martingale characterization in a quite different form, which is a natural but nontrivial generalization of the classical case in a probability space.

**Theorem 4.1 (Martingale characterization of the \( G \)-Brownian motion).**

Let \( \{M_t\} \) be a symmetric \( G \)-martingale with \( M_T \in L^\alpha_G(\Omega_T) \) for some \( \alpha > 2 \) and \( \{\langle M \rangle_t\} \) a process with stationary and independent increments (w.r.t. the filtration). Then \( \{M_t\} \) is a \( G \)-Brownian motion:

**Proof.** By Corollary 5.2 in [11], there exists \( h \in M^2_G(0, T) \) such that \( M_t = \int_0^t h_r \, dB_r \). So \( \langle M \rangle_t = \int_0^t h_r^2 \, dB_r \). By the assumption, we know that \( \langle M \rangle_T \in L^\beta_G(\Omega_T) \) for some \( \beta > 1 \). By Theorem 3.6, there exists some constant \( c \geq 0 \) such that \( h^2 \equiv c \). Thus by Theorem 2.12 and Remark 2.13, \( \{M_t\} \) is a \( G \)-Brownian motion with \( M_t \) distributed as \( N(0, [c\sigma^2 t, c\sigma^2 t]) \).

On the other hand, if \( \{M_t\} \) is a \( G \)-Brownian motion on \( (\Omega_T, L^1_G(\Omega_T), \hat{E}) \), then \( \{M_t\} \) is a symmetric \( G \)-martingale. By the above arguments, we have \( \langle M \rangle_t = c\langle B \rangle_t \) for some positive constant \( c \).

\[ \square \]

Let
\[ \mathcal{H} = \left\{ a \bigg| a(t) = \sum_{k=0}^{n-1} a_k 1_{[t_k, t_{k+1})}(t), n \in \mathbb{N}, 0 = t_0 < t_1 < \cdots < t_n = T \right\}, \]
and \( H = \{a \in \mathcal{H} | a(0) = 0\} \), where \( \lambda \) is the Lebesgue measure.

**Lemma 4.2.** Let \( \{L_t\} \) be a process with absolutely continuous paths. Assume that there exist real numbers \( c \leq \bar{c} \) such that \( c(t - s) \leq L_t - L_s \leq \bar{c}(t - s) \) for any \( s < t \). Let \( C(a) = \bar{c}a^+ - ca^- \) for any \( a \in \mathbb{R} \). If
\[ \hat{E} \left( \int_0^T a(s) \, dL_s \right) = \int_0^T C(a(s)) \, ds \quad \text{for all } a \in \mathcal{H}, \]
we have that \( \{L_t\} \) is a process with stationary and independent increments such that \( cT = -\hat{E}(-L_t) \leq \hat{E}(L_t) = \bar{c}t \), i.e., its distribution is determined by \( c, \bar{c} \).
Proof. It suffices to prove the lemma for the case $c < \bar{c}$. For any $a \in H$, let
\[ \theta^a_s = \mathbb{1}_{[a(s) \geq 0]} + \mathbb{1}_{[a(s) < 0]}. \]
By assumption,
\[ \hat{E} \left( \int_0^T a(s) \, dL_s \right) = \int_0^T a(s) \theta^a_s \, ds. \]
On the other hand, by Theorem 2.12, there exists some weakly compact subset $P \subset M_1(\Omega_T)$ such that
\[ \hat{E}(\xi) = \max_{P \in \mathcal{P}} E_P(\xi) \quad \text{for all } \xi \in L^1_G(\Omega_T), \]
which means that there exists $P_a \in \mathcal{P}$ such that
\[ E_{P_a} \left( \int_0^T a(s) \, dL_s \right) = \int_0^T a(s) \theta^a_s \, ds. \]
By the assumption for $\{L_t\}$, we have $P_a \{L_t = \int_0^t \theta^a_s \, ds, \text{ for all } t \in [0, T]\} = 1$. From this we have
\[ \hat{E} \left[ \varphi(L_{t_1} - L_{t_0}, \ldots, L_{t_n} - L_{t_{n-1}}) \right] \geq \varphi \left( \int_{t_0}^{t_1} \theta^a_s \, ds, \ldots, \int_{t_{n-1}}^{t_n} \theta^a_s \, ds \right) \]
for any $\varphi \in C_b(R^n)$ and $n \in N$. Consequently,
\[ \hat{E} \left[ \varphi(L_{t_1} - L_{t_0}, \ldots, L_{t_n} - L_{t_{n-1}}) \right] \geq \sup_{a \in H} \varphi \left( \int_{t_0}^{t_1} \theta^a_s \, ds, \ldots, \int_{t_{n-1}}^{t_n} \theta^a_s \, ds \right) = \sup_{c_1, \ldots, c_n \in [\underline{c}, \bar{c}]} \varphi(c_1(t_1 - t_0), \ldots, c_n(t_n - t_{n-1})). \]
The converse inequality is obvious. Thus $\{L_t\}$ is a process with stationary and independent increments such that $c_t = -\hat{E}(-L_t) \leq \hat{E}(L_t) = \bar{c}. \quad \square$

Lemma 4.3. Let $\{L_t\}$ be a $G$-martingale with finite variation and $L_T \in L^\beta_G(\Omega_T)$ for some $\beta > 1$. Then $\{L_t\}$ is nonincreasing. Particularly, $L_t \leq L_0 = \hat{E}(L_T)$.

Proof. By Theorem 4.5 in [10], we know $\{L_t\}$ has the following decomposition
\[ L_t = \hat{E}(L_T) + M_t + K_t, \]
where $\{M_t\}$ is a symmetric $G$-martingale and $\{K_t\}$ is a nonpositive, nonincreasing $G$-martingale. Since both $\{L_t\}$ and $\{K_t\}$ are processes with finite variation, we get $M_t \equiv 0$. Therefore, we have $L_t = \hat{E}(L_T) + K_t \leq \hat{E}(L_T) = L_0. \quad \square$

Theorem 4.4. Let $\{X_t\}$ be a generalized $G$-Brownian motion with zero mean. Then we have the following decomposition:
\[ X_t = M_t + L_t, \]
where $\{M_t\}$ is a symmetric $G$-Brownian motion, and $\{L_t\}$ is a nonpositive, nonincreasing $G$-martingale with stationary and independent increments.
Proof. Clearly $\{X_t\}$ is a $G$-martingale. By Theorem 4.5 in [10], we have the following decomposition

$$X_t = M_t + L_t,$$

where $\{M_t\}$ is a symmetric $G$-martingale, and $\{L_t\}$ is a nonpositive, nonincreasing $G$-martingale. Noting that $X_t \in L_\beta^\infty(\Omega_T)$ from the definition of generalized $G$-Brownian motion, we know that $M_t, L_t \in L_\beta^\infty(\Omega_T)$ for any $1 \leq \beta < 3$ by Theorem 4.5 in [10].

In the sequel, we first prove that $\{L_t\}$ is a process with stationary and independent increments. Noting that

$$\hat{E}(-L_t) = \hat{E}(-X_t) = ct$$

for some positive constant $c$ since $\{X_t\}$ is a process with stationary and independent increments, we claim that $-L_t - ct$ is a $G$-martingale. To prove this, it suffices to show that for any $t > s$,

$$\hat{E}_s[-(L_t - L_s)] = c(t - s).$$

In fact, since $\{M_t\}$ is a symmetric $G$-martingale, we have

$$\hat{E}_s[-(L_t - L_s)] = \hat{E}_s[-(X_t - X_s + M_s)] = \hat{E}_s[-(X_t - X_s)].$$

Noting that $\{X_t\}$ is a process with independent increments (w.r.t. the filtration),

$$\hat{E}_s[-(X_t - X_s)] = \hat{E}[-(X_t - X_s)] = c(t - s).$$

Combining this with Lemma 4.3, we have $-(L_t - L_s) - c(t - s) \leq 0$ for any $s < t$. On the other hand, for any $a \in \mathcal{H}$, noting that $\{M_t\}$ is a symmetric $G$-martingale, we have

$$\hat{E} \left[ \int_0^T a(s) \, dL_s \right] = \hat{E} \left[ \int_0^T a(s) \, dX_s \right] = \hat{E} \left[ \sum_{k=0}^{n-1} a_{t_k} (X_{t_{k+1}} - X_{t_k}) \right].$$

Since $\{X_t\}$ is a process with stationary, independent increments, we have

$$\hat{E} \left[ \int_0^T a(s) \, dL_s \right] = \sum_{k=0}^{n-1} \hat{E} \left[ a_{t_k} (X_{t_{k+1}} - X_{t_k}) \right]$$

$$= \sum_{k=0}^{n-1} c a_{t_k}^+ (t_{k+1} - t_k)$$

$$= \int_0^T c a^-(s) \, ds = \int_0^T C(a(s)) \, ds,$$

where $C(a(s))$ is defined as in Lemma 4.2 with $\varpi = 0$, $c = -c$. By Lemma 4.2, $\{L_t\}$ is a process with stationary and independent increments.

Now we are in a position to show that $\{M_t\}$ is a (symmetric) $G$-Brownian motion. To this end, by Theorem 4.1, it suffices to prove that $\{\langle M \rangle_t\}$ is a process with stationary and independent increments (w.r.t. the filtration). For $n \in \mathbb{N}$, let

$$X^n_t = \sum_{k=0}^{2^n - 1} X_{kT/2^n} 1_{[kT/2^n, (k+1)T/2^n]}(t)$$

and

$$\Omega^n_t(X) = \sum_{k=0}^{2^n - 1} (X_{(k+1)T/2^n} - X_{kT/2^n})^2.$$
Observing that \( \Omega_t^n(X) = X_t^n - 2 \int_0^t X_s^n \, dX_s \), we have

\[
\begin{align*}
\left| \Omega_t^n(X) - \Omega_t^{m+n}(X) \right| & \leq 2 \left( \left| \int_0^t (X_s^n - X_s^{m+n}) \, dM_s \right| + \left| \int_0^t (X_s^n - X_s^{m+n}) \, dL_s \right| \right) \\
& = 2 (|I| + |II|)
\end{align*}
\]

for any \( n, m \in \mathbb{N} \). It’s easy to check that

\[
\hat{E}(|II|) \leq c \int_0^t \hat{E}(\left| X_s^n - X_s^{m+n} \right|) \, ds \to 0 \quad \text{as} \; m, n \to \infty.
\]

Noting that

\[
I = \sum_{i=0}^{2^n-1} \sum_{j=0}^{2^m-1} (X_{it/2^n + j} - X_{it/2^n}) (M_{it/2^n + (j+1)t/2^n} - M_{it/2^n + j + 1/2^n})
\]

we get

\[
\hat{E}(I^2) \leq \sum_{i=0}^{2^n-1} \sum_{j=0}^{2^m-1} \hat{E}([I_i^j]^2).
\]

Let’s estimate the expectation \( \hat{E}([I_i^j]^2) \):

\[
\hat{E}([I_i^j]^2) = \hat{E}((X_{it/2^n + j} - X_{it/2^n})^2 (M_{it/2^n + (j+1)t/2^n} - M_{it/2^n + j + 1/2^n})^2) \\
\leq 2 \hat{E}((X_{it/2^n + j} - X_{it/2^n})^2 (X_{it/2^n + (j+1)t/2^n} - X_{it/2^n + j + 1/2^n})^2) \\
+ \hat{E}((L_{it/2^n + (j+1)t/2^n} - L_{it/2^n + j + 1/2^n})^2)
\]

Noting that \(-c(t-s) \leq L_t - L_s \leq 0\), we have

\[
\hat{E}([I_i^j]^2) \leq \hat{E}((X_{it/2^n + j} - X_{it/2^n})^2 (X_{it/2^n + (j+1)t/2^n} - X_{it/2^n + j + 1/2^n})^2 + c^2 \frac{i^2}{2(2^n+m))})
\]

By (2.2), \( \hat{E}([X_t - X_s]^2) \leq C_1 |t-s| \) for some constant \( C_1 \). From the condition of independent increments of \( X \), we have \( \hat{E}([I_i^j]^2) \leq C \frac{i^2}{2(2^n+m)} \) for some constant \( C \), hence that \( \hat{E}(I^2) \to 0 \), and finally that \( \hat{E}(|\Omega_t^n(X) - \Omega_t^{m+n}(X)|) \to 0 \) as \( m, n \to \infty \). Then

\[
\langle X \rangle_t := \lim_{\Omega_t^n \to \Omega_t} \Omega_t^n
\]

is a process with stationary and independent increments (w.r.t. the filtration). Noting that \( \langle M \rangle_t = \langle X \rangle_t \), \( \langle M \rangle_t \) is also a process with stationary and independent increments (w.r.t. the filtration). □
5. $G$-martingales with finite variation

**Proposition 5.1.** Let $\eta \in M^1_G(0, T)$ with $|\eta| \equiv c$ for some constant $c$. Then

$$K_t := \int_0^t \eta_s \, d\langle B \rangle_s - \int_0^t 2G(\eta_s) \, ds$$

(5.1)

is a process with stationary and independent increments. Moreover, for fixed $c$, all processes in the above form have the same distribution.

**Proof.** Since $-c(\sigma^2 - \sigma^2)(t - s) \leq K_t - K_s \leq 0$ for any $s < t$, by Lemma 4.2, it suffices to prove that for any $a \in \mathcal{H}$

$$\hat{E}\left(\int_0^T a_s \, dK_s\right) = \int_0^T C(a_s) \, ds,$$

where $C(a_s)$ is defined as in Lemma 4.2 with $\xi = -c(\sigma^2 - \sigma^2)$. In fact, noting that

$$\int_0^T a_s \, dK_s \leq \int_0^T 2G(a_s \eta_s) \, ds - \int_0^T 2a_s G(\eta_s) \, ds = \int_0^T C(a_s) \, ds,$$

we have

$$\hat{E}\left(\int_0^T a_s \, dK_s\right) \leq \int_0^T C(a_s) \, ds.$$

On the other hand, we have

$$\hat{E}\left(\int_0^T a_s \, dK_s\right) \geq -\hat{E}\left\{-\int_0^T 2G(a_s \eta_s) \, ds - \int_0^T 2a_s G(\eta_s) \, ds\right\} = \int_0^T C(a_s) \, ds.$$

So $\{K_t\}$ is a process with stationary and independent increments and its distribution is determined by $c$. \hfill $\square$

Just like the conjecture by Shige Peng for the representation of $G$-martingales with finite variation, we guess that any $G$-martingale with stationary, independent increments and finite variation should have the form of (5.1). At the end we present a characterization for $G$-martingales with finite variation.

**Proposition 5.2.** Let $\{M_t\}$ be a $G$-martingale with $M_T \in L^\beta_G(\Omega_T)$ for some $\beta > 1$. Then $\{M_t\}$ is a $G$-martingale with finite variation if and only if $\{f(M_t)\}$ is a $G$-martingale for any nondecreasing $f \in C_{b, \text{Lip}}(R)$.

**Proof.** Necessity. Assume $\{M_t\}$ is a $G$-martingale with finite variation. By Lemma 4.3, we know that $\{M_t\}$ is nonincreasing. By Theorem 5.4 in [11], there exists a sequence $\{\eta^n_t\} \subset H^0_G(0, T)$ such that

$$\hat{E}\left[\sup_{t \in [0, T]} |M_t - L_t(\eta^n)|^\beta\right] \to 0$$

as $n$ goes to infinity, where $L_t(\eta^n) = \int_0^t \eta^n_s \, d\langle B \rangle_s - \int_0^t 2G(\eta^n_s) \, ds$. It suffices to prove that for any $\eta \in H^0_G(0, T)$ and nondecreasing $f \in C^2_b(R)$, $f(L_t(\eta))$ is a $G$-martingale. In fact,

$$f(L_t(\eta)) = f(L_0) + \int_0^t f'(L_s(\eta)) \, dL_s(\eta)$$

$$= f(L_0) + \int_0^t f'(L_s(\eta)) \eta_s \, d\langle B \rangle_s - \int_0^t 2f'(L_s(\eta)) G(\eta_s) \, ds.$$
Since \(f'(L_s(\eta)) \geq 0\) and \(f'(L_s(\eta))\eta_s \in M^L_V(0, T)\), we conclude that

\[
f(L_t(\eta)) = f(L_0) + L_t(f'(L(\eta))\eta)
\]
is a \(G\)-martingale.

Sufficiency. Assume \(\{f(M_t)\}\) is a \(G\)-martingale for any nondecreasing \(f \in C_b, \text{Lip}(R)\). Let \(X_t := \arctan M_t\). Then \(\{X_t\}\) is a bounded \(G\)-martingale and \(\{f(X_t)\}\) is a \(G\)-martingale for any nondecreasing \(f \in C_b, \text{Lip}(R)\). By Theorem 4.5 in [10], we know \(\{X_t\}\) has the following decomposition

\[
X_t = \hat{E}(X_T) + N_t + K_t,
\]
where \(\{N_t\}\) is a symmetric \(G\)-martingale and \(\{K_t\}\) is a nonpositive, nonincreasing \(G\)-martingale. Then by Itô’s formula

\[
e^{\alpha X_t} = e^{\alpha X_0} + \alpha \int_0^t e^{\alpha X_s} dX_s + \frac{\alpha^2}{2} \int_0^t e^{\alpha X_s} d\langle N \rangle_s.
\]

For any \(\alpha > 0\), by assumption, \(e^{\alpha X_t}\) is a \(G\)-martingale. So \(L_t := \int_0^t e^{\alpha X_s} dK_s + \frac{\alpha^2}{2} \int_0^t e^{\alpha X_s} d\langle N \rangle_s\) is a \(G\)-martingale with finite variation. By Lemma 4.3, \(L_t\) is nonincreasing, by which we conclude that \(K_t + \frac{\alpha}{2} \langle N \rangle_t\) is nonincreasing. So

\[
\frac{\alpha}{2} \hat{E}(\langle N \rangle_T) \leq \hat{E}(-K_T) \quad \text{for all } \alpha > 0.
\]

By this, we conclude that \(\hat{E}(\langle N \rangle_T) = 0\) and \(N_t \equiv 0\). Then \(X_t = \hat{E}(X_T) + K_t\) is nonincreasing, and consequently, \(M_t\) is nonincreasing.

Particularly, Proposition 5.2 provides a method to convert \(G\)-martingales with finite variation into bounded \(G\)-martingales with finite variation.

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References


