Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2012, Vol. 48, No. 4, 947–972
DOI: 10.1214/11-AIHP422

ANNALES DE L'INSTITUT HENRI POINCARÉ
PROBABILITÉS ET STATISTIQUES
www.imstat.org/aihp

Invariance of Poisson measures under random transformations

Nicolas Privault[1]

Division of Mathematical Sciences, School of Physical and Mathematical Sciences, Nanyang Technological University, SPMS-MAS, 21 Nanyang Link, Singapore 637371. E-mail: nprivault@ntu.edu.sg

Received 28 September 2010; revised 10 February 2011; accepted 23 February 2011

Abstract. We prove that Poisson measures are invariant under (random) intensity preserving transformations whose finite difference gradient satisfies a cyclic vanishing condition. The proof relies on moment identities of independent interest for adapted and anticipating Poisson stochastic integrals, and is inspired by the method of Üstünel and Zakai (*Probab. Theory Related Fields* **103** (1995) 409–429) on the Wiener space, although the corresponding algebra is more complex than in the Wiener case. The examples of application include transformations conditioned by random sets such as the convex hull of a Poisson random measure.

Résumé. Nous montrons que les mesures de Poisson sont invariantes par les transformations aléatoires qui préservent les mesures d'intensité, et dont le gradient aux différences finies satisfait une condition d'annulation cyclique. La preuve de ce résultat repose sur des identités de moments d'intérêt indépendant pour les intégrales stochastiques de Poisson adaptées et anticipantes, et est inspirée par la méthode de Üstünel et Zakai (*Probab. Theory Related Fields* **103** (1995) 409–429) sur l'espace de Wiener, bien que l'algèbre correspondante soit plus compliquée que dans le cas Wiener. Les exemples d'application incluent des transformations conditionnées par des ensembles aléatoires tels que l'enveloppe convexe d'une mesure aléatoire de Poisson.

MSC: 60G57; 60G30; 60G55; 60H07; 28D05; 28C20; 37A05

Keywords: Poisson measures; Random transformations; Invariance; Skorohod integral; Moment identities

1. Introduction

Poisson random measures on metric spaces are known to be quasi-invariant under deterministic transformations satisfying suitable conditions, cf. e.g. [20,24]. For Poisson processes on the real line this quasi-invariance property also holds under adapted transformations, cf. e.g. [4,11]. The quasi-invariance of Poisson measures on the real line with respect to anticipative transformations has been studied in [13] and in the general case of metric spaces in [1]. In the Wiener case, random non-adapted transformations of Brownian motion have been considered by several authors using the Malliavin calculus, cf. [23] and references therein.

On the other hand, the *invariance* property of the law of stochastic processes has important applications, for example to the construction of identically distributed samples of antithetic random variables that can be used for variance reduction in the Monte Carlo method, cf. e.g. Section 4.5 of [3]. Invariance results for the Wiener measure under quasi-nilpotent random isometries have been obtained in [21,22], by means of the Malliavin calculus, based on the duality between gradient and divergence operators on the Wiener space. In comparison with invariance results, quasi-invariance in the anticipative case usually requires more smoothness on the considered transformation. Somehow surprisingly, the invariance of Poisson measures under non-adapted transformations does not seem to have been the object of many studies to date.

[1] Supported by the Grant GRF 102309 from the Research Grants Council of the Hong Kong Special Administrative Region of the People's Republic of China.

The classical invariance theorem for Poisson measures states that given a *deterministic* transformation $\tau : X \to Y$ between measure spaces (X, σ) and (Y, μ) sending σ to μ, the corresponding transformation on point processes maps the Poisson distribution π_σ with intensity $\sigma(\mathrm{d}x)$ on X to the Poisson distribution π_μ with intensity $\mu(\mathrm{d}y)$ on Y. As a simple deterministic example in the case of Poisson jumps times $(T_k)_{k\geq 1}$ on the half line $X = Y = \mathbb{R}_+$ with $\sigma(\mathrm{d}x) = \mu(\mathrm{d}x) = \mathrm{d}x/x$, the homothetic transformation $\tau(x) = rx$ leaves π_σ invariant for all fixed $r > 0$. However, the random transformation of the Poisson process jump times according to the mapping $\tau(x) = x/T_1$ does not yield a Poisson process since the first jump time of the transformed point process is constantly equal to 1.

In this paper we obtain sufficient conditions for the invariance of *random* transformations $\tau : \Omega^X \times X \to Y$ of Poisson random measures on metric spaces X, Y. Here the almost sure isometry condition on $\mathbb{R}^d$ assumed in the Gaussian case will be replaced by a pointwise condition on the preservation of intensity measures, and the quasi-nilpotence hypothesis will be replaced by a cyclic condition on the finite difference gradient of the transformation, cf. Relation (3.7) below. In particular, this condition is satisfied by predictable transformations of Poisson measures, as noted in Example 1 of Section 4.

In the case of the Wiener space $W = \mathcal{C}_0(\mathbb{R}_+; \mathbb{R}^d)$ one considers almost surely defined random isometries

$$R(\omega) : L^2\big(\mathbb{R}_+; \mathbb{R}^d\big) \to L^2\big(\mathbb{R}_+; \mathbb{R}^d\big), \quad \omega \in W,$$

given by $R(\omega)h(t) = U(\omega, t)h(t)$ where $U(\omega, t) : \mathbb{R}^d \to \mathbb{R}^d$, $t \in \mathbb{R}_+$, is a random process of isometries of $\mathbb{R}^d$. The Gaussian character of the measure transformation induced by R is then given by checking for the Gaussianity of the (anticipative) Wiener–Skorohod integral $\delta(Rh)$ of Rh, for all $h \in L^2(\mathbb{R}_+; \mathbb{R}^d)$. In the Poisson case we consider random isometries

$$R(\omega) : L^2_\mu(Y) \to L^2_\sigma(X)$$

given by $R(\omega)h(x) = h(\tau(\omega, x))$ where $\tau(\omega, \cdot) : (X, \sigma) \to (Y, \mu)$ is a random transformation that maps $\sigma(\mathrm{d}x)$ to $\mu(\mathrm{d}y)$ for all $\omega \in \Omega^X$. Here, the Poisson character of the measure transformation induced by R is obtained by showing that the Poisson–Skorohod integral $\delta_\sigma(Rh)$ of Rh has same distribution under π_σ as the compensated Poisson stochastic integral $\delta_\mu(h)$ of h under π_μ, for all $h \in \mathcal{C}_c(Y)$.

For this we will use the Malliavin calculus under Poisson measures, which relies on a finite difference gradient D and a divergence operator δ that extends the Poisson stochastic integral. Our results and proofs are to some extent inspired by the treatment of the Wiener case in [22], see [15] for a recent simplified proof on the Wiener space. However, the use of *finite difference* operators instead of *derivation* operators as in the continuous case makes the proofs and arguments more complex from an algebraic point of view.

As in the Wiener case, we will characterize probability measures via their moments. Recall that the moment $E_\lambda[Z^n]$ of order n of a Poisson random variable Z with intensity λ can be written as

$$E_\lambda\big[Z^n\big] = T_n(\lambda),$$

where $T_n(\lambda)$ is the Touchard polynomial of order n, defined by $T_0(\lambda) = 1$ and the recurrence relation

$$T_{n+1}(\lambda) = \lambda \sum_{k=0}^{n} \binom{n}{k} T_k(\lambda), \quad n \geq 0, \tag{1.1}$$

also called the exponential polynomials, cf. e.g. Section 11.7 of [6], Replacing the Touchard polynomial $T_n(\lambda)$ by its centered version $\tilde{T}_n(\lambda)$ defined by $\tilde{T}_0(\lambda) = 1$ and

$$\tilde{T}_{n+1}(\lambda) = \lambda \sum_{k=0}^{n-1} \binom{n}{k} \tilde{T}_k(\lambda), \quad n \geq 0, \tag{1.2}$$

yields the moments of the *centered* Poisson random variable with intensity $\lambda > 0$ as

$$\tilde{T}_n(\lambda) = E_\lambda\big[(Z - \lambda)^n\big], \quad n \geq 0.$$

Our characterization of Poisson measures will use recurrence relations similar to (1.2), cf. (2.12) below, and identities for the moments of compensated Poisson stochastic integrals which are another motivation for this paper, cf. Theorem 5.1 below.

The paper is organized as follows. The main results (Corollary 3.2 and Theorem 3.3) on the invariance of Poisson measures are stated in Section 3 after recalling the definition of the finite difference gradient D and the Skorohod integral operator δ under Poisson measures in Section 2. Section 4 contains examples of transformations satisfying the required conditions which include the classical adapted case and transformations acting inside the convex hull generated by Poisson random measures, given the positions of the extremal vertices. Section 5 contains the moment identities for Poisson stochastic integrals of all orders that are used in this paper, cf. Theorem 5.1. In Section 6 we prove the main results of Section 3 based on the lemmas on moment identities established in Section 5. In the Appendix we prove some combinatorial results that are needed in the proofs. Some of the results of this paper have been presented in [14].

2. Poisson measures and finite difference operators

In this section we recall the construction of Poisson measures, finite difference operators and Poisson–Skorohod integrals, cf. e.g. [9] and [16], Chapter 6, for reviews. We also introduce some other operators that will be needed in the sequel, cf. Definition 2.5 below.

Let X be a σ-compact metric space with Borel σ-algebra $\mathcal{B}(X)$ and a σ-finite diffuse measure σ. Let Ω^X denote the configuration space on X, i.e. the space of at most countable and locally finite subsets of X, defined as

$$\Omega^X = \big\{\omega = (x_i)_{i=1}^N \subset X, x_i \neq x_j \ \forall i \neq j, N \in \mathbb{N} \cup \{\infty\}\big\}.$$

Each element ω of Ω^X is identified with the Radon point measure

$$\omega = \sum_{i=1}^{\omega(X)} \epsilon_{x_i},$$

where ϵ_x denotes the Dirac measure at $x \in X$ and $\omega(X) \in \mathbb{N} \cup \{\infty\}$ denotes the cardinality of ω. The Poisson random measure $N(\omega, \mathrm{d}x)$ is defined by

$$N(\omega, \mathrm{d}x) = \omega(\mathrm{d}x) = \sum_{k=1}^{\omega(X)} \epsilon_{x_k}(\mathrm{d}x), \qquad \omega \in \Omega^X. \tag{2.1}$$

The Poisson probability measure π_σ on X can be characterized as the only probability measure on Ω^X under which for all compact disjoint subsets $A_1, \ldots, A_n$ of X, $n \geq 1$, the mapping

$$\omega \mapsto \big(\omega(A_1), \ldots, \omega(A_n)\big)$$

is a vector of independent Poisson distributed random variables on $\mathbb{N}$ with respective intensities $\sigma(A_1), \ldots, \sigma(A_n)$.

The Poisson measure π_σ is also characterized by its Fourier transform

$$\psi_\sigma(f) = E_\sigma\bigg[\exp\bigg(\mathrm{i}\int_X f(x)\big(\omega(\mathrm{d}x) - \sigma(\mathrm{d}x)\big)\bigg)\bigg], \qquad f \in L^2_\sigma(X),$$

where E_σ denotes expectation under π_σ, which satisfies

$$\psi_\sigma(f) = \exp\bigg(\int_X \big(\mathrm{e}^{\mathrm{i}f(x)} - \mathrm{i}f(x) - 1\big)\sigma(\mathrm{d}x)\bigg), \qquad f \in L^2_\sigma(X), \tag{2.2}$$

where the compensated Poisson stochastic integral $\int_X f(x)(\omega(\mathrm{d}x) - \sigma(\mathrm{d}x))$ is defined by the isometry

$$E_\sigma\bigg[\bigg(\int_X f(x)\big(\omega(\mathrm{d}x) - \sigma(\mathrm{d}x)\big)\bigg)^2\bigg] = \int_X \big|f(x)\big|^2 \sigma(\mathrm{d}x), \qquad f \in L^2_\sigma(X). \tag{2.3}$$

We refer to [8,10,12], for the following definition.

Definition 2.1. *Let D denote the finite difference gradient defined as*

$$D_x F(\omega) = \varepsilon_x^+ F(\omega) - F(\omega), \quad \omega \in \Omega^X, x \in X, \tag{2.4}$$

for any random variable $F : \Omega^X \to \mathbb{R}$, where

$$\varepsilon_x^+ F(\omega) = F\big(\omega \cup \{x\}\big), \quad \omega \in \Omega^X, x \in X.$$

The operator D is continuous on the space $\mathbb{D}_{2,1}$ defined by the norm

$$\|F\|_{2,1}^2 = \|F\|_{L^2(\Omega^X, \pi_\sigma)}^2 + \|DF\|_{L^2(\Omega^X \times X, \pi_\sigma \otimes \sigma)}^2, \quad F \in \mathbb{D}_{2,1}.$$

We refer to Corollary 1 of [12] for the following definition.

Definition 2.2. *The Skorohod integral operator δ_σ is defined on any measurable process $u : \Omega^X \times X \to \mathbb{R}$ by the expression*

$$\delta_\sigma(u) = \int_X u_t\big(\omega \setminus \{t\}\big)\big(\omega(\mathrm{d}t) - \sigma(\mathrm{d}t)\big), \tag{2.5}$$

provided $E_\sigma[\int_X |u(\omega, t)| \sigma(\mathrm{d}t)] < \infty$.

Relation (2.5) between δ_σ and the Poisson stochastic integral will be used to characterize the distribution of the perturbed configuration points. Note that if $D_t u_t = 0$, $t \in X$, and in particular when applying (2.5) to $u \in L^1_\sigma(X)$ a deterministic function, we have

$$\delta_\sigma(u) = \int_X u(t)\big(\omega(\mathrm{d}t) - \sigma(\mathrm{d}t)\big) \tag{2.6}$$

i.e. $\delta_\sigma(u)$ with the compensated Poisson–Stieltjes integral of u. In addition if $X = \mathbb{R}_+$ and $\sigma(\mathrm{d}t) = \lambda_t \,\mathrm{d}t$ we have

$$\delta_\sigma(u) = \int_0^\infty u_t(\mathrm{d}N_t - \lambda_t \,\mathrm{d}t) \tag{2.7}$$

for all square-integrable predictable processes $(u_t)_{t \in \mathbb{R}_+}$, where $N_t = \omega([0, t])$, $t \in \mathbb{R}_+$, is a Poisson process with intensity $\lambda_t > 0$, cf. e.g. the Example, p. 518, of [12].

The next proposition can be obtained from Corollaries 1 and 5 in [12].

Proposition 2.3. *The operators D and δ_σ are closable and satisfy the duality relation*

$$E_\sigma\big[\langle DF, u\rangle_{L^2_\sigma(X)}\big] = E_\sigma\big[F \delta_\sigma(u)\big] \tag{2.8}$$

on their L^2 domains $\mathrm{Dom}(\delta_\sigma) \subset L^2(\Omega^X \times X, \pi_\sigma \otimes \sigma)$ and $\mathrm{Dom}(D) = \mathbb{D}_{2,1} \subset L^2(\Omega^X, \pi_\sigma)$ under the Poisson measure π_σ with intensity σ.

The operator δ_σ is continuous on the space $\mathbb{L}_{2,1} \subset \mathrm{Dom}(\delta_\sigma)$ defined by the norm

$$\|u\|_{2,1}^2 = E_\sigma\left[\int_X |u_t|^2 \sigma(\mathrm{d}t)\right] + E_\sigma\left[\int_X |D_s u_t|^2 \sigma(\mathrm{d}s)\sigma(\mathrm{d}t)\right],$$

and for any $u \in \mathbb{L}_{2,1}$ we have the Skorohod isometry

$$E_\sigma\big[\delta_\sigma(u)^2\big] = E_\sigma\big[\|u\|_{L^2_\sigma(X)}^2\big] + E_\sigma\left[\int_X \int_X D_s u_t D_t u_s \sigma(\mathrm{d}s)\sigma(\mathrm{d}t)\right], \tag{2.9}$$

cf. Corollary 4 and pp. 517–518 of [12].

In addition, from (2.5) we have the commutation relation

$$\varepsilon_t^+ \delta_\sigma(u) = \delta_\sigma\big(\varepsilon_t^+ u\big) + u_t, \quad t \in X, \tag{2.10}$$

provided $D_t u \in \mathbb{L}_{2,1}$, $t \in X$.

The moments identities for Poisson stochastic integrals proved in this paper rely on the decomposition stated in the following lemma.

Lemma 2.4. *Let $u \in \mathbb{L}_{2,1}$ be such that $\delta_\sigma(u)^n \in \mathbb{D}_{2,1}$, $D_t u \in \mathbb{L}_{2,1}$, $\sigma(\mathrm{d}t)$-a.e., and*

$$E_\sigma\left[\int_X |u_t|^{n-k+1} \big|\delta_\sigma\big(\varepsilon_t^+ u\big)\big|^k \sigma(\mathrm{d}t)\right] < \infty, \qquad E_\sigma\left[\big|\delta_\sigma(u)\big|^k \int_X |u_t|^{n-k+1} \sigma(\mathrm{d}t)\right] < \infty,$$

$0 \le k \le n$. Then we have

$$\begin{aligned} E_\sigma\big[\delta_\sigma(u)^{n+1}\big] = {} & \sum_{k=0}^{n-1} \binom{n}{k} E_\sigma\left[\delta_\sigma(u)^k \int_X u_t^{n-k+1} \sigma(\mathrm{d}t)\right] \\ & + \sum_{k=1}^{n} \binom{n}{k} E_\sigma\left[\int_X u_t^{n-k+1} \big(\delta_\sigma\big(\varepsilon_t^+ u\big)^k - \delta_\sigma(u)^k\big) \sigma(\mathrm{d}t)\right] \end{aligned}$$

for all $n \ge 1$.

Proof. We have, applying (2.10) to $F = \delta_\sigma(u)^n$,

$$\begin{aligned} E_\sigma\big[\delta_\sigma(u)^{n+1}\big] &= E_\sigma\left[\int_X u_t D_t \delta_\sigma(u)^n \sigma(\mathrm{d}t)\right] \\ &= E_\sigma\left[\int_X u_t \big(\big(\varepsilon_t^+ \delta_\sigma(u)\big)^n - \delta_\sigma(u)^n\big) \sigma(\mathrm{d}t)\right] \\ &= E_\sigma\left[\int_X u_t \big(\big(u_t + \delta_\sigma\big(\varepsilon_t^+ u\big)\big)^n - \delta_\sigma(u)^n\big) \sigma(\mathrm{d}t)\right] \\ &= \sum_{k=0}^{n-1} \binom{n}{k} E_\sigma\left[\int_X u_t^{n-k+1} \delta_\sigma\big(\varepsilon_t^+ u\big)^k \sigma(\mathrm{d}t)\right] \\ &= \sum_{k=0}^{n-1} \binom{n}{k} E_\sigma\left[\int_X u_t^{n-k+1} \delta_\sigma(u)^k \sigma(\mathrm{d}t)\right] \\ &\quad + \sum_{k=1}^{n} \binom{n}{k} E_\sigma\left[\int_X u_t^{n-k+1} \big(\delta_\sigma\big(\varepsilon_t^+ u\big)^k - \delta_\sigma(u)^k\big) \sigma(\mathrm{d}t)\right]. \end{aligned}$$

□

From Relation (2.6) and Lemma 2.4 we find that the moments of the compensated Poisson stochastic integral $\int_X f(t)(\omega(\mathrm{d}t) - \sigma(\mathrm{d}t))$ of $f \in \bigcap_{p=1}^{N+1} L_\sigma^p(X)$ satisfy the recurrence identity

$$\begin{aligned} & E_\sigma\left[\left(\int_X f(t)\big(\omega(\mathrm{d}t) - \sigma(\mathrm{d}t)\big)\right)^{n+1}\right] \\ & = \sum_{k=0}^{n-1} \binom{n}{k} \int_X f^{n-k+1}(t) \sigma(\mathrm{d}t) E_\sigma\left[\left(\int_X f(t)\big(\omega(\mathrm{d}t) - \sigma(\mathrm{d}t)\big)\right)^k\right], \end{aligned} \tag{2.11}$$

$n = 0, \dots, N$, which is analog to Relation (1.2) for the centered Touchard polynomials and coincides with (2.3) for $n = 1$.

The Skorohod isometry (2.9) shows that δ_σ is continuous on $\mathbb{L}_{2,1}$, and that its moment of order two of $\delta_\sigma(u)$ satisfies

$$E_\sigma\big[\delta_\sigma(u)^2\big] = E_\sigma\big[\|u\|^2_{L^2_\sigma(X)}\big],$$

provided

$$\int_X \int_X D_s u_t D_t u_s \sigma(\mathrm{d}s)\sigma(\mathrm{d}t) = 0$$

as in the Wiener case [22]. This condition is satisfied when

$$D_t u_s D_s u_t = 0, \quad s, t \in X,$$

i.e. u is adapted in the sense of e.g. [18], Definition 4, or predictable when $X = \mathbb{R}_+$.

The computation of moments of higher orders turns out to be more technical, cf. Theorem 5.1 below, and will be used to characterize the Poisson distribution. From (2.11), in order for $\delta_\sigma(u) \in L^{n+1}_\sigma(\Omega^X)$ to have the same moments as the compensated Poisson integral of $f \in \bigcap_{p=2}^{n+1} L^p_\sigma(X)$, it should satisfy the recurrence relation

$$E_\sigma\big[\delta_\sigma(u)^{n+1}\big] = \sum_{k=0}^{n-1} \binom{n}{k} \int_X f^{n-k+1}(t)\sigma(\mathrm{d}t) E_\sigma\big[\delta_\sigma(u)^k\big], \tag{2.12}$$

$n \geq 0$, which is an extension of Relation (2.11) to the moments of compensated Poisson stochastic integrals, and characterizes their distribution by Carleman's condition [5] when $\sup_{p \geq 1} \|f\|_{L^p_\sigma(Y)} < \infty$.

In order to simplify the presentation of moment identities for the Skorohod integral δ_σ it will be convenient to use the following symbolic notation in the sequel.

Definition 2.5. *For any measurable process $u : \Omega^X \times X \to \mathbb{R}$, let*

$$\Delta_{s_0} \cdots \Delta_{s_j} \prod_{p=0}^{n} u_{s_p} = \sum_{\substack{\Theta_0 \cup \dots \cup \Theta_n = \{s_0, s_1, \dots, s_j\} \\ s_0 \notin \Theta_0, \dots, s_j \notin \Theta_j}} D_{\Theta_0} u_{s_0} \cdots D_{\Theta_n} u_{s_n}, \tag{2.13}$$

$s_0, \dots, s_n \in X$, $0 \leq j \leq n$, where $D_\Theta := \prod_{s_j \in \Theta} D_{s_j}$ when $\Theta \subset \{s_0, s_1, \dots, s_j\}$.

Note that the sum in (2.13) includes empty sets. For example we have

$$\Delta_{s_0} \prod_{p=0}^{n} u_{s_p} = u_{s_0} \sum_{\substack{\Theta_1 \cup \dots \cup \Theta_n = \{s_0\} \\ s_0 \notin \Theta_0, \dots, s_j \notin \Theta_j}} D_{\Theta_1} u_{s_1} \cdots D_{\Theta_n} u_{s_n} = u_{s_0} D_{s_0} \prod_{p=1}^{n} u_{s_p},$$

and $\Delta_{s_0} u_{s_0} = 0$. The use of this notation allows us to rewrite the Skorohod isometry (2.9) as

$$E_\sigma\big[\delta_\sigma(u)^2\big] = E_\sigma\left[\int_X u_s^2 \sigma(\mathrm{d}s)\right] + E_\sigma\left[\int_X \int_X \Delta_s \Delta_t (u_s u_t) \sigma(\mathrm{d}s)\sigma(\mathrm{d}t)\right],$$

since by definition we have

$$\Delta_s \Delta_t (u_s u_t) = D_s u_t D_t u_s, \quad s, t \in X.$$

As a consequence of Theorem 5.1 and Relation (6.1) of Proposition 6.1 below, the third moment of $\delta_\sigma(u)$ is given by

$$\begin{aligned} E_\sigma\left[\delta_\sigma(u)^3\right] = {} & E_\sigma\left[\int_X u_s^3 \sigma(\mathrm{d}s)\right] + 3E_\sigma\left[\delta(u)\int_X u_s^2\sigma(\mathrm{d}s)\right] \\ & + 3E_\sigma\left[\int_{X^3} \Delta_{s_1}\Delta_{s_2}\left(u_{s_1}u_{s_2}^2\right)\sigma(\mathrm{d}s_1)\sigma(\mathrm{d}s_2)\right] \\ & + E_\sigma\left[\int_{X^3} \Delta_{s_1}\Delta_{s_2}\Delta_{s_3}(u_{s_1}u_{s_2}u_{s_3})\sigma(\mathrm{d}s_1)\sigma(\mathrm{d}s_2)\sigma(\mathrm{d}s_3)\right], \end{aligned} \tag{2.14}$$

cf. (5.4) and (6.2) below, which reduces to

$$E_\sigma\left[\delta_\sigma(u)^3\right] = E_\sigma\left[\int_X u_s^3 \sigma(\mathrm{d}s)\right] + 3E_\sigma\left[\delta(u)\int_X u_s^2\sigma(\mathrm{d}s)\right]$$

when u satisfies the cyclic conditions

$$D_{t_1}u_{t_2}D_{t_2}u_{t_1} = 0 \quad \text{and} \quad D_{t_1}u_{t_2}D_{t_2}u_{t_3}D_{t_3}u_{t_1} = 0, \quad t_1,\dots,t_3 \in X,$$

of Lemma A.2 in the Appendix, which shows that (2.13) vanishes, see also (6.4) below for moments of higher orders. When $X = \mathbb{R}_+$, (A.2) is satisfied in particular when u is predictable with respect to the standard Poisson process filtration.

3. Main results

The main results of this paper are stated in this section under the form of Corollary 3.2 and Theorem 3.3.

Let (Y,μ) denote another measure space with associated configuration space Ω^Y and σ-finite diffuse intensity measure $\mu(\mathrm{d}y)$. Given an everywhere defined measurable random mapping

$$\tau : \Omega^X \times X \to Y, \tag{3.1}$$

indexed by X, let $\tau_*(\omega)$, $\omega \in \Omega^X$, denote the image measure of ω by τ, i.e.

$$\tau_* : \Omega^X \to \Omega^Y \tag{3.2}$$

maps

$$\omega = \sum_{i=1}^{\omega(X)} \epsilon_{x_i} \in \Omega^X \quad \text{to} \quad \tau_*(\omega) = \sum_{i=1}^{\omega(X)} \epsilon_{\tau(\omega,x_i)} \in \Omega^Y.$$

In other terms, the random mapping $\tau_* : \Omega^X \to \Omega^Y$ shifts each configuration point $x \in \omega$ according to $x \mapsto \tau(\omega,x)$, and in the sequel we will be interested in finding conditions for $\tau_* : \Omega^X \to \Omega^Y$ to map π_σ to π_μ. This question is well known to have an affirmative answer when the transformation $\tau : X \to Y$ is deterministic and maps σ to μ, as can be checked from the Lévy–Khintchine representation (2.2) of the characteristic function of π_σ. In the random case we will use the moment identity of the next Proposition 3.1, which is a direct application of Proposition 6.2 below with $u = Rh$. We apply the convention that $\sum_{i=1}^{0} l_i = 0$, so that $\{l_0, l_1 \geq 0 \colon \sum_{i=1}^{0} l_i = 0\}$ is an arbitrary singleton.

Proposition 3.1. *Let $N \geq 0$ and let $R(\omega) : L_\mu^p(Y) \to L_\sigma^p(X)$, $\omega \in \Omega^X$, be a random isometry for all $p = 1,\dots,N+1$. Then for all $h \in \bigcap_{p=1}^{N+1} L_\mu^p(Y)$ such that $Rh \in \mathbb{L}_{2,1}$ is bounded and*

$$E_\sigma\left[\int_{X^{a+1}} \left|\Delta_{s_0}\cdots\Delta_{s_a}\left(\prod_{p=0}^{a}\left(Rh(s_p)\right)^{l_p}\right)\right|\sigma(\mathrm{d}s_0)\cdots\sigma(\mathrm{d}s_a)\right] < \infty, \tag{3.3}$$

$l_0+\cdots+l_a \le N+1$, $l_0,\dots,l_a \ge 1$, $a \ge 0$, *we have* $\delta_\sigma(Rh) \in L^{n+1}(\Omega^X,\pi_\sigma)$ *and*

$$\begin{aligned} E_\sigma\big[\delta_\sigma(Rh)^{n+1}\big] &= \sum_{k=0}^{n-1}\binom{n}{k}\int_Y h^{n-k+1}(y)\mu(\mathrm{d}y)E_\sigma\big[\delta_\sigma(Rh)^k\big] \\ &\quad + \sum_{a=0}^{n}\sum_{j=0}^{a}\sum_{b=a}^{n}\sum_{\substack{l_0+\cdots+l_a=n-b \\ l_0,\dots,l_a\ge 0 \\ l_{a+1},\dots,l_b=0}}\binom{a}{j}C^{l_0,n}_{\mathfrak{L}_a,b} \\ &\quad \times\left(\prod_{q=j+1}^{b}\int_Y h^{1+l_q}(y)\mu(\mathrm{d}y)\right)E_\sigma\left[\int_{X^{j+1}}\Delta_{s_0}\cdots\Delta_{s_j}\left(\prod_{p=0}^{j}\big(Rh(s_p)\big)^{1+l_p}\right)\mathrm{d}\sigma^{j+1}(\mathfrak{s}_j)\right], \end{aligned}$$

$n=0,\dots,N$, *where* $\mathrm{d}\sigma^{j+1}(\mathfrak{s}_j)=\sigma(\mathrm{d}s_0)\cdots\sigma(\mathrm{d}s_j)$, $\mathfrak{L}_a=(l_1,\dots,l_a)$, *and*

$$C^{l_0,n}_{\mathfrak{L}_a,a+c} = (-1)^c\binom{n}{l_0}\sum_{0=r_{c+1}<\cdots<r_0=a+c+1}\prod_{q=0}^{c}\prod_{p=r_{q+1}+1-(c-q)}^{r_q-1-(c-q)}\binom{l_1+\cdots+l_p+p+q-1}{l_1+\cdots+l_{p-1}+p+q-1}. \tag{3.4}$$

As a consequence of Proposition 3.1, if in addition $R(\omega): L^p_\mu(Y)\to L^p_\sigma(X)$ satisfies the condition

$$\int_{X^{j+1}}\Delta_{t_0}\cdots\Delta_{t_j}\left(\prod_{p=0}^{j}\big(Rh(\omega,t_p)\big)^{l_p}\right)\sigma(\mathrm{d}t_0)\cdots\sigma(\mathrm{d}t_j)=0, \tag{3.5}$$

$\pi_\sigma(\omega)$-a.s. for all $l_0+\cdots+l_j\le N+1$, $l_0\ge 1,\dots,l_j\ge 1$, $j=1,\dots,N$, then we have

$$E_\sigma\big[\delta_\sigma(Rh)^{n+1}\big] = \sum_{k=0}^{n-1}\binom{n}{k}\int_Y h^{n-k+1}(y)\mu(\mathrm{d}y)E_\sigma\big[\delta_\sigma(Rh)^k\big], \tag{3.6}$$

$n=0,\dots,N$, i.e. the moments of $\delta_\sigma(Rh)$ satisfy the extended recurrence relation (2.11) of the Touchard type.

Hence Proposition 3.1 and Lemma A.2 yield the next corollary in which the sufficient condition (3.7) is a strengthened version of the Wiener space condition $\mathrm{trace}(DRh)^n=0$ of Theorem 2.1 in [22].

Corollary 3.2. *Let* $R: L^p_\mu(Y)\to L^p_\sigma(X)$ *be a random isometry for all* $p\in[1,\infty]$. *Assume that* $h\in\bigcap_{p=1}^\infty L^p_\mu(Y)$ *is such that* $\sup_{p\ge 1}\|h\|_{L^p_\mu(Y)}<\infty$, *and that* Rh *satisfies* (3.3) *and the cyclic condition*

$$D_{t_1}Rh(t_2)\cdots D_{t_k}Rh(t_1)=0, \quad t_1,\dots,t_k\in X, \tag{3.7}$$

$\pi_\sigma\otimes\sigma^{\otimes k}$-*a.e. for all* $k\ge 2$. *Then, under* π_σ, $\delta_\sigma(Rh)$ *has same distribution as the compensated Poisson integral* $\delta_\mu(h)$ *of* h *under* π_μ.

Proof. Lemma A.2 below shows that Condition (3.5) holds under (3.7) since

$$D_s\big(Rh(t)\big)^l=\varepsilon_s^+\big(Rh(t)\big)^l-\big(Rh(t)\big)^l=\sum_{k=1}^{l}\binom{l}{k}\big(Rh(t)\big)^{l-k}\big(D_s\big(Rh(t)\big)\big)^k=0,$$

$s,t\in X$, $l\ge 1$, hence by Proposition 3.1, Relation (3.6) holds for all $n\ge 1$, and this shows by induction from (2.12) that under π_σ, $\delta_\sigma(Rh)$ has same moments as $\delta_\mu(h)$ under π_μ. In addition, since $\sup_{p\ge 1}\|h\|_{L^p_\mu(Y)}<\infty$, Relation (3.6) also shows by induction that the moments of $\delta_\sigma(Rh)$ satisfy the bound $E_\sigma[|\delta_\sigma(Rh)|^n] \le (Cn)^n$ for some $C > 0$ and all $n \ge 1$, hence they characterize its distribution by the Carleman condition

$$\sum_{k=1}^{\infty} \big(E_\sigma\big[\delta_\sigma(Rh)^{2n}\big]\big)^{-1/(2n)} = +\infty,$$

cf. [5] and p. 59 of [19]. $\square$

We will apply Corollary 3.2 to the random isometry $R : L^p_\mu(Y) \to L^p_\sigma(X)$ is given as

$$Rh = h \circ \tau, \quad h \in L^p_\mu(Y),$$

where $\tau : \Omega^X \times X \to Y$ is the random transformation (3.1) of configuration points considered at the beginning of this section. As a consequence we obtain the following invariance result for Poisson measures when $(X, \sigma) = (Y, \mu)$.

Theorem 3.3. *Let $\tau : \Omega^X \times X \to Y$ be a random transformation such that $\tau(\omega, \cdot) : X \to Y$ maps σ to μ for all $\omega \in \Omega^X$, i.e.*

$$\tau_*(\omega, \cdot)\sigma = \mu, \quad \omega \in \Omega^X,$$

and satisfying the cyclic condition

$$D_{t_1}\tau(\omega, t_2) \cdots D_{t_k}\tau(\omega, t_1) = 0 \quad \forall \omega \in \Omega^X, \forall t_1, \dots, t_k \in X, \tag{3.8}$$

for all $k \ge 1$. Then $\tau_* : \Omega^X \to \Omega^Y$ maps π_σ to π_μ, i.e.

$$\tau_*\pi_\sigma = \pi_\mu$$

is the Poisson measure with intensity $\mu(\mathrm{d}y)$ on Y.

Proof. We first show that, under π_σ, $\delta_\sigma(h \circ \tau)$ has same distribution as the compensated Poisson integral $\delta_\mu(h)$ of h under π_μ, for all $h \in \mathcal{C}_c(Y)$.

Let $(K_r)_{r \ge 1}$ denote an increasing family of compact subsets of X such that $X = \bigcup_{r \ge 1} K_r$, and let $\tau_r : \Omega^X \times X \to Y$ be defined for $r \ge 1$ by

$$\tau_r(\omega, x) = \tau(\omega \cap K_r, x), \quad x \in X, \omega \in \Omega^X.$$

Letting $R_r h = h \circ \tau_r$ defines a random isometry $R_r : L^p_\mu(Y) \to L^p_\sigma(X)$ for all $p \ge 1$, which satisfies the assumptions of Corollary 3.2. Indeed we have

$$\begin{aligned} D_s R_r h(t) &= D_s h\big(\tau_r(\omega, t)\big) \\ &= \mathbf{1}_{K_r}(s)\big(h\big(\tau_r(\omega, t) + D_s\tau_r(\omega, t)\big) - h\big(\tau_r(\omega, t)\big)\big) \\ &= \mathbf{1}_{K_r}(s)\big(h\big(\tau_r\big(\omega \cup \{s\}, t\big)\big) - h\big(\tau_r(\omega, t)\big)\big), \quad s, t \in X, \end{aligned}$$

hence (3.8) implies that Condition (3.7) holds, and Corollary 3.2 shows that we have

$$E_\sigma\big[\mathrm{e}^{\mathrm{i}\lambda\delta_\mu(h\circ\tau_r)}\big] = E_\mu\big[\mathrm{e}^{\mathrm{i}\lambda\delta_\mu(h)}\big] \tag{3.9}$$

for all $\lambda \in \mathbb{R}$. Next we note that Condition (3.8) implies that

$$D_t\tau_r(\omega, t) = 0 \quad \forall \omega \in \Omega^X, \forall t \in X, \tag{3.10}$$

i.e. $\tau_r(\omega, t)$ does not depend on the presence or absence of a point in ω at t, and in particular,

$$\tau_r(\omega, t) = \tau_r\big(\omega \cup \{t\}, t\big), \quad t \notin \omega,$$

and

$$\tau_r(\omega,t)=\tau_r\big(\omega\setminus\{t\},t\big),\quad t\in\omega.$$

Hence by (2.6) we have

$$\begin{aligned}\delta_\mu(h)\circ\tau_{r*}&=\int_Y h(y)\big(\tau_{r*}\omega(\mathrm{d}y)-\mu(\mathrm{d}y)\big)\\&=\int_X h\big(\tau_r(\omega,x)\big)\big(\omega(\mathrm{d}x)-\sigma(\mathrm{d}x)\big)\\&=\int_X h\big(\tau_r\big(\omega\setminus\{x\},x\big)\big)\big(\omega(\mathrm{d}x)-\sigma(\mathrm{d}x)\big)\\&=\delta_\sigma(h\circ\tau_r),\end{aligned}$$

and by (3.9) we get

$$\begin{aligned}&E_\sigma\left[\exp\left(\mathrm{i}\int_Y h(y)\big(\tau_{r*}\omega(\mathrm{d}y)-\mu(\mathrm{d}y)\big)\right)\right]\\&\quad=E_\sigma\left[\exp\left(\mathrm{i}\int_Y h(y)\big(\omega(\mathrm{d}y)-\mu(\mathrm{d}y)\big)\right)\circ\tau_{r*}\right]\\&\quad=E_\sigma\big[\mathrm{e}^{\mathrm{i}\delta_\mu(h)\circ\tau_{r*}}\big]\\&\quad=E_\mu\big[\mathrm{e}^{\mathrm{i}\delta_\mu(h)}\big]\\&\quad=E_\mu\left[\exp\left(\mathrm{i}\int_Y h(y)\big(\omega(\mathrm{d}y)-\mu(\mathrm{d}y)\big)\right)\right].\end{aligned}$$

Next, letting r go to infinity we get

$$E_\sigma\left[\exp\left(\mathrm{i}\int_Y h(y)\big(\tau_{r*}\omega(\mathrm{d}y)-\mu(\mathrm{d}y)\big)\right)\right]=E_\mu\left[\exp\left(\mathrm{i}\int_Y h(y)\big(\omega(\mathrm{d}y)-\mu(\mathrm{d}y)\big)\right)\right]$$

for all $h\in\mathcal{C}_c(Y)$, hence the conclusion. □

In Theorem 3.3 above the Identity (3.8) is interpreted for $k\geq 2$ by stating that $\omega\in\Omega^X$, and $t_1,\ldots,t_k\in X$, the k-tuples

$$\big(\tau\big(\omega\cup\{t_1\},t_2\big),\tau\big(\omega\cup\{t_2\},t_3\big),\ldots,\tau\big(\omega\cup\{t_{k-1}\},t_k\big),\tau\big(\omega\cup\{t_k\},t_1\big)\big)$$

and

$$\big(\tau(\omega,t_2),\tau(\omega,t_3),\ldots,\tau(\omega,t_k),\tau(\omega,t_1)\big)$$

coincide on at least one component n^o $i\in\{1,\ldots,k\}$ in Y^k, i.e. $D_{t_i}\tau(\omega,t_{i+1\,\mathrm{mod}\,k})=0$.

4. Examples

In this section we consider some examples of transformations satisfying the hypotheses of Section 3, in case $X=Y$ for σ-finite measures σ and μ. Using various binary relations on X we consider successively the adapted case, and transformations that are conditioned by a random set such as the convex hull of a Poisson random measure. Such results are consistent with the fact that given the position of its extremal vertices, a Poisson random measure remains Poisson within its convex hull, cf. the unpublished manuscript [7], see also [25] for a related use of stopping sets.

Example 1. *First, we remark that if X is endowed with a total binary relation $\preceq$ and if $\tau : \Omega^X \times X \to Y$ is (backward) predictable in the sense that*

$$x \preceq y \quad \Longrightarrow \quad D_x \tau(\omega, y) = 0, \tag{4.1}$$

i.e.

$$\tau\big(\omega \cup \{x\}, y\big) = \tau(\omega, y), \quad x \preceq y, \tag{4.2}$$

then the cyclic condition (3.8) is satisfied, i.e. we have

$$D_{x_1} \tau(\omega, x_2) \cdots D_{x_k} \tau(\omega, x_1) = 0, \quad x_1, \ldots, x_k \in X, \omega \in \Omega^X, \tag{4.3}$$

for all $k \geq 1$. Indeed, for all $x_1, \ldots, x_k \in X$ there exists $i \in \{1, \ldots, k\}$ such that $x_i \preceq x_j$, for all $1 \leq j \leq k$, hence $D_{x_i} \tau(\omega, x_j) = 0$, $1 \leq j \leq k$, by the predictability condition (4.1), hence (4.3) holds. Consequenly, $\tau_* : \Omega^X \to \Omega^Y$ maps π_σ to π_μ by Theorem 3.3, provided $\tau(\omega, \cdot) : X \to Y$ maps σ to μ for all $\omega \in \Omega^X$.

Such binary relations on X can be defined via an increasing family $(C_\lambda)_{\lambda \in \mathbb{R}}$ of subsets whose reunion is X and such that for all $x \neq y \in X$ there exists $\lambda_x, \lambda_y \in \mathbb{R}$ with $x \in C_{\lambda_x} \setminus C_{\lambda_y}$ and $y \in C_{\lambda_y}$, or $y \in C_{\lambda_y} \setminus C_{\lambda_x}$ and $x \in C_{\lambda_x}$, which is equivalent to $y \preceq x$ or $x \preceq y$, respectively.

This framework includes the classical adaptedness condition when X has the form $X = \mathbb{R}_+ \times Y$. For example, if X and Y are of the form $X = Y = \mathbb{R}_+ \times Z$, consider the filtration $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$, where $\mathcal{F}_t$ is generated by

$$\big\{\sigma\big([0, s] \times A\big) : 0 \leq s < t, A \in \mathcal{B}_b(Z)\big\},$$

where $\mathcal{B}_c(Z)$ denotes the compact Borel subsets of Z. In this case it is well-known that $\omega \mapsto \tau_* \omega$ is Poisson distributed with intensity μ under π_σ, provided $\tau(\omega, \cdot) : \mathbb{R}_+ \times Z \to \mathbb{R}_+ \times Z$ is predictable in the sense that $\omega \mapsto \tau(s, z)$ is $\mathcal{F}_t$-measurable for all $0 \leq s \leq t$, $z \in Z$, cf. e.g. Theorem 3.10.21 of [2]. Here, Condition (4.1) holds for the partial order

$$(s, x) \preceq (t, y) \quad \Longleftrightarrow \quad s \geq t \tag{4.4}$$

on $Z \times \mathbb{R}_+$ by taking $C_\lambda = [\lambda, \infty) \times X$, $\lambda \in \mathbb{R}_+$, and the cyclic condition (3.8) is satisfied when $\tau(\omega, \cdot) : \mathbb{R}_+ \times Z \to \mathbb{R}_+ \times Z$ is predictable in the sense of (4.1).

Next, we consider other examples in which the binary relation $\preceq$ is configuration dependent. This includes in particular transformations of Poisson measures within their convex hull, given the positions of extremal vertices.

Example 2. *Let $X = \bar{B}(0, 1) \setminus \{0\}$ denote the closed unit ball in $\mathbb{R}^d$. For all $\omega \in \Omega^X$, let $\mathcal{C}(\omega)$ denote the convex hull of ω in $\mathbb{R}^d$ with interior $\dot{\mathcal{C}}(\omega)$, and let $\omega_{\mathrm{e}} = \omega \cap (\mathcal{C}(\omega) \setminus \dot{\mathcal{C}}(\omega))$ denote the extremal vertices of $\mathcal{C}(\omega)$. Consider a measurable mapping $\tau : \Omega^X \times X \to X$ such that for all $\omega \in \Omega^X$, $\tau(\omega, \cdot)$ is measure preserving, maps $\dot{\mathcal{C}}(\omega)$ to $\dot{\mathcal{C}}(\omega)$, and for all $\omega \in \Omega^X$,*

$$\tau(\omega, x) = \begin{cases} \tau(\omega_{\mathrm{e}}, x), & x \in \dot{\mathcal{C}}(\omega), \\ x, & x \in X \setminus \dot{\mathcal{C}}(\omega), \end{cases} \tag{4.5}$$

i.e. the points of $\dot{\mathcal{C}}(\omega)$ are shifted by $\tau(\omega, \cdot)$ depending on the positions of the extremal vertices of the convex hull of ω, which are left invariant by $\tau(\omega, \cdot)$. Figure 1 shows an example of a transformation that modifies only the interior of the convex hull generated by the random measure, in which the number of points is taken to be finite for simplicity of illustration.

Next we prove the invariance of such transformations as a consequence of Theorem 3.3. This invariance property is related to the intuitive fact that given the positions of the extreme vertices, the distribution of the inside points remains Poisson when they are shifted according to the data of the vertices, cf. e.g. [7].

Here we consider the binary relation $\preceq_\omega$ given by

$$x \preceq_\omega y \quad \Longleftrightarrow \quad x \in \mathcal{C}\big(\omega \cup \{y\}\big), \quad \omega \in \Omega^X, x, y \in X.$$

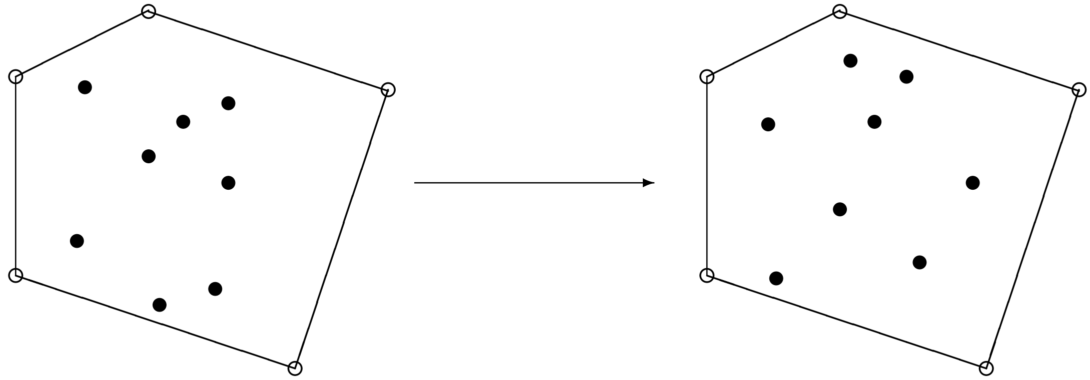

Fig. 1. Example of random transformation.

The relation $\preceq_\omega$ is clearly reflexive, and it is transitive since $x \preceq_\omega y$ and $y \preceq_\omega z$ implies

$$x \in \mathcal{C}\big(\omega \cup \{y\}\big) \subset \mathcal{C}\big(\omega \cup \{z\}\big),$$

hence $x \preceq_\omega z$. Note that $\preceq_\omega$ is also total on $\mathcal{C}(\omega)$ and it is an order relation on $X \setminus \mathcal{C}(\omega)$, since it is also antisymmetric on that set, i.e. if $x, y \notin \mathcal{C}(\omega)$ then

$$x \preceq_\omega y \quad \text{and} \quad y \preceq_\omega x$$

means $x \in \mathcal{C}(\omega \cup \{y\})$ and $y \in \mathcal{C}(\omega \cup \{x\})$, which implies $x = y$. We will need the following lemma.

Lemma 4.1. *For all $x, y \in X$ and $\omega \in \Omega^X$ we have*

$$x \preceq_\omega y \implies D_x \tau(\omega, y) = 0 \tag{4.6}$$

and

$$x \not\preceq_\omega y \implies D_y \tau(\omega, x) = 0. \tag{4.7}$$

Proof. Let $x, y \in X$ and $\omega \in \Omega^X$. First, if $x \not\preceq_\omega y$ then we have $x \notin \mathcal{C}(\omega \cup \{y\})$ hence $\tau(\omega \cup \{y\}, x) = \tau(\omega, x) = x$ by (4.5). Next, if $x \preceq_\omega y$, i.e. $x \in \mathcal{C}(\omega \cup \{y\})$, we can distinguish two cases:

(a) $x \in \mathcal{C}(\omega)$. In this case we have $\mathcal{C}(\omega \cup \{x\}) = \mathcal{C}(\omega)$, hence $\tau(\omega \cup \{x\}, y) = \tau(\omega, y)$ for all $y \in X$.
(b) $x \in \mathcal{C}(\omega \cup \{y\}) \setminus \mathcal{C}(\omega)$. If $y \in \mathcal{C}(\omega \cup \{x\})$ then $x = y \notin \dot{\mathcal{C}}(\omega \cup \{x\})$, hence $\tau(\omega \cup \{x\}, y) = \tau(\omega, y)$. On the other hand if $y \notin \mathcal{C}(\omega \cup \{x\})$ then $y \not\preceq_\omega x$ and $\tau(\omega \cup \{x\}, y) = \tau(\omega, y) = y$ as above.

We conclude that $D_x \tau(\omega, y) = 0$ in both cases. □

Let us now show that $\tau : \Omega^X \times X \to \Omega^X$ satisfies the cyclic condition (3.8). Let $t_1, \ldots, t_k \in X$. First, if $t_i \in \mathcal{C}(\omega)$ for some $i \in \{1, \ldots, k\}$, then for all $j = 1, \ldots, k$ we have $t_i \preceq_\omega t_j$ and by Lemma 4.1 we get

$$D_{t_i} \tau(\omega, t_j) = 0,$$

thus (3.8) holds, and we may assume that $t_i \notin \mathcal{C}(\omega)$ for all $i = 1, \ldots, k$. In this case, if $t_{i+1 \bmod k} \not\preceq_\omega t_i$ for some $i = 1, \ldots, k$, then by Lemma 4.1 we have

$$D_{t_i} \tau(\omega, t_{i+1 \bmod k}) = 0,$$

which shows that (3.8) holds. Finally, if $t_1 \preceq_\omega t_k \preceq_\omega \cdots \preceq_\omega t_2 \preceq_\omega t_1$, then by transitivity of $\preceq_\omega$ we have $t_1 \preceq_\omega t_k \preceq_\omega t_1$, which implies $t_1 = t_k \notin \mathcal{C}(\omega)$ by antisymmetry on $X \setminus \mathcal{C}(\omega)$, hence $D_{t_k} \tau(\omega, t_1) = 0$, and $\tau : \Omega^X \times X \to X$ satisfies the cyclic condition (3.8) for all $k \geq 2$. Hence τ satisfies the hypotheses of Theorem 3.3, and $\tau_* \pi_\sigma = \pi_\mu$ provided $\tau(\omega, \cdot) : X \to Y$ maps σ to μ for all $\omega \in \Omega^X$.

5. Moment identities for stochastic integrals

In this section we prove a moment identity for Poisson stochastic integrals of arbitrary orders in Theorem 5.1, whose application will be to prove Proposition 3.1. More precisely, given $F : \Omega^X \to \mathbb{R}$ a random variable and $u : \Omega^X \times X \to \mathbb{R}$ a measurable process, we aim at decomposing $E_\sigma[\delta_\sigma(u)^n F]$ in terms of the gradient D, while removing all occurrences of δ_σ using the integration by parts formula (2.8).

In Theorem 5.1 and in the rest of this section we will use the notation

$$\varepsilon^+_{\mathfrak{s}_b} = \varepsilon^+_{s_1} \cdots \varepsilon^+_{s_b}, \quad \mathfrak{s}_b = (s_1, \dots, s_b) \in X^b, b \geq 1.$$

Moreover, by saying that $u : \Omega^X \times X \to \mathbb{R}$ has a compact support in X we mean that there exists a compact subset K of X such that $u(\omega, x) = 0$ for all $\omega \in \Omega^X$ and $x \in X \setminus K$.

Theorem 5.1. *Let $F : \Omega^X \to \mathbb{R}$ be a bounded random variable and let $u : \Omega^X \times X \to \mathbb{R}$ be a bounded process with compact support in X. For all $n \geq 0$ we have*

$$E_\sigma\big[\delta_\sigma(u)^n F\big] = \sum_{a=0}^{n} \sum_{b=a}^{n} (-1)^{b-a} \sum_{\substack{l_1+\cdots+l_a=n-b \\ l_1,\dots,l_a \geq 0 \\ l_{a+1},\dots,l_b=0}} C_{\mathfrak{L}_a,b} E_\sigma\left[\int_{X^b} \varepsilon^+_{\mathfrak{s}_a} F \prod_{p=1}^{b} \varepsilon^+_{\mathfrak{s}_a \setminus s_p} u_{s_p}^{1+l_p} \, \mathrm{d}\sigma^b(\mathfrak{s}_b)\right], \tag{5.1}$$

where $\mathrm{d}\sigma^b(\mathfrak{s}_b) = \sigma(\mathrm{d}s_1) \cdots \sigma(\mathrm{d}s_b)$, $\mathfrak{L}_a = (l_1, \dots, l_a)$, *and*

$$C_{\mathfrak{L}_a,a+c} = \sum_{0=r_{c+1}<\cdots<r_0=a+c+1} \prod_{q=0}^{c} \prod_{p=r_{q+1}+q-c+1}^{r_q+q-c-1} \binom{l_1+\cdots+l_p+p+q-1}{l_1+\cdots+l_{p-1}+p+q-1}. \tag{5.2}$$

Before turning to the proof of Theorem 5.1 we consider some examples.

Example 1. *For $n = 2$ and $F = 1$, Theorem 5.1 recovers the Skorohod isometry (2.9) as follows:*

$$\begin{aligned}
E_\sigma\big[\delta_\sigma(u)^2\big] &= E_\sigma\left[\int_{X^2} u_{s_1} u_{s_2} \sigma(\mathrm{d}s_1)\sigma(\mathrm{d}s_2)\right] && [a=0, b=2] \\
&\quad - 2E_\sigma\left[\int_{X^2} u_{s_1}(I + D_{s_1}) u_{s_2} \sigma(\mathrm{d}s_1)\sigma(\mathrm{d}s_2)\right] && [a=1, b=2] \\
&\quad + E_\sigma\left[\int_X |u_{s_1}|^2 \sigma(\mathrm{d}s_1)\right] && [a=1, b=1] \\
&\quad + E_\sigma\left[\int_{X^2} (I + D_{s_1}) u_{s_2} (I + D_{s_2}) u_{s_1} \sigma(\mathrm{d}s_1)\sigma(\mathrm{d}s_2)\right] && [a=2, b=2] \\
&= E_\sigma\left[\int_X |u_s|^2 \sigma(\mathrm{d}s)\right] + E_\sigma\left[\int_{X^2} \Delta_{s_1}\Delta_{s_2}(u_{s_1}u_{s_2}) \sigma(\mathrm{d}s_1)\sigma(\mathrm{d}s_2)\right]. && (5.3)
\end{aligned}$$

Example 2. *For $n = 3$ and $F = 1$, Theorem 5.1 yields the following third moment identity:*

$$\begin{aligned}
E_\sigma\big[\delta_\sigma(u)^3\big] &= E_\sigma\left[\int_X u_{s_1}^3 \sigma(\mathrm{d}s_1)\right] && [a=1, b=1] \\
&\quad - 3E_\sigma\left[\int_{X^2} u_{s_1}^2 (I + D_{s_1}) u_{s_2} \sigma(\mathrm{d}s_1)\sigma(\mathrm{d}s_2)\right] && [a=1, b=2] \\
&\quad + 3E_\sigma\left[\int_{X^2} (I + D_{s_2}) u_{s_1} (I + D_{s_1}) u_{s_2}^2 \sigma(\mathrm{d}s_1)\sigma(\mathrm{d}s_2)\right] && [a=2, b=2]
\end{aligned}$$

$$
\begin{aligned}
&- E_\sigma\left[\int_{X^3} u_{s_1}u_{s_2}u_{s_3}\sigma(\mathrm{d}s_1)\sigma(\mathrm{d}s_2)\sigma(\mathrm{d}s_3)\right] && [a=0,b=3]\\
&+ 3E_\sigma\left[\int_{X^3} u_{s_1}(I+D_{s_1})u_{s_3}(I+D_{s_1})u_{s_2}\sigma(\mathrm{d}s_1)\sigma(\mathrm{d}s_2)\sigma(\mathrm{d}s_3)\right] && [a=1,b=3]\\
&- 3E_\sigma\left[\int_{X^3} (I+D_{s_1})(I+D_{s_2})u_{s_3}(I+D_{s_1})u_{s_2}(I+D_{s_2})u_{s_1}\sigma(\mathrm{d}s_1)\sigma(\mathrm{d}s_2)\sigma(\mathrm{d}s_3)\right] \\
& && [a=2,b=3]\\
&+ E_\sigma\left[\int_{X^3} (I+D_{s_1})(I+D_{s_2})u_{s_3}(I+D_{s_1})(I+D_{s_3})\right.\\
&\left.\times u_{s_2}(I+D_{s_2})(I+D_{s_3})u_{s_1}\sigma(\mathrm{d}s_1)\sigma(\mathrm{d}s_2)\sigma(\mathrm{d}s_3)\right] && [a=3,b=3]\\
= {}& E_\sigma\left[\int_X u_{s_1}^3\sigma(\mathrm{d}s_1)\right] + 3E_\sigma\left[\int_{X^2} u_{s_1}D_{s_1}u_{s_2}^2\sigma(\mathrm{d}s_1)\sigma(\mathrm{d}s_2)\right]\\
&+ 3E_\sigma\left[\int_{X^3} \Delta_{s_1}\Delta_{s_2}\big(u_{s_1}u_{s_2}^2\big)\sigma(\mathrm{d}s_1)\sigma(\mathrm{d}s_2)\right]\\
&+ E_\sigma\left[\int_{X^3} \Delta_{s_1}\Delta_{s_2}\Delta_{s_3}(u_{s_1}u_{s_2}u_{s_3})\sigma(\mathrm{d}s_1)\sigma(\mathrm{d}s_2)\sigma(\mathrm{d}s_3)\right]. && (5.4)
\end{aligned}
$$

Example 3. *Noting that $C_{\mathfrak{L}_a,c}$ defined in (5.2) represents the number of partitions of a set of $l_1+\cdots+l_a+a+c$ elements into a subsets of lengths $1+l_1,\dots,1+l_a$ and c singletons, we find that when $F=1$ and $u=\mathbf{1}_A$ is a deterministic indicator function, and Theorem 5.1 reads*

$$E_\sigma\big[(Z-\lambda)^n\big]=\sum_{a=0}^{n}\lambda^a\sum_{c=0}^{a}(-1)^c\binom{n}{c}S(n-c,a-c)$$

for $Z-\lambda=\delta(\mathbf{1}_A)=\omega(A)-\sigma(A)$ a compensated Poisson random variable with intensity $\lambda=\sigma(A)$, where $S(n,c)$ denotes the Stirling number of the second kind, i.e. the number of ways to partition a set of n objects into c non-empty subsets. This coincides with the moment formula

$$E_\lambda\big[(Z-\lambda)^n\big]=\sum_{a=0}^{n}\lambda^a S_2(n,a),$$

where $S_2(n,a)$ denotes the number of partitions of a set of size n into a non-singleton subsets, which can be obtained from the sequence $(0,\lambda,\lambda,\dots)$ of cumulants of the compensated Poisson distribution, through the combinatorial identity

$$S_2(n,a)=\sum_{c=0}^{a}(-1)^c\binom{n}{c}S(n-c,a-c),\quad 0\le a\le n,$$

which is the binomial dual of

$$S(m,n)=\sum_{k=0}^{n}\binom{m}{k}S_2(m-k,n-k),$$

cf. [17] *for details.*

The proof of Theorem 5.1 will be done by induction based on the following lemma.

Lemma 5.2. *Let $G:\Omega^X\to\mathbb{R}$ be a bounded random variable and let $u:\Omega^X\times X\to\mathbb{R}$ be a bounded process with compact support in X. For all $n\geq 0$ we have*

$$
\begin{aligned}
E_\sigma\big[\delta_\sigma(u)^n G\big] = {}& \sum_{\substack{0=k_d<\cdots<k_0=n\\ 0\leq d\leq n}} c_{\mathfrak{K}_d} E_\sigma\left[\int_{X^d}\varepsilon^+_{\mathfrak{s}_d} G\prod_{p=1}^d \varepsilon^+_{\mathfrak{s}_d\setminus s_p} u_{s_p}^{k_{p-1}-k_p}\,\mathrm{d}\sigma^d(\mathfrak{s}_d)\right]\\
&- \sum_{\substack{0=k_d<\cdots<k_0=n\\ 1\leq d\leq n}} c_{\mathfrak{K}_d} E_\sigma\left[\int_{X^d}\delta_\sigma\big(\varepsilon^+_{\mathfrak{s}_{d-1}}u\big)^{k_{d-1}-1}\varepsilon^+_{\mathfrak{s}_{d-1}}(u_{s_d}G)\prod_{p=1}^{d-1}\varepsilon^+_{\mathfrak{s}_{d-1}\setminus s_p}u_{s_p}^{k_{p-1}-k_p}\,\mathrm{d}\sigma^d(\mathfrak{s}_d)\right],
\end{aligned}
\tag{5.5}
$$

where $c_{\mathfrak{K}_d}=\prod_{p=0}^{d-1}\binom{k_p-1}{k_{p+1}}$, $\mathfrak{K}_d=(k_0,\dots,k_d)\in\mathbb{N}^{d+1}$.

Proof. The formula clearly holds when $n=0$, while when $n\geq 1$, the first summation in (5.5) actually starts from $d=1$. The proof follows by application to $l=n-1$ or $l=n$ of the following identity:

$$
\begin{aligned}
&E_\sigma\big[\delta_\sigma(u)^n G\big]\\
&\quad= \sum_{0=k_{l+1}<\cdots<k_0=n} c_{\mathfrak{K}_{l+1}} E_\sigma\left[\int_{X^{l+1}}\delta_\sigma\big(\varepsilon^+_{\mathfrak{s}_l}u\big)^{k_l-1}\varepsilon^+_{\mathfrak{s}_{l+1}}G\varepsilon^+_{\mathfrak{s}_l}u_{s_{l+1}}\varepsilon^+_{s_{l+1}}\prod_{p=1}^{l}\varepsilon^+_{\mathfrak{s}_{l+1}\setminus s_p}u_{s_p}^{k_{p-1}-k_p}\,\mathrm{d}\sigma^{l+1}(\mathfrak{s}_{l+1})\right]\\
&\qquad+\sum_{d=1}^{l}\sum_{0=k_d<\cdots<k_0=n} c_{\mathfrak{K}_d}E_\sigma\left[\int_{X^d}\varepsilon^+_{\mathfrak{s}_d}G\prod_{p=1}^{d}\varepsilon^+_{\mathfrak{s}_d\setminus s_p}u_{s_p}^{k_{p-1}-k_p}\,\mathrm{d}\sigma^d(\mathfrak{s}_d)\right]\\
&\qquad-\sum_{d=1}^{l+1}\sum_{0=k_d<\cdots<k_0=n} c_{\mathfrak{K}_d}E_\sigma\left[\int_{X^d}\delta_\sigma\big(\varepsilon^+_{\mathfrak{s}_{d-1}}u\big)^{k_{d-1}-1}\varepsilon^+_{\mathfrak{s}_{d-1}}(u_{s_d}G)\prod_{p=1}^{d-1}\varepsilon^+_{\mathfrak{s}_{d-1}\setminus s_p}u_{s_p}^{k_{p-1}-k_p}\,\mathrm{d}\sigma^d(\mathfrak{s}_d)\right]
\end{aligned}
\tag{5.6}
$$

$$
\quad= \mathcal{A}_l+\sum_{d=1}^{l}\mathcal{B}_d-\sum_{d=1}^{l+1}\mathcal{C}_d,
\tag{5.7}
$$

which will be proved by induction on $l=0,\dots,n$. First, note that (5.6) holds for $l=0$ as by (2.4) and (2.8) we have

$$
\begin{aligned}
E_\sigma\big[\delta_\sigma(u)^n G\big] &= E_\sigma\left[\int_X u_{s_1}D_{s_1}\big(\delta_\sigma(u)^{n-1}G\big)\sigma(\mathrm{d}s_1)\right]\\
&= E_\sigma\left[\int_X u_{s_1}\varepsilon^+_{s_1}\delta_\sigma(u)^{n-1}\varepsilon^+_{s_1}G\sigma(\mathrm{d}s_1)\right]-E_\sigma\left[G\int_X u_{s_1}\delta_\sigma(u)^{n-1}\sigma(\mathrm{d}s_1)\right]\\
&= E_\sigma\left[\int_X u_{s_1}\big(u_{s_1}+\delta_\sigma\big(\varepsilon^+_{s_1}u\big)\big)^{n-1}\varepsilon^+_{s_1}G\sigma(\mathrm{d}s_1)\right]-E_\sigma\left[G\int_X u_{s_1}\delta_\sigma(u)^{n-1}\sigma(\mathrm{d}s_1)\right]\\
&= \mathcal{A}_0-\mathcal{C}_1,
\end{aligned}
$$

which also proves the lemma in case $n=1$. Next, when $n\geq 2$, for $l=0,\dots,n-1$, using the duality formula (2.8) and the relations $\varepsilon^+_{s_{l+1}}\delta_\sigma(\varepsilon^+_{\mathfrak{s}_l}u)=\varepsilon^+_{\mathfrak{s}_l}u_{s_{l+1}}+\delta_\sigma(\varepsilon^+_{\mathfrak{s}_{l+1}}u)$, cf. (2.10), and $D_{s_{l+2}}=\varepsilon^+_{s_{l+2}}-I$, we rewrite the first term in (5.6) as

$$
\begin{aligned}
\mathcal{A}_l &= \sum_{0=k_{l+1}<\cdots<k_0=n} c_{\mathfrak{K}_{l+1}} \\
&\quad \times E_\sigma\left[\int_{X^{l+1}} \varepsilon^+_{\mathfrak{s}_{l+1}} G \varepsilon^+_{\mathfrak{s}_l} u_{s_{l+1}} \left(\varepsilon^+_{\mathfrak{s}_l} u_{s_{l+1}} + \delta_\sigma\left(\varepsilon^+_{\mathfrak{s}_{l+1}} u\right)\right)^{k_l-1} \prod_{p=1}^{l} \varepsilon^+_{\mathfrak{s}_{l+1}\setminus s_p} u_{s_p}^{k_{p-1}-k_p} \, d\sigma^{l+1}(\mathfrak{s}_{l+1})\right] \\
&= \sum_{0\le k_{l+1}<\cdots<k_0=n} c_{\mathfrak{K}_{l+1}} E_\sigma\left[\int_{X^{l+1}} \delta_\sigma\left(\varepsilon^+_{\mathfrak{s}_{l+1}} u\right)^{k_{l+1}} \varepsilon^+_{\mathfrak{s}_{l+1}} G \varepsilon^+_{\mathfrak{s}_l} u_{s_{l+1}}^{k_l-k_{l+1}} \prod_{p=1}^{l} \varepsilon^+_{\mathfrak{s}_{l+1}\setminus s_p} u_{s_p}^{k_{p-1}-k_p} \, d\sigma^{l+1}(\mathfrak{s}_{l+1})\right] \\
&= \sum_{1\le k_{l+1}<\cdots<k_0=n} c_{\mathfrak{K}_{l+1}} E_\sigma\left[\int_{X^{l+1}} \delta_\sigma\left(\varepsilon^+_{\mathfrak{s}_{l+1}} u\right)^{k_{l+1}} \varepsilon^+_{\mathfrak{s}_{l+1}} G \prod_{p=1}^{l+1} \varepsilon^+_{\mathfrak{s}_{l+1}\setminus s_p} u_{s_p}^{k_{p-1}-k_p} \, d\sigma^{l+1}(\mathfrak{s}_{l+1})\right] \\
&\quad + \sum_{0=k_{l+1}<\cdots<k_0=n} c_{\mathfrak{K}_{l+1}} E_\sigma\left[\int_{X^{l+1}} \varepsilon^+_{\mathfrak{s}_{l+1}} G \prod_{p=1}^{l+1} \varepsilon^+_{\mathfrak{s}_{l+1}\setminus s_p} u_{s_p}^{k_{p-1}-k_p} \, d\sigma^{l+1}(\mathfrak{s}_{l+1})\right] \\
&= \sum_{1\le k_{l+1}<\cdots<k_0=n} c_{\mathfrak{K}_{l+1}} \\
&\quad \times E_\sigma\left[\int_{X^{l+2}} \varepsilon^+_{\mathfrak{s}_{l+1}} u_{s_{l+2}} D_{s_{l+2}} \left(\delta_\sigma\left(\varepsilon^+_{\mathfrak{s}_{l+1}} u\right)^{k_{l+1}-1} \varepsilon^+_{\mathfrak{s}_{l+1}} G \prod_{p=1}^{l+1} \varepsilon^+_{\mathfrak{s}_{l+1}\setminus s_p} u_{s_p}^{k_{p-1}-k_p}\right) d\sigma^{l+2}(\mathfrak{s}_{l+2})\right] \\
&\quad + \sum_{0=k_{l+1}<\cdots<k_0=n} c_{\mathfrak{K}_{l+1}} E_\sigma\left[\int_{X^{l+1}} \varepsilon^+_{\mathfrak{s}_{l+1}} G \prod_{p=1}^{l+1} \varepsilon^+_{\mathfrak{s}_{l+1}\setminus s_p} u_{s_p}^{k_{p-1}-k_p} \, d\sigma^{l+1}(\mathfrak{s}_{l+1})\right] \\
&= \sum_{1\le k_{l+1}<\cdots<k_0=n} c_{\mathfrak{K}_{l+1}} \\
&\quad \times E_\sigma\left[\int_{X^{l+2}} \varepsilon^+_{\mathfrak{s}_{l+1}} u_{s_{l+2}} \varepsilon^+_{s_{l+2}} \left(\delta_\sigma\left(\varepsilon^+_{\mathfrak{s}_{l+1}} u\right)^{k_{l+1}-1} \varepsilon^+_{\mathfrak{s}_{l+1}} G \prod_{p=1}^{l+1} \varepsilon^+_{\mathfrak{s}_{l+1}\setminus s_p} u_{s_p}^{k_{p-1}-k_p}\right) d\sigma^{l+2}(\mathfrak{s}_{l+2})\right] \\
&\quad - \sum_{1\le k_{l+1}<\cdots<k_0=n} c_{\mathfrak{K}_{l+1}} E_\sigma\left[\int_{X^{l+2}} \delta_\sigma\left(\varepsilon^+_{\mathfrak{s}_{l+1}} u\right)^{k_{l+1}-1} \varepsilon^+_{\mathfrak{s}_{l+1}} (u_{s_{l+2}} G) \prod_{p=1}^{l+1} \varepsilon^+_{\mathfrak{s}_{l+1}\setminus s_p} u_{s_p}^{k_{p-1}-k_p} \, d\sigma^{l+2}(\mathfrak{s}_{l+2})\right] \\
&\quad + \sum_{0=k_{l+1}<\cdots<k_0=n} c_{\mathfrak{K}_{l+1}} E_\sigma\left[\int_{X^{l+1}} \varepsilon^+_{\mathfrak{s}_{l+1}} G \prod_{p=1}^{l+1} \varepsilon^+_{\mathfrak{s}_{l+1}\setminus s_p} u_{s_p}^{k_{p-1}-k_p} \, d\sigma^{l+1}(\mathfrak{s}_{l+1})\right] \\
&= \mathcal{A}_{l+1} + \mathcal{B}_{l+1} - \mathcal{C}_{l+2},
\end{aligned}
$$

which proves (5.6) by induction on $l = 1, \ldots, n-1$, as

$$
E_\sigma\left[\delta_\sigma(u)^n G\right] = \mathcal{A}_0 - \mathcal{C}_1 = -\mathcal{C}_1 + \sum_{d=0}^{n-1} \mathcal{A}_d - \mathcal{A}_{d+1} = -\mathcal{C}_1 + \sum_{d=0}^{n-1} \mathcal{B}_{d+1} - \mathcal{C}_{d+2} = \sum_{d=1}^{n} \mathcal{B}_d - \sum_{d=1}^{n+1} \mathcal{C}_d. \qquad \square
$$

Proof of Theorem 5.1. We check that in (5.1), all terms with $a = 0$ and $0 \le b \le n-1$ vanish, hence in particular the formula also holds when $n = 0$. When $n \ge 1$ the proof of (5.1) is obtained by application to $c = n$ or $c = n+1$ of the following identity:

$$
\begin{aligned}
E_\sigma\big[\delta_\sigma(u)^n F\big] &= (-1)^c \sum_{a=0}^{n-c} \sum_{\substack{l_1+\cdots+l_{a+1}=n-c-a\\ l_1,\dots,l_{a+1}\geq 0}} C_{\mathfrak{L}_{a+1},a+c} \\
&\quad \times E_\sigma\left[\int_{X^{a+c}} \delta_\sigma\big(\varepsilon^+_{\mathfrak{s}_a} u\big)^{l_{a+1}} \varepsilon^+_{\mathfrak{s}_a} F \prod_{q=a+1}^{a+c} \varepsilon^+_{\mathfrak{s}_a} u_{s_q} \prod_{p=1}^{a} \varepsilon^+_{\mathfrak{s}_a\setminus s_p} u_{s_p}^{1+l_p}\, \mathrm{d}\sigma^{a+c}(\mathfrak{s}_{a+c})\right] \\
&\quad + \sum_{b=0}^{c-1} (-1)^b \sum_{a=0}^{n-b} \sum_{\substack{l_1+\cdots+l_a=n-b-a\\ l_1,\dots,l_a\geq 0}} C_{\mathfrak{L}_a,a+b} \\
&\quad \times E_\sigma\left[\int_{X^{a+b}} \varepsilon^+_{\mathfrak{s}_a} F \prod_{q=a+1}^{a+b} \varepsilon^+_{\mathfrak{s}_a} u_{s_q} \prod_{p=1}^{a} \varepsilon^+_{\mathfrak{s}_a\setminus s_p} u_{s_p}^{1+l_p}\, \mathrm{d}\sigma^{a+b}(\mathfrak{s}_{a+b})\right] \\
&= \mathcal{D}_c + \sum_{b=0}^{c-1} \mathcal{E}_b,
\end{aligned}
\tag{5.8}
$$

which will be proved by induction on $c = 1, \dots, n+1$. First, we note that since

$$
C_{\mathfrak{L}_a,a} = \prod_{p=1}^{a} \binom{l_1+\cdots+l_p+p-1}{l_1+\cdots+l_{p-1}+p-1},
$$

the Identity (5.8) holds for $c = 1$ from Lemma 5.2. Next, for all $c = 1, \dots, n-1$, applying Lemma 5.2 with $n = l_{a+1}$ and

$$
G = \varepsilon^+_{\mathfrak{s}_a} F \prod_{q=a+1}^{a+c} \varepsilon^+_{\mathfrak{s}_a} u_{s_q} \prod_{p=1}^{a} \varepsilon^+_{\mathfrak{s}_a\setminus s_p} u_{s_p}^{1+l_p}
$$

and fixing $s_1, \dots, s_{a+c}$, we rewrite the first term in (5.8) using (5.2) and the change of index

$$
k_{d-p} = p + m_1 + \cdots + m_p, \quad 0 \leq p \leq d,
$$

as

$$
\begin{aligned}
\mathcal{D}_c &= (-1)^c \sum_{a=0}^{n-c} \sum_{\substack{l_1+\cdots+l_{a+1}=n-c-a\\ l_1,\dots,l_{a+1}\geq 0}} C_{\mathfrak{L}_{a+1},a+c} \\
&\quad \times E_\sigma\left[\int_{X^{a+c}} \delta_\sigma\big(\varepsilon^+_{\mathfrak{s}_a} u\big)^{l_{a+1}} \varepsilon^+_{\mathfrak{s}_a} F \prod_{q=a+1}^{a+c} \varepsilon^+_{\mathfrak{s}_a} u_{s_q} \prod_{p=1}^{a} \varepsilon^+_{\mathfrak{s}_a\setminus s_p} u_{s_p}^{1+l_p}\, \mathrm{d}\sigma^{a+c}(\mathfrak{s}_{a+c})\right] \\
&= (-1)^c \sum_{a=0}^{n-c} \sum_{\substack{l_1+\cdots+l_{a+1}=n-c-a\\ l_1,\dots,l_{a+1}\geq 0}} C_{\mathfrak{L}_{a+1},a+c} \sum_{d=0}^{l_{a+1}} \sum_{\substack{m_1+\cdots+m_d=l_{a+1}-d\\ m_1,\dots,m_d\geq 0}} \prod_{p=1}^{d} \binom{m_1+\cdots+m_p+p-1}{m_1+\cdots+m_{p-1}+p-1} \\
&\quad \times E_\sigma\left[\int_{X^{a+c+d}} \varepsilon^+_{\mathfrak{s}_{a+d}} F \prod_{q=a+d+1}^{a+c+d} \varepsilon^+_{\mathfrak{s}_{a+d}} u_{s_q} \prod_{p=1}^{a} \varepsilon^+_{\mathfrak{s}_{a+d}\setminus s_p} u_{s_p}^{1+l_p} \prod_{k=a+1}^{a+d} \varepsilon^+_{\mathfrak{s}_{a+d}\setminus s_p} u_{s_k}^{1+m_{k-a}}\, \mathrm{d}\sigma^{a+c+d}(\mathfrak{s}_{a+c+d})\right]
\end{aligned}
$$

$$
\begin{aligned}
& - (-1)^c \sum_{a=0}^{n-c} \sum_{\substack{l_1+\cdots+l_{a+1}=n-c-a \\ l_1,\ldots,l_{a+1}\geq 0}} C_{\mathfrak{L}_{a+1},a+c} \sum_{d=1}^{l_{a+1}} \sum_{\substack{m_1+\cdots+m_d=l_{a+1}-d \\ m_1,\ldots,m_d\geq 0}} \prod_{p=1}^{d} \binom{m_1+\cdots+m_p+p-1}{m_1+\cdots+m_{p-1}+p-1} \\
& \times E_\sigma\Bigg[\int_{X^{a+c+d}} \delta_\sigma\big(\varepsilon^+_{\mathfrak{s}_{a+d-1}} u\big)^{m_d} \varepsilon^+_{\mathfrak{s}_{a+d-1}} F \prod_{q=a+d}^{a+c+d} \varepsilon^+_{\mathfrak{s}_{a+d-1}} u_{s_q} \\
& \times \prod_{p=1}^{a} \varepsilon^+_{\mathfrak{s}_{a+d-1}\setminus s_p} u_{s_p}^{1+l_p} \prod_{k=a+c+1}^{a+c+d-1} \varepsilon^+_{\mathfrak{s}_{a+d-1}\setminus s_p} u_{s_k}^{1+m_{k-a-c}} \, \mathrm{d}\sigma^{a+c+d}(\mathfrak{s}_{a+c+d})\Bigg] \\
= {} & (-1)^c \sum_{a'=0}^{n-c} \sum_{\substack{l'_1+\cdots+l'_{a'}=n-c-a' \\ l'_1,\ldots,l'_{a'}\geq 0}} C_{\mathfrak{L}_{a'},a'+c} \\
& \times E_\sigma\Bigg[\int_{X^{a'+c}} \varepsilon^+_{\mathfrak{s}_{a'}} F \prod_{q=a'+1}^{a'+c} \varepsilon^+_{\mathfrak{s}_{a'+c}} u_{s_q} \prod_{p=1}^{a'} \varepsilon^+_{\mathfrak{s}_{a'}\setminus s_p} u_{s_p}^{1+l'_p} \, \mathrm{d}\sigma^{a'+c}(\mathfrak{s}_{a'+c})\Bigg] && (5.9) \\
& + (-1)^{c+1} \sum_{a'=0}^{n-c} \sum_{\substack{l'_1+\cdots+l'_{a'+1}=n-c-a'-1 \\ l'_1,\ldots,l'_{a'+1}\geq 0}} C_{\mathfrak{L}_{a'+1},a'+c+1} \\
& \times E_\sigma\Bigg[\int_{X^{a'+c+1}} \delta_\sigma\big(\varepsilon^+_{\mathfrak{s}_{a'}} u\big)^{l'_{a'+1}} \varepsilon^+_{\mathfrak{s}_{a'}} F \prod_{q=a'+1}^{a'+c+1} \varepsilon^+_{\mathfrak{s}_{a'}} u_{s_q} \prod_{p=1}^{a'} \varepsilon^+_{\mathfrak{s}_{a'}\setminus s_p} u_{s_p}^{1+l'_p} \, \mathrm{d}\sigma^{a'+c+1}(\mathfrak{s}_{a'+c+1})\Bigg] && (5.10) \\
= {} & \mathcal{E}_c + \mathcal{D}_{c+1}
\end{aligned}
$$

under the changes of indices

$$l'_1+\cdots+l'_{a'} = l_1+\cdots+l_a+m_1+\cdots+m_d, \quad a' = a+d,$$

in (5.9) when $d = 0,\ldots,l_{a+1}$, and

$$l'_1+\cdots+l'_{a'+1} = l_1+\cdots+l_a+m_1+\cdots+m_d, \quad a'+1 = a+d,$$

in (5.10) when $d = 1,\ldots,l_{a+1}$. Noting that in (5.10), the summation on a' actually ends at $a' = n-c-1$ when $c < n$. We conclude the proof by induction, as

$$E_\sigma\big[\delta_\sigma(u)^n F\big] = \mathcal{D}_1 - \mathcal{D}_{n+1} + \mathcal{E}_0 = \mathcal{E}_0 + \sum_{b=1}^{n} \mathcal{D}_b - \mathcal{D}_{b+1} = \sum_{b=0}^{n} \mathcal{E}_b,$$

and by the change of indices $(a,b) \to (a,b-a)$ in (5.1). □

6. Recursive moment identities

The main results of this section are Propositions 6.1 and 6.2. Their proofs are stated using Lemma 2.4 above and Proposition 6.3 below, and they are used to prove the main results of Section 3. In the next theorem we use the notation Δ_s of Definition 2.5 and let

$$\Delta_{\mathfrak{s}_j} = \Delta_{\mathfrak{s}_0}\cdots\Delta_{\mathfrak{s}_j}, \quad \mathfrak{s}_j = (s_0,\ldots,s_j),$$

and

$$\mathrm{d}\sigma^{b+1}(\mathfrak{s}_b) = \sigma(\mathrm{d}s_0)\cdots\sigma(\mathrm{d}s_b), \quad \mathfrak{s}_b = (s_0, \dots, s_b),$$

$0 \le j \le b$.

Proposition 6.1. *Let $N \ge 0$ and let $u \in \mathbb{L}_{2,1}$ be bounded with $u \in \bigcap_{p=1}^{N+1} L^\infty(\Omega^X, L^p_\sigma(X))$ and*

$$E_\sigma\left[\int_{X^{b+1}} \left|\Delta_{s_0}\cdots\Delta_{s_j}\left(\prod_{q=a+1}^{b} u_{s_q}\prod_{p=0}^{a} u_{s_p}^{l_p}\right)\right| \mathrm{d}\sigma^{b+1}(\mathfrak{s}_b)\right] < \infty,$$

$l_0 + \cdots + l_a \le N+1$, $l_0, \dots, l_a \ge 1$, $0 \le j \le a \le b \le N$. Then for all $n = 0, \dots, N$ we have $\delta_\sigma(u) \in L^{n+1}(\Omega^X, \pi_\sigma)$ and

$$\begin{aligned} E_\sigma\left[\delta_\sigma(u)^{n+1}\right] &= \sum_{k=0}^{n-1}\binom{n}{k} E_\sigma\left[\delta_\sigma(u)^k \int_X u_t^{n-k+1}\sigma(\mathrm{d}t)\right] \\ &\quad + \sum_{a=0}^{n}\sum_{j=0}^{a}\sum_{b=a}^{n}\sum_{\substack{l_0+\cdots+l_a=n-b\\ l_0,\dots,l_a\ge 0}}\binom{a}{j} C^{l_0,n}_{\mathfrak{L}_a,b} \\ &\quad \times E_\sigma\left[\int_{X^{b+1}} \Delta_{\mathfrak{s}_j}\left(\prod_{q=a+1}^{b} u_{s_q}\prod_{p=0}^{a} u_{s_p}^{1+l_p}\right)\mathrm{d}\sigma^{b+1}(\mathfrak{s}_b)\right], \end{aligned} \tag{6.1}$$

where

$$C^{l_0,n}_{\mathfrak{L}_a,b} = (-1)^{b-a}\binom{n}{l_0} C_{\mathfrak{L}_a,b},$$

and $C_{\mathfrak{L}_a,b}$ is defined in (5.2).

Proof. When $u : \Omega^X \times X \to \mathbb{R}$ is a bounded process with compact support in X this result is a direct consequence of Lemma 2.4 and Proposition 6.3 below applied with $n = k$ and $l_0 + k - b = n - b$. We conclude the proof by induction and a limiting argument, as follows. Let $(K_r)_{r\ge 1}$ denote an increasing family of compact subsets of X such that $X = \bigcup_{r\ge 1} K_r$. The family of processes $u_x^{(r)}(\omega) := u_x \mathbf{1}_{K_r}(x)$, $r \ge 1$, converges in $\mathbb{L}_{2,1}$ to u as r goes to infinity, hence $\delta_\sigma(u^{(r)})$ converges to $\delta(u)$ in $L^2(\Omega^X, \pi_\sigma)$ as r goes to infinity. Clearly the result holds for $N = 0$ by applying the formula to the process $u^{(r)}$ which is bounded with compact support by letting r go to infinity. Next, letting $N \ge 0$ and assuming that $\delta_\sigma(u) \in L^{n+1}(\Omega^X, \pi_\sigma)$ and that (6.1) holds for all $n = 0, \dots, N$, we note that for all even integer $m \in \{2, \dots, N+1\}$ we have the bound

$$\begin{aligned} E_\sigma\left[\delta_\sigma(u)^m\right] &\le \sum_{k=0}^{m-2}\binom{m-1}{k} E_\sigma\left[\delta_\sigma(u)^{m-2}\right]^{k/(m-2)} \left\|\int_X |u_t|^{m-k}\sigma(\mathrm{d}t)\right\|_\infty \\ &\quad + \sum_{a=0}^{m-1}\sum_{j=0}^{a}\sum_{b=a}^{m-1}\sum_{\substack{l_0+\cdots+l_a=m-b-1\\ l_0,\dots,l_a\ge 0}}\binom{a}{j} C^{l_0,m-1}_{\mathfrak{L}_a,b} \\ &\quad \times E_\sigma\left[\int_{X^{b+1}} \left|\Delta_{\mathfrak{s}_j}\left(\prod_{q=a+1}^{b} u_{s_q}\prod_{p=0}^{a} u_{s_p}^{1+l_p}\right)\right|\mathrm{d}\sigma^{b+1}(\mathfrak{s}_b)\right], \end{aligned}$$

which, applied to $u^{(r)}(\omega)$, allows us to extend (6.1) to the order $N+1$ by uniform integrability after taking the limit as r goes to infinity. □

Let us consider some particular cases of Proposition 6.1. For $n=1$, Relation (6.1) reads

$$\begin{aligned}
E_\sigma\big[\delta_\sigma(u)^2\big] &= E_\sigma\left[\int_X |u_s|^2\sigma(\mathrm{d}s)\right] \\
&\quad - E_\sigma\left[\int_{X^2}\Delta_{s_1}(u_{s_1}u_{s_2})\sigma(\mathrm{d}s_1)\sigma(\mathrm{d}s_2)\right] && [a=0,b=1,\ j=0] \\
&\quad + E_\sigma\left[\int_{X^2}\Delta_{s_1}(u_{s_1}u_{s_2})\sigma(\mathrm{d}s_1)\sigma(\mathrm{d}s_2)\right] && [a=1,b=1,\ j=0] \\
&\quad + E_\sigma\left[\int_{X^2}\Delta_{s_1}\Delta_{s_2}(u_{s_1}u_{s_2})\sigma(\mathrm{d}s_1)\sigma(\mathrm{d}s_2)\right] && [a=1,b=1,\ j=1] \\
&= E_\sigma\left[\int_X |u_s|^2\sigma(\mathrm{d}s)\right] + E_\sigma\left[\int_{X^2}\Delta_{s_1}\Delta_{s_2}(u_{s_1}u_{s_2})\sigma(\mathrm{d}s_1)\sigma(\mathrm{d}s_2)\right],
\end{aligned}$$

which coincides with (5.3). On the other hand for $n=2$ Relation (6.1) yields the third moment

$$\begin{aligned}
E_\sigma\big[\delta_\sigma(u)^3\big] &= E_\sigma\left[\int_X u_s^3\sigma(\mathrm{d}s)\right] + 2E_\sigma\left[\delta(u)\int_X u_s^2\sigma(\mathrm{d}s)\right] \\
&\quad - 2E_\sigma\left[\int_{X^2}\Delta_{s_0}\big(u_{s_0}^2u_{s_1}\big)\sigma(\mathrm{d}s_0)\sigma(\mathrm{d}s_1)\right] && [a=0,b=1,\ j=0] \\
&\quad + E_\sigma\left[\int_{X^2}\Delta_{s_0}(u_{s_0}u_{s_1}u_{s_2})\sigma(\mathrm{d}s_0)\sigma(\mathrm{d}s_1)\right] && [a=0,b=2,\ j=0] \\
&\quad + E_\sigma\left[\int_{X^2}\Delta_{s_0}\big(u_{s_0}u_{s_1}^2\big)\sigma(\mathrm{d}s_0)\sigma(\mathrm{d}s_1)\right] + 2E_\sigma\left[\int_{X^2}\Delta_{s_0}\big(u_{s_0}^2u_{s_1}\big)\sigma(\mathrm{d}s_0)\sigma(\mathrm{d}s_1)\right] \\
&\quad && [a=1,b=1,\ j=0] \\
&\quad + 3E_\sigma\left[\int_{X^2}\Delta_{s_0}\Delta_{s_1}\big(u_{s_0}u_{s_0}^2\big)\sigma(\mathrm{d}s_0)\sigma(\mathrm{d}s_1)\right] && [a=1,b=1,\ j=1] \\
&\quad - E_\sigma\left[\int_{X^2}\Delta_{s_0}(u_{s_0}u_{s_1}u_{s_2})\sigma(\mathrm{d}s_0)\sigma(\mathrm{d}s_1)\right] && [a=1,b=2,\ j=0] \\
&\quad - E_\sigma\left[\int_{X^3}\Delta_{s_0}\Delta_{s_1}(u_{s_0}u_{s_1}u_{s_2})\sigma(\mathrm{d}s_0)\sigma(\mathrm{d}s_1)\sigma(\mathrm{d}s_2)\right] && [a=1,b=2,\ j=1] \\
&\quad + E_\sigma\left[\int_{X^3}\Delta_{s_0}(u_{s_0}u_{s_1}u_{s_2})\sigma(\mathrm{d}s_0)\sigma(\mathrm{d}s_1)\sigma(\mathrm{d}s_2)\right] && [a=2,b=2,\ j=0] \\
&\quad + E_\sigma\left[\int_{X^3}\Delta_{s_0}\Delta_{s_1}(u_{s_0}u_{s_1}u_{s_2})\sigma(\mathrm{d}s_0)\sigma(\mathrm{d}s_1)\sigma(\mathrm{d}s_2)\right] && [a=2,b=2,\ j=1] \\
&\quad + E_\sigma\left[\int_{X^3}\Delta_{s_0}\Delta_{s_1}\Delta_{s_2}(u_{s_0}u_{s_1}u_{s_2})\sigma(\mathrm{d}s_0)\sigma(\mathrm{d}s_1)\sigma(\mathrm{d}s_2)\right] && [a=2,b=2,\ j=2] \\
&= E_\sigma\left[\int_X u_s^3\sigma(\mathrm{d}s)\right] + 2E_\sigma\left[\delta(u)\int_X u_s^2\sigma(\mathrm{d}s)\right] + E_\sigma\left[\int_{X^2}u_{s_0}D_{s_0}u_{s_1}^2\sigma(\mathrm{d}s_0)\sigma(\mathrm{d}s_1)\right] \\
&\quad + 3E_\sigma\left[\int_{X^2}\Delta_{s_0}\Delta_{s_1}\big(u_{s_0}u_{s_1}^2\big)\sigma(\mathrm{d}s_0)\sigma(\mathrm{d}s_1)\right] \\
&\quad + E_\sigma\left[\int_{X^3}\Delta_{s_0}\Delta_{s_1}\Delta_{s_2}(u_{s_0}u_{s_1}u_{s_2})\sigma(\mathrm{d}s_0)\sigma(\mathrm{d}s_1)\sigma(\mathrm{d}s_2)\right], && (6.2)
\end{aligned}$$

which recovers (5.4) by the duality relation (2.8). As a consequence of Proposition 6.1 and Lemma A.2 in the Appendix, when the process u satisfies the cyclic condition

$$D_{t_1}u_{t_2}(\omega)\cdots D_{t_k}u_{t_1}(\omega)=0,\quad \omega\in\Omega^X, t_1,\dots,t_k\in X, \tag{6.3}$$

$k\geq 2$, Relation (6.1) becomes

$$\begin{aligned} E_\sigma\big[\delta_\sigma(u)^{n+1}\big] &= \sum_{k=0}^{n-1}\binom{n}{k}E_\sigma\Big[\delta_\sigma(u)^k\int_X u_t^{n-k+1}\sigma(\mathrm{d}t)\Big] \\ &\quad+\sum_{a=0}^{n}\sum_{j=0}^{a\wedge(b-1)}\sum_{b=a}^{n}\sum_{\substack{l_0+\cdots+l_a=n-b\\ l_0,\dots,l_a\geq 0}}\binom{a}{j}C_{\mathfrak{L}_a,b}^{l_0,n} \\ &\quad\times E_\sigma\Big[\int_{X^{b+1}}\Delta_{\mathfrak{s}_j}\Big(\prod_{q=a+1}^{b}u_{s_q}\prod_{p=0}^{a}u_{s_p}^{1+l_p}\Big)\mathrm{d}\sigma^{b+1}(\mathfrak{s}_b)\Big], \end{aligned}\tag{6.4}$$

i.e. the last two terms of (6.2) vanish when $n=2$. In case $X=\mathbb{R}_+\times Z$, Condition (6.3) is satisfied when u is predictable, by the same argument as the one leading to (4.3).

The next proposition follows from Proposition 6.1 and is used to prove Proposition 3.1.

Proposition 6.2. *Let $N\geq 0$ and let $u\in\mathbb{L}_{2,1}$ be a bounded process such that $u\in\bigcap_{p=1}^{N+1}L^\infty(\Omega^X,L_\sigma^p(X))$ and the integral $\int_X u_t^n\sigma(\mathrm{d}t)$ is deterministic, for all $n=1,\dots,N+1$, and*

$$E_\sigma\Big[\int_{X^{a+1}}\Big|\Delta_{s_0}\cdots\Delta_{s_a}\Big(\prod_{p=0}^{a}u_{s_p}^{l_p}\Big)\Big|\mathrm{d}\sigma^{a+1}(\mathfrak{s}_a)\Big]<\infty,$$

$l_0+\cdots+l_a\leq N+1$, $l_0,\dots,l_a\geq 1$, $0\leq a\leq N+1$. *Then for all $n=0,\dots,N$ we have $\delta_\sigma(u)\in L^{n+1}(\Omega^X,\pi_\sigma)$ and*

$$\begin{aligned} E_\sigma\big[\delta_\sigma(u)^{n+1}\big] &= \sum_{k=0}^{n-1}\binom{n}{k}\int_X u_t^{n-k+1}\sigma(\mathrm{d}t)E_\sigma\big[\delta_\sigma(u)^k\big] \\ &\quad+\sum_{0\leq j\leq a\leq b\leq n}\sum_{\substack{l_0+\cdots+l_a=n-b\\ l_0,\dots,l_a\geq 0\\ l_{a+1},\dots,l_b=0}}\binom{a}{j}C_{\mathfrak{L}_a,b}^{l_0,n} \\ &\quad\times E_\sigma\Big[\int_{X^{j+1}}\Delta_{\mathfrak{s}_j}\prod_{p=0}^{j}u_{s_p}^{1+l_p}\,\mathrm{d}\sigma^{j+1}(\mathfrak{s}_j)\Big]\prod_{q=j+1}^{b}\int_X u_t^{1+l_q}\sigma(\mathrm{d}t). \end{aligned}$$

Proof. We apply Proposition 6.1 after integrating in $s_{j+1},\dots,s_a$ and using (2.13). □

Consequently if $u:\Omega^X\to\mathbb{R}$ satisfy the hypotheses of Proposition 6.2 and is such that

$$\int_{X^{j+1}}\Delta_{s_0}\cdots\Delta_{s_j}\Big(\prod_{p=0}^{j}u_{s_p}^{l_p}\Big)\mathrm{d}\sigma^{j+1}(\mathfrak{s}_j)=0, \tag{6.5}$$

π_σ-a.s., for all $l_0+\cdots+l_j\leq N+1$, $l_0\geq 1,\dots,l_j\geq 1$, $j=1,\dots,N$, or simply the cyclic condition

$$D_{t_0}u_{t_1}(\omega)\cdots D_{t_j}u_{t_0}(\omega)=0,\quad \omega\in\Omega^X, t_1,\dots,t_j\in X,$$

$j = 1, \ldots, N$, cf. Lemma A.2 below, then we have

$$E_\sigma\left[\delta_\sigma(u)^{n+1}\right] = \sum_{k=0}^{n-1} \binom{n}{k} \int_X u_t^{n-k+1} \sigma(\mathrm{d}t) E_\sigma\left[\delta_\sigma(u)^k\right], \quad n = 0, \ldots, N,$$

i.e. the moments of $\delta_\sigma(u)$ satisfy the same recurrence relation (5.8) as the moments of compensated Poisson integrals.

The next proposition is used to prove Proposition 6.1 with the help of Lemma 2.4.

Proposition 6.3. *Let $u : \Omega^X \times X \to \mathbb{R}$ and $v : \Omega^X \times X \to \mathbb{R}$ be bounded processes with compact support in X. For all $k \geq 0$ we have*

$$\begin{aligned} E_\sigma\left[\int_X v_s \delta_\sigma\left(\varepsilon_s^+ u\right)^k \sigma(\mathrm{d}s)\right] &= E_\sigma\left[\delta_\sigma(u)^k \int_X v_s \sigma(\mathrm{d}s)\right] \\ &\quad + \sum_{a=0}^{k} \sum_{j=0}^{a} \binom{a}{j} \sum_{b=a}^{k} (-1)^{b-a} \sum_{\substack{l_1+\cdots+l_a=k-b \\ l_1,\ldots,l_a \geq 0}} C_{\mathfrak{L}_a,b} \\ &\quad \times E_\sigma\left[\int_{X^{b+1}} \Delta_{\mathfrak{s}_j}\left(v_{s_0} \prod_{q=a+1}^{b} u_{s_q} \prod_{p=1}^{a} u_{s_p}^{1+l_p}\right) \mathrm{d}\sigma^{b+1}(\mathfrak{s}_b)\right], \end{aligned}$$

where $C_{\mathfrak{L}_a,b}$ is defined in (5.2).

Proof. This proof is an application of Theorem 5.1 with $F = v_s$. Using Proposition A.1 below and the expansion

$$\prod_{i=0}^{j} (I + \Delta_{s_i}) = \sum_{l=0}^{j} \sum_{0 \leq i_0 < \cdots < i_l \leq j} \Delta_{s_{i_0}} \cdots \Delta_{s_{i_l}},$$

we have, up to the symmetrization due to the integral in $\sigma(\mathrm{d}s_0) \cdots \sigma(\mathrm{d}s_a)$ and the summation on $l_1, \ldots, l_a$,

$$\begin{aligned} &\varepsilon_{\mathfrak{s}_a}^+ v_{s_0} (I + D_{s_0})\left(\prod_{q=a+1}^{b} \varepsilon_{\mathfrak{s}_a}^+ u_{s_q} \prod_{p=1}^{a} \varepsilon_{\mathfrak{s}_a \setminus s_p}^+ u_{s_p}^{1+l_p}\right) \\ &\quad = \left(\prod_{i=1}^{a} (I + \Delta_{s_i})\right)\left(v_{s_0} (I + D_{s_0})\left(\prod_{p=1}^{a} u_{s_p}^{1+l_p} \prod_{q=a+1}^{b} u_{s_q}\right)\right) \\ &\quad = \left(\prod_{i=1}^{a} (I + \Delta_{s_i})\right)\left(v_{s_0} \prod_{p=1}^{a} u_{s_p}^{1+l_p} \prod_{q=a+1}^{b} u_{s_q}\right) \\ &\qquad + \left(\prod_{i=1}^{a} (I + \Delta_{s_i})\right)\left(v_{s_0} D_{s_0}\left(\prod_{p=1}^{a} u_{s_p}^{1+l_p} \prod_{q=a+1}^{b} u_{s_q}\right)\right) \\ &\quad = \left(\prod_{i=1}^{a} (I + \Delta_{s_i})\right)\left(v_{s_0} \prod_{p=1}^{a} u_{s_p}^{1+l_p} \prod_{q=a+1}^{b} u_{s_q}\right) \\ &\qquad + \sum_{j=0}^{a} \binom{a}{j} \Delta_{s_1} \cdots \Delta_{s_j}\left(v_{s_0} D_{s_0}\left(\prod_{p=1}^{a} u_{s_p}^{1+l_p} \prod_{q=a+1}^{b} u_{s_q}\right)\right) \\ &\quad = \varepsilon_{\mathfrak{s}_a}^+ v_{s_0} \prod_{q=a+1}^{b} \varepsilon_{\mathfrak{s}_a}^+ u_{s_q} \prod_{p=1}^{a} \varepsilon_{\mathfrak{s}_a \setminus s_p}^+ u_{s_p}^{1+l_p} + \sum_{j=0}^{a} \binom{a}{j} \Delta_{s_0} \cdots \Delta_{s_j}\left(v_{s_0} \prod_{p=1}^{a} u_{s_p}^{1+l_p} \prod_{q=a+1}^{b} u_{s_q}\right), \end{aligned}$$

hence by Theorem 5.1 applied to $G = v_s$ with fixed $s \in X$ we have

$$
\begin{aligned}
&E_\sigma\left[\int_X v_s \delta_\sigma\big((I + D_s)u\big)^k \sigma(\mathrm{d}s)\right] \\
&\quad = \sum_{a=0}^{k}\sum_{b=0}^{k-a}(-1)^{b-a} \sum_{\substack{l_1+\cdots+l_a=k-b\\ l_1,\dots,l_a\ge 0}} C_{\mathfrak{L}_a,b} \\
&\qquad \times E_\sigma\left[\int_{X^{b+1}} \varepsilon^+_{\mathfrak{s}_a} v_{s_0}(I + D_{s_0})\left(\prod_{q=a+1}^{b} \varepsilon^+_{\mathfrak{s}_a} u_{s_q} \prod_{p=1}^{a} \varepsilon^+_{\mathfrak{s}_a\setminus s_p} u_{s_p}^{1+l_p}\right) \mathrm{d}\sigma^{b+1}(\mathfrak{s}_b)\right] \\
&\quad = \sum_{a=0}^{k}\sum_{b=0}^{k-a}(-1)^{b-a} \sum_{\substack{l_1+\cdots+l_a=k-b\\ l_1,\dots,l_a\ge 0}} C_{\mathfrak{L}_a,b} E_\sigma\left[\int_{X^{b+1}} \varepsilon^+_{\mathfrak{s}_a} v_{s_0} \prod_{q=a+1}^{b} \varepsilon^+_{\mathfrak{s}_a} u_{s_q} \prod_{p=1}^{a} \varepsilon^+_{\mathfrak{s}_a\setminus s_p} u_{s_p}^{1+l_p}\, \mathrm{d}\sigma^{b+1}(\mathfrak{s}_b)\right] \\
&\qquad + \sum_{a=1}^{k}\sum_{j=0}^{a}\binom{a}{j}\sum_{b=0}^{k-a}(-1)^{b-a} \sum_{\substack{l_1+\cdots+l_a=k-b\\ l_1,\dots,l_a\ge 0}} C_{\mathfrak{L}_a,b} \\
&\qquad \times E_\sigma\left[\int_{X^{b+1}} \Delta_{s_0}\cdots\Delta_{s_j}\left(v_{s_0}\prod_{q=a+1}^{b} u_{s_q} \prod_{p=1}^{a} u_{s_p}^{1+l_p}\right) \mathrm{d}\sigma^{b+1}(\mathfrak{s}_b)\right] \\
&\quad = E_\sigma\left[\delta_\sigma(u)^k \int_X v_s \sigma(\mathrm{d}s)\right] \\
&\qquad + \sum_{a=1}^{k}\sum_{j=0}^{a}\binom{a}{j}\sum_{b=0}^{k-a}(-1)^{b-a} \sum_{\substack{l_1+\cdots+l_a=k-b\\ l_1,\dots,l_a\ge 0}} C_{\mathfrak{L}_a,b} \\
&\qquad \times E_\sigma\left[\int_{X^{b+1}} \Delta_{\mathfrak{s}_j}\left(v_{s_0}\prod_{q=a+1}^{b} u_{s_q} \prod_{p=1}^{a} u_{s_p}^{1+l_p}\right) \mathrm{d}\sigma^{b+1}(\mathfrak{s}_b)\right],
\end{aligned}
\tag{6.6}
$$

where we identified $E_\sigma[\delta_\sigma(u)^k \int_X v_s\sigma(\mathrm{d}s)]$ to (6.6) on the last step, by another application of Theorem 5.1 to $F = \int_X v_s \sigma(\mathrm{d}s)$. □

Appendix

In this appendix we state some combinatorial results that have been used above.

Proposition A.1. *Let $u : \Omega^X \times X \to \mathbb{R}$ be a measurable process. For all $0 \le j, p \le n$ we have the relation*

$$
\prod_{p=0}^{n} \varepsilon^+_{\mathfrak{s}_j\setminus s_p} u_{s_p} = \left(\prod_{i=0}^{j}(I + \Delta_{s_i})\right)\prod_{p=0}^{n} u_{s_p} \tag{A.1}
$$

for mutually different $\mathfrak{s}_n = (s_0, \dots, s_n) \subset X$.

Proof. We will prove Relation (A.1) for all $n \geq 0$ by induction on $j \in \{0, 1, \ldots, n\}$. Clearly for $j = 0$ the relation holds since

$$
\begin{aligned}
u_{s_0} \prod_{p=1}^{n} \varepsilon_{s_0}^{+} u_{s_p} &= u_{s_0} \prod_{p=1}^{n} u_{s_p} + u_{s_0} \sum_{\Xi_1 \cup \cdots \cup \Xi_n = \{s_0\}} D_{\Xi_1} u_{s_1} \cdots D_{\Xi_n} u_{s_n} \\
&= u_{s_0} \prod_{p=1}^{n} u_{s_p} + \sum_{\substack{\Xi_0 \cup \cdots \cup \Xi_n = \{s_0\} \\ s_0 \notin \Xi_0}} D_{\Xi_0} u_{s_0} \cdots D_{\Xi_n} u_{s_n} \\
&= (I + \Delta_{s_0}) \prod_{p=0}^{n} u_{s_p}.
\end{aligned}
$$

Next, assuming that (A.1) holds at the rank $j \in \{0, 1, \ldots, n-1\}$ and taking $\{s_0, \ldots, s_n\} \subset X$ mutually different we have

$$
\begin{aligned}
\prod_{p=0}^{n} \varepsilon_{\mathfrak{s}_{j+1} \setminus s_p}^{+} u_{s_p} &= \prod_{p=0}^{n} \left((I + \mathbf{1}_{\{p \neq j+1\}} D_{s_{j+1}}) \prod_{\substack{i=0 \\ i \neq p}}^{j} (I + D_{s_i}) u_{s_p} \right) \\
&= \sum_{\substack{\Xi_0 \cup \cdots \cup \Xi_n \subset \{s_{j+1}\} \\ s_{j+1} \notin \Xi_{j+1}}} \prod_{p=0}^{n} \left(\prod_{\substack{i=0 \\ i \neq p}}^{j} (I + D_{s_i}) D_{\Xi_p} u_{s_p} \right) \\
&= \sum_{\substack{\Xi_0 \cup \cdots \cup \Xi_n \subset \{s_{j+1}\} \\ s_{j+1} \notin \Xi_{j+1}}} \left(\prod_{i=0}^{j} (I + \Delta_{s_i}) \right) \prod_{p=0}^{n} D_{\Xi_p} u_{s_p} \\
&= \sum_{\substack{\Xi_0 \cup \cdots \cup \Xi_n \subset \{s_{j+1}\} \\ s_{j+1} \notin \Xi_{j+1}}} \sum_{\substack{\Theta_0 \cup \cdots \cup \Theta_n = \{s_0, s_1, \ldots, s_j\} \\ s_0 \notin \Theta_0, \ldots, s_j \notin \Theta_j}} D_{\Theta_0} D_{\Xi_0} u_{s_0} \cdots D_{\Theta_n} D_{\Xi_n} u_{s_n} \\
&= \sum_{\substack{\Theta_0 \cup \cdots \cup \Theta_n = \{s_0, s_1, \ldots, s_j\} \\ s_0 \notin \Theta_0, \ldots, s_j \notin \Theta_j}} D_{\Theta_0} u_{s_0} \cdots D_{\Theta_n} u_{s_n} \\
&\quad + \sum_{\substack{\Xi_0 \cup \cdots \cup \Xi_n = \{s_{j+1}\} \\ s_{j+1} \notin \Xi_{j+1}}} \sum_{\substack{\Theta_0 \cup \cdots \cup \Theta_n = \{s_0, s_1, \ldots, s_j\} \\ s_0 \notin \Theta_0, \ldots, s_j \notin \Theta_j}} D_{\Xi_0} D_{\Theta_0} u_{s_0} \cdots D_{\Xi_n} D_{\Theta_n} u_{s_n} \\
&= \left(\prod_{i=0}^{j} (I + \Delta_{s_i}) \right) \prod_{p=0}^{n} u_{s_p} + \Delta_{s_{j+1}} \left(\prod_{i=0}^{j} (I + \Delta_{s_i}) \right) \prod_{p=0}^{n} u_{s_p} \\
&= \left(\prod_{i=0}^{j+1} (I + \Delta_{s_i}) \right) \prod_{p=0}^{n} u_{s_p}.
\end{aligned}
$$

□

Finally in the next lemma, which is used to prove Corollary 3.2, we show that Relation (6.5) is satisfied provided $D_s u_t(\omega)$ satisfies the cyclic condition (A.2).

Lemma A.2. *Let $N \geq 1$ and assume that $u : \Omega^X \times X \to \mathbb{R}$ satisfies the cyclic condition*

$$D_{t_0} u_{t_1}(\omega) \cdots D_{t_j} u_{t_0}(\omega) = 0, \quad \omega \in \Omega^X, t_0, t_1, \ldots, t_j \in X, \tag{A.2}$$

for $j = 1, \ldots, N$. Then we have

$$\Delta_{t_0} \cdots \Delta_{t_j} \big(u_{t_0}(\omega) \cdots u_{t_j}(\omega) \big) = 0, \quad \omega \in \Omega^X, t_0, t_1, \ldots, t_j \in X,$$

for $j = 1, \ldots, N$.

Proof. By Definition 2.5 we have

$$\Delta_{t_0} \cdots \Delta_{t_j} \prod_{p=0}^{j} u_{t_p} = \sum_{\substack{\Theta_0 \cup \cdots \cup \Theta_j = \{t_0, t_1, \ldots, t_j\} \\ t_0 \notin \Theta_0, \ldots, t_j \notin \Theta_j}} D_{\Theta_0} u_{t_0} \cdots D_{\Theta_j} u_{t_j}, \tag{A.3}$$

$t_0, \ldots, t_j \in X$, $j = 2, \ldots, N$. Without loss of generality we may assume that $\{t_0, t_1, \ldots, t_j\}$ are not equal to eachother and that $\Theta_0 \neq \varnothing, \ldots, \Theta_j \neq \varnothing$ and $\Theta_k \cap \Theta_l = \varnothing$, $0 \leq k \neq l \leq j$, in the above sum. In this case we can construct a sequence $(k_1, \ldots, k_i)$ by choosing

$$t_0 \neq t_{k_1} \in \Theta_0, \qquad t_{k_2} \in \Theta_{k_1}, \qquad \ldots, \qquad t_{k_{i-1}} \in \Theta_{k_{i-2}},$$

until $t_{k_i} = t_0 \in \Theta_{k_{i-1}}$ for some $i \in \{2, \ldots, j\}$ since $\Theta_0 \cap \cdots \cap \Theta_j = \varnothing$ and $\Theta_0 \cup \cdots \cup \Theta_j = \{t_0, t_1, \ldots, t_j\}$. Hence by (A.2) we have

$$D_{t_{k_1}} u_{t_0} D_{t_{k_2}} u_{t_{k_1}} \cdots D_{t_{k_{i-1}}} u_{t_{k_{i-2}}} D_{t_{k_0}} u_{t_{k_{i-1}}} = 0$$

by (A.2), which implies

$$D_{\Theta_0} u_{t_0} D_{\Theta_{k_1}} u_{t_{k_1}} \cdots D_{\Theta_{k_{i-1}}} u_{t_{k_{i-1}}} = 0,$$

since

$$(t_{k_1}, t_{k_2}, \ldots, t_{k_{i-1}}, t_0) \in \Theta_0 \times \Theta_{k_1} \times \cdots \times \Theta_{k_{i-1}},$$

hence

$$D_{\Theta_0} u_{t_0} D_{\Theta_{k_1}} u_{t_{k_1}} \cdots D_{\Theta_{k_j}} u_{t_j} = 0,$$

and (A.3) vanishes. □

Again, in case $X = \mathbb{R}_+$, Condition (A.2) holds in particular when either $D_s u_t = 0$, $0 \leq s \leq t$, as in (4.2), resp. $D_t u_s = 0$, $0 \leq s \leq t$, which is the case when u is backward, resp. forward, predictable.

References

[1] S. Albeverio and N. V. Smorodina. A distributional approach to multiple stochastic integrals and transformations of the Poisson measure. *Acta Appl. Math.* **94** (2006) 1–19. MR2271674

[2] K. Bichteler. *Stochastic Integration with Jumps. Encyclopedia of Mathematics and Its Applications* **89**. Cambridge Univ. Press, Cambridge, 2002. MR1906715

[3] P. Brandimarte. *Numerical Methods in Finance and Economics*, second edition. *Statistics in Practice*. Wiley-Interscience, Hoboken, NJ, 2006. MR2255817

[4] P. Brémaud. *Point Processes and Queues, Martingale Dynamics*. Springer, New York, 1981. MR0636252

[5] T. Carleman. *Les fonctions quasi analytiques*. Gauthier-Villars, Éditeur, Paris, 1926.

[6] C. A. Charalambides. *Enumerative combinatorics. CRC Press Series on Discrete Mathematics and Its Applications*. Chapman & Hall/CRC, Boca Raton, FL, 2002. MR1937238
[7] Y. Davydov and S. Nagaev. On the convex hulls of point processes. Manuscript, 2000.
[8] A. Dermoune, P. Krée and L. Wu. Calcul stochastique non adapté par rapport à la mesure de Poisson. In *Séminaire de Probabilités XXII* 477–484. *Lecture Notes in Math.* **1321**. Springer, Berlin, 1988. MR0960543
[9] G. Di Nunno, B. Øksendal and F. Proske. *Malliavin Calculus for Lévy Processes with Applications to Finance*. Springer, Berlin, 2009. MR2460554
[10] Y. Ito. Generalized Poisson functionals. *Probab. Theory Related Fields* **77** (1988) 1–28 . MR0921816
[11] J. Jacod and A. N. Shiryaev. *Limit Theorems for Stochastic Processes*, second edition. *Grundlehren der Mathematischen Wissenschaften* **288**. Springer, Berlin, 2003. MR1943877
[12] J. Picard. Formules de dualité sur l'espace de Poisson. *Ann. Inst. Henri Poincaré Probab. Stat.* **32** (1996) 509–548. MR1411270
[13] N. Privault. Girsanov theorem for anticipative shifts on Poisson space. *Probab. Theory Related Fields* **104** (1996) 61–76. MR1367667
[14] N. Privault. Moment identities for Poisson–Skorohod integrals and application to measure invariance. *C. R. Math. Acad. Sci. Paris* **347** (2009) 1071–1074. MR2554579
[15] N. Privault. Moment identities for Skorohod integrals on the Wiener space and applications. *Electron. Commun. Probab.* **14** (2009) 116–121 (electronic). MR2481671
[16] N. Privault. *Stochastic Analysis in Discrete and Continuous Settings*. *Lecture Notes in Math*. **1982**. Springer, Berlin, 2009. MR2531026
[17] N. Privault. Generalized Bell polynomials and the combinatorics of Poisson central moments. *Electron. J. Combin.* **18** (2011) P54.
[18] N. Privault and J. L. Wu. Poisson stochastic integration in Hilbert spaces. *Ann. Math. Blaise Pascal* **6** (1999) 41–61. MR1735278
[19] J. A. Shohat and J. D. Tamarkin. *The Problem of Moments*. *American Mathematical Society Mathematical Surveys* **II**. Amer. Math. Soc., New York, 1943. MR0008438
[20] Y. Takahashi. Absolute continuity of Poisson random fields. *Publ. Res. Inst. Math. Sci.* **26** (1990) 629–647. MR1081507
[21] A. S. Üstünel and M. Zakai. Analyse de rotations aléatoires sur l'espace de Wiener. *C. R. Acad. Sci. Paris Sér. I Math.* **319** (1994) 1069–1073. MR1305678
[22] A. S. Üstünel and M. Zakai. Random rotations of the Wiener path. *Probab. Theory Related Fields* **103** (1995) 409–429. MR1358084
[23] A. S. Üstünel and M. Zakai. *Transformation of Measure on Wiener Space*. *Springer Monogr. Math*. Springer, Berlin, 2000.
[24] A. M. Veršik, I. M. Gel'fand and M. I. Graev. Representations of the group of diffeomorphisms. *Uspekhi Mat. Nauk* **30** (1975) 1–50. MR0399343
[25] S. Zuyev. Stopping sets: Gamma-type results and hitting properties. *Adv. in Appl. Probab.* **31** (1999) 355–366. MR1724557