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Excursions of diffusion processes and continued fractions

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Abstract. It is well-known that the excursions of a one-dimensional diffusion process can be studied by considering a certain Riccati equation associated with the process. We show that, in many cases of interest, the Riccati equation can be solved in terms of an infinite continued fraction. We examine the probabilistic significance of the expansion. To illustrate our results, we discuss some examples of diffusions in deterministic and in random environments.

Résumé. Il est bien connu que les excursions d'un processus de diffusion peuvent être etudiées en considérant une certaine équation de Riccati associée au processus. On montre que, dans beaucoup de cas intéressants, certaines solutions de cette équation de Riccati peuvent être développées en fraction continue. On examine le contenu probabiliste de ce développement. Ces résultats sont illustrés par quelques exemples de diffusions en milieux aléatoires et déterministes.

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1. Introduction

Recently, Marklof et al. [35,36] studied the random continued fraction

$$\frac{2\lambda}{u_1+}\,\frac{2\lambda}{u_2+}\,\frac{2\lambda}{u_3+\cdots}, \tag{1.1}$$

where $\lambda \in \mathbb{C}$ and where $u_1, u_2, \ldots$ are independent random variables with the same gamma distribution, i.e. for every Lebesgue-measurable set $A \subset \mathbb{R}_+$,

$$\mathbb{P}(u_n \in A) = \int_A \frac{1}{2^\mu \Gamma(\mu)} y^{\mu-1} \mathrm{e}^{-y/2}\,\mathrm{d}y, \quad \mu > 0. \tag{1.2}$$

Here, and throughout the rest of the paper, we make use of the compact notation

$$\frac{a_1}{b_1+}\,\frac{a_2}{b_2+}\,\frac{a_3}{b_3+\cdots} := \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3+\cdots}}}.$$

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An earlier work of Letac and Seshadri [33] had shown that, in the case of positive λ, the continued fraction (1.1) is a generalised inverse Gaussian random variable. One of the results obtained by Marklof et al. is an explicit formula for the probability density function for every complex λ.

The genesis of the work reported here was the curious observation that the *same distribution* – not the continued fraction itself – appears also in the articles of Bouchaud et al. [11] ($\lambda < 0$) and Kawazu and Tanaka [31] ($\lambda > 0$) on diffusion in a Brownian environment with drift. More precisely, consider a process X with infinitesimal generator

$$\mathscr{G}_W := \frac{a}{2}\mathrm{e}^{-2W}\frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{e}^{2W}\frac{\mathrm{d}}{\mathrm{d}x}\right], \tag{1.3}$$

where

$$W(x) := \int^x \frac{b(y)}{a(y)}\,\mathrm{d}y$$

and the functions a and b are the instantaneous variance and instantaneous drift, respectively. Many fundamental quantities associated with the process can be expressed in terms of the solutions of the equation

$$\mathscr{G}_W\phi(\cdot,\lambda) = \lambda\phi(\cdot,\lambda). \tag{1.4}$$

The order of the equation may be lowered by introducing the Riccati variable

$$U := \frac{\phi'}{\phi},$$

where the prime symbol denotes differentiation with respect to x. This yields the *Riccati equation*

$$\frac{\mathrm{d}U}{\mathrm{d}x} + U^2 + 2W'(x)U = 2\lambda/a(x). \tag{1.5}$$

When the state space is $\mathbb{R}$ and W is a Brownian motion with drift, this Riccati equation admits a positive stationary solution[2] whose law is the same as that of the continued fraction (1.1). This formulation in terms of the Riccati variable goes back to the pioneering work of Frisch and Lloyd [22] and has since been used extensively, both in the physics [11,34] and the mathematics [30,31] literatures.

In this work, we show that, for many diffusions – be they in a fixed or a random environment W – a solution of this Riccati equation may be found in terms of the infinite continued fraction

$$U(x,\lambda) = u_0(x) + \frac{2\lambda/a(x)}{u_1(x)} \underset{+}{} \frac{2\lambda/a(x)}{u_2(x)} \underset{+}{} \frac{2\lambda/a(x)}{u_3(x)} \underset{+\cdots}{} . \tag{1.6}$$

Continued fractions arise in the study of many random processes on a discrete state space, notably birth-and-death processes (see, for instance, [9,17,21,23,27] and the references therein) and random walks in a random environment [3,5,16]; their occurrence in the context of processes on a continuous state space is comparatively rare, though not unknown [8]. In the remainder of this introduction, we provide a brief explanation of the origin and significance of the continued fraction in the context of diffusion processes, and summarise our main results.

[2] Equation (1.5) is a stochastic equation in which the "noise" W' appears as a multiplicative term. In this paper, such equations will always be understood in the sense of Stratonovich; see [2,41] for a precise definition of the Stratonovich stochastic integral, and [38] for a comparison of the Itô and Stratonovich interpretations of stochastic equations. The *Itô stochastic equation* satisfied by this same positive stationary solution is

$$\frac{\mathrm{d}U}{\mathrm{d}x} + U^2 + 2\big[W'(x) - 1\big]U = 2\lambda/a(x).$$

This equation corresponds to the one used by Kawazu and Tanaka [31]. We prefer to work with the Stratonovich equation because its form is the same as that of the Riccati equation associated with (1.4) in a deterministic environment $W(x)$.

1.1. *Probabilistic interpretation of the Riccati variable*

Equation (1.4), supplemented with the boundary conditions that may be required in order to specify the process X uniquely, has two non-negative solutions, say $\phi_-(\cdot,\lambda)$ and $\phi_+(\cdot,\lambda)$, that are of particular importance: they are characterised (up to a constant factor) by the fact that $\phi_-(\cdot,\lambda)$ is *non-decreasing* and $\phi_+(\cdot,\lambda)$ *non-increasing*. Let x denote the *starting point* of the diffusion and let y be any other point in the state space. Then the *first hitting time* $H(y)$ of y is the random variable defined by

$$H(y) := \inf\{t \ge 0:\ X_t = y\} \tag{1.7}$$

and we have the following well-known formula for its Laplace transform [10]:

$$\mathbb{E}_x\bigl(\mathrm{e}^{-\lambda H(y)}\bigr) = \begin{cases} \frac{\phi_-(x,\lambda)}{\phi_-(y,\lambda)} & \text{if } x \le y, \\ \frac{\phi_+(x,\lambda)}{\phi_+(y,\lambda)} & \text{if } x \ge y. \end{cases} \tag{1.8}$$

We deduce

$$U_\pm(x,\lambda) := \frac{\phi_\pm'(x,\lambda)}{\phi_\pm(x,\lambda)} = \frac{\mathrm{d}}{\mathrm{d}y}\mathbb{E}_y\bigl(\mathrm{e}^{-\lambda H(x)}\bigr)\Big|_{y=x\pm}. \tag{1.9}$$

This expresses two particular solutions of the Riccati equation – one positive (U_-) and one negative (U_+) – in simple probabilistic terms. The choice of sign in this notation is somewhat disconcerting, but its justification will soon become manifest.

Following Pitman and Yor [40], one can gain further insight by defining a local time process $L(x)$ at x by

$$L_t(x) := \lim_{\varepsilon\to 0+}\frac{a(x)}{2\varepsilon}\int_0^t \mathbf{1}_{(x-\varepsilon,x+\varepsilon)}(X_\tau)\,\mathrm{d}\tau. \tag{1.10}$$

This process is proportional to the time that X spends in the vicinity of the starting point up to time t. Let ζ denote the lifetime of X. If X is recurrent (resp., transient), then $L_\zeta(x)$ is infinite (resp., finite) almost surely. The *inverse local time* $L^{-1}(x)$ at the starting point x is the process defined by

$$L_t^{-1}(x) := \inf\bigl\{\tau:\ L_\tau(x) > t\bigr\}, \quad 0 \le t < L_\zeta(x). \tag{1.11}$$

It is a measure of the time that elapses before X has spent a total time t in the vicinity of its starting point. Using the trivial identity

$$t = \int_0^t \mathbf{1}_{(-\infty,x]}(X_\tau)\,\mathrm{d}\tau + \int_0^t \mathbf{1}_{(x,\infty)}(X_\tau)\,\mathrm{d}\tau,$$

we can write

$$L^{-1}(x) = T_-(x) + T_+(x), \tag{1.12}$$

where

$$T_{-,t}(x) = \int_0^{L_t^{-1}(x)} \mathbf{1}_{(-\infty,x]}(X_\tau)\,\mathrm{d}\tau \quad \text{and} \quad T_{+,t}(x) = \int_0^{L_t^{-1}(x)} \mathbf{1}_{(x,\infty)}(X_\tau)\,\mathrm{d}\tau.$$

The processes $T_-(x)$ and $T_+(x)$ have obvious interpretations as occupation times; they are subordinators, i.e. non-decreasing Lévy processes, and we have

$$\mathbb{E}\bigl(\exp\bigl[-\lambda T_{\pm,t}(x)\bigr]\bigr) = \exp\bigl[-t\psi_\pm(x,\lambda)\bigr], \quad \lambda > 0, \tag{1.13}$$

for some function $\psi_\pm(x,\cdot)$ called the *Laplace exponent* of $T_\pm(x)$ [6]. By the Lévy–Khintchine formula, the Laplace exponent can be written in the form

$$\psi_\pm(x,\lambda) = \int_{(0,\infty]} \bigl(1-\mathrm{e}^{-\lambda y}\bigr)\nu_\pm(x,\mathrm{d}y) \tag{1.14}$$

for some measure $\nu_\pm(x)$ on $(0,\infty]$ such that

$$\int_{(0,\infty)} \min\{1,y\}\nu_\pm(x,\mathrm{d}y) < \infty.$$

The measure $\nu_\pm(x,\cdot)$ is called the *Lévy measure* of the process $T_\pm(x)$; it describes the distribution of the heights of its jumps. Each jump corresponds to an *excursion* of the process X, i.e. to the path followed by the process between two successive visits to the starting point x, and the jump height is the duration of the excursion. The $+$ (resp., $-$) case picks out the "upward" (resp., "downward") excursions, i.e. those whose path lies entirely above (resp., below or at) x. In the transient case, one or both of $\nu_+(x,\cdot)$ and $\nu_-(x,\cdot)$ will have an atom at infinity, reflecting the fact that X begins an excursion of infinite duration at the time of its last visit to x. Pitman and Yor [40] show that

$$\psi_\pm(x,\lambda) = \mp\frac{1}{2}U_\pm(x,\lambda). \tag{1.15}$$

A proof may also be found in [25], p. 215. Knowing $U_\pm$, one can therefore recover the Lévy measure $\nu_\pm(x,\cdot)$ by inverting a Laplace transform.

1.2. *Stieltjes functions*

Knight [28] and Kotani and Watanabe [32] make the observation that Krein's theory of strings implies the existence of a measure, say $\sigma_\pm(x)$, on $[0,\infty)$ such that

$$U_\pm(x,\lambda) = \mp 2\lambda \int_{[0,\infty)} \frac{\sigma_\pm(x,\mathrm{d}z)}{\lambda+z}. \tag{1.16}$$

So, knowing $U_\pm$, we can obtain $\sigma_\pm$ (and hence $\nu_\pm$) from

$$\sigma_\pm\bigl(x,\{0\}\bigr) = \frac{\mp 1}{2}U_\pm(x,0) \tag{1.17}$$

and the *Stieltjes–Perron inversion formula*

$$\sigma_\pm(x,A) = \lim_{\varepsilon\to 0+}\frac{1}{2\pi}\int_A \operatorname{Im}\left[\frac{\mp U_\pm(x,-\lambda-\mathrm{i}\varepsilon)}{-\lambda-\mathrm{i}\varepsilon}\right]\mathrm{d}\lambda \tag{1.18}$$

which holds for every $\sigma_\pm$-measurable set $A\subset(0,\infty)$. Then

$$\nu_\pm\bigl(x,\{\infty\}\bigr) = \sigma_\pm\bigl(x,\{0\}\bigr) \quad\text{and}\quad \nu_\pm(x,\mathrm{d}y) = \int_0^\infty z\mathrm{e}^{-yz}\sigma_\pm(x,\mathrm{d}z)\,\mathrm{d}y. \tag{1.19}$$

A function of the form

$$S(\omega) = \int_0^\infty \frac{\sigma(\mathrm{d}z)}{\omega+z}, \quad \omega\in\mathbb{C}\setminus\mathbb{R}_-,$$

is called a *Stieltjes function* (or *transform*). Stieltjes [44] considered the problem of recovering a measure on $\mathbb{R}_+$ from its moments, and the continued fraction

$$\frac{1}{m_0\omega}+\frac{1}{\ell_1}+\frac{1}{m_1\omega}+\frac{1}{\ell_2}+\cdots \tag{1.20}$$

played an important part in his solution of this moment problem, as we proceed to explain. Suppose that the measure σ satisfies the following moment condition:

$$\text{For every non-negative integer } n \quad \int_0^\infty z^n \sigma(\mathrm{d}z) \text{ exists.} \tag{1.21}$$

For simplicity, suppose also that σ has infinitely many points of growth. Then S has an asymptotic expansion in decreasing powers of ω, namely

$$S(\omega) \sim \sum_{n=0}^{\infty} \left[\int_0^\infty (-z)^n \sigma(\mathrm{d}z) \right] \omega^{-n-1} \quad \text{as } \omega \to +\infty,$$

and one may associate with S two sequences $\{m_n\}$ and $\{\ell_n\}$ of positive numbers by requiring that the finite truncations of (1.20) have asymptotic expansions that agree with the above series up to some order [1,4]. Importantly, only the first $2n+1$ of the moments are required in order to calculate the coefficients m_n and ℓ_n. Stieltjes showed how, knowing these coefficients, a measure with the required moments can be constructed [1,37,44]. Furthermore, he showed that if the series

$$\sum_{n=1}^{\infty} m_n \quad \text{or} \quad \sum_{n=0}^{\infty} \ell_n$$

diverges, then there is only one measure σ with the given moments, and the infinite continued fraction converges to $S(\omega)$.

1.3. *Statement of the main results and outline of the paper*

Our first task is to bring out the relationship between the continued fraction of Stieltjes and our own. To this end, and with a view to making the paper reasonably self-contained, we recall in Section 2 some of the results from Krein's theory of strings that are most relevant to diffusion processes. In Krein's terminology, Eq. (1.20) is the characteristic function of a string with a discrete distribution of masses m_n such that m_0 is at 0 and the spacing between m_n and m_{n-1} is ℓ_n. Then, in Section 3, we prove the following theorem.

Theorem 1. *Let X be a non-singular diffusion process started at x. Suppose that $\sigma_\pm(x,\cdot)$ has infinitely many points of growth and that the following moment condition holds*:

$$\textit{For every positive integer } n \quad \int_0^\infty z^{-n} \sigma_\pm(x, \mathrm{d}z) < \infty. \tag{M}$$

Then there is a sequence of positive numbers

$$\mp u_{1,\pm}(x), \qquad \mp u_{2,\pm}(x), \ldots$$

such that, for every, $n = 0, 1, 2, \ldots$

$$U_\pm(x,\lambda) - U_\pm(x,0) = \frac{2\lambda/a(x)}{u_{1,\pm}(x)} \,+\, \frac{2\lambda/a(x)}{u_{2,\pm}(x)} \,+ \cdots +\, \frac{2\lambda/a(x)}{u_{n,\pm}(x)} + \mathrm{O}\big(\lambda^{n+1}\big) \quad \textit{as } \lambda \to 0+.$$

Conversely, suppose that the identity

$$U_\pm(x,\lambda) - U_\pm(x,0) = \frac{2\lambda/a(x)}{u_{1,\pm}(x)} \,+\, \frac{2\lambda/a(x)}{u_{2,\pm}(x)} \,+\, \frac{2\lambda/a(x)}{u_{3,\pm}(x)} \,+ \cdots$$

holds for some infinite sequence $\mp u_{1,\pm}(x), \mp u_{2,\pm}(x), \ldots$ of positive numbers such that the series

$$\sum_{n=1}^{\infty} u_{n,\pm}(x)$$

diverges. Then $\sigma_\pm(x, \cdot)$ *has infinitely many points of growth and the moment condition* (M) *holds.*

By using (1.19) and Fubini's theorem, it is easy to see that the moment condition (M) can be written in the equivalent form

$$\text{For every positive integer } n \quad \int_0^\infty y^n \nu_\pm(x, \mathrm{d}y) < \infty. \tag{1.22}$$

So when the coefficients are positive, we immediately glean some information about the tail behaviour of the Lévy measure. In this case, the continued fraction (1.6) is related to that used by Stieltjes in his study of the moment problem, and the Riccati solution $U_\pm(x, \lambda)$ has an expansion in increasing powers of λ, from which the continued fraction coefficients can be computed by a well-known algorithm [4].

The moment condition (M), or its equivalent form (1.22), is by no means necessary for the Riccati solution $U_\pm$ to have a continued fraction expansion *if we allow the coefficients* $u_{n,\pm}$ *to be of arbitrary sign*. In such cases, $U_\pm(x, \lambda)$ is no longer asymptotic to a power series in λ. Nevertheless, the continued fraction coefficients may be computed by an algorithm, presented in a more general form in Common and Roberts [13], which we describe in Section 4.

Theorem 2. *Suppose that the Riccati equation* (1.5) *has a solution expressible in the form* (1.6). *Then, for* $n \in \mathbb{N}$, u_n *solves the homogeneous Riccati equation*

$$\frac{\mathrm{d}u_n}{\mathrm{d}x} + u_n^2 + 2W_n' u_n = 0, \tag{1.23}$$

where

$$W_n' = \frac{a'}{2a} - u_{n-1} - W_{n-1}', \qquad W_0' := W'. \tag{1.24}$$

In some cases, this expansion algorithm admits a probabilistic interpretation in terms of a sequence of diffusion processes, where the nth process corresponds to the environment W_n. In Section 5, we study the relationship between adjacent processes in this sequence.

In Section 6, we consider some simple examples of diffusions in a deterministic environment which illustrate these results. This reveals a surprising connection with the Ciesielski–Taylor theorem generalised by Biane [7,14], which exhibits a large class of ordered pairs of diffusions such that some occupation time of the first process has the same law as some hitting time of the second process.

Finally, in Section 7, we return to the problem that provided the initial motivation for this work, and consider the case where the environment W is itself a diffusion process.

Theorem 3. *Let* W *be a standard Brownian motion with positive drift. Then the system of Eqs* (1.23) *and* (1.24), *understood in the sense of Stratonovich, has a stationary solution such that* $u_0 = 0$ *and* $u_1, u_2, \ldots$ *are independent random variables with the same gamma distribution.*

Thus, we recover the continued fraction of [33,35,36].

As this outline indicates, the present paper deals exclusively with what may be called the *direct problem*: given the characteristics a and b of the diffusion, together with appropriate boundary conditions, we use a continued fraction expansion to compute the Lévy measure of the excursions. The *inverse problem*, namely that of finding the diffusion given the Lévy measure, is also of great interest [18]. The relevance of our results to the solution of this inverse problem will be developed in a separate publication.

2. Krein's theory of strings and diffusions

This section provides a succinct review of some well-known results concerning strings and diffusions. Kotani and Watanabe's account of these topics in [32] is particularly well suited to our purpose, and we follow them very closely. The reader familiar with this material need only take note of Definition 2.2 before proceeding to the next section.

2.1. *Strings*

A *string*, say $\mathbf{m}$, is a function from $[0,\infty]$ to $[0,\infty]$ that is non-decreasing, right-continuous and infinite at infinity. Let $\mathbf{m}$ be a string and suppose that $\mathbf{m}$ is not identically infinite. Set

$$c := \inf\{\mathbf{x}\colon \mathbf{m}(\mathbf{x}) > 0\} \quad \text{and} \quad \ell := \sup\{\mathbf{x}\colon \mathbf{m}(\mathbf{x}) < \infty\}.$$

Following [19], we then call the number $\ell - c$ the *length* of the string.

By setting $\mathbf{m}(0-) = 0$, we obtain from $\mathbf{m}$ a (Stieltjes) measure $\mathrm{d}\mathbf{m}$ on $[0,\infty)$. Let $\omega > 0$ and denote by $\xi(\cdot,\omega)$ and $\eta(\cdot,\omega)$ the (unique!) solutions of the following integral equations on $[0,\ell)$:

$$\xi(\mathbf{x},\omega) = 1 + \omega \int_0^{\mathbf{x}} \left[\int_{0-}^{y+} \xi(z,\omega)\,\mathrm{d}\mathbf{m}(z)\right]\mathrm{d}y$$

and

$$\eta(\mathbf{x},\omega) = \mathbf{x} + \omega \int_0^{\mathbf{x}} \left[\int_{0-}^{y+} \eta(z,\omega)\,\mathrm{d}\mathbf{m}(z)\right]\mathrm{d}y.$$

ξ (resp., η) can be viewed as the particular solution of the generalised differential equation

$$\frac{\mathrm{d}^2}{\mathrm{d}\mathbf{m}\,\mathrm{d}\mathbf{x}} f(\cdot,\omega) = \omega f(\cdot,\omega), \quad 0 < \mathbf{x} < \ell, \tag{2.1}$$

subject to the conditions

$$f(0,\omega) = 1, \qquad \frac{\mathrm{d}f}{\mathrm{d}\mathbf{x}}(0,\omega) = \mathbf{m}(0+)\omega \quad \left(\text{resp., } f(0,\omega) = 0,\ \frac{\mathrm{d}f}{\mathrm{d}\mathbf{x}}(0,\omega) = 1\right).$$

We shall refer to (ξ,η) as the pair of *fundamental solutions* associated with the string.

The function

$$S(\omega) := \lim_{\mathbf{x}\to\ell-} \frac{\eta(\mathbf{x},\omega)}{\xi(\mathbf{x},\omega)}$$

is called the *characteristic function* of the string. It may be shown that S is necessarily of the form

$$S(\omega) = c + \int_{[0,\infty)} \frac{\sigma(\mathrm{d}z)}{\omega + z} \tag{2.2}$$

for some measure σ on $[0,\infty)$ such that

$$\int_{[0,\infty)} \frac{\sigma(\mathrm{d}y)}{1+y} < \infty.$$

Conversely, Krein showed that every function of the form (2.2) is the characteristic function of a unique string.

The right-continuous inverse, denoted $\mathbf{m}^{-1}$, of a string $\mathbf{m}$ is called the *dual string* of $\mathbf{m}$. Denote by

$$S^d(\omega) = c^d + \int_{[0,\infty)} \frac{\sigma^d(\mathrm{d}z)}{z + \omega}$$

the characteristic function of the dual string. Then

$$c^d = \mathbf{m}(0+) \quad \text{and} \quad S^d(\omega) = \frac{1}{\omega S(\omega)}.$$

2.2. *Generalised diffusion processes*

Given a pair $(\mathbf{m}_-, \mathbf{m}_+)$ of strings such that $\mathbf{m}_-(0+) = 0$ and $\ell_\pm > 0$, let $\mathrm{d}\tilde{\mathbf{m}}_-$ be the image measure of $\mathrm{d}\mathbf{m}_-$ under the map $\mathbf{x} \mapsto -\mathbf{x}$ and set

$$\mathrm{d}\mathbf{m} := \begin{cases} \mathrm{d}\tilde{\mathbf{m}}_- & \text{on } (-\infty, 0), \\ \mathrm{d}\mathbf{m}_+ & \text{on } [0, \infty). \end{cases}$$

The support of $\mathrm{d}\mathbf{m}$, denoted $I_{\mathbf{m}}$, is a subset of $(-\ell_-, \ell_+)$. Let B be a standard Brownian motion, denote by $L_t^B(\mathbf{x})$ its local time and set

$$\varphi(t) := \int_{\mathbb{R}} L_t^B(\mathbf{x})\, \mathrm{d}\mathbf{m}.$$

Then

$$\mathbf{X}_t := B_{\varphi^{-1}(t)}$$

defines a Markov process on $I_{\mathbf{m}}$ whose lifetime is the first hitting time of $-\ell_-$ or ℓ_+. $\mathbf{X}$ is called the *generalised diffusion* corresponding to the pair $(\mathbf{m}_-, \mathbf{m}_+)$.

Definition 2.1. *A* one-sided *diffusion is a generalised diffusion such that either* $\mathbf{m}_-$ *or* $\mathbf{m}_+$ *is the zero string.*

Thus a generalised diffusion may be thought of as an ordered pair of one-sided diffusions.
The following non-standard definition will also be helpful later on:

Definition 2.2. *The* dual $\mathbf{X}^d$ *of the generalised diffusion process* $\mathbf{X}$ *corresponding to the pair* $(\mathbf{m}_-, \mathbf{m}_+)$ *is the generalised diffusion process corresponding to the pair* $(\mathbf{m}_-^{-1}, \mathbf{m}_+^{-1})$.

2.3. *Non-singular diffusion processes*

Next, consider a diffusion process X with generator (1.3) whose state space I is an interval with left endpoint l and right endpoint r. For the sake of greater clarity, we shall, in this subsection, depart from our usual notation and use x_0 instead of x to denote the starting point of the process; this leaves us free to use x to denote some generic point in the closure of I. We say that X is *non-singular* [29,32] or *regular* [10] if, for every $x_0, y \in I$, $H(y)$ is finite with positive probability. We shall show that such a process, after a suitable transformation of the state space, is also a generalised diffusion process in the sense of the previous subsection.

A *scale function* s and a *speed measure* m associated with the diffusion are given respectively by

$$s(x) = \int_{x_0}^{x} \mathrm{e}^{-2W(y)}\, \mathrm{d}y \quad \text{and} \quad m(\mathrm{d}x) = \frac{2}{a(x)} \mathrm{e}^{2W(x)}\, \mathrm{d}x.$$

The infinitesimal generator may then be expressed in the equivalent form

$$\mathscr{G}_W = \frac{\mathrm{d}}{\mathrm{d}m}\left(\frac{1}{s'(x)}\frac{\mathrm{d}}{\mathrm{d}x}\right), \quad l < x < r.$$

In general, the scale function and the speed measure are not enough to determine completely (up to a constant factor) the non-negative monotonic solutions ϕ_- and ϕ_+ of Eq. (1.4) that characterise the process. A boundary condition for $\phi_+(\cdot, \lambda)$ (resp., $\phi_-(\cdot, \lambda)$) will be required if and only if r (resp., l) is non-singular, and then only at r (resp., l); we use here the same terminology as Borodin and Salminen [10].

We will now show that the process $\mathbf{X} = s(X)$ is a generalised diffusion by constructing its associated pair of strings. Set $\mathbf{x} = s(x)$. Then

$$\mathscr{G}_W = \frac{\mathrm{d}^2}{\mathrm{d}\mathbf{m}\,\mathrm{d}\mathbf{x}}, \quad \mathbf{l} < \mathbf{x} < \mathbf{r},$$

where

$$\mathbf{m}(\mathrm{d}\mathbf{x}) = m(\mathrm{d}x), \qquad \mathbf{l} := \lim_{x \to l} s(x) \quad \text{and} \quad \mathbf{r} := \lim_{x \to r} s(x).$$

Let $\mathbf{m}_\pm$ be as follows: for $0 \le \mathbf{x} < \mathbf{r}$ (resp., $0 \le \mathbf{x} < -\mathbf{l}$), set

$$\mathbf{m}_+(\mathbf{x}) = \mathbf{m}\big([0, \mathbf{x}]\big) \quad \big(\text{resp., } \mathbf{m}_-(\mathbf{x}) = \mathbf{m}\big([-\mathbf{x}, 0]\big)\big).$$

To make $\mathbf{m}_\pm$ into a string, we need to define it over the whole of $[0, \infty]$. It will suffice to discuss the extension of $\mathbf{m}_+$ into a string of length ℓ_+; the extension of $\mathbf{m}_-$ into a string of length ℓ_- is analogous and will be immediately obvious.

If r is singular, then it must be that $\mathbf{r} + \mathbf{m}([0, \mathbf{r}]) = \infty$. This forces $\ell_+ := \mathbf{r}$ and

$$\mathbf{m}_+(\mathbf{x}) = \infty \quad \text{for } \mathbf{x} \ge \ell_+.$$

On the other hand, if r is non-singular, we need to bring in the boundary condition satisfied by $\phi_+(\cdot, \lambda)$:

(1) If r belongs to I, we use the Feller-type boundary condition

$$\alpha_+ \phi_+(r, \lambda) + \beta_+ s'(r) \phi_+'(r, \lambda) + \gamma_+ \lambda \phi_+(r, \lambda) = 0,$$

where

$$\alpha_+ \ge 0, \qquad \beta_+ > 0, \qquad \gamma_+ \ge 0 \quad \text{and} \quad \alpha_+ + \beta_+ + \gamma_+ = 1.$$

This boundary condition is implemented by setting $\ell_+ := \mathbf{r} + \frac{\beta_+}{\alpha_+}$ and

$$\mathbf{m}_+(\mathbf{x}) = \begin{cases} \mathbf{m}\big([0, \mathbf{r}]\big) + \frac{\gamma_+}{\beta_+} & \text{for } \mathbf{r} \le \mathbf{x} < \ell_+, \\ \infty & \text{for } \mathbf{x} \ge \ell_+. \end{cases}$$

(2) If r does not belong to I, we use the killing boundary condition

$$\phi_+(r-, \lambda) \quad \big(\text{resp., } \phi_-(l+, \lambda)\big) = 0.$$

This is implemented by setting $\ell_+ = \mathbf{r}$ and

$$\mathbf{m}_+(\mathbf{x}) = \infty \quad \text{if } \mathbf{x} \ge \ell_+.$$

By using the representation of $\mathbf{X}$ as a standard Brownian motion with a random clock (see, for instance, [25], Chapter 5), it is readily verified that it is the generalised diffusion process corresponding to the pair of strings thus constructed. In particular, by expressing $\phi_\pm$ in terms of the fundamental solutions corresponding to the strings $\mathbf{m}_-$ and $\mathbf{m}_+$, we deduce easily that

$$\mp U_\pm(x_0, \lambda) = \frac{s'(x_0)}{S_\pm(\lambda)} = \lambda s'(x_0) S_\pm^d(\lambda), \tag{2.3}$$

where $S_\pm$ is the characteristic function of $\mathbf{m}_\pm$. It is also clear from the foregoing construction that $c_\pm^d = 0$; so this formula is equivalent to our earlier Eq. (1.16), from which (1.19) follows immediately.

Remark 2.1. *In the case where the process X is started at a reflecting boundary, then one of $\mathbf{m}_+$ or $\mathbf{m}_-$ is the zero string, and so the corresponding generalised diffusion process is a one-sided diffusion.*

3. Proof of Theorem 1

Suppose that $\sigma_\pm(x,\cdot)$ has infinitely many points of growth and that the moment condition (M) holds. Define a measure σ on $\mathbb{R}_+$ by

$$\sigma(A) = \int_{\tilde{A}} z^{-1}\sigma_\pm(x,\mathrm{d}z), \qquad \tilde{A} := \{1/z\colon\ z \in A\}. \tag{3.1}$$

Then σ has infinitely many points of growth and satisfies the Stieltjes moment condition (1.21). So we can write

$$\int_0^\infty \frac{\sigma(\mathrm{d}z)}{\omega+z} = \frac{1}{\omega}\int_0^\infty \frac{\sigma(\mathrm{d}z)}{1+z/\omega} \sim \sum_{j=0}^\infty \left[\int_0^\infty (-z)^j\sigma(\mathrm{d}z)\right]\omega^{-j-1} \quad \text{as } \omega \to +\infty.$$

Following Stieltjes [44] (see also [1,32,37]), one can construct from this series two sequences $\{m_k\}$ and $\{\ell_k\}$ of positive numbers such that, for every $n \in \mathbb{N}$,

$$\frac{1}{m_0\omega+}\,\frac{1}{\ell_1+}\,\frac{1}{m_1\omega+}\,\frac{1}{\ell_2+\cdots+}\,\frac{1}{r_n} = \sum_{j=0}^{n'}\left[\int_0^\infty (-z)^j\sigma(\mathrm{d}z)\right]\omega^{-j-1} + \mathrm{O}\big(\omega^{-n'-2}\big)$$

as $\omega \to +\infty$, where

$$n' = \begin{cases} 2n-1 & \text{if } r_n = \ell_n, \\ 2n & \text{if } r_n = m_n\omega. \end{cases}$$

Thus, in the same limit, we can write

$$\begin{aligned}
&\frac{1}{m_0\omega+}\,\frac{1}{\ell_1+}\,\frac{1}{m_1\omega+}\,\frac{1}{\ell_2+\cdots+}\,\frac{1}{r_n} + \mathrm{O}\big(\omega^{-n'-2}\big) \\
&\quad = \int_0^\infty \frac{\sigma(\mathrm{d}z)}{\omega+z} \overset{z\to 1/z}{=} \int_0^\infty \frac{\sigma_\pm(x,\mathrm{d}z)}{1+z\omega} = 1/\omega\int_0^\infty \frac{\sigma_\pm(x,\mathrm{d}z)}{1/\omega+z} = 1/2\big[\mp U_\pm(x,1/\omega) \pm U_\pm(x,0)\big],
\end{aligned}$$

where we have made use of Eq. (1.16) to obtain the last equality. The first statement in the theorem follows if we take, for $n = 0, 1, \ldots,$

$$\mp u_{2n+1,\pm}(x) = \frac{m_n}{a(x)} \quad \text{and} \quad \mp u_{2n+2,\pm}(x) = 2\ell_n. \tag{3.2}$$

To prove the second statement, construct from the given $u_{n,\pm}(x)$ two sequences $\{m_n\}$ and $\{\ell_n\}$ of positive numbers via Eq. (3.2). The hypothesis implies that the expansion

$$\frac{1}{m_0\omega+}\,\frac{1}{\ell_1+}\,\frac{1}{m_1\omega+}\,\frac{1}{\ell_2+\cdots}$$

is a well-defined function of $\omega \in \mathbb{C}\setminus\mathbb{R}_-$. As shown by Stieltjes [44], since

$$\sum_{n=0}^\infty m_n \quad \text{or} \quad \sum_{n=1}^\infty \ell_n$$

diverges, there is one and only one measure σ on $\mathbb{R}_+$ such that

$$\int_0^\infty \frac{\sigma(\mathrm{d}z)}{\omega+z} = \frac{1}{m_0\omega+}\,\frac{1}{\ell_1+}\,\frac{1}{m_1\omega+}\,\frac{1}{\ell_2+\cdots}.$$

Furthermore, σ satisfies the moment condition (1.21). The uniqueness of σ implies easily that Eq. (3.1) holds, and so $\sigma_\pm(x,\cdot)$ satisfies the condition (M). The fact that it has infinitely many points of growth follows from the fact that the sequence of the $u_{n,\pm}(x)$ is infinite.

4. The expansion algorithm

We now describe a method of obtaining the continued fraction which is based on the fact that the *form* of the Riccati equation does not change under a certain linear fractional transformation. The idea of the method can be traced back to Euler [20]; it appears also, for different purposes, in the works of Darboux [15] and Crum [12] on isospectral transformations of linear differential operators and, much more recently, in Common and Roberts [13].

Set

$$U(x,\lambda) =: U_0(x,\lambda) = u_0(x) + \frac{2\lambda/a(x)}{U_1(x,\lambda)}, \tag{4.1}$$

where u_0 and U_1 are some functions which we shall specify presently. Substitution in Eq. (1.5) yields

$$u_0' - \frac{2\lambda/a}{U_1}\frac{a'}{a} - \frac{2\lambda/a}{U_1^2}U_1' + u_0^2 + 2u_0\frac{2\lambda/a}{U_1} + 2\lambda/a\frac{2\lambda/a}{U_1^2} + 2W'u_0 + 2W'\frac{2\lambda/a}{U_1} = 2\lambda/a.$$

Now *choose* u_0 so that it solves the *homogeneous* Riccati equation

$$u_0' + u_0^2 + 2W'u_0 = 0.$$

Then the equation satisfied by U_1 is

$$U_1' + U_1^2 + 2\left(\frac{a'}{2a} - u_0 - W'\right)U_1 = 2\lambda/a.$$

This is of the same form as Eq. (1.5), save that W' has been replaced by $a'/(2a) - u_0 - W'$. By iterating, we deduce the expansion

$$U(x,\lambda) = u_0(x) + \frac{2\lambda/a(x)}{u_1(x)} {\atop +\cdots+} \frac{2\lambda/a(x)}{u_n(x)} {\atop +} \frac{2\lambda/a(x)}{U_{n+1}(x,\lambda)}. \tag{4.2}$$

In this expression, u_n satisfies the homogeneous Riccati equation

$$\frac{\mathrm{d}u_n}{\mathrm{d}x} + u_n^2 + 2W_n'u_n = 0, \tag{4.3}$$

the remainder U_n satisfies the *inhomogeneous* Riccati equation

$$\frac{\mathrm{d}U_n}{\mathrm{d}x} + U_n^2 + 2W_n'U_n = 2\lambda/a \tag{4.4}$$

and

$$W_n' = \frac{a'}{2a} - u_{n-1} - W_{n-1}', \qquad W_0' := W'.$$

Theorem 2 is thus proved.

The calculation of the non-trivial solutions of the homogeneous Riccati equation, i.e. those solutions that are not identically zero, is sometimes facilitated by the following proposition.

Proposition 4.1. *The non-trivial solution u_n of the homogeneous Riccati equation of index n satisfies the recurrence formula*: *for every $n \ge k \ge 0$,*

$$u_n(x) = \frac{\mathrm{d}}{\mathrm{d}x}\ln\left|\int^x \frac{\mathrm{e}^{-2W_k(y)}\,\mathrm{d}y}{\prod_{j=k}^{n-1}[a(y)u_j^2(y)]}\right|.$$

Proof. By assumption, $u_j \neq 0$ for $k \leq j \leq n$, and so the homogeneous Eq. (4.3) is readily integrated to yield

$$u_j(x) = \frac{e^{-2W_j(x)}}{\int^x e^{-2W_j(y)}\,dy} = \frac{d}{dx}\ln\left|\int^x e^{-2W_j(y)}\,dy\right|.$$

Let us write

$$W_n(x) = \int^x W_n'(y)\,dy,$$

where the indefinite integral means that W_n is any primitive of the integrand. For $n > k$,

$$\begin{aligned}-2W_n(x) &= -2\int^x W_n'(y)\,dy = -2\int^x \left[\frac{1}{2}\frac{a'(y)}{a(y)} - u_{n-1}(y) - W_{n-1}'(y)\right]dy \\ &= -\ln a(x) + 2W_{n-1}(x) + 2\ln\left|\int^x e^{-2W_{n-1}(y)}\,dy\right|\end{aligned}$$

and so

$$e^{-2W_n(x)} = \frac{1}{a(x)}e^{2W_{n-1}(x)}\left|\int^x e^{-2W_{n-1}(y)}\,dy\right|^2 = \frac{e^{-2W_{n-1}(x)}}{a(x)u_{n-1}^2(x)}.$$

The required formula follows easily by iterating. □

5. A probabilistic interpretation of the algorithm

It is straightforward to give a probabilistic interpretation of the first few coefficients in the continued fraction. Indeed, from Eqs (1.14) and (1.15) and Eq. (15) of [40],

$$u_{0,-}(x) - u_{0,+}(x)$$

is inversely proportional to the mean of the local time spent at x. Also,

$$\frac{1}{\mp a(x)u_{1,\pm}(x)} = \int_0^\infty y\nu_\pm(x, dy)$$

and hence $\mp u_{1,\pm}(x)$ is inversely proportional to the average duration of the finite excursions to the right (+) or left (−).

Our aim in this section is to gain some insight into the probabilistic content of the *expansion algorithm* itself. If the initial diffusion X has a Riccati variable U with a continued fraction expansion, then, as can be seen from the proof of Theorem 2 given in the previous section, the algorithm produces a sequence of W_n and a sequence of remainders U_n. Each U_n can be thought of as a Riccati variable of some diffusion, say X_n, corresponding to the "environment" W_n, and it is then natural to investigate the relationship between adjacent diffusions in this sequence. In pursuing this line of thought, it is important to bear in mind the following points:

(1) The W_n depend on which of the two Riccati variables U_+ and U_- is being expanded. In particular, if *both* U_+ and U_- have a continued fraction expansion, then there are *two* sequences of W_n.
(2) It takes *two* strings to specify a regular diffusion – each string corresponding to a one-sided diffusion (see Definition 2.1 and Remark 2.1). Each Riccati variable specifies one string, and the regular diffusion is obtained by welding the strings together.

In the remainder of this section, we will show how, from *one* continued fraction expansion of a Riccati variable of the regular diffusion X, one can construct a particular sequence of diffusions X_n such that the remainder U_n is a Riccati variable of X_n. It should be clear from the points just made that sequences with this property cannot be unique. Our particular construction is inspired by the following key formula, proved by Pitman and Yor [40]:

$$\psi_\pm(x,\lambda) = \psi_\pm(x,0) \mp \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}y}\bigg|_{y=x\pm} \mathbb{E}_y\big(\mathrm{e}^{-\lambda H(x)} | H(x) < \infty\big). \tag{5.1}$$

To simplify matters, we assume that the speed measure m of X is absolutely continuous with respect to the Lebesgue measure and, with some abuse of notation, write

$$m(A) = \int_A m(x)\,\mathrm{d}x$$

for every measurable set A. The instantaneous variance a and the instantaneous drift b of X are then given by

$$a = \frac{2}{ms'} \quad \text{and} \quad b = \frac{1}{m}\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{s'}.$$

5.1. *The h-transform*

For definiteness, let us suppose that we work with a continued fraction expansion of the Riccati variable U_+. Taking the $+$ sign in Eq. (5.1) and making use of the identity (1.15), we obtain

$$U_+(x,\lambda) = U_+(x,0) - \frac{\mathrm{d}}{\mathrm{d}y}\bigg|_{y=x+} \mathbb{E}_y\big(\mathrm{e}^{-\lambda H(x)} | H(x) < \infty\big).$$

We will show that the second term on the right-hand side is the Riccati variable of a diffusion, say Y, obtained from X by conditioning, and we will express the characteristics of Y in terms of the characteristics of X. In particular, it will follow from our construction that the Riccati variables $U_\pm^Y$ of Y are related to those of X via

$$U_+(x,\lambda) = U_+(x,0) + U_+^Y(x,\lambda) \quad \text{and} \quad U_-(x,\lambda) = U_+(x,0) + U_-^Y(x,\lambda). \tag{5.2}$$

If $U_+(\cdot,0)$ vanishes, then we take $Y = X$. Otherwise, X is transient and the event

$$\lim_{t\to\zeta} X_t = r,$$

where ζ denotes the lifetime of X, occurs with positive probability. Let $l < z < x$ and denote by Y the process obtained from X by conditioning on the event

$$H(z) < \infty.$$

We have (see [10], Section II.12)

$$\mathbb{P}_x\big(H(z) < \infty\big) = \phi_+(x,0) =: h(x),$$

where $\mathbb{P}_x$ is the probability measure associated with X started at x and ϕ_+ is normalised so that $\phi_+(z,0) = 1$.

For $z < y < x$, denote by $\mathrm{d}y$ an interval of infinitesimal length centered on y. Then

$$\begin{aligned}\mathbb{P}_x(Y_t \in \mathrm{d}y) = \mathbb{P}_x\big(X_t \in \mathrm{d}y | H(z) < \infty\big) &= \frac{\mathbb{P}_x(X_t \in \mathrm{d}y, H(z) < \infty)}{\mathbb{P}_x(H(z) < \infty)} \\ &= \frac{\mathbb{P}_x(X_t \in \mathrm{d}y)\mathbb{P}_y(H(z) < \infty)}{\mathbb{P}_x(H(z) < \infty)} = \frac{\mathbb{P}_x(X_t \in \mathrm{d}y)h(y)}{h(x)}.\end{aligned}$$

Since h is excessive (see [10], Section II.30), this calculation shows that Y is the h-transform of X. This result does not depend on the particular choice of z as long as $l < z < x$. Clearly,

$$\phi_+^Y = \frac{\phi_+}{h}.$$

Table 1
The transformation $\mathcal{T}_h$ and its effect on the characteristics of the diffusion; see Section 5.1 for the choice of h and how it depends on the particular Riccati variable that is being expanded

X	$\xrightarrow{h\text{-transform}}$	Y	$\xrightarrow{\text{Krein duality}}$	$\mathcal{T}_h(X)$
m		$h^2 m$		$h^{-2}s'$
s'		$h^{-2}s'$		$h^2 m$
a		a		a
b		$b + ah'/h$		$a'/2 - b - ah'/h$
$U_\pm(\cdot,\lambda)$		$U_\pm(\cdot,\lambda) - h'/h$		$\frac{2\lambda/a}{U_\pm(\cdot,\lambda)-h'/h}$

The characteristics of Y are displayed in Table 1. It follows in particular from (5.2) that $U_+^Y(\cdot,0) = 0$; this is consistent with the well-known fact that, in the transient case, Y converges almost surely to the boundary point ℓ [39,42].

The foregoing discussion assumed that one is working with the Riccati variable U_+. If, instead, one is working with the other Riccati variable U_-, then one should use $h = \phi_-(\cdot,\lambda)$ to define Y. It is then easily verified that

$$U_+(x,\lambda) = U_-(x,0) + U_+^Y(x,\lambda) \quad \text{and} \quad U_-(x,\lambda) = U_-(x,0) + U_-^Y(x,\lambda)$$

holds instead of Eq. (5.2).

5.2. *Krein duality*

Given Eq. (5.2), we ask next for a diffusion, say Z, whose Riccati variables are related to those of Y via

$$U_\pm^Z(x,\lambda)U_\pm^Y(x,\lambda) = 2\lambda/a(x). \tag{5.3}$$

We emphasise that this equation should be understood as specifying *both* U_+^Z and U_-^Z. Equation (2.3) points to the key rôle played by the concept of duality. However, as defined in Section 2, duality requires the non-singular diffusion to be in its natural scale, and so we shall use instead the following closely related concept.

Definition 5.1. *Let X be a non-singular diffusion process with speed measure m and scale function s. The process*

$$X^* := m^{-1}\big(\mathbf{X}^d\big),$$

where $\mathbf{X} = s(X)$ and $\mathbf{X}^d$ is the dual of the generalised diffusion $\mathbf{X}$, is called the Krein dual *of X.*

To put it simply, this new Krein dual is the old dual of $s(X)$ after a certain transformation of the state space. Our justification for using this non-standard terminology is contained in the following proposition.

Proposition 5.1. *Let X be a non-singular diffusion process, started at x, on an interval I that includes no non-singular boundary points – except possibly reflecting ones. Let X^* be the Krein dual of X. Then*

(1) *X^* is a non-singular diffusion process, started at x, on an interval I^* with the same endpoints as I and that includes no non-singular boundary points – except possibly reflecting ones*;
(2) *the speed measure of X is the scale function of X^* and vice-versa*;
(3) *the Riccati variables U_+^* and U_-^* of X^* are related to the Riccati variables U_+ and U_- of X via*

$$U_\pm(x,\lambda)U_\pm^*(x,\lambda) = 2\lambda/a(x),$$

where a is the common infinitesimal variance of X and X^*.

Proof. We work "backwards," i.e. from X, we define a non-singular process $\hat{X}$ on an interval I^* with the same endpoints as I, started at x. Then we show that its associated pair of strings is the same as that of the generalised diffusion process $m(X^*)$. For the sake of convenience, we shall use in the proof the same notation for the process X as in Section 2.3.

In the interior of I^*, the process $\hat{X}$ is determined by the generator

$$\mathscr{G}_W^* := \frac{1}{s'(x)}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{m(x)}\frac{\mathrm{d}}{\mathrm{d}x}\right).$$

The boundary behaviour of $\hat{X}$ is as follows: for every non-singular $p \in \{l, r\}$, if p is reflecting for X then it is killing for $\hat{X}$ and $p \notin I^*$; if p is killing for X, then p is reflecting for $\hat{X}$ and $p \in I^*$.

We remark that, for $\mathbf{l} < \mathbf{x} < \mathbf{r}$,

$$\mathscr{G}_W^* \overset{\mathbf{x}=s(x)}{\overset{\downarrow}{=}} \frac{\mathrm{d}}{\mathrm{d}\mathbf{x}}\left(\frac{s'(x)}{m(x)}\frac{\mathrm{d}}{\mathrm{d}\mathbf{x}}\right) = \frac{\mathrm{d}^2}{\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{m}(\mathbf{x})},$$

where $\mathrm{d}\mathbf{m}(\mathbf{x}) = \mathbf{m}(\mathrm{d}\mathbf{x}) = m(\mathrm{d}x)$. Recalling the definition of the string $\mathbf{m}_\pm$, this shows that the natural scale of $\hat{X}$ is

$$\mathbf{y} = \mathbf{m}(\mathbf{x}) = \begin{cases} \mathbf{m}_+(\mathbf{x}) & \text{if } 0 < \mathbf{x} < \mathbf{r}, \\ -\mathbf{m}_-(-\mathbf{x}) & \text{if } \mathbf{l} < \mathbf{x} \le 0 \end{cases}$$

and we deduce

$$\hat{\mathbf{m}}_+(\mathbf{y}) = \mathbf{m}_+^{-1}(\mathbf{y}), \quad 0 < \mathbf{y} < \mathbf{m}_+(\mathbf{r}), \qquad \hat{\mathbf{m}}_-(\mathbf{y}) = \mathbf{m}_-^{-1}(\mathbf{y}), \quad 0 \le \mathbf{y} < \mathbf{m}_-(-\mathbf{l}). \tag{5.4}$$

To show that $m(\hat{X}) = \mathbf{X}^d$, there only remains to extend the validity of these two equalities to every $\mathbf{y} \in [0, \infty]$. Consider the extension of the first equality. As explained in Section 2.3, the definition of $\hat{\mathbf{m}}_+$ in $[\mathbf{m}_+(\mathbf{r}), \infty]$ depends on the nature of the boundary point r. We bear in mind that, since the speed measure of $\hat{X}$ is the scale function of X and vice-versa, r is entrance for $\hat{X}$ if and only if it is exit for X and vice-versa. There are three cases to consider:

(i) If r is singular for $\hat{X}$, then $\hat{\mathbf{m}}_+(\mathbf{y}) = \infty$ for $\mathbf{y} \ge \mathbf{m}_+(\mathbf{r})$. Since r is also singular for X, we have $\mathbf{r} + \mathbf{m}_+(\mathbf{r}) = \infty$. If $\mathbf{m}_+(\mathbf{r}) = \infty$, there is nothing to extend; if $\mathbf{m}_+(\mathbf{r}) < \infty$, then $\mathbf{r} = \infty$ and, since $\mathbf{m}_+^{-1}$ is by definition the right-continuous inverse of $\mathbf{m}_+$,

$$\mathbf{m}_+^{-1}(\mathbf{y}) = \infty = \hat{\mathbf{m}}_+(\mathbf{y}) \quad \text{for } \mathbf{y} \ge \mathbf{m}_+(\mathbf{r}).$$

(ii) If r is reflecting for $\hat{X}$, then

$$\hat{\mathbf{m}}_+(\mathbf{y}) = \begin{cases} \mathbf{r} & \text{for } \mathbf{m}_+(\mathbf{r}) \le \mathbf{y} < \infty, \\ \infty & \text{for } \mathbf{y} = \infty. \end{cases}$$

On the other hand, r is killing for X, i.e. $\mathbf{m}_+(\mathbf{x}) = \infty$ for $\mathbf{x} \ge \mathbf{r}$. Again, since $\mathbf{m}_+^{-1}$ is by definition the right-continuous inverse of $\mathbf{m}_+$, we deduce $\mathbf{m}_+^{-1}(\mathbf{y}) = \hat{\mathbf{m}}_+(\mathbf{y})$ for $y \ge \mathbf{m}_+(\mathbf{r})$.

(iii) If r is killing for $\hat{X}$, then it is reflecting for X, and the desired result follows from (ii) by symmetry.

The extension of the second equality in (5.4) follows along the same lines. We have thus shown that $\hat{X} = X^*$.

The first two statements in the proposition follow immediately. For the last statement, we apply Eq. (2.3) to X and to X^*; this gives

$$U_\pm(x,\lambda)U_\pm^*(x,\lambda) = \frac{s'(x)}{S_\pm(\lambda)}\frac{m(x)}{S_\pm^d(\lambda)} = \frac{2/a(x)}{1/\lambda}. \qquad \square$$

This concept of duality is analogous to that used by Jansons [26], Soucaliuc [43] and Tóth [45] (see the definition of *conjugate diffusion* in his Appendix 1); all we have done is to bring out its connection with Krein's theory of strings. Table 1 shows how the characteristics of a process transform under Krein duality in the situation envisaged by Proposition 5.1.

To summarise the foregoing discussion, one can – at least in some cases – describe the first iteration of the expansion algorithm in terms of a map $\mathcal{T}_h$ obtained by composing two transformations:

$$X \xrightarrow{h\text{-transform}} Y \xrightarrow{\text{Krein duality}} Z =: \mathcal{T}_h(X).$$

Of course, this description only makes sense if the diffusion Y, obtained from X after conditioning, satisfies the hypothesis of Proposition 5.1. If X is an arbitrary non-singular diffusion, there is no guarantee that this will be the case, and so we have to be content to *assume* that the hypothesis is satisfied. Then, expanding the Riccati variable U_+ of X leads, after one step, to the Riccati variable of the new diffusion $\mathcal{T}_h(X)$ where $h = \phi_+(\cdot, 0)$. If, instead, one works with the Riccati variable U_-, then $h = \phi_-(\cdot, 0)$ is used to construct the next diffusion $\mathcal{T}_h(X)$.

6. Some deterministic examples

There are two methods for calculating the continued fraction coefficients: the first uses the algorithm of Section 4; the second proceeds by tracking the diffusions produced by the recurrence

$$X \to \mathcal{T}_{h_0}(X) \to \mathcal{T}_{h_1} \circ \mathcal{T}_{h_0}(X) \to \cdots.$$

The latter method provides greater insight but is not always applicable; the former method can always be used, but the determination of the constants of integration can be tedious. In this section, we study some well-known diffusions in deterministic environments which will serve to illustrate the two approaches. The processes that we consider have the remarkable property of belonging to a parametrised class that is closed under both the $\phi_\pm(\cdot, 0)$-transform and Krein duality; see Table 2. Our main reference is [10], Appendix I.

6.1. *Brownian motion with drift*

Let B^μ be a Brownian motion with drift μ and set $X = B^\mu$. The generator is

$$\mathcal{G}_W = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \mu\frac{\mathrm{d}}{\mathrm{d}x}$$

and the state space is $\mathbb{R}$; the endpoints are natural and so there are no boundary conditions. We deduce that the Riccati equation has a unique positive solution and a unique negative solution. We find

$$\phi_\pm(x,\lambda) = \exp\bigl[x\bigl(-\mu \mp \sqrt{\mu^2 + 2\lambda}\bigr)\bigr] \quad \text{and so} \quad U_\pm(x,\lambda) = -\mu \mp \sqrt{\mu^2 + 2\lambda}.$$

Table 2
The $\phi_\pm(\cdot, 0)$-transform $X^\pm$ and the Krein dual X^* for some processes X in a deterministic environment: B^μ denotes Brownian motion with drift μ and $\mathrm{BES}(p)$ denotes a Bessel process of parameter p

X	Parameters	X^-	X^+	X^*
B^μ	$\mu < 0$	$B^{-\mu}$	B^μ	$B^{-\mu}$
B^μ	$\mu > 0$	B^μ	$B^{-\mu}$	$B^{-\mu}$
$\mathrm{BES}(p)$	$p \le -1$	$\mathrm{BES}(-p)$	$\mathrm{BES}(p)$	$\mathrm{BES}(-p-1)$
$\mathrm{BES}(p)$	$-1 < p < 0$, 0 killing	$\mathrm{BES}(-p)$	$\mathrm{BES}(p)$, 0 killing	$\mathrm{BES}(-p-1)$, 0 reflecting
$\mathrm{BES}(p)$	$-1 < p < 0$, 0 reflecting	$\mathrm{BES}(p)$, 0 reflecting	$\mathrm{BES}(p)$, 0 reflecting	$\mathrm{BES}(-p-1)$, 0 killing
$\mathrm{BES}(p)$	$p = 0$	$\mathrm{BES}(0)$	$\mathrm{BES}(0)$	$\mathrm{BES}(-1)$
$\mathrm{BES}(p)$	$0 < p < 1$	$\mathrm{BES}(p)$	$\mathrm{BES}(-p)$, 0 killing	$\mathrm{BES}(-p-1)$
$\mathrm{BES}(p)$	$p \ge 1$	$\mathrm{BES}(p)$	$\mathrm{BES}(-p)$	$\mathrm{BES}(-p-1)$

The Stieltjes–Perron inversion formula yields

$$\sigma_{\pm}(x,\, \mathrm{d}z) = \frac{1}{\pi} \frac{\sqrt{2z-\mu^2}}{2z}\, \mathrm{d}z, \quad z > \mu^2/2,$$

and

$$\sigma_{\pm}\big(x, \{0\}\big) = \frac{1}{2}\big[|\mu| \pm \mu\big].$$

Condition (M) is therefore satisfied unless $\mu = 0$ (standard Brownian motion). In fact, the expansion of $U_{\pm}$ is easily obtained by elementary means: assuming for definiteness that $\mu > 0$, we have

$$U_{-}(x,\lambda) = \sqrt{\mu^2 + 2\lambda} - \mu = \frac{2\lambda}{2\mu + U_{-}(x,\lambda)} = \frac{2\lambda}{2\mu} \frac{2\lambda}{+ 2\mu} \frac{2\lambda}{+ 2\mu} + \cdots$$

and

$$-U_{+}(x,\lambda) = \sqrt{\mu^2 + 2\lambda} + \mu = 2\mu + U_{-}(x,\lambda) = 2\mu + \frac{2\lambda}{2\mu} \frac{2\lambda}{+ 2\mu} \frac{2\lambda}{+ 2\mu} + \cdots.$$

Let us demonstrate how these expansions can be obtained by the algorithm of Section 4. Again, for definiteness, suppose that $\mu > 0$. We need to distinguish two cases, namely $u_0 = 0$ and $u_0 \neq 0$.

Take $u_0 = 0$. Then $W_1' = -\mu$, $W_1 = -\mu x$ and so

$$u_1(x) = \frac{\mathrm{e}^{2\mu x}}{\mathrm{e}^{2\mu x}/(2\mu) + c},$$

where c is a constant of integration. There is no possibility of choosing this constant so that u_1 is negative. But by taking $c \geq 0$, we obtain $u_1 > 0$. Furthermore,

$$u_2(x) = \frac{\mathrm{e}^{2\mu x}/u_1^2(x)}{\int^x \mathrm{e}^{2\mu y}/u_1^2(y)\, \mathrm{d}y} = \frac{(1/(4\mu^2))\mathrm{e}^{2\mu x} + c/\mu + c^2 \mathrm{e}^{-2\mu x}}{(1/(8\mu^3))\mathrm{e}^{2\mu x} + (c/\mu)x - (c^2/(2\mu))\mathrm{e}^{-2\mu x} + C},$$

where C is another constant of integration. The only way to ensure that u_2 is also positive is by taking $c = 0$. Hence

$$u_1(x) = 2\mu \quad \text{and} \quad u_2(x) = \frac{(1/(4\mu^2))\mathrm{e}^{2\mu x}}{(1/(8\mu^3))\mathrm{e}^{2\mu x} + C}.$$

By iterating, we obtain the expansion of U_{-}.

The other possibility is to take $u_0 \neq 0$. Since $W_0 = \mu x$, we find

$$u_0(x) = \frac{\mathrm{e}^{-2\mu x}}{(-1/(2\mu))\mathrm{e}^{-2\mu x} + c},$$

where c is a constant of integration. No choice of c can make u_0 positive but, by taking $c \leq 0$, we can make it negative. Then

$$u_1(x) = \frac{\mathrm{e}^{-2\mu x}/u_0^2(x)}{\int^x \mathrm{e}^{-2\mu y}/u_0^2(y)\, \mathrm{d}y}.$$

The numerator is positive; the denominator is

$$-\frac{1}{8\mu^3}\mathrm{e}^{-2\mu x} - \frac{c}{\mu}x + \frac{c^2}{2\mu}\mathrm{e}^{2\mu x} + C,$$

where C is another constant of integration. This expression cannot be negative for every x unless $c = 0$. We deduce

$$u_0(x) = -2\mu \quad \text{and} \quad u_1(x) = \frac{(1/(4\mu^2))\mathrm{e}^{-2\mu x}}{(-1/(8\mu^3))\mathrm{e}^{-2\mu x} + C}.$$

By iterating, we obtain the expansion of U_+.

Next, we demonstrate how the expansion may be interpreted in terms of a sequence of diffusions obtained via h-transforms and Krein duality. We have

$$\phi_\pm(x,0) = \exp\bigl[\bigl(-\mu \mp |\mu|\bigr)x\bigr].$$

Suppose that $\mu > 0$. Then $\phi_-(\cdot,0) = 1$, $\phi_+(x,0) = \mathrm{e}^{-2\mu x}$ and, from Table 1, it is easily deduced that the $\phi_+(\cdot,0)$-transform of B^μ is $B^{-\mu}$. On the other hand, if $\mu < 0$, then $\phi_+(\cdot,0) = 1$, $\phi_-(x,0) = \mathrm{e}^{-2\mu x}$, and so it is the $\phi_-(\cdot,0)$-transform of B^μ that yields $B^{-\mu}$. Also, from Table 1, it is immediate that the Krein dual of B^μ is $B^{-\mu}$. Putting these results together, we find, for example, that the expansion of U_+ for $\mu > 0$ corresponds to the sequence

$$\begin{aligned} X = B^\mu &\xrightarrow{\phi_+(\cdot,0)\text{-transform}} B^{-\mu} \xrightarrow{\text{Krein duality}} \mathcal{T}_{h_0}(X) \\ = B^\mu &\xrightarrow{\phi_+(\cdot,0)\text{-transform}} B^{-\mu} \xrightarrow{\text{Krein duality}} \mathcal{T}_{h_1} \circ \mathcal{T}_{h_0}(X) = B^\mu, \quad \text{etc.} \end{aligned}$$

6.2. *A Bessel process*

Let $\mathrm{BES}(p)$ denote the Bessel process with parameter p and, for $p > 0$, let $X = \mathrm{BES}(p)$. Then

$$\mathcal{G}_W = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{p+1/2}{x}\frac{\mathrm{d}}{\mathrm{d}x}, \quad x > 0,$$

0 is entrance-not-exit, ∞ is natural,

$$\phi_-(x,\lambda) = x^{-p} I_p\bigl(\sqrt{2\lambda}x\bigr), \qquad \phi_+(x,\lambda) = x^{-p} K_p\bigl(\sqrt{2\lambda}x\bigr)$$

and

$$U_-(x,\lambda) = \sqrt{2\lambda}\frac{I_{p+1}(\sqrt{2\lambda}x)}{I_p(\sqrt{2\lambda}x)}, \qquad -U_+(x,\lambda) = \sqrt{2\lambda}\frac{K_{p+1}(\sqrt{2\lambda}x)}{K_p(\sqrt{2\lambda}x)}.$$

To obtain these expressions for the Riccati solutions, we have made use of the three-term recurrence relations satisfied by the Bessel functions I_p and K_p:

$$I_{p-1}(z) - I_{p+1}(z) = \frac{2p}{z} I_p(z), \qquad 2I_p'(z) = I_{p-1}(z) + I_{p+1}(z)$$

and

$$K_{p-1}(z) - K_{p+1}(z) = -\frac{2p}{z} K_p(z), \qquad -2K_p'(z) = K_{p-1}(z) + K_{p+1}(z).$$

The Stieltjes measures are [24]:

$$\sigma_-\bigl(x,\{0\}\bigr) = 0, \qquad \sigma_-(x,\mathrm{d}z) = \sum_{k=1}^\infty \frac{1}{x}\delta_{z_k},$$

where δ_{z_k} is the probability measure with all the mass at $z_k := j_{p,k}^2/(2x^2)$, $j_{p,k}$ is the kth positive zero of the Bessel function J_p, and

$$\sigma_+\bigl(x,\{0\}\bigr) = \frac{p}{x}, \qquad \sigma_+(x,\mathrm{d}z) = \frac{1}{\pi^2 z x}\frac{\mathrm{d}z}{J_p^2(\sqrt{2z}x) + Y_p^2(\sqrt{2z}x)}.$$

Since $j_{p,k}$ increases linearly with k as $k \to \infty$, $\sigma_-(x)$ satisfies the moment condition. On the other hand, since

$$\sigma_+(x, \mathrm{d}z) \sim c(p, x) z^{p-1}\,\mathrm{d}z \quad \text{as } z \to 0,$$

the moment condition is not satisfied by $\sigma_+(x, \cdot)$. By using the recurrence relations satisfied by the Bessel functions, it is straightforward to verify that

$$U_-(x, \lambda) = \frac{2\lambda}{2(p+1)/x +} \frac{2\lambda}{2(p+2)/x +} \frac{2\lambda}{2(p+3)/x + \cdots}$$

and

$$-U_+(x, \lambda) = \frac{2p}{x} + \frac{2\lambda}{2(p-1)/x +} \frac{2\lambda}{2(p-2)/x +} \frac{2\lambda}{2(p-3)/x + \cdots}.$$

This last expansion does not contradict Theorem 1, for the coefficients cannot all be positive.

Next, we examine the sequence of diffusions associated with these expansions. With the help of Table 2, we see that the diffusions associated with U_- are:

$$\begin{aligned} X = \mathrm{BES}(p) &\xrightarrow{\phi_-(\cdot,0)\text{-transform}} \mathrm{BES}(p) \xrightarrow{\text{Krein duality}} \mathcal{T}_{h_0}(X) \\ &= \mathrm{BES}(-p-1) \xrightarrow{\phi_-(\cdot,0)\text{-transform}} \mathrm{BES}(p+1) \xrightarrow{\text{Krein duality}} \mathcal{T}_{h_1} \circ \mathcal{T}_{h_0}(X) = \mathrm{BES}(-p-2), \quad \text{etc.} \end{aligned}$$

When applied to U_+, using $u_{0,+} = -2p/x$, the first iteration of the algorithm yields

$$X = \mathrm{BES}(p) \xrightarrow{\phi_+(\cdot,0)\text{-transform}} \mathrm{BES}(-p) \xrightarrow{\text{Krein duality}} \mathcal{T}_{h_0}(X) = \mathrm{BES}(p-1). \tag{6.1}$$

It is also possible to obtain both expansions from the algorithm of Section 4, by arguing as in the previous example.

Other well-studied diffusions that lead to simple continued fraction expansions are exponential Brownian motions and squared Bessel processes.

6.3. *The Ciesielski–Taylor theorem*

Ciesielski and Taylor [14] noticed that the total time spent by a $(d+2)$-dimensional standard Brownian motion inside the unit ball in $\mathbb{R}^{d+2}$ has the same distribution as the first hitting time of the unit ball in $\mathbb{R}^d$ by a d-dimensional standard Brownian motion. This result can be expressed in terms of one-dimensional diffusions as an identity in law between some occupation time of a Bessel process, say X, of parameter p and a hitting time of another Bessel process, say Z, of parameter $p-1$. On the other hand, Eq. (6.1) says

$$Z = \mathcal{T}_h(X), \quad \text{where } h = \phi_+(\cdot, 0). \tag{6.2}$$

Biane [7] generalised the Ciesielski–Taylor theorem and found many other pairs (X, Z) of diffusions such that some occupation time of X has the same law as some hitting time of Z. It turns out that his construction of these pairs can be described by Eq. (6.2). A similar observation has already been made by Tóth in [45]. A proof of this statement along purely analytical lines is not difficult but would require a fairly detailed account of Biane's result and techniques, and so we will not pursue this here. The deeper probabilistic content – if there is one – of this intriguing connection between continued fractions and the Ciesielski–Taylor theorem has yet to be elucidated.

7. Diffusion in a Brownian environment with positive drift

Now, let $a \equiv 1$ and suppose that

$$W(x) = \mu x + B_x, \quad \mu > 0,$$

where B is a standard Brownian motion. For every realisation of W, the process with generator (1.3) is a linear diffusion and the algorithm of Section 4 produces a continued fraction whose coefficients $u_n(x)$ are random variables. We shall be interested in the stationary distributions of U and the u_n.

In order to find these stationary distributions, we consider the "master" system of stochastic equations

$$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}x} = \mathbf{a}(\mathbf{y}) + B'\mathbf{b}\mathbf{y}, \tag{7.1}$$

where $\mathbf{y}:\mathbb{R} \to \mathbb{R}^d$, $\mathbf{a}:\mathbb{R}^d \to \mathbb{R}^d$ is a vector-valued function with components a_i, and $\mathbf{b}$ is a (fixed) diagonal $d \times d$ matrix given by

$$\mathbf{b} := \operatorname{diag}(b_i).$$

The solution of this Stratonovich equation is a multidimensional diffusion whose infinitesimal generator $\mathscr{G}$ is given by (see [2], Chapter 6)

$$\mathscr{G} f = \sum_{i=1}^d \left\{ a_i + \frac{1}{2} b_i^2 y_i \right\} \frac{\partial f}{\partial y_i} + \frac{1}{2} \sum_{i=1}^d b_i y_i \sum_{j=1}^d b_j y_j \frac{\partial^2 f}{\partial y_i \, \partial y_j}. \tag{7.2}$$

Its adjoint $\mathscr{G}^\dagger$ is the operator

$$\mathscr{G}^\dagger f = \sum_{i=1}^d \frac{\partial}{\partial y_i} \left\{ -a_i f + \frac{1}{2} b_i y_i \sum_{j=1}^d \frac{\partial}{\partial y_j} [b_j y_j f] \right\}. \tag{7.3}$$

If f is a normalisable solution of the Fokker–Planck (forward Kolmogorov) equation

$$\mathscr{G}^\dagger f = 0, \tag{7.4}$$

then it is the density of a stationary solution $\mathbf{y}$ of the stochastic Eq. (7.1); see for instance, [2], Section 3.5.3.

7.1. *The stationary distribution of the continued fraction*

The equation satisfied by the Riccati variable is

$$\frac{\mathrm{d}U}{\mathrm{d}x} = 2\lambda - 2\mu U - U^2 - 2B'U. \tag{7.5}$$

This is of the form (7.1) with $d = 1$,

$$a = 2\lambda - 2\mu y - y^2, \qquad b = -2.$$

Denote by f_U the density of the distribution of U. The Fokker–Planck equation is then

$$\frac{\mathrm{d}}{\mathrm{d}y} \left\{ -\left(2\lambda - 2\mu y - y^2\right) f_U + 2y \frac{\mathrm{d}}{\mathrm{d}y}(y f_U) \right\} = 0.$$

This has one normalisable solution, namely

$$f_U(y) = c y^{-\mu-1} \exp\left[-\frac{y}{2} - \frac{\lambda}{y} \right], \quad y > 0. \tag{7.6}$$

This is the density of the inverse Gaussian distribution.

7.2. *Proof of Theorem* 3

The equation for the continued fraction coefficient u_0 is

$$\frac{\mathrm{d}u_0}{\mathrm{d}x} = -2\mu u_0 - u_0^2 - 2B' u_0.$$

This is the homogeneous version of Eq. (7.5); for $\mu > 0$, it has no normalisable solution. Hence $u_0 = 0$. The equation for u_1 is then

$$\frac{\mathrm{d}u_1}{\mathrm{d}x} = 2\mu u_1 - u_1^2 + 2B' u_1. \tag{7.7}$$

This is of the form (7.1) with $d = 1$,

$$a = 2\mu y - y^2, \qquad b = 2.$$

Denote by f the density of the distribution of u_1. The Fokker–Planck equation is then

$$\frac{\mathrm{d}}{\mathrm{d}y}\left\{\left(-2\mu y + y^2\right) f + 2y \frac{\mathrm{d}}{\mathrm{d}y}(yf)\right\} = 0. \tag{7.8}$$

This has one normalisable solution, namely

$$f(y) = c y^{\mu-1} \mathrm{e}^{-y/2}, \quad y > 0. \tag{7.9}$$

This is the density of the gamma distribution.

More generally, for $d \in \mathbb{N}$, the stochastic equation satisfied by the first d of the non-zero u_n is of the form (7.1) with

$$a_i(y) = 2(-1)^{i-1} y_i \left[\mu + \sum_{k=1}^{i-1} (-1)^k y_k\right] - y_i^2 \quad \text{and} \quad b_i = 2(-1)^{i-1}.$$

Equation (7.1) for the density, say f_d, of the joint stationary distribution of the random variables $u_1, \ldots, u_d$ becomes

$$\sum_{i=1}^{d} \frac{\partial}{\partial y_i}\left\{-a_i f_d + 2(-1)^{i-1} y_i \sum_{j=1}^{d} (-1)^{j-1} \frac{\partial}{\partial y_j}[y_j f_d]\right\} = 0. \tag{7.10}$$

We are now in a position to prove Theorem 3. Let f be defined as in Eq. (7.9). We will show that the probability density function

$$f_d(y_1, \ldots, y_d) := \prod_{i=1}^{d} f(y_i)$$

solves the Fokker–Planck equation (7.10).

Proof of Theorem 3. We proceed by induction on d. The case $d = 1$ has already been dealt with. We make the induction hypothesis: namely, we set

$$f_d(y_1, \ldots, y_d) := \prod_{i=1}^{d} f(y_i)$$

and suppose that Eq. (7.10) holds for some d. For $d+1$, the left-hand side of the Fokker–Planck equation may be written as the sum

$$\mathbf{A}+\mathbf{B}+\mathbf{C},$$

where

$$\mathbf{A} := \sum_{i=1}^{d} \frac{\partial}{\partial y_i}\left\{-a_i f_{d+1} + 2(-1)^{i-1} y_i \sum_{j=1}^{d}(-1)^{j-1}\frac{\partial}{\partial y_j}[y_j f_{d+1}]\right\}, \tag{7.11}$$

$$\mathbf{B} := \sum_{i=1}^{d} \frac{\partial}{\partial y_i}\left\{2(-1)^{i-1} y_i (-1)^d \frac{\partial}{\partial y_{d+1}}[y_{d+1} f_{d+1}]\right\} \tag{7.12}$$

and

$$\mathbf{C} := \frac{\partial}{\partial y_{d+1}}\left\{-a_{d+1} f_{d+1} + 2(-1)^{d} y_{d+1} \sum_{j=1}^{d+1}(-1)^{j-1}\frac{\partial}{\partial y_j}[y_j f_{d+1}]\right\}. \tag{7.13}$$

First, we note that

$$f_{d+1}(y_1,\dots,y_{d+1}) = f_d(y_1,\dots,y_d) f(y_{d+1}).$$

So

$$\mathbf{A} = f(y_{d+1}) \sum_{i=1}^{d} \frac{\partial}{\partial y_i}\left\{-a_i f_d + 2(-1)^{i-1} y_i \sum_{j=1}^{d}(-1)^{j-1}\frac{\partial}{\partial y_j}[y_j f_d]\right\}$$

and since, by the induction hypothesis, the sum vanishes, we deduce that $\mathbf{A}=0$. Next, we remark that

$$\begin{aligned}\mathbf{B} &= \frac{\partial}{\partial y_{d+1}} \sum_{i=1}^{d} \frac{\partial}{\partial y_i}\left\{2(-1)^{i-1} y_i (-1)^d [y_{d+1} f_{d+1}]\right\}\\ &= \frac{\partial}{\partial y_{d+1}}\left\{(-1)^d y_{d+1} f(y_{d+1}) \sum_{i=1}^{d}(-1)^{i-1} 2\frac{\partial}{\partial y_i}[y_i f_d]\right\}.\end{aligned}$$

Hence

$$\begin{aligned}\mathbf{A}+\mathbf{B}+\mathbf{C} = \frac{\partial}{\partial y_{d+1}}\Bigg\{&(-1)^d y_{d+1} f(y_{d+1}) \sum_{i=1}^{d}(-1)^{i-1} 2\frac{\partial}{\partial y_i}[y_i f_d]\\ &- a_{d+1} f(y_{d+1}) f_d + (-1)^d y_{d+1} f(y_{d+1}) \sum_{i=1}^{d}(-1)^{i-1} 2\frac{\partial}{\partial y_i}[y_i f_d]\\ &+ 2(-1)^d y_{d+1} (-1)^d f_d \frac{\partial}{\partial y_{d+1}}\big[y_{d+1} f(y_{d+1})\big]\Bigg\}\\ = \frac{\partial}{\partial y_{d+1}}\Bigg\{&(-1)^d y_{d+1} f(y_{d+1}) \sum_{i=1}^{d}(-1)^{i-1} 2\frac{\partial}{\partial y_i}[y_i f_d]\\ &- a_{d+1} f(y_{d+1}) f_d + 2 y_{d+1} f_d \frac{\partial}{\partial y_{d+1}}\big[y_{d+1} f(y_{d+1})\big]\Bigg\}.\end{aligned}$$

Now,

$$\frac{\partial}{\partial y_i}[y_i f_d] = \left(\prod_{\substack{j=1\\ j\neq i}}^{d} f(y_j)\right)\frac{\partial}{\partial y_i}\big[y_i f(y_i)\big] = \left(\frac{1}{2y_i}\prod_{\substack{j=1\\ j\neq i}}^{d} f(y_j)\right)2y_i\frac{\partial}{\partial y_i}\big[y_i f(y_i)\big]$$

and so

$$\frac{\partial}{\partial y_i}[y_i f_d] = \left(\frac{1}{2y_i}\prod_{\substack{j=1\\ j\neq i}}^{d} f(y_j)\right)y_i(2\mu - y_i)f(y_i) = \frac{1}{2}(2\mu - y_i)f_d,$$

where we have used the fact that $f(y_i)$ solves Eq. (7.8) with 1 replaced by i. Therefore

$$\begin{aligned}\mathbf{A}+\mathbf{B}+\mathbf{C} = f_d\frac{\partial}{\partial y_{d+1}}\Bigg\{&2(-1)^d y_{d+1} f(y_{d+1})\sum_{i=1}^{d}(-1)^{i-1}(2\mu - y_i)\\ &- a_{d+1} f(y_{d+1}) + 2y_{d+1}\frac{\partial}{\partial y_{d+1}}\big[y_{d+1} f(y_{d+1})\big]\Bigg\}.\end{aligned}$$

By using the definition of a_{d+1}, it is easy to verify that

$$2(-1)^d y_{d+1}\sum_{i=1}^{d}(-1)^{i-1}(2\mu - y_i) - a_{d+1} = y_{d+1}(y_{d+1} - 2\mu).$$

Hence

$$\mathbf{A}+\mathbf{B}+\mathbf{C} = f_d\frac{\partial}{\partial y_{d+1}}\left\{2y_{d+1}\frac{\partial}{\partial y_{d+1}}\big[y_{d+1} f(y_{d+1})\big] - y_{d+1}(2\mu - y_{d+1})f(y_{d+1})\right\}.$$

By virtue of Eq. (7.8), the expression in curly brackets vanishes, and so the proof is complete. □

If we assume that this Fokker–Planck equation admits no more than one smooth solution, then it follows that u_1, $u_2, \ldots$ are, in the stationary regime, independent and have the same gamma distribution. The formula (7.6) reproduces the result obtained by Letac and Seshadri [33]. Since all the coefficients are positive, this continued fraction yields twice the Laplace exponent ψ_-. In particular, we can assert that, in a Brownian environment with positive drift, in the stationary regime, the reciprocal of the mean duration of the excursions below the starting point is gamma-distributed. The corresponding random measure σ_- was studied by Marklof et al. in [36]; they found that the essential spectrum is $[0, \infty)$, with an empty absolutely continuous part.

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