Kolmogorov equation associated to the stochastic reflection problem on a smooth convex set of a Hilbert space II

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Abstract. This work is concerned with the existence and regularity of solutions to the Neumann problem associated with an Ornstein–Uhlenbeck operator on a bounded and smooth convex set $K$ of a Hilbert space $H$. This problem is related to the reflection problem associated with a stochastic differential equation in $K$.

Résumé. Dans cet article nous étudions l’existence et la régularité des solutions d’un problème de Neumann associé à un opérateur d’Ornstein–Uhlenbeck défini sur un domaine convexe $K$, borné et régulier dans un espace de Hilbert $H$. Le problème est lié à un problème de réflexion associé à une équation différentielle stochastique dans le domaine $K$.

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1. Introduction

We are given a non-degenerate Gaussian measure $\mu = N_Q$ with mean 0 and covariance operator $Q$ in a separable Hilbert space $H$ (with scalar product $\langle \cdot, \cdot \rangle$ and norm $| \cdot |$). We fix $\alpha \in [0, 1]$ and consider the following Neumann problem on a regular convex subset $K$ of $H$,

$$ \begin{cases} \lambda \varphi - L_\alpha \varphi = f & \text{in } K, \\ \frac{\partial \varphi}{\partial n} = 0 & \text{on } \Sigma, \end{cases} $$

where $\lambda > 0$, $\Sigma$ is the boundary of $K$, $f : H \to \mathbb{R}$ is a given function on $H$ and $L$ is the Ornstein–Uhlenbeck operator

$$ L_\alpha \varphi := \frac{1}{2} \text{Tr} [ Q^{1-\alpha} D^2 \varphi ] - \frac{1}{2} \langle x, Q^{-\alpha} D \varphi \rangle. $$

We shall denote by $A$ the self-adjoint operator $A := Q^{-1}$. Since $\mu$ is not degenerate, there exists $\delta > 0$ such that $\langle A x, x \rangle \geq \delta |x|^2$, $\forall x \in D(A)$ for some $\delta > 0$. Of course we have also that $\text{Tr} A^{-1} < \infty$.

Concerning $K$, we shall assume that

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Hypothesis 1.1. There exists a convex $C^\infty$-function $g : H \to [0, \infty)$ with $g(0) = 0$, $g'(0) = 0$ and $D^2g$ positively defined, i.e., $\langle D^2g(x)h, h \rangle \geq \kappa |h|^2$, $\forall h \in H, x \in H$, where $\kappa > 0$, such that
\[ K = \{ x \in H : g(x) \leq 1 \}, \quad \Sigma = \{ x \in H : g(x) = 1 \}. \]
Moreover, we also suppose that $D^2g$ is bounded on $K$ and that $g$ and all its derivatives grow at infinity at the most polynomially.

We denote by $\mu_\Sigma$ the surface measure induced by $\mu$ on $\Sigma$ (see [5,11,12]) and by $n(y)$ the inner normal to $K$ at $y$, that is
\[ n(y) = \frac{Dg(y)}{|Dg(y)|} \quad \forall y \in \Sigma. \quad (1.3) \]

By Hypothesis 1.1 it follows that

Lemma 1.2. $K$ is convex, closed and bounded. Moreover there are $\gamma$, $\rho$, $\delta > 0$ such that
\[ \langle Dg(x), x \rangle \geq \gamma |x|^2 \quad \forall x \in H, \quad |Dg(x)| \leq \delta \quad \forall x \in K, \quad (1.4) \]
\[ g(x) \geq \frac{\gamma}{2} |x|^2 \quad \forall x \in H, \quad (1.5) \]
\[ |Dg(x)| \geq \rho \quad \forall x \in \Sigma. \quad (1.6) \]

Proof. We have
\[ Dg(x) = \int_0^1 D^2g(tx)x \, dt \quad \forall x \in H. \]

Therefore
\[ \langle Dg(x), x \rangle = \int_0^1 \langle D^2g(tx)x, x \rangle \, dt \geq \kappa |x|^2 \quad \forall x \in H, \]
which implies the first estimate in (1.4) and also that $Dg$ is bounded on $K$.

Similarly by
\[ g(x) = \int_0^1 \langle Dg(tx), x \rangle \, dt \quad \forall x \in H \]
and (1.4) it follows (1.5). This implies that $K$ is bounded and $0 \in \overset{\circ}{K}$, where $\overset{\circ}{K}$ is the interior of $K$. Finally by (1.4) it follows (1.6) otherwise there is $\{x_n\} \subset \Sigma$ such that $Dg(x_n) \to 0$. Taking into account that $0 < g(x) \leq \langle Dg(x), x \rangle$ and that $\{x_n\}$ is bounded the latter implies that $1 = g(x_n) \to 0$ which is of course absurd. $\square$

It is easy to see that $\mu$ is the unique invariant measure of the Ornstein–Uhlenbeck process in $H$,
\[
\begin{aligned}
\begin{cases}
 dX(t) + \frac{1}{2} A^\alpha X(t) \, dt = A^{(\alpha-1)/2} \, dW(t), \\
 X(0) = x \in H,
\end{cases}
\end{aligned}
\]
where $W$ is a cylindrical Wiener process in a filtered probability space
\[
(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})
\]
of the form
\[ \{W(t), z\} = \sum_{k=1}^\infty \beta_k(t)(z, e_k), \quad t \geq 0 \quad \forall z \in H. \]
Here \( \{\beta_k\} \) is a sequence of mutually independent real Brownian motions on \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}) \) (see, e.g., [9]) and \( \{e_k\} \) is an orthonormal basis in \( H \) which will be taken as a system of eigen-functions for \( A \) for simplicity, i.e.,

\[
A e_k = a_k e_k \quad \forall k \in \mathbb{N},
\]

where obviously \( a_k \geq \delta \).

Let us describe the results of the paper. First we consider the symmetric Dirichlet form

\[
a(\varphi, \psi) = \int_{K} \left( A^{(\alpha-1)/2} D\varphi, A^{(\alpha-1)/2} D\psi \right) \, d\nu \quad \forall \varphi, \psi \in C^1(K),
\]

where \( \nu = \frac{1}{\mu(K)} \mu \) and show that \( a \) is closable (equivalently continuous) in the space \( W_{A^{\alpha-1}}^{1,2}(K, \nu) \) (see Section 2). We notice that for \( \alpha = 0 \) this space reduces to the Malliavin space \( D^{1,2}(K, \nu) \). Here we use a recent result about an integration by parts formula on \( K \) proved in [4].

Then we define a weak solution of the Neumann problem (1.1) in the usual way as a solution \( \varphi \in W_{A^{\alpha-1}}^{1,2}(K, \nu) \) of the equation

\[
\lambda \int_{H} \varphi \psi \, d\mu + \frac{1}{2} a(\varphi, \psi) = \int_{H} f \psi \, d\nu \quad \forall \psi \in W_{A^{\alpha-1}}^{1,2}(K, \nu),
\]

\[
(1.9)
\]

where \( f \in L^2(K, \nu) \).

If we denote by \( N \) the Kolmogorov operator corresponding to the Dirichlet form (1.8) then (1.9) can be equivalently written as \( \lambda \varphi - N \varphi = f \). The second-order regularity of \( \varphi \) as well as the proof that it satisfies the Neumann boundary condition on \( \Sigma \) in the sense of trace is one of the main results of this work (Theorem 3.5). In the previous work [4] this result was proved in the case \( \alpha = 1 \). It should be emphasized that, though the treatment closely follows [4], there are, however, some notable differences which will be mentioned later on. The nice feature of problem (1.1) is that for all \( \alpha \) the corresponding Ornstein–Uhlenbeck operators (1.7) have the same invariant measure \( \mu = N_Q \) and this allows a unified treatment. Moreover, since the trace assumption on \( A^{-\alpha} \) is weaker than that on \( A^{-1} \) we can treat into this general functional setting reflection problem not treatable for \( \alpha = 1 \).

We note that in specific situations \( A \) is a linear elliptic operator with suitable boundary conditions on a bounded and open subset \( \partial \) of \( \mathbb{R}^d \). (See Section 5 below.)

The second part of the paper is devoted to the construction of a process \( X(t, x) \) such that the semigroup \( P_t \) generated by \( N \) is expressed as \( P_t \varphi(x) = \mathbb{E}[\varphi(X(t, x))] \) where \( X \) is formally the solution to the following stochastic variational inequality

\[
\begin{align*}
\frac{dX}{dt} + \frac{1}{2} A^{\alpha} X \, dt + A^{\alpha-1} N_K(X) \, dt & \equiv A^{(\alpha-1)/2} \, dW_t, \\
X(0) & = x,
\end{align*}
\]

\[
(1.10)
\]

where \( N_K \) is the normal cone to \( K \), i.e.,

\[
\begin{align*}
N_K(x) & = \partial \quad \text{if } x \in \partial, \\
N_K(x) & = \{ \lambda n(x), \lambda \geq 0 \} \quad \text{if } x \in \Sigma.
\end{align*}
\]

When \( \alpha = 1 \) this problem is known in literature as the stochastic reflection problem on convex set \( K \) and was studied in finite-dimensional spaces \( H \) by [2,3,6,8]. If \( H \) is infinite-dimensional, however, no results concerning existence and uniqueness of strong solutions with the notable exception of the 1992 work of Nualart and Pardoux [14] which treats this problem in \( H = L^2(0, 1) \) and for \( K = \{ y \in L^2(0, 1): y \geq 0 \, \text{a.e. in} \, (0, 1) \} \).

The transition semigroup

\[
(\mathcal{P}_t \varphi)(x) = \mathbb{E}[\varphi(X(t, x))] \quad \forall \varphi \in C_b(K), \, t \geq 0
\]

\[
(1.11)
\]

formally relates the Neumann problem (1.1) and Eq. (1.10) but no rigorous proof of this conjecture exists except the cases mentioned above (see also [16]). However, in [1] this is proven for \( \alpha = 1 \) via some sharp arguments involving
theory of Langrangian flows. In particular, it is proven the existence and uniqueness of a martingale solution in sense of Stroock and Varadhan.

When \( \alpha \in [0, 1) \) the operator \( A^{\alpha-1}N_k \) is not monotone in \( H \), so no existence results in the literature for Eq. (1.10) seems to be available. The second part of the paper is concerned with representation of semigroup \( P_t \) as a transition Markov semigroup in the special case where \( K \) is a ball and \( \text{Tr}[A^{2\delta-1}] < \infty \) for some \( \delta > 0 \). The proof of existence of the process is constructive and relies on some sharp \( BV \)-estimates on solutions to approximating equation associated with (1.10) and the Skorohod theorem.

2. Notations and preliminary results

Everywhere in the following \( D\varphi \) is the derivative of a function \( \varphi : H \to \mathbb{R} \). By \( D^2\varphi : H \to L(H, H) \) we shall denote the second derivative of \( \varphi \). We shall denote also by \( C_b(H) \) and \( C_b^k(H), \alpha \in \mathbb{N} \), the spaces of all continuous and bounded functions on \( H \) and, respectively, of \( k \)-times differentiable functions with continuous and bounded derivatives. The space \( C^k(K), \alpha \in \mathbb{N} \), is defined as the space of restrictions of functions of \( C^k_b(H) \) to the subset \( K \). Also we refer to [7,9] for notations and basic results on infinite-dimensional processes.

We denote by \( \{e_k\} \) the orthonormal basis in \( H \) of eigenfunctions of \( Q \), i.e.

\[
Qe_k = \lambda_k e_k \quad \forall k \in \mathbb{N},
\]

(2.1)

where \( \lambda_k = \frac{1}{\lambda_k} \) with \( \{\lambda_k, k \in \mathbb{N}\} \) the eigenvalues of \( A \), by \( D_k \) the derivative in the direction \( e_k \) and set \( x_k = \langle x, e_k \rangle \) for all \( x \in H, k \in \mathbb{N} \). We denote by \( \mathcal{E}(H) \) the linear span of all exponential functions \( \{e^{\langle x, e_k \rangle}, h \in \mathbb{N}\} \).

Then we recall a basic integration by parts formula in \( H \).

\[
\int_H D_k \varphi \, d\mu = \frac{1}{\lambda_k} \int_H x_k \varphi \, d\mu \quad \forall k \in \mathbb{N}, \varphi \in C_b^1(H).
\]

(2.2)

We denote by \( M_\alpha : C_b^1(H) \subset L^2(H, \mu) \to L^2(H, \mu; H) \)

\[
M_\alpha \varphi := A^{(\alpha-1)/2} D\varphi, \quad \varphi \in C_b^1(H).
\]

Here \( M_0 \) is the Malliavin derivative [12]. It is well known (and easy to show thanks to (2.2)) that \( M_\alpha \) is closable. We shall denote its closure by \( M_\alpha \) and also by \( A^{(\alpha-1)/2} D \).

The domain of the closure of \( M_\alpha \) will be denoted by \( W_{A^{\alpha-1}}^{1,2}(H, \mu) \). It is a Hilbert space with the inner product

\[
\langle \varphi, \psi \rangle_{W_{A^{\alpha-1}}^{1,2}(H, \mu)} = \int_H \varphi \psi \, d\mu + \int_H \langle A^{(\alpha-1)/2} D\varphi, A^{(\alpha-1)/2} D\psi \rangle \, d\mu
\]

\[
= \int_H \varphi \psi \, d\mu + \sum_{k=1}^{\infty} \int_H \lambda_k^{\alpha-1} D_k \varphi D_k \psi \, d\mu.
\]

Denote by \( L^2(H, \mu) \) and \( L^2(K, v) \) the space of \( \mu \)-square integrable functions (\( v \)-square integrable functions) on \( H \) and \( K \), respectively.

In a similar way we define the space \( W_{A^{\alpha-1}}^{2,2}(H, \nu) \). The corresponding inner product is defined by (see [4,7,10])

\[
\langle \varphi, \psi \rangle_{W_{A^{\alpha-1}}^{2,2}(H, \mu)} = \langle \varphi, \psi \rangle_{W_{A^{\alpha-1}}^{1,2}(H, \mu)} + \int_H \text{Tr}[A^{2(\alpha-1)} D^2 \varphi D^2 \psi] \, d\mu
\]

\[
= \langle \varphi, \psi \rangle_{W_{A^{\alpha-1}}^{1,2}(H, \mu)} + \sum_{h,k=1}^{\infty} \int_H \lambda_h^{1-\alpha} \lambda_k^{1-\alpha} D_{h,k} \varphi D_{h,k} \psi \, d\mu.
\]
2.1. The integration by parts formula on $K$

The following result is proved in [4]. For reader’s convenience we recall it here, deferring to the Appendix for a proof (Theorem A.2).

**Lemma 2.1.** Let $K = \{ x \in H : g(x) \leq 1 \}$ where $g \in C^2(H)$ is convex and $|Dg(x)|^{-1} \in L^p(H, \mu)$ for all $p \geq 1$. Then

$$\int_K D_h \varphi(x) \mu(dx) = \frac{1}{\mu(K)} \int_{\Sigma} n_h(y) \varphi(y) \mu_{\Sigma}(dy)$$

$$+ \frac{1}{\lambda_h} \int_K x_h \varphi(x) \mu(dx) \quad \forall h \in H, \varphi \in C^1_b(H), \quad (2.3)$$

where $n_h(y) = \langle n(y), e_h \rangle$.

With the help of this result we can define the spaces $W^{1,2}_{A_{\alpha-1}}(K, \nu)$ and $W^{2,2}_{A_{\alpha-1}}(K, \nu)$ as in [4].

Moreover, we can define the trace of a function $\varphi \in W^{1,2}_{A_{\alpha-1}}(K, \nu)$ thanks to the following result.

**Proposition 2.2.** For any $\varphi \in C^1_b(H)$ we have

$$\int_{\Sigma} |Q^{1/2} n(y)|^2 \varphi^2(y) \mu_{\Sigma}(dy)$$

$$\leq C \left( \int_K \varphi^2(x) \mu(dx) + \int_K |Q^{1/2} D\varphi(x)|^2 \mu(dx) \right). \quad (2.4)$$

**Proof.** Let $\varphi \in C^1_b(H)$ and $h \in \mathbb{N}$. Replacing in (2.3) $\varphi$ with $\lambda_h D_h g \varphi^2$ and then $D_h \varphi$ with $2 \lambda_h D_h g \varphi D_h \varphi + \lambda_h D_h^2 g \varphi^2$, yields

$$2 \int_K \lambda_h D_h g \varphi D_h \varphi \ d\mu + \int_K \lambda_h D_h^2 g \varphi^2 \ d\mu$$

$$= \frac{1}{\mu(K)} \int_{\Sigma} \lambda_h n_h(y) D_h g \varphi^2 \ d\mu_{\Sigma} + \int_K x_h D_h g \varphi^2 \ d\mu.$$

Summing up on $h$ yields

$$2 \int_K \langle Q D\varphi, Dg \rangle \varphi^2 \ d\mu + \int_K \text{Tr}[Q D^2 g] \varphi^2 \ d\mu$$

$$= \frac{1}{\mu(K)} \int_{\Sigma} \langle Qn(y), Dg \rangle \varphi^2 \ d\mu_{\Sigma} + \int_K \langle x, Dg \rangle \varphi^2 \ d\mu.$$

But, taking into account (1.3), (1.6) we have

$$\langle Qn(y), Dg(y) \rangle = |Dg(y)| \langle Qn(y), n(y) \rangle$$

$$\geq \rho \langle Qn(y), n(y) \rangle \quad \forall y \in \Sigma.$$

Substituting in the previous identity yields

$$\frac{1}{\rho \mu(K)} \int_{\Sigma} \langle Qn(y), n(y) \rangle \varphi^2 \ d\mu_{\Sigma} + \int_K \langle x, Dg \rangle \varphi^2 \ d\mu$$

$$\leq 2 \int_K \langle Q D\varphi, Dg \rangle \varphi \ d\mu + \int_K \text{Tr}[Q D^2 g] \varphi^2 \ d\mu.$$
Taking into account that $K$ is bounded and that $Dg, D^2g$ are bounded on $K$, the conclusion follows.

We can now define the trace of a function $\phi \in W^{1,2}_{A^{\alpha-1}}(K, \nu)$. Let $\{\phi_j\} \subset C_b(K)$ be such that
\[
\lim_{n \to \infty} \phi_j = \phi \quad \text{in} \quad L^2(K),
\]
\[
\lim_{n \to \infty} A^{(\alpha-1)/2}D\phi_j = A^{(\alpha-1)/2}D\phi \quad \text{in} \quad L^2(K).
\]

Then by (2.4) it follows that the sequence $\{|Q^{1/2}n(y)|\gamma_0(\phi_j)|\}$, where $\gamma_0(\phi_j)$ denotes the trace of $\phi_j$, is convergent in $L^2(\Sigma, \mu_\Sigma)$ to a function $\psi \in L^2(\Sigma, \mu_\Sigma)$. Then we define the trace $\gamma_0(\phi)$ of $\phi$ as
\[
\gamma_0(\phi) = \psi \left|Q^{1/2}n(y)\right|.
\]

2.2. Trace of the normal derivative

**Proposition 2.3.** Assume that $\phi \in W^{2,2}_{A^{\alpha-1}}(K, \nu)$. Then the following estimate holds,
\[
\int_\Sigma |Q^{1/2}n(y)|^2 |A^{(\alpha-1)/2}D\phi|^2(y)\mu_\Sigma(dy) \leq C \left( \int_K |A^{(\alpha-1)/2}D\phi(x)|^2\mu(dx) + \int_K \text{Tr}[\left(A^{\alpha-1}D^2\phi(x)\right)^2]\mu(dx) \right).
\]

**Proof.** Let $\phi \in W^{2,2}_{A^{\alpha-1}}(K, \nu)$ and let $\{\phi_j\} \subset C^2(K)$ be convergent to $\phi$ in $W^{2,2}_{A^{\alpha-1}}(K, \nu)$. For $i \in \mathbb{N}$ we apply (2.3) to $a^{(\alpha-1)/2}_i D_i \phi_j$. We have
\[
\int_\Sigma |Q^{1/2}n(y)|^2 |a^{(\alpha-1)/2}_i D_i \phi_j|^2(y)\mu_\Sigma(dy) \leq C a^{(\alpha-1)/2}_i \left( \int_K |D_i \phi_j(x)|^2\mu(dx) + a^{(\alpha-1)/2}_i \int_K |A^{(\alpha-1)/2}DD_i \phi_j(x)|^2\mu(dx) \right).
\]
Summing up on $i$ yields
\[
\int_\Sigma |Q^{1/2}n(y)|^2 |A^{(\alpha-1)/2}D\phi|^2(y)\mu_\Sigma(dy) \leq C \left( \int_K |A^{(\alpha-1)/2}D\phi_j(x)|^2\mu(dx) + \int_K \text{Tr}[\left(A^{\alpha-1}D^2\phi_j(x)\right)^2]\mu(dx) \right).
\]
Now the conclusion follows letting $j \to \infty$. 

3. The penalized problem

We are here concerned for any $\varepsilon > 0$ with the penalized equation
\[
\begin{cases}
\d X_{\varepsilon}(t) + \left[\frac{1}{2} A^\alpha X_{\varepsilon}(t) + A^{\alpha-1} \beta_{\varepsilon} \left(X_{\varepsilon}(t)\right)\right] dt = A^{(\alpha-1)/2}dW_t, \\
X_{\varepsilon}(0) = x,
\end{cases}
\]
where
\[
\beta_{\varepsilon}(x) = \frac{1}{\varepsilon} \left(x - \Pi_K(x)\right) \quad \forall x \in H.
\]
Since $\beta_\varepsilon$ is Lipschitz continuous, it is easily seen that Eq. (3.1) which can be equivalently be written as

$$X_\varepsilon(t) = e^{-tA^{\alpha}/2}x - \int_0^t A^{\alpha-1}e^{-A^{\alpha}(t-s)/2}\beta_\varepsilon(X_\varepsilon(s))\,ds + \int_0^t e^{-A^{\alpha}(t-s)/2}A^{(\alpha-1)/2}\,dW_s$$

has a unique mild solution

$$X_\varepsilon(\cdot, x) \in L^2(\Omega, C([0, +\infty); H)).$$

Moreover, it is easy to see that there is a unique invariant probability measure $\nu_\varepsilon$ for $X_\varepsilon$ given by

$$\nu_\varepsilon(dx) = Z_\varepsilon^{-1}e^{-d_K^2(x)/\varepsilon}, \quad \text{where } d_K \text{ is the distance to } K \text{ and }$$

$$Z_\varepsilon = \int_H e^{-d_K^2(y)/\varepsilon}\mu(dy).$$

The corresponding Kolmogorov operator reads as follows,

$$N_\varepsilon \varphi = L\varphi - \langle A^{\alpha-1}\beta_\varepsilon(x), D\varphi \rangle, \quad \varphi \in C_c(H) \quad \forall \varepsilon > 0,$$

where $L$ is the Ornstein–Uhlenbeck operator

$$L\varphi = \frac{1}{2} \text{Tr}[A^{\alpha-1}D^2\varphi] - \frac{1}{2}\langle x, A^{\alpha}D\varphi \rangle \quad \forall \varphi \in C_c(H).$$

One can easily check that $\nu_\varepsilon$ (as defined in (3.2) and (3.3)) is an invariant measure for $N_\varepsilon$ and that

$$\int_H N_\varepsilon \varphi \psi \,d\nu_\varepsilon = -\frac{1}{2} \int_H \langle A^{\alpha-1}D\varphi, D\psi \rangle \,d\nu_\varepsilon \quad \forall \varphi, \psi \in C_c(H).$$

Moreover, since $\beta_\varepsilon$ is Lipschitz continuous, the operator $N_\varepsilon$ is essentially $m$-dissipative in $L^2(H, \nu_\varepsilon)$ (we still denote by $N_\varepsilon$ its closure) and $C_c(H)$ is a core for $N_\varepsilon$, see [7].

Section 3.1 below is devoted to several estimates for $(\lambda I - N_\varepsilon)^{-1}f$ where $f \in L^2(H, \nu_\varepsilon)$. Then these estimates are used in Section 3.2 to prove that $(\lambda I - N_\varepsilon)^{-1}f$ converges to $(\lambda I - N)^{-1}f$ as $\varepsilon \to 0$, where $N$ is the self-adjoint operator corresponding to the Dirichlet form (1.8) (see (3.32) below), for any $f \in L^2(K, \nu)$. Moreover, we shall end up the section by proving a few sharp properties of the domain $D(N)$ of $N$.

3.1. Estimates for $(\lambda I - N_\varepsilon)^{-1}f$

Let $\lambda > 0, \varepsilon > 0, \varphi \in C_c(H)$. We set

$$f_\varepsilon = \lambda \varphi - N_\varepsilon \varphi.$$

We are going to prove for later use a few estimates of the first and second derivatives of $\varphi$. To this purpose, since $\beta_\varepsilon$ is not differentiable, we need a further approximation $\beta_{\varepsilon, \eta}$ of $\beta_\varepsilon$.

More precisely, for any $\varepsilon > 0, \eta > 0$ we consider the penalized equation

$$\begin{cases}
\frac{dX_{\varepsilon, \eta}(t)}{dt} + \left(\frac{1}{2}A^{\alpha}X_{\varepsilon, \eta}(t) + A^{\alpha-1}\beta_{\varepsilon, \eta}(X_{\varepsilon, \eta}(t))\right)\,dt = A^{-(1-\alpha)/2}\,dW_t, \\
X_{\varepsilon, \eta}(0) = x,
\end{cases}$$

where $\beta_{\varepsilon, \eta}$ is the regularization of $\beta_\varepsilon$ given by the infinite-dimensional mollifier

$$\beta_{\varepsilon, \eta}(x) = e^{-\eta A} \int_H \beta_\varepsilon(e^{-\eta A}x + y)\mu_\eta(dy), \quad x \in H, \eta > 0.$$
Here $\mu_\eta$ is the Gaussian measure on $H$ with mean 0 and covariance operator

$$Q_\eta := \frac{1}{2} A^{-1}(1 - e^{-2\eta A}).$$

Notice that $\beta_{\varepsilon, \eta}$ is of class $C^\infty$ and its derivatives of all order are bounded. Moreover, $\beta_{\varepsilon, \eta}$ is a monotone mapping in $H$ and

$$\lim_{\eta \to \infty} \beta_{\varepsilon, \eta}(x) = \beta_\varepsilon(x) \quad \text{in } H \forall \varepsilon > 0, x \in H. \tag{3.9}$$

Since $\beta_{\varepsilon, \eta}$ is Lipschitz, Eq. (3.7) has a unique mild solution $X_{\varepsilon, \eta}(t, x)$. Moreover, it is easy to see that there is a unique invariant probability measure $\nu_{\varepsilon, \eta}$ for (3.7) given by

$$\nu_{\varepsilon, \eta}(dx) = Z_{\varepsilon, \eta}^{-1} e^{-\frac{1}{2} d_{K, \eta}(x)^2}/\varepsilon, \tag{3.10}$$

where

$$Z_{\varepsilon, \eta} = \int_H e^{-\frac{1}{2} d_{K, \eta}(y)^2}/\varepsilon \mu(dy). \tag{3.11}$$

$\frac{1}{2\varepsilon} d_{K, \eta}^2$ is the potential associated with $\beta_{\varepsilon, \eta}$, that is

$$\frac{1}{2\varepsilon} d_{K, \eta}^2(x) = \beta_{\varepsilon, \eta}(x) \quad \forall x \in H, \tag{3.12}$$

equivalently

$$\frac{1}{2\varepsilon} d_{K, \eta}^2(x) = \int_0^1 \langle \beta_{\varepsilon, \eta}(tx), x \rangle \, dt \quad \forall x \in H.$$

The corresponding Kolmogorov operator reads as follows,

$$N_{\varepsilon, \eta} \varphi = L \varphi - \langle A^{\alpha-1} \beta_{\varepsilon, \eta}(x), D \varphi \rangle, \quad \varphi \in \mathcal{E}(H), \varepsilon > 0, \tag{3.13}$$

where $L$ is the Ornstein–Uhlenbeck operator introduced before. Then $\nu_{\varepsilon, \eta}$ is an invariant measure for $N_{\varepsilon, \eta}$ and

$$\int_H N_{\varepsilon, \eta} \varphi \psi \, d\nu_{\varepsilon, \eta} = -\frac{1}{2} \int_H \langle A^{\alpha-1} D \varphi, D \psi \rangle \, d\nu_{\varepsilon, \eta} \quad \forall \varphi, \psi \in \mathcal{E}(H). \tag{3.14}$$

Moreover, since $\beta_{\varepsilon, \eta}$ is Lipschitz continuous, the operator $N_{\varepsilon, \eta}$ is essentially $m$-dissipative in $L^2(H, \nu_{\varepsilon, \eta})$ and $\mathcal{E}(H)$ is a core for $N_{\varepsilon, \eta}$ (see [10]). We shall denote again by $N_{\varepsilon, \eta}$ the closure of $N_{\varepsilon, \eta}$ in $L^2(H, \nu_{\varepsilon, \eta})$. Moreover, we have

$$\lim_{\eta \to 0} \left| X_{\varepsilon, \eta}(t, x) - X_\varepsilon(t, x) \right| = 0 \quad \forall t \geq 0, x \in H, \mathbb{P}\text{-a.s.} \tag{3.15}$$

Indeed by (3.1) and (3.7) we have for all $t \geq 0, \varepsilon > 0, \eta > 0$,

$$X_{\varepsilon, \eta}(t, x) - X_\varepsilon(t, x)$$

$$= -\int_0^t A^{1-\alpha} e^{-A\eta(t-s)/2} \left( \beta_{\varepsilon, \eta} \left( X_{\varepsilon, \eta}(t, x) \right) - \beta_\varepsilon \left( X_\varepsilon(t, x) \right) \right) \, ds \quad \mathbb{P}\text{-a.s.}$$

and this yields

$$\left| X_{\varepsilon, \eta}(t, x) - X_\varepsilon(t, x) \right| \leq C \int_0^t \left| \beta_{\varepsilon, \eta} \left( X_{\varepsilon, \eta}(t, x) \right) - \beta_\varepsilon \left( X_\varepsilon(t, x) \right) \right| \, ds$$

$$+ C \int_0^t \left| X_{\varepsilon, \eta}(t, x) - X_\varepsilon(t, x) \right| \, ds \quad \forall t \geq 0, \varepsilon, \eta > 0 \mathbb{P}\text{-a.s.},$$
because
\[ \| \beta_{\epsilon, \eta} \|_{\text{Lip}} \leq \frac{1}{\epsilon} \quad \forall \eta > 0. \]

Since
\[ \lim_{\eta \to 0} \beta_{\epsilon, \eta}(X_{\epsilon}(t, x)) = \beta_{\epsilon}(X_{\epsilon}(t, x)), \]
we obtain by Gronwall’s lemma that (3.15) holds.

**Lemma 3.1.** Let \( \lambda > 0, \epsilon > 0, \eta > 0, \varphi \in \mathcal{E}(H) \) and set
\[ f_{\epsilon, \eta} = \lambda \varphi - N_{\epsilon, \eta} \varphi. \] (3.16)
Then the following estimates hold
\[ \int_H \varphi^2 \, d\nu_{\epsilon, \eta} \leq \frac{1}{\lambda^2} \int_H f_{\epsilon, \eta}^2 \, d\nu_{\epsilon, \eta}, \] (3.17)
\[ \int_H |A^{(\alpha-1)/2} D\varphi|^2 \, d\nu_{\epsilon, \eta} \leq \frac{2}{\lambda} \int_H f_{\epsilon, \eta}^2 \, d\nu_{\epsilon, \eta}, \] (3.18)
\[ \lambda \int_H |A^{(\alpha-1)/2} D\varphi|^2 \, d\nu_{\epsilon, \eta} + \frac{1}{2} \int_H \text{Tr}\left[\left(A^{\alpha-1} D^2 \varphi\right)^2\right] \, d\nu_{\epsilon, \eta} \]
\[ + \frac{1}{2} \int_H |A^{\alpha/2} D\varphi|^2 \, d\nu_{\epsilon, \eta} \leq 4 \int_H f_{\epsilon, \eta}^2 \, d\nu_{\epsilon, \eta}. \] (3.19)

**Proof.** Multiplying both sides of (3.16) by \( \varphi \), taking into account (3.14) and integrating in \( \nu_{\epsilon, \eta} \) over \( H \), yields
\[ \lambda \int_H \varphi^2 \, d\nu_{\epsilon, \eta} + \frac{1}{2} \int_H |A^{(\alpha-1)/2} D\varphi|^2 \, d\nu_{\epsilon, \eta} = \int_H \varphi f_{\epsilon, \eta} \, d\nu_{\epsilon, \eta}. \] (3.20)
Now (3.17) and (3.18) follow easily from the Hölder inequality. To prove (3.19) we differentiate both sides of (3.16) in the direction of \( e_k \) and obtain that
\[ \lambda D_k \varphi - N_{\epsilon, \eta} D_k \varphi + \frac{1}{2} a_k D_k \varphi + \sum_{h=1}^{\infty} \langle D_k \beta_{\epsilon, \eta} e_h, e_k \rangle D_{\eta} \varphi = D_k f_{\epsilon, \eta}. \]
Next we multiply both sides of latter equation by \( a_k^{\alpha-1} D_k \varphi \). Taking into account (3.14), integrating in \( \nu_{\epsilon, \eta} \) over \( H \) and summing up over \( k \), yields
\[ \lambda \int_H |A^{(\alpha-1)/2} D\varphi|^2 \, d\nu_{\epsilon, \eta} + \frac{1}{2} \int_H \text{Tr}\left[\left(A^{\alpha-1} D^2 \varphi\right)^2\right] \, d\nu_{\epsilon, \eta} \]
\[ + \frac{1}{2} \int_H |A^{\alpha/2} D\varphi|^2 \, d\nu_{\epsilon, \eta} + \int_K \langle D\beta_{\epsilon, \eta} A^{(\alpha-1)/2} D\varphi, A^{(\alpha-1)/2} D\varphi \rangle \, d\nu_{\epsilon, \eta} \]
\[ = \int_H |A^{(\alpha-1)/2} D\varphi, A^{(\alpha-1)/2} Df_{\epsilon, \eta}| \, d\nu_{\epsilon, \eta}. \] (3.21)
Noting finally that, again in view of (3.14),
\[ \int_H |A^{(\alpha-1)/2} D\varphi, A^{(\alpha-1)/2} Df_{\epsilon, \eta}| \, d\nu_{\epsilon, \eta} \]
\[ = 2 \int_H f_{\epsilon, \eta}^2 \, d\nu_{\epsilon, \eta} - 2\lambda \int_H f_{\epsilon, \eta} \varphi \, d\nu_{\epsilon, \eta} \leq 4 \int_H f_{\epsilon, \eta}^2 \, d\nu_{\epsilon, \eta}, \]
the conclusion follows. □

Taking into account (3.15) and that
\[
\lim_{\eta \to 0} N_{\varepsilon, \eta} \varphi(x) = N_{\varepsilon} \varphi(x) \quad \forall \varepsilon > 0,
\]
letting \( \eta \to 0 \) we obtain the following result.

**Corollary 3.2.** Let \( \lambda > 0, \varepsilon > 0, \varphi \in \mathcal{D}(H) \) and let
\[
f_\varepsilon = \lambda \varphi - N_{\varepsilon} \varphi.
\]
Then the following estimates hold
\[
\int_{H} \varphi_{\varepsilon}^2 \, dv_{\varepsilon} \leq \frac{1}{\lambda^2} \int_{H} f_\varepsilon^2 \, dv_{\varepsilon},
\]
\[
\int_{H} |A^{(\alpha-1)/2} D\varphi_\varepsilon|^2 \, dv_{\varepsilon} \leq \frac{2}{\lambda} \int_{H} f_\varepsilon^2 \, dv_{\varepsilon},
\]
\[
\lambda \int_{H} |A^{(\alpha-1)/2} D\varphi_\varepsilon|^2 \, dv_{\varepsilon} + \frac{1}{2} \int_{H} \text{Tr} \left[ (A^{\alpha-1} D^2 \varphi_\varepsilon)^2 \right] \, dv_{\varepsilon}
\]
\[
+ \frac{1}{2} \int_{H} |A^{\alpha/2} D\varphi_\varepsilon|^2 \, dv_{\varepsilon} \leq 4 \int_{H} f_\varepsilon^2 \, dv_{\varepsilon}.
\]

Now we are able to prove.

**Proposition 3.3.** Let \( \lambda > 0, f \in L^2(H, \nu_{\varepsilon}) \) and let \( \varphi_{\varepsilon} \) be the solution of the equation
\[
\lambda \varphi_{\varepsilon} - N_{\varepsilon} \varphi_{\varepsilon} = f.
\]
Then \( \varphi_{\varepsilon} \in W^{2,2}_{A^{\alpha-1}}(H, \nu_{\varepsilon}) \), \( A^{\alpha/2} D\varphi_{\varepsilon} \in L^2(H, \nu_{\varepsilon}) \) and the following estimates hold
\[
\int_{H} \varphi_{\varepsilon}^2 \, dv_{\varepsilon} \leq \frac{1}{\lambda^2} \int_{H} f^2 \, dv_{\varepsilon},
\]
\[
\int_{H} |A^{(\alpha-1)/2} D\varphi_{\varepsilon}|^2 \, dv_{\varepsilon} \leq \frac{2}{\lambda} \int_{H} f^2 \, dv_{\varepsilon},
\]
\[
\lambda \int_{H} |A^{(\alpha-1)/2} D\varphi_{\varepsilon}|^2 \, dv_{\varepsilon} + \frac{1}{2} \int_{H} \text{Tr} \left[ (A^{\alpha-1} D^2 \varphi_{\varepsilon})^2 \right] \, dv_{\varepsilon}
\]
\[
+ \frac{1}{2} \int_{H} |A^{\alpha/2} D\varphi_{\varepsilon}|^2 \, dv_{\varepsilon} \leq 4 \int_{H} f^2 \, dv_{\varepsilon}.
\]

**Proof.** Inequality (3.27) is obvious since by (3.5), \( N_{\varepsilon} \) is dissipative in \( L^2(H, \nu_{\varepsilon}) \). Let us prove (3.28). Let \( \lambda > 0, f \in L^2(H, \nu_{\varepsilon}) \) and let \( \varphi_{\varepsilon} \) be the solution to Eq. (3.26). Since \( \mathcal{D}(H) \) is a core for \( N_{\varepsilon} \) there exists a sequence \( \{\varphi_{\varepsilon,n}\}_{n \in \mathbb{N}} \subset \mathcal{D}(H) \) such that
\[
\lim_{n \to \infty} \varphi_{\varepsilon,n} \to \varphi_{\varepsilon}, \quad \lim_{n \to \infty} N_{\varepsilon} \varphi_{\varepsilon,n} \to N_{\varepsilon} \varphi_{\varepsilon} \quad \text{in} \quad L^2(H, \nu_{\varepsilon}).
\]
We set \( f_{\varepsilon,n} = \lambda \varphi_{\varepsilon,n} - N_{\varepsilon} \varphi_{\varepsilon,n} \). Clearly, \( f_{\varepsilon,n} \to f \) in \( L^2(H, \nu_{\varepsilon}) \) as \( n \to \infty \). We claim that \( \varphi_{\varepsilon} \in W^{1,2}_{A^{\alpha-1}}(H, \nu_{\varepsilon}) \) and that
\[
\lim_{n \to \infty} A^{(\alpha-1)/2} D\varphi_{\varepsilon,n} \to A^{(\alpha-1)/2} D\varphi_{\varepsilon} \quad \text{in} \quad L^2(H, \nu_{\varepsilon}; H),
\]
which will imply (3.28).

Let \( m, n \in \mathbb{N} \); then by (3.24) it follows that

\[
\int_H |A^{(a-1)/2} \varphi_{e,n} - A^{(a-1)/2} \varphi_{e,m}|^2 \, d\nu_e \leq \frac{2}{\lambda} \int_H |f_{e,n} - f_{e,m}|^2 \, d\nu_e.
\]

Therefore the sequence \( \{ \varphi_{e,n} \}_{n \in \mathbb{N}} \) is Cauchy in \( W^{1,2}_{A^\alpha}(H, \nu_e) \) and the conclusion follows. The estimate (3.29) follows similarly by (3.25). \( \square \)

We conclude this subsection with an integration by parts formula needed later. We set

\[ V := \left\{ \psi \in C^1_b(K) : |Q^{1/2} n(y)|^{-1} \psi \in C_b(K) \right\}. \tag{3.30} \]

**Lemma 3.4.** Let \( \varphi \in D(N_e) \) and \( \psi \in V \). Then the following identity holds.

\[
\int_K N_{\varphi} \psi \, dv = -\frac{1}{2} \int_K \{ A^{(a-1)/2} \varphi, A^{(a-1)/2} \psi \} \, dv + \frac{1}{\mu(K)} \int_\Sigma \{ A^{a-1} \gamma(D\varphi), n(y) \} \psi \, d\mu_\Sigma. \tag{3.31}
\]

**Proof.** We first notice that the last integral in (3.31) is meaningful since

\[
\left| \int_\Sigma \{ A^{a-1} \gamma(D\varphi), n(y) \} \psi \, d\mu_\Sigma \right| \leq \| A^{a-1} \| \int_\Sigma \left| A^{(a-1)/2} \gamma(D\varphi) \right|^2 |Q^{1/2} n(y)|^2 \, d\mu_\Sigma \int_\Sigma \psi^2 |Q^{1/2} n(y)|^{-2} \, d\mu_\Sigma < \infty
\]

by (2.5).

Now, taking in account that \( \mathcal{E}(H) \) is a core for \( N_e \), it is sufficient to prove (3.31) for \( \varphi \in \mathcal{E}(H) \). By the basic integration by parts formula (2.2) we deduce, for any \( i \in \mathbb{N} \) and \( \psi \in V \) that

\[
\int_K D_i \varphi D_i \psi \, dv = -\int_K D_i^2 \varphi \psi \, dv + \frac{1}{\mu(K)} \int_\Sigma \gamma(D_i \varphi)(n(y))_i \psi \, d\mu_\Sigma + \frac{1}{\lambda_i} \int_K x_i D_i \psi \, dv.
\]

It follows that

\[
a_i^{a-1} \int_K D_i \varphi D_i \psi \, dv = -a_i^{a-1} \int_K D_i^2 \varphi \psi \, dv + \frac{1}{\mu(K)} a_i^{a-1} \int_\Sigma \gamma(D_i \varphi)(n(y))_i \psi \, d\mu_\Sigma + \frac{1}{2} a_i^a \int_K x_i D_i \psi \, dv.
\]

Now, summing up on \( i \) yields

\[
\int_K \{ A^{(a-1)/2} \varphi, A^{(a-1)/2} \psi \} \, dv = -\int_K \text{Tr}[A^{a-1} D^2 \varphi] \psi \, dv + \frac{1}{\mu(K)} \int_\Sigma \{ A^{a-1} \gamma(D\varphi), n(y) \} \psi \, d\mu_\Sigma + 2 \int_K \{ x, A^a D\varphi \} \psi \, dv,
\]

which is precisely Eq. (3.31). \( \square \)
3.2. Convergence of \( \{ \varphi_\varepsilon \} \) as \( \varepsilon \to 0 \)

Let \( N : D(N) \subset L^2(K, \nu) \to L^2(K, \nu) \) be the operator defined by

\[
\{ \langle N\varphi, \psi \rangle_{L^2(K, \nu)} = -\frac{1}{2} a(\varphi, \psi), \quad \forall \psi \in W^{1,2}_{A(\alpha-1/2)}(K, \nu), \varphi \in D(N), \}
\]

\[
D(N) = \{ \varphi \in W^{1,2}_{A(\alpha-1/2)}(K, \nu); \quad |a(\varphi, \psi)| \leq C|\varphi|_{L^2(K, \nu)}|\psi|_{L^2(K, \nu)}, \quad \forall \psi \in W^{1,2}_{A(\alpha-1/2)}(K, \nu) \}. \tag{3.32}
\]

The operator \( L \) is self-adjoint in \( L^2(K, \nu) \) and the Neumann problem (1.1) (or equivalently (1.9)) reduces to

\[
\lambda \varphi - N \varphi = f. \tag{3.33}
\]

We are going to show that for each \( f \in L^2(K, \nu) \) and \( \varepsilon \to 0 \), \( \varphi_\varepsilon = (\lambda I - N_\varepsilon)^{-1} f \) is convergent in \( L^2(K, \nu) \) to \( \varphi = (\lambda I - N)^{-1} f \) and derive so, via the estimate proven in Proposition 3.3, high order regularity properties for the solution \( \varphi \) to (3.33).

We first note that for \( f \in C_b(H) \) we have

\[
\varphi_\varepsilon(x) = E \int_0^\infty e^{-\lambda t} f(X_\varepsilon(t, x)) \, dt \quad \forall x \in H. \tag{3.34}
\]

Now, by a standard argument it follows that from (3.34) if \( f \in C^1_b(H) \) we have

\[
\sup_{x \in H} |D\varphi_\varepsilon(x)| \leq \frac{1}{\lambda} \| Df \|_{C_b(H)} \quad \forall \lambda, \varepsilon > 0. \tag{3.35}
\]

Theorem 3.5 below is the main result of this section.

**Theorem 3.5.** Let \( \lambda > 0 \), \( f \in L^2(K, \nu) \) and let \( \varphi_\varepsilon \) be the solution of Eq. (3.26). Then \( \{ \varphi_\varepsilon \} \) is strongly convergent in \( L^2(K, \nu) \) to \( \varphi = (\lambda I - N)^{-1} f \) where \( N \) is defined by (3.32).

Moreover, the following statements hold.

1. \( \lim_{\varepsilon \to 0} A^{(\alpha-1)/2} D\varphi_\varepsilon = A^{(\alpha-1)/2} D\varphi \) in \( L^2(K, \nu; H) \),
2. \( \varphi \in W^{1,2}_{A(\alpha-1)}(K, \nu) \) and \( |A^{\alpha/2} D\varphi| \in L^2(K, \nu) \),
3. \( \varphi \) fulfills the Neumann condition

\[
\langle A^{\alpha-1} \gamma(D\varphi(x)), n(x) \rangle = 0, \quad \mu_\Sigma \text{ a.e. on } \Sigma, \tag{3.36}
\]

where \( \gamma(D\varphi(x)) \) is defined by Proposition 2.3.

In particular, since \( N \) is dissipative Theorem 3.5 amounts to say that for each \( f \in L^2(K, \nu) \) the equation \( \lambda \varphi - N \varphi = f \) has a unique solution \( \varphi \) satisfying (ii), (iii).

**Proof of Theorem 3.5.** Without danger of confusion we shall denote again by \( f \) the restriction \( f|_K \) of \( f \) to \( K \). In fact each \( f \in L^2(K, \nu) \) can be extended by 0 outside \( K \) to a function in \( L^2(H, \nu) \). By this convention, everywhere in the sequel \( (\lambda I - N)^{-1} f \) for \( f \in L^2(H, \nu) \) means \( (\lambda I - N)^{-1} f|_K \).

**Step 1.** We have

\[
\lim_{\varepsilon \to 0} \varphi_\varepsilon = (\lambda I - N)^{-1} f \quad \text{in } L^2(K, \nu). \tag{3.37}
\]

In fact by (3.28), (3.29) it follows that there exist a sequence \( \{ \varepsilon_k \} \to 0 \) and \( \varphi \in W^{1,2}_{A(\alpha-1)}(K, \nu) \) such that

\[
\varphi_{\varepsilon_k} \to \varphi, \quad \text{weakly in } L^2(K, \nu),
\]

\[
A^{(\alpha-1)/2} D\varphi_{\varepsilon_k} \to A^{(\alpha-1)/2} D\varphi, \quad \text{weakly in } L^2(K, \nu; H).
\]
Let $\psi \in C^1_b(H)$ and notice that by (3.5) and by (3.26) we have the identity

$$\frac{1}{2} \int_H \langle A^{(\alpha-1)/2} D\varphi_\varepsilon, A^{(\alpha-1)/2} D\psi \rangle \, d\nu_\varepsilon = \int_H (f - \lambda \varphi_\varepsilon) \psi \, d\nu_\varepsilon.$$  
Equivalently

$$\frac{1}{2} \int_K \langle A^{(\alpha-1)/2} D\varphi_\varepsilon, A^{(\alpha-1)/2} D\psi \rangle \, d\nu + \frac{1}{2} \int_K \langle A^{(\alpha-1)/2} D\varphi_\varepsilon, A^{(\alpha-1)/2} D\psi \rangle \, d\nu_\varepsilon$$

$$= \int_H (f - \lambda \varphi_\varepsilon) \psi \, d\nu_\varepsilon. \quad (3.38)$$

Since by (3.28) we have

$$\left| \int_{K^c} \langle A^{(\alpha-1)/2} D\varphi_\varepsilon, A^{(\alpha-1)/2} D\psi \rangle \, d\nu_\varepsilon \right|^2 \leq \int_H |A^{(\alpha-1)/2} D\varphi_\varepsilon|^2 \, d\nu_\varepsilon \int_{K^c} |A^{(\alpha-1)/2} D\psi|^2 \, d\nu_\varepsilon \leq \frac{2}{\lambda} \int_H f^2 \, d\nu_\varepsilon \int_{K^c} |A^{(\alpha-1)/2} D\psi|^2 \, d\nu_\varepsilon \to 0,$$

as $\varepsilon \to 0$, it follows by (3.38) that

$$\frac{1}{2} \int_K \langle A^{(\alpha-1)/2} D\varphi, A^{(\alpha-1)/2} D\psi \rangle \, d\nu = \int_K (f - \lambda \varphi) \psi \, d\nu \quad \forall \psi \in C^1_b(K).$$

Obviously, this identity extends to all $\psi \in W^{1,2}_{A^{\alpha-1}}(K, \nu)$, which implies that $\varphi_\varepsilon \to (\lambda I - N)^{-1} f$ weakly in $L^2(K, \nu)$ as $\varepsilon \to 0$.

**Step 2.** We have

$$\lim_{\varepsilon \to 0} \varphi_\varepsilon = \varphi$$

$$\lim_{\varepsilon \to 0} A^{(\alpha-1)/2} D\varphi_\varepsilon = A^{(\alpha-1)/2} D\varphi \quad \text{in } L^2(K, \nu),$$

which follows from (3.5). By (3.26) and (3.39) we see that

$$\frac{1}{2} \int_H |A^{(\alpha-1)/2} D\varphi_\varepsilon|^2 \, d\nu_\varepsilon = -\int_H (\lambda \varphi_\varepsilon - f) \varphi_\varepsilon \, d\nu_\varepsilon, \quad (3.40)$$

which implies in virtue of (3.32), (3.33)

$$\lim_{\varepsilon \to 0} \int_K \left( \frac{1}{2} |A^{(\alpha-1)/2} D\varphi_\varepsilon|^2 + \lambda \varphi_\varepsilon^2 \right) \, d\nu_\varepsilon = \int_K f \varphi \, d\nu$$

$$= -(N\varphi, \varphi) + \lambda \int_K \varphi^2 \, d\nu$$

$$= \int_K \left( \frac{1}{2} |A^{(\alpha-1)/2} D\varphi|^2 + \lambda \varphi^2 \right) \, d\nu. \quad (3.41)$$

Here we have used the fact that

$$\lim_{\varepsilon \to 0} \int_{K^c} |A^{(\alpha-1)/2} D\varphi_\varepsilon|^2 \, d\nu_\varepsilon = 0.$$
which follows taking into account (3.35).

Therefore, there exists a sequence \( \{\varepsilon_k\} \downarrow 0 \) such that

\[
\begin{align*}
\varphi_{\varepsilon_k} & \rightarrow \varphi \\
A^{(\alpha-1)/2} \varphi_{\varepsilon_k} & \rightarrow A^{(\alpha-1)/2} \varphi \quad \text{weakly in } L^2(K, v), \\
\lim_{k \rightarrow \infty} \int_K \left( \lambda \varphi_{\varepsilon_k}^2 + \frac{1}{2} A^{(\alpha-1)/2} \varphi_{\varepsilon_k}^2 \right) dv = \int_K \left( \lambda \varphi^2 + \frac{1}{2} A^{(\alpha-1)/2} \varphi^2 \right) dv.
\end{align*}
\]

This implies that \( \varphi_{\varepsilon_k} \rightarrow \varphi \) strongly in \( L^2(K, v) \) and \( A^{(\alpha-1)/2} \varphi_{\varepsilon_k} \rightarrow A^{(\alpha-1)/2} \varphi \) strongly in \( L^2(K, v; H) \).

We finally assume that \( f \in L^2(H, v) \). Since \( C_b^1(K) \) is dense in \( L^2(K, v) \), there exists a sequence \( \{f_n\} \subset C_b^1(H) \) strongly convergent in \( L^2(K, v) \) to \( f \). Set \( \varphi_{n, \varepsilon} = (\lambda I - N) f_n \). By (3.28) we have

\[
\int_H |A^{(\alpha-1)/2} \varphi_{n, \varepsilon} - A^{(\alpha-1)/2} \varphi_{n, \varepsilon}|^2 dv \leq \frac{2}{\lambda} \int_K |f - f_n|^2 dv,
\]

which implies

\[
\int_K |A^{(\alpha-1)/2} \varphi_{n, \varepsilon} - A^{(\alpha-1)/2} \varphi_{n, \varepsilon}|^2 dv \leq \frac{2}{\lambda} \int_K |f - f_n|^2 dv.
\]

So, again \( A^{(\alpha-1)/2} \varphi_{n, \varepsilon} \rightarrow A^{(\alpha-1)/2} \varphi \) strongly in \( L^2(K, v; H) \) as claimed.

**Step 3.** We have \( \varphi \in W^{2,2}_{A^{\alpha-1}}(K, v) \) and \( A^{\alpha/2} \varphi \in L^2(K, v) \).

By estimate (3.29) we have that \( \{\varphi_{\varepsilon}\} \) is bounded in \( W^{2,2}_{A^{\alpha-1}}(K, v) \). Therefore there is a subsequence, still denoted \( \{\varphi_{\varepsilon}\} \) which converges to \( \varphi \) in \( W^{2,2}_{A^{\alpha-1}}(K, v) \). In the same way we show that \( A^{\alpha/2} \varphi \in L^2(K, v) \).

**Step 4.** Checking the Neumann condition for \( \varphi \).

We recall that (see from (3.31))

\[
\int_K N_{\varepsilon} \varphi_{\varepsilon} \psi dv = -\frac{1}{2} \int_K \left( A^{(\alpha-1)/2} \varphi_{\varepsilon}, A^{(\alpha-1)/2} \varphi \right) dv + \frac{1}{\mu(K)} \int_{\Sigma} \psi \left( A^{\alpha-1} \gamma(D\varphi_{\varepsilon}) \cdot n(y) \right) d\mu_{\Sigma}.
\]

(3.42)

Recalling that for \( \varepsilon \rightarrow 0 \), \( N_{\varepsilon} \varphi_{\varepsilon} = \lambda \varphi_{\varepsilon} - f \rightarrow \lambda \varphi - f = N \varphi \) in \( L^2(K, v) \) and by Proposition 2.3 we have

\[
|Q^{1/2} n(y)||\left( A^{\alpha-1} \gamma(D\varphi_{\varepsilon}) \cdot n(y) \right) | \rightarrow |Q^{1/2} n(y)||\left( A^{\alpha-1} \gamma(D\varphi) \cdot n(y) \right) |
\]

in \( L^2(\Sigma, \mu_{\Sigma}) \), it follows by (3.42) that

\[
\int_{\Sigma} \left( A^{\alpha-1} \gamma(D\varphi) \cdot n(y) \right) \psi d\mu_{\Sigma} = 0 \quad \forall \psi \in V,
\]

where \( V \) is defined by (3.30). Since \( V \) is dense in \( L^2(\Sigma, \mu_{\Sigma}) \) the conclusion follows.

This completes the proof of the theorem.

\[
\square
\]

4. The process associated with the reflection problem

Throughout this section the following hypothesis will be assumed.

**Hypothesis 4.1.**

(i) \( \alpha \in [0, \frac{1}{3}] \) and there is \( \delta \in (0, 1) \) such that \( \text{Tr}[A^{2\delta-1}] < \infty \).

(ii) \( K = B(0, 1) = \{x \in H: |x| \leq 1\} \).
We are going to construct a stochastic process \( X = X(t, x) \) on a probability space \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})\) associated with the semigroup \( P_t \) generated by \( N \) on \( L^2(K, \nu) \), i.e.,

\[
(P_t f)(x) = \tilde{\mathbb{E}}[ f(X(t, x)) ] \quad \forall f \in C_b(H), \ x \in H.
\]

The main result, Theorem 4.10 below amounts to saying that there is a cadlag \( H \)-valued process \( X \) with this property.

To this aim we need first some sharp estimates on solution \( X_\varepsilon(t, x) \) to approximating Eq. (3.1), that is

\[
\begin{cases}
    dX_\varepsilon + \frac{1}{2} A^\alpha X_\varepsilon \, dt + A^{\alpha-1} \beta_\varepsilon(X_\varepsilon) \, dt = A^{(\alpha-1)/2} \, dW_t, & t \geq 0, \\
    X_\varepsilon(0) = x.
\end{cases}
\]

(4.1)

4.1. Estimates for \( X_\varepsilon \)

We set

\[
|x|_a = |A^a x|, \quad \langle x, y \rangle_a = \langle A^a x, A^a y \rangle, \quad \forall x, y \in D(A^a), 0 < a < 1
\]

and

\[
W_A(t) = \int_0^t e^{-A^a(t-s)/2} A^{(\alpha-1)/2} \, dW_s, \quad t \geq 0.
\]

**Lemma 4.2.** The following estimates hold

\[
\mathbb{E} \left[ \sup_{t \in [0, T]} \left| W_A(t) \right|_\delta^{2m} \right] \leq CT^{m+1/m+1} \quad \forall T > 0,
\]

(4.2)

\[
\mathbb{E} \left[ \sup_{t \in [T-h, T]} \left| W_A(t) - W_A(t-h) \right|_\delta^{2m} \right] \leq Ch^\rho T^{m+1/m+1} \quad \forall T > 0, \forall h > 0,
\]

(4.3)

where \( m > 1 \) and \( 1 < \rho < m \).

Here \( C \) is a positive constant independent of \( \omega, T \) and \( \varepsilon \).

**Proof.** Since the proof is identical with Theorem 2.9 in [7] we shall sketch it only for convenience. We have (see [7], p. 25)

\[
W_A(t) = \frac{\sin(\pi \gamma)}{\pi} \int_0^t e^{-(t-s)A^{\alpha/2}/2}(t-s)^{-\gamma} Y(s) \, ds,
\]

(4.4)

where \( 0 < \gamma < 1 \) and

\[
Y(t) = \int_0^t e^{-(t-s)A^{\alpha/2}/2}(t-s)^{-\gamma} A^{(\alpha-1)/2} \, dW_s.
\]

In the following we shall fix \( m > \frac{1}{2\gamma} \) and \( 0 < \gamma < \frac{1}{2} \).

We have

\[
\left| \int_0^t e^{-(t-s)A^{\alpha/2}/2}(t-s)^{-\gamma} f(s) \, ds \right| \leq C t^{\gamma-1/2(m)} |f|_{L^2(0,T; H)}
\]

(4.5)

and therefore

\[
\sup_{t \in [0, T]} \left| W_A(t) \right|_\delta^{2m} \leq CT^{2m(\gamma-1/2m)} \int_0^T \left| Y(s) \right|_\delta^{2m} \, ds.
\]
On the other hand, under Hypothesis 4.1 we have
\[ \mathbb{E}( |X(s)|^{2m \delta} ) \leq C s^m \quad \forall s > 0 \]
and this implies (4.2) as claimed.

As regards (4.3), we have by (4.4) that
\[ W_A(t) - W_A(t - h) = \sin(\pi y) \int_0^{t-h} e^{-(t-h-s)A\alpha/2} [(t-s)\gamma^{-1} - (t-h-s)\gamma^{-1}e^{-hA\alpha/2}] Y(s) \, ds \]
\[ + \sin(\pi y) \int_{t-h}^t e^{-(t-s)A\alpha/2} (t-s)^{-1} Y(s) \, ds. \]

Then by (4.5) we have that
\[ \sup_{t \in [h, T-h]} |W_A(t) - W_A(t - h)|^{2m} \]
\[ \leq C \left( h^{2m\gamma} \int_0^T |Y(s)|^{2m} \, ds + \int_0^T |(1 - e^{-hA\alpha/2}) Y(s)|^{2m} \, ds + h^{2m-1} \int_0^T |Y(s)|^{2m} \, ds \right) \]
\[ \leq C \left( h^{2m\gamma} + h^{-2m-1} + h^m \right) \int_0^T |Y(s)|^{2m} \, ds \]
because \( |1 - e^{-hA\alpha/2}| \leq Ch^{1/2} |Y|_{\alpha/2} \). Then we get as above that (4.3) holds.

In the following we set \( y_\varepsilon = X_\varepsilon - W_A \) and notice that \( y_\varepsilon \) is the solution to equation
\[
\begin{align*}
\frac{dy_\varepsilon}{dt}(t) + \frac{1}{2} A^{\alpha} y_\varepsilon(t) + A^{\alpha-1} \beta_\varepsilon(y_\varepsilon(t) + W_A(t)) &= 0, \quad t \geq 0, \quad P\text{-a.s.} \quad (4.6)
\end{align*}
\]
Equivalently
\[
\begin{align*}
A^{1-\alpha} \frac{dy_\varepsilon}{dt}(t) + \frac{1}{2} A y_\varepsilon(t) + \beta_\varepsilon(y_\varepsilon(t) + W_A(t)) &= 0, \quad t \geq 0, \quad P\text{-a.s.} 
\end{align*}
\]
Denote by \( BV([0, T]; H) \) the space of all \( H \)-valued functions with bounded variation on \([0, T]\) and denote by \( \|y\|_{BV([0, T]; H)} \) the total variation of \( y \in BV([0, T]; H) \). We set \( \eta = \frac{1-\alpha}{2} \).

**Lemma 4.3.** Assume that \( x \in D(A^\eta) \), then there exists a constant \( C > 0 \) independent of \( \omega \in \Omega, T > 0 \) and \( \varepsilon, h \) such that
\[
\int_0^T |y_\varepsilon(t)|^{2/1/2} \, dt + \sup_{t \in [0, T]} |y_\varepsilon(t)|_\eta + \int_0^T |\beta_\varepsilon(y_\varepsilon(t) + W_A(t))| \, dt \]
\[
\leq C \left( |x|_\eta + \sup_{t \in [0, T]} |W_A(t)|_\eta \right)^{1/2} \left( 1 - h^p \sup_{s,t \in [0, T]} |W_A(t) - W_A(s)| |t-s|^{-p} \right) \]
\[
- \mu^\delta \sup_{s \in [0, T]} |W_A(s)|_\delta^{-1},
\]
\[
\|y_\varepsilon\|_{BV([0, T]; H)} \leq C \left( |x|_\eta + \int_0^T |\beta_\varepsilon(y_\varepsilon(t) + W_A(t))| \, dt + \left( \int_0^T |y_\varepsilon(t)|_1^{1/2} \, dt \right)^{1/2} \right),
\]
where \( p = \frac{\alpha}{2m} \).
Proof. We have

$$\{ \beta_\varepsilon (y_\varepsilon + W_A), y_\varepsilon + W_A - \theta \}$$

$$= \frac{1}{\varepsilon} \left( 1 - \frac{1}{|y_\varepsilon + W_A|} \right)^+ \langle y_\varepsilon + W_A, y_\varepsilon + W_A - \theta \rangle \quad \forall \theta \in H.$$ 

This yields

$$\{ \beta_\varepsilon (y_\varepsilon + W_A), y_\varepsilon + W_A - \theta \} \geq 0 \quad \forall \theta \in H \text{ such that } |\theta| \leq 1.$$ 

In particular, the latter holds for

$$\theta = \frac{\beta_\varepsilon (y_\varepsilon + W_A)}{|\beta_\varepsilon (y_\varepsilon + W_A)|}$$

and so we get, for any $\varepsilon > 0$ and $t \in [0, T]$

$$\int_0^t |\beta_\varepsilon (y_\varepsilon + W_A)| \, ds \leq \int_0^t \langle \beta_\varepsilon (y_\varepsilon + W_A), y_\varepsilon + W_A \rangle \, ds.$$ 

(4.10)

On the other hand, by (4.7) we see that

$$\int_0^t \langle \beta_\varepsilon (y_\varepsilon + W_A), y_\varepsilon \rangle \, ds + \frac{1}{2} |y_\varepsilon (t)|^2 + \int_0^t |A^{1/2} y_\varepsilon (s)|^2 \, ds = \frac{1}{2} |x|^2 \quad \forall t \geq 0$$

and so (4.10) yields

$$\int_0^t |\beta_\varepsilon (y_\varepsilon + W_A)| \, ds + \frac{1}{2} |y_\varepsilon (t)|^2 + \int_0^t |A^{1/2} y_\varepsilon (s)|^2 \, ds$$

$$\leq \frac{1}{2} |x|^2 + \int_0^t \langle \beta_\varepsilon (y_\varepsilon + W_A), W_A \rangle \, ds.$$ 

(4.11)

Now we consider $W_\mu = (1 + \mu A)^{-1} W_A$. We have

$$|W_\mu (t) - W_\mu (s)| \leq |W_A (t) - W_A (s)| \quad \forall t, s > 0,$$

$$|W_\mu (t) - W_A (t)| \leq \mu |A (1 + \mu A)^{-1} W_A| \leq \mu^\delta |W_A| \delta,$$

(4.12)

$$|A W_\mu (t)| \leq \left( 1 + \frac{1}{\mu} \right) |W_A (t)| \quad \forall t \geq 0, \mu > 0.$$ 

Then we have

$$\int_0^t \langle \beta_\varepsilon (y_\varepsilon + W_A), W_A \rangle \, ds$$

$$\leq \int_0^t \langle \beta_\varepsilon (y_\varepsilon + W_A), W_A - W_\mu \rangle \, ds + \int_0^t \langle \beta_\varepsilon (y_\varepsilon + W_A), W_\mu \rangle \, ds$$

$$\leq \sup_{s \in (0, t)} |W_A (s) - W_\mu (s)| \int_0^t |\beta_\varepsilon (y_\varepsilon + W_A)| \, ds + \int_0^t \langle \beta_\varepsilon (y_\varepsilon + W_A), W_\mu \rangle \, ds$$

$$\leq \mu^\delta \sup_{s \in (0, t)} |W_A (s)| \int_0^t |\beta_\varepsilon (y_\varepsilon + W_A)| \, ds + \int_0^t \langle \beta_\varepsilon (y_\varepsilon + W_A), W_\mu \rangle \, ds.$$
On the other hand, we have
\[
\int_0^t \langle \beta \nu(y + W_A), W_{\mu} \rangle \, ds = \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \langle \beta \nu(y + W_A), W_{\mu}(s) - W_{\mu}(t_i) \rangle \, ds
\]
\[
+ \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \langle \beta \nu(y + W_A), W_{\mu}(t_i) \rangle \, ds,
\]
(4.13)
where 0 = t_0 \leq t_1 \leq \cdots \leq t_N = t are chosen in such a way that max(t_{i+1} - t_i) \leq h. We have therefore by (4.12) that
\[
\left| \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \langle \beta \nu(y + W_A), W_{\mu}(s) - W_{\mu}(t_i) \rangle \, ds \right|
\]
\[
\leq h^{p} \sup_{s, \bar{s} \in [0, t]} \left[ |W_A(s) - W_A(\bar{s})| |s - \bar{s}|^{-p} \right] \int_0^t |\beta \nu(y + W_A)| \, ds
\]
(4.14)
and by (4.6) it follows that
\[
\left| \sum_{i=0}^{N-1} \left( W_{\mu}(t_i), \int_{t_i}^{t_{i+1}} \beta \nu(y + W_A) \, ds \right) \right|
\]
\[
\leq \sum_{i=0}^{N-1} \left| \left( W_{\mu}(t_i), A^{2q} y_{\nu}(t_{i+1}) - A^{2q} y_{\nu}(t_i) - \frac{1}{2} \int_{t_i}^{t_{i+1}} A y_{\nu}(s) \, ds \right) \right|
\]
\[
\leq \sum_{i=0}^{N-1} \left| W_{\mu}(t_i) \right| \eta \left( |y_{\nu}(t_{i+1})|_\eta + |y_{\nu}(t_i)|_\eta \right)
\]
\[
+ \sum_{i=0}^{N-1} \left| W_{\mu}(t_i) \right|_{1/2} \int_{t_i}^{t_{i+1}} |y_{\nu}(s)|_{1/2} \, ds
\]
\[
\leq 2N \left( 1 + \frac{1}{\mu} \right) \sup_{s \in [0, t]} |W_A(s)| \sup_{s \in [0, t]} |y_{\nu}(s)|_{\eta}
\]
\[
+ \left( 1 + \frac{1}{\mu} \right) \sup_{s \in [0, t]} |W_A(s)| \int_0^t |y_{\nu}(s)|_{1/2} \, ds.
\]
(4.15)
because \(|W_{\mu}|_{\eta} \leq |A W_A| \leq (1 + \frac{1}{\mu}) |W_A|.

Then substituting into (4.13) yields
\[
\int_0^t \langle \beta \nu(y + W_A), W_{\mu} \rangle \, ds \leq \frac{1}{4} \left( \sup_{s \in (0, t)} |y_{\nu}(s)|_\eta^2 + \int_0^t |y_{\nu}(s)|_{1/2}^2 \, ds \right) + C \left( 1 + \frac{T}{\mu^2} \right) \sup_{s \in (0, t)} |W_A(s)|^2
\]
and substituting into (4.11) we get by (4.13) that
\[
\int_0^t |\beta \nu(y + W_A)| \, ds \leq \frac{1}{4} \left( \sup_{s \in (0, t)} |y_{\nu}(s)|_\eta^2 + \int_0^t |y_{\nu}(s)|_{1/2}^2 \, ds \right)
\]
\[
\leq C \left( |x|^2_\eta + \left( 1 + \frac{T}{\mu^2} \right) \sup_{s \in (0, t)} |W_A(s)|^2 \right)
\]
\[
+ \left( h^p \sup_{s, \bar{s} \in (0, t)} |W_A(s) - W_A(\bar{s})| |s - \bar{s}|^{-p} + \mu^\delta \sup_{s \in (0, t)} |W_A(s)|_\delta \right) \int_0^t |\beta \nu(y + W_A)| \, ds
\]
which implies (4.8) as claimed. By (4.6) we see that (recall that $0 \leq \alpha \leq \frac{1}{2}$),
\[
\int_0^T \left| \frac{dy_\varepsilon}{dt} \right| dt \leq C \int_0^T \left( |A^\varepsilon y_\varepsilon| + |\beta_\varepsilon (y_\varepsilon + W_A)| \right) dt \leq C \left( \left( \int_0^T |y_\varepsilon(t)|^2 dt \right)^{1/2} + \int_0^T |\beta_\varepsilon (y_\varepsilon + W_A)| dt \right)
\]
which clearly implies (4.9).

Now combining (4.8) and (4.9) yields
\[
\sup_{t \in [0,T]} \left| y_\varepsilon(t) \right|_\eta + \|y_\varepsilon\|_{BV([0,T];H)} \leq C \left( |x|_\eta^2 + \frac{T^2}{\mu} \sup_{t \in [0,T]} |W_A(t)|_\delta \right) \left( 1 - h^p H(T) - \mu^\delta H_1(T) \right),
\]
where
\[
H(T) = \sup_{s,t \in [0,T]} \left[ |W_A(t) - W_A(s)| \|t - s\|^{-p} \right],
\]
\[
H_1(T) = \sup_{t \in [0,T]} |W_A(t)|_\delta.
\]

An immediate corollary is Lemma 4.4 below.

**Lemma 4.4.** For each $N > 0$ and $T > 0$ there is $\Omega_{T,N} \subset \Omega$ such that
\[
\mathbb{P}(\Omega_{T,N}) \geq 1 - \frac{C_1^* N}{N} \tag{4.18}
\]
and
\[
\|y_\varepsilon\|_{BV([0,T];H)} + \sup_{t \in [0,T]} |y_\varepsilon(t)|_\eta^2 \leq C_2^* \left( |x|_\eta^2 + N^{1/2} T^6 \right) \quad \forall \omega \in \Omega_{T,N}, \tag{4.19}
\]
where $C_i^*$, $i = 1, 2$, are independent of $\varepsilon$, $T$, $N$ and $\omega$.

**Proof.** By (4.2) and respectively (4.3) we have for all $M > 0$ and $m = 2$
\[
\mathbb{P}\left( \sup_{t \in [0,T]} |W_A(t)|_\delta \leq M \right) \geq 1 - \frac{C}{M^4 T^3} \tag{4.20}
\]
and
\[
\mathbb{P}\left( h^p H(T) \leq \frac{1}{4} \right) \geq 1 - C T^3 h^{2p} \quad \forall h > 0, \tag{4.21}
\]
\[
\mathbb{P}\left( \mu^\delta H_1(T) \leq \frac{1}{4} \right) \geq 1 - C T^3 \mu^{4\delta}.
\]

On the other hand, by (4.8), (4.9) and (4.16) we have
\[
\sup_{t \in [0,T]} |y_\varepsilon|_\eta^2 + \|y_\varepsilon\|_{BV([0,T];H)} \leq 2C \left( |x|_\eta^2 + M^2 \right)
\]
in
\[
\omega: h^p H(T) \leq \frac{1}{4} \right) \cap \left\{ \omega: \sup_{t \in [0,T]} |W_A(t)|_\eta \leq M \right\} \cap \left\{ \omega: \mu^\delta H_1(T) \leq \frac{1}{4} \right\}.
\]

\[
\omega: h^p H(T) \leq \frac{1}{4} \right) \cap \left\{ \omega: \sup_{t \in [0,T]} |W_A(t)|_\eta \leq M \right\} \cap \left\{ \omega: \mu^\delta H_1(T) \leq \frac{1}{4} \right\}.
\]
If we choose $M = N^{1/4}T^{3}$, $h = (NT^{3})^{-2/p}$ and
\[
\Omega_{T,N} = \left\{ \omega: \sup_{t \in [0,T]} |W_{A}(t)|_{\eta} \leq M \right\} \\
\cap \left\{ \omega: h^{p}H(T) \leq \frac{1}{4} \right\} \cap \left\{ \omega: \mu H_{T}(T) \leq \frac{1}{4} \right\}
\]
we obtain (4.18) and (4.19) as desired. \[\square\]

The convergence in law
We denote by $BV(0, \infty; H)$ the space of $H$-valued functions $u : [0, \infty) \to H$ which have bounded variation on each interval $[0, T]$. This is a locally convex space with the family of seminorms
\[
|u|_{T} = \|u\|_{BV([0,T]; H)} \quad \forall T > 0.
\]

We shall construct below a space of cadlag trajectories which is a Polish space in an appropriate topology. To this end we consider the family of spaces \{$X_{N}$\}$_{N=1}^{\infty}$ defined by
\[
X_{N} = \left\{ u \in BV(0, \infty; H) \cap L_{\text{loc}}^{\infty}(0, \infty; D(A^{0})): \right. \left. |u|_{T} + |u|_{L_{\text{loc}}^{\infty}(0,T,D(A^{0}))}^{2} \leq 2C_{e}^{2} \left( |x|_{\eta}^{2} + N^{1/2}T^{6} \right) \forall T > 0 \right\}. 
\]
(Here $C_{e}^{2}$ is the constant arising in (4.19).)

Each $X_{N}$ is a closed and bounded subset of $BV([0, T]; H)$. We shall introduce on $X_{N}$ the topology (in fact a pseudo-topology) defined by the convergence in measure, i.e., we say that $u_{n} \to u$ in $X_{N}$ if for each $T > 0$
\[
\lim_{n \to \infty} \int_{0}^{T} f(t, u_{n}(t)) \, dt = \int_{0}^{T} f(t, u(t)) \, dt
\]
for all bounded and continuous functions $f \in C_{b}([0, \infty) \times H)$.

It turns out that this topology is just given by the metric
\[
d(u, v) = \frac{1}{2^{j}} \frac{d_{T_{j}}(u, v)}{1 + d_{T_{j}}(u, v)}, \quad j = 1, 2, \ldots
\]
where \{$T_{j}$\} is an increasing sequence of times that goes to infinity and
\[
d_{T_{j}}(u, v) = \sum_{k=1}^{\infty} \frac{1}{2^{k}} \frac{|\int_{0}^{T_{j}} (f_{k}^{j}(t, u(t)) - f_{k}^{j}(t, v(t))) \, dt|}{1 + |\int_{0}^{T_{j}} (f_{k}^{j}(t, u(t)) - f_{k}^{j}(t, v(t))) \, dt|},
\]
where, for each $j$, \{$f_{k}^{j}$\}$_{k=1}^{\infty}$ is a dense subset of $C([0, T_{j}] \times H)$.

Lemma 4.5. The space $X_{N}$ endowed with the metric $d$ is a compact complete metric space and the convergence induced by this topology coincides with that induced by convergence in measure (4.24).

Proof. It is immediate that $d$ is a metric on $X_{N}$ and that $u_{n} \to u$ if and only if $\lim_{n \to \infty} d(u_{n}, u) = 0$. Moreover, by the infinite-dimensional Helly theorem the set $X_{N}$ is compact in topology $\to$ (or equivalently that induced by the distance $d$). This implies that the metric $d$ is complete and the space $X_{N}$ is compact and so also separable. \[\square\]

Now we shall define the space $\mathcal{X}^{\circ} \subset BV(0, \infty; H) \cap L_{\text{loc}}^{\infty}(0, \infty; D(A^{0}))$ by
\[
\mathcal{X}^{\circ} = \bigcup_{N=1}^{\infty} X_{N}. \quad (4.26)
\]
In other words, \( u \in \mathcal{X} \) if and only if \( u \in \mathcal{X}_N \) for some \( N \in \mathbb{N} \). (Recall that \( \eta = \frac{1-\alpha}{2} \).

We shall denote by \( \mathcal{X} \) the completion of \( \mathcal{X}_N \) with respect to the topology induced by \( d \). Clearly \( \mathcal{X} \) is a separable complete metric space.

For each \( u \in \mathcal{X} \) we can associate its pseudo-path which is a probability law \( \mu_u \) on \([0, \infty) \times H\). Then for each \( f \in C_b([0, \infty) \times H) \) we have

\[
\int f(t, u(t)) \, dt = \int f \, d\mu_u \quad \forall f \in C_b([0, \infty) \times H)
\]

and so the convergence \((4.24)\) (respectively the topology induced by it) reduces to the convergence in measure or to the so-called pseudo-path topology (see [13]). Since the space \( \mathbb{D} \) of cadlag \( H \)-valued functions is closed in this topology and

\[
\mathcal{X} \subset BV(0, \infty; H) \cap L^\infty_{\text{loc}}(0, \infty; D(A^\eta)) \subset \mathbb{D}
\]

we conclude that

**Lemma 4.6.** Any \( u \in \mathcal{X} \) is a cadlag \( H \)-valued function, i.e., \( u \) is right continuous with left limit.

**Remark 4.7.** Of course the previous analysis of cadlag function spaces refer to real valued functions but it extends mutatis mutandis to \( H \)-valued functions considering first weakly cadlag functions \( u : [0, \infty) \to H \), i.e., \( t \mapsto (u(t), x) \) is cadlag for each \( x \in H \) and after to strong cadlag functions via compactness \( D(A^\eta) \subset H \).

Now we consider the family of probability measures \( \{\mathcal{P}_\varepsilon\} \subset \mathcal{P}(\mathcal{X}) \) defined by

\[
\mathcal{P}_\varepsilon(\Gamma) = \mathbb{P}(X_\varepsilon \in \Gamma), \quad \Gamma \subset \mathcal{X} \text{ Borelian}. \tag{4.27}
\]

**Lemma 4.8.** The family \( \{\mathcal{P}_\varepsilon\}_{\varepsilon > 0} \) is tight.

**Proof.** Taking into account that \( X_\varepsilon = y_\varepsilon + W_A \) it suffices to prove that the family \( \{\tilde{\mathcal{P}}_\varepsilon\} \), where \( \tilde{\mathcal{P}}_\varepsilon(\Gamma) = \mathbb{P}(y_\varepsilon \in \Gamma) \), is tight. By the Prohorov theorem it suffices to show that for each \( \xi > 0 \) there is a compact subset \( K_\xi \subset \mathcal{X} \) such that

\[
\mathbb{P}(y_\varepsilon \in K_\xi) \geq 1 - \xi. \tag{4.28}
\]

We take

\[
K_\xi = \{u \in BV(0, \infty; H) \cap L^\infty_{\text{loc}}(0, \infty; D(A^\eta)) : \\
|u|^2 + |u|^2_{L^\infty(0, T; D(A^\eta))} \leq 2C_\varepsilon^2 (|x|^2 + (C_\varepsilon^1 \xi^{-1})^{1/2} T^6) \quad \forall T > 0\}.
\]

By Lemma 3.4 we see that \(4.28\) holds. On the other hand, since \( K_\eta \subset \mathcal{X}_N \) for \( N = C_\varepsilon^1 \xi^{-1} \) it follows that \( K_\eta \) is compact in \( \mathcal{X}_N \) and therefore in \( \mathcal{X} \) as well. This completes the proof of Lemma 4.8. \(\Box\)

Then there is \( \mathcal{P} \in \mathcal{P}(\mathcal{X}) \) such that on a subsequence \( \varepsilon \to 0 \)

\[
\mathcal{P}_\varepsilon \to \mathcal{P} \quad \text{weakly in } \mathcal{P}(\mathcal{X}).
\]

Moreover, by the Skorohod theorem (see, e.g., [15]), we have

**Proposition 4.9.** There is a probability space \((\tilde{\Omega}, \tilde{\mathcal{X}}, \tilde{\mathcal{P}})\) and a sequence \(\{\tilde{X}_\varepsilon\}\) of \(\mathcal{X}\)-valued processes on \((\tilde{\Omega}, \tilde{\mathcal{X}}, \tilde{\mathcal{P}})\) and \(\mathcal{X}\)-valued stochastic process \(X\) such that

\[
\mathcal{P}_\varepsilon(\Gamma) = \mathbb{P}(\tilde{X}_\varepsilon \in \Gamma), \tag{4.29}
\]

\[
\tilde{X}_\varepsilon \to X \quad \text{\tilde{P}-a.s. in } \mathcal{X}, \tag{4.30}
\]

\[
\mathcal{P}(\Gamma) = \mathbb{P}(X \in \Gamma) \tag{4.31}
\]
for all Borelian set \( \Gamma \subset \mathcal{X} \).

By Lemma 4.6, \( X = X(t, x) \) is a cadlag \( H \)-valued process.

Let \( N \) be the Kolmogorov operator associated with the Neumann problem and let \( P_t \) the semigroup generated by \( N \). We have

**Theorem 4.10.** Let Hypothesis 4.1 holds. Let \( X = X(t) : [0, \infty) \to H \) be the process defined by Proposition 4.9. Then

\[
(P_t \varphi)(x) = \int_{\tilde{\Omega}} \varphi(X(t, x)) \, d\tilde{\mathbb{P}}(\omega) \quad \forall t \geq 0, x \in D(A^\delta), \varphi \in C_b(H). \tag{4.32}
\]

**Proof.** We have by Proposition 4.9

\[
\left( P_\varepsilon(t) \varphi \right)(x) = \int_{\tilde{\Omega}} \varphi(\tilde{X}_\varepsilon(t, x)) \, d\tilde{\mathbb{P}}(\omega) \quad \forall t \geq 0, x \in D(A^\delta), \varphi \in C_b(H)
\]

\[
\lim_{\varepsilon \to 0} \left( P_\varepsilon(t) \varphi \right)(x) = \int_{\tilde{\Omega}} \varphi(X(t, x)) \, d\tilde{\mathbb{P}}(\omega).
\tag{4.33}
\]

On the other hand, we know by Theorem 3.5 that

\[
(\lambda I - N)^{-1} \varphi = \lim_{\varepsilon \to 0} (\lambda I - N_\varepsilon)^{-1} \varphi = \lim_{\varepsilon \to 0} \int_0^\infty e^{-\lambda t} P_\varepsilon(t) \varphi \, dt \quad \forall \lambda > 0.
\tag{4.34}
\]

By (4.33), (4.34) we see that

\[
\int_0^\infty e^{-\lambda t} (P_t \varphi)(x) \, dt = \int_0^\infty e^{-\lambda t} dt \int_{\tilde{\Omega}} \varphi(X(t, x)) \, d\tilde{\mathbb{P}}(\omega) \quad \forall \lambda > 0
\]

which clearly implies (4.32) as claimed.

**Proposition 4.11.** We have

\[
X(t, x) \in K \quad \tilde{\mathbb{P}}\text{-a.s.} \quad \forall t > 0.
\tag{4.35}
\]

**Proof.** By Lemma 4.4 we have that for each \( N \),

\[
\int_0^T |\beta_\varepsilon (X_\varepsilon(t))| \, dt \leq C \left( 1 + N^{1/2} T^6 \right) \quad \forall \omega \in \Omega_{T,N},
\]

where \( \mathbb{P}(\Omega_{T,N}) \geq 1 - \frac{C_1}{N} \).

This yields

\[
\int_0^T |X_\varepsilon(t) - \Pi_K X_\varepsilon(t)| \, dt \leq C \varepsilon \left( 1 + N^{1/2} T^6 \right) \quad \forall \varepsilon > 0, \omega \in \Omega_{T,N}
\]

and therefore

\[
\int_0^T |\tilde{X}_\varepsilon(t) - \Pi_K \tilde{X}_\varepsilon(t)| \, dt \leq C \varepsilon \left( 1 + N^{1/2} T^6 \right) \quad \forall \varepsilon > 0, \omega \in \tilde{\Omega}_{T,N},
\]

where \( \tilde{\Omega}_{T,N} \subset \tilde{\Omega} \), and \( \tilde{\mathbb{P}}(\tilde{\Omega}_{T,N}) \geq 1 - \frac{C_1}{N} \).

Letting \( \varepsilon \) tend to zero we obtain that \( |X(t) - \Pi_K X(t)| = 0, \forall t \geq 0, \tilde{\mathbb{P}}\text{-a.s.} \) as claimed.
Remark 4.12. We recall that \( X \) is a martingale solution to (1.10), if
\[
\hat{\mathbb{P}}(X(t) \in K, \forall t \geq 0) = 1, \quad \hat{\mathbb{P}}(X(0, x) = x) = 1
\]
and for any smooth function \( \varphi \) in a core \( D(N_0) \) of \( N \),
\[
\varphi(X(t)) - \int_0^t N(\varphi(X(s))) \, ds - \varphi(x) =: \tilde{M}(t)
\]
is a martingale with respect to natural filtration \( \hat{\mathcal{F}}_t = \sigma(X(s), s \leq t), t \geq 0 \).

It is easily seen by Theorem 4.10 and (3.5) that if \( N \) has a core \( D(N_0) \) then the process \( X \) constructed above is the unique martingale solution to (1.1). However the existence of a core for \( N \) is still open.

5. An example

Consider the stochastic variational inequality (see (1.10))
\[
dX(t) - \Delta X(t) \, dt - \Delta N_K(X(t)) \, dt \ni A_0^{-1} \, dW_t \quad \text{in} \ (0, \infty) \times \partial \mathcal{O},
\]
\[
X(t) = 0 \quad \text{on} \ (0, \infty) \times \partial \mathcal{O},
\]
\[
X(0) = x \quad \text{in} \ \mathcal{O},
\]
where \( \mathcal{O} \) is a bounded open subset of \( \mathbb{R}^d \) with smooth boundary \( \partial \mathcal{O} \) and
\[
K = \left\{ x \in L^2(\mathcal{O}) : \int_{\partial \mathcal{O}} j(x(\xi)) \, d\xi \leq 1 \right\},
\]
where \( j : \mathbb{R} \to \mathbb{R} \) is a \( C^\infty \)-convex function such that \( 0 < c \leq j''(r) \leq c_1, \forall r \in \mathbb{R}, j(0) = j'(0) = 0 \) and \( A_0 = -\Delta, D(A_0) = H^1(\mathcal{O}) \cap H^2(\mathcal{O}) \).

Formally, (5.1) reduces to the stochastic reflection problem
\[
dX(t) - \Delta X(t) \, dt = A_0^{-1} \, dW_t \quad \text{in} \ \left\{ x \in L^2(\mathcal{O}) : \int_{\partial \mathcal{O}} j(x(\xi)) \, d\xi < 1 \right\},
\]
\[
dX(t) - \Delta X(t) \, dt \in \left\{ \lambda \Delta j'(X(t)) \right\}_{\lambda > 0} \, dt + A_0^{-1} \, dW_t \quad \text{in} \ \left\{ x \in L^2(\mathcal{O}) : \int_{\partial \mathcal{O}} j(x(\xi)) \, d\xi = 1 \right\},
\]
\[
X(t) = 0 \quad \text{on} \ (0, \infty) \times \partial \mathcal{O},
\]
\[
X(0) = x \quad \text{in} \ \mathcal{O}.
\]
The results of Sections 1–3 and in particular, Theorem 3.5 apply with \( \alpha = \frac{1}{2}, H = L^2(\mathcal{O}), A = \Delta^2, D(A) = \{ u \in H^2(\mathcal{O}) \cap H^1_0(\mathcal{O}), \Delta u \in H^1_0(\mathcal{O}), \Delta^2 u \in L^2(\mathcal{O}) \} \) on \( K \) defined by (5.2). Then \( A^{1/2} = A_0 \) and \( \text{Tr} \, A^{-1/2} < \infty \) if \( 1 \leq d \leq 3 \) and \( \delta \) is small.

Then the corresponding Kolmogorov operator \( N \) defined by (3.32) satisfies the regularity properties in Theorem 3.5 and the Markov semigroup \( P_t \) generated by \( N \) is given by
\[
(P_t \varphi)(x) = \varphi(t, x) \quad \forall t \geq 0, x \in L^2(\mathcal{O}),
\]
where \( \varphi \) is the solution to infinite-dimensional parabolic problem
\[
\frac{d}{dt} \int_K \varphi(t, x) \psi(x) \, d\mu(x) - \frac{1}{2} \int_K \left( \int_{\partial \mathcal{O}} \Delta \varphi(t, X(\xi)) \psi(X(\xi)) \, d\xi \right) \psi(x) \, d\mu(x) \quad \forall t \geq 0, \forall \psi \in C^1(K),
\]
\[
\varphi(0, x) = \varphi_0(x).
\]
Moreover, if $d = 1$ and $j(r) = r^2$ then Hypothesis 4.1 holds and so by Theorem 4.10 there is a cadlag process $X(t):[0, \infty) \to L^2(\mathcal{G})$ in a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ such that

$$(P_t \phi)(x) = \int_{\tilde{\Omega}} \phi(X(t,x)) \, d\tilde{P}(\omega) \quad \forall x \in D(A^\delta)$$

for $\delta > 0$.

As mentioned earlier we may view $X$ as a martingale solution to problem (5.1).

**Remark 5.1.** This example illustrates the fact that considering the class of problems (1.7) with $\alpha \in [0, 1]$ one might study reflection problems of the form (5.1) which otherwise are untractable in more dimensions.

**Appendix**

We recall again the following well-known integration by parts formula for the measure $\mu$ (see, e.g., [10]). For any $\varphi, \psi \in W^{1,2}(H, \mu)$ and $z \in H$,

$$\int_H \langle D\varphi, Q^{1/2}z \rangle \psi \, d\mu = - \int_H \langle D\psi, Q^{1/2}z \rangle \varphi \, d\mu + \int_H W_z \varphi \psi \, d\mu,$$

where $W_z$ represents the *white noise* function,

$$W_z(x) = \sum_{k=1}^{\infty} \frac{1}{\sqrt{\lambda_k}} \langle x, e_k \rangle \langle z, e_k \rangle \quad \forall z \text{ and } \mu-\text{a.e. } x \in H.$$

We recall that $W_z$ is a Gaussian random variable in $L^2(H, \mu)$ with mean 0 and covariance $|z|^2$. We notice that, thanks to Hypothesis 1.1(ii) the surface measure $\mu_{\Sigma}$ is well defined (see [12]).

We want now to prove an integration by parts formula in a subdomain $K$ of $H$ which generalizes (A.1). $K$ is defined by a function $g$ as stated in the Introduction. It is convenient to introduce a sequence of suitable measures $\{\mu_\varepsilon\}_{\varepsilon > 0}$ defined by

$$\mu_\varepsilon(dx) = \rho_\varepsilon(x) \mu(dx), \quad x \in H,$$

where

$$\rho_\varepsilon(x) = e^{-(g(x) - 1)^2/\varepsilon^2} 1_{g(x) \geq 1}.$$

Notice that,

$$\lim_{\varepsilon \to 0} \rho_\varepsilon(x) = \begin{cases} 1 & \text{if } x \in K, \\ 0 & \text{if } x \notin K. \end{cases}$$

So, we have

$$\lim_{\varepsilon \to 0} \mu_\varepsilon = \mu(K) \nu \quad \text{weakly in } \mathcal{P}(H),$$

where $\nu$ is the measure introduced previously. Moreover,

$$D\rho_\varepsilon(x) = -\frac{2}{\varepsilon} \rho_\varepsilon(x) 1_{g(x) \geq 1} Dg(x) (g(x) - 1),$$

so that $\rho_\varepsilon \in W^{1,2}(H, \mu)$.
The integration by parts formula

Here we are going to derive from (A.1), an integration by parts formula for the measure \( \mu_\varepsilon \). Let \( \varphi \in C^1_b(H) \), \( z \in H \), then, since \( \rho_\varepsilon \in W^{1,2}(H, \mu) \), we find from (A.1) that

\[
\int_H \langle D\varphi, Q^{1/2}z \rangle d\mu_\varepsilon = \int_H \langle D\varphi, Q^{1/2}z \rangle \rho_\varepsilon d\mu
\]

\[
= -\int_H \varphi \langle D\log \rho_\varepsilon, Q^{1/2}z \rangle d\mu_\varepsilon + \int_H W_\varepsilon \varphi d\mu_\varepsilon.
\]

Since,

\[
D \log \rho_\varepsilon(x) = -\frac{2}{\varepsilon} 1_{g(x) \geq 1} Dg(x)(g(x) - 1),
\]

we find the formula,

\[
\int_H \langle D\varphi, Q^{1/2}z \rangle \mu_\varepsilon(dx) = \frac{1}{2} \int H \varphi(y) |n(y)| Q^{1/2}z \mu_\Sigma(dy),
\]

(A.2)

Lemma A.1. Let \( \varphi \in C^1_b(H) \), \( z \in H \). Then there exists the limit,

\[
\lim_{\varepsilon \to 0} J_\varepsilon^z(\varphi) := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_H \varphi(x) 1_{g(x) \geq 1}(g(x) - 1) [Dg(x), Q^{1/2}z] \mu_\varepsilon(dx)
\]

\[
= \frac{1}{2} \int \varphi(y) |n(y)| Q^{1/2}z \mu_\Sigma(dy),
\]

(A.3)

where \( n(y) = \frac{Dg(y)}{|Dg(y)|} \) is the exterior normal to \( \Sigma \) at \( y \) and \( \mu_\Sigma \) is the surface measure on \( \Sigma \) induced by \( \mu \) (see [12]).

Proof. First we notice that

\[
J_\varepsilon^z(\varphi) = \frac{1}{\varepsilon} \int_{g(x) > 1} \varphi(x)(g(x) - 1) [Dg(x), Q^{1/2}z] e^{-(g(x)-1)^2/\varepsilon} \mu(dx).
\]

By the co-area formula (see [12], p. 140)\(^1\) we have

\[
\int_H f \mu(dx) = \int_0^\infty \left[ \int_{g=r} f(y) \frac{1}{|Dg(y)|} \mu_\Sigma(dy) \right] dr.
\]

(A.4)

(By (1.4) we know that \( |Dg(x)| \geq \gamma |x| \) and so \( |Dg(x)|^{-1} \in L^p(H, \mu) \) for all \( p \geq 1 \).) Notice that the surface measure is defined for all \( r \geq 0 \) taking into account [12], Theorem 6.2, Chapter V, moreover, [12], Theorem 1.1, Corollary 6.3.2, Chapter V, give the continuity property in Theorem 6.3.1 of Chapter V of [12]. Setting in (A.4)

\[
f = \mathbb{1}_{g \geq 1} \varphi(x)(g(x) - 1) [Dg(x), Q^{1/2}z] e^{-(g(x)-1)^2/\varepsilon}
\]

we get

\[
\int_{g \geq 1} \varphi(x)(g(x) - 1) [Dg(x), Q^{1/2}z] e^{-(g(x)-1)^2/\varepsilon} \mu(dx)
\]

\[
= \int_1^\infty (r - 1) e^{-(r-1)^2/\varepsilon} \left[ \int_{g=r} \varphi(y) [Dg(y), Q^{1/2}z] \frac{1}{|Dg(y)|} \mu_\Sigma(dy) \right] dr.
\]

\(^1\)Here, we have extended the validity of (A.4) to functions \( f \), continuous and in \( L^p(H, \mu) \) for any \( p \geq 1 \), by a density argument.
Hence, setting \( r = 1 + \sqrt{\varepsilon s} \), yields
\[
J_z^\varepsilon (\varphi) = \int_0^\infty s e^{-s^2} ds \int_{\mathbb{R}^d} \varphi(y) \left[ \frac{Dg(y)}{|Dg(y)|} \right] Q^{1/2} \mu_{\mathbb{R}^d} (dy).
\]
So (A.3) follows.

We are now in position to prove the announced integration by parts formula.

**Theorem A.2.** Let \( \varphi \in C^1_b(H) \), \( z \in H \). Then for any \( z \in H \) we have
\[
\int_K \langle D\varphi(x), Q^{1/2}z \rangle \mu(dx) = \int_\Sigma \varphi(y) \langle n(y), Q^{1/2}z \rangle \mu_\Sigma (dy)
\tag{A.5}
\]
\[
+ \int_K W_z(x) \varphi(x) \mu(dx).
\tag{A.6}
\]

**Proof.** The conclusion of the theorem follows letting \( \varepsilon \to 0 \) in (A.2) and taking into account Lemma A.1.

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**References**