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A stochastic min-driven coalescence process and its hydrodynamical limit

Anne-Laure Basdevant[a], Philippe Laurençot[b], James R. Norris[c] and Clément Rau[b]

[a]*Laboratoire MODAL'X, Université Paris 10, F–92000 Nanterre, France.*
E-mail: anne-laure.basdevant@u-paris10.fr
[b]*Institut de Mathématiques de Toulouse, CNRS UMR 5219, Université de Toulouse, F–31062 Toulouse Cedex 9, France.*
E-mails: philippe.laurencot@math.univ-toulouse.fr; clement.rau@math.univ-toulouse.fr
[c]*Statistical Laboratory, Centre for Mathematical Sciences, Wilberforce Road, Cambridge CB3 0WB, UK.*
E-mail: j.r.norris@statslab.cam.ac.uk

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Abstract. A stochastic system of particles is considered in which the sizes of the particles increase by successive binary mergers with the constraint that each coagulation event involves a particle with minimal size. Convergence of a suitably renormalized version of this process to a deterministic hydrodynamical limit is shown and the time evolution of the minimal size is studied for both deterministic and stochastic models.

Résumé. L'évolution d'un système aléatoire de particules est étudiée lorsque la taille des particules croît par coagulation binaire, chaque réaction de coagulation impliquant nécessairement une particule de taille minimale. Nous montrons qu'une version renormalisée du processus stochastique associé converge vers une limite déterministe et étudions l'évolution temporelle de la taille minimale pour les modèles stochastique et déterministe.

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1. Introduction

Coagulation models describe the evolution of a population of particles increasing their sizes by successive binary mergers, the state of each particle being fully determined by its size. Well-known examples of such models are the Smoluchowski coagulation equation [21,22] and its stochastic counterpart, the Marcus–Lushnikov process [16,17], and both have been extensively studied in recent years (see [1,3,13,15,20,23] and the references therein). Another class of coagulation models has also received some interest, the main feature of these models being that the particles with the smallest size play a more important role than the others. A first example are the Becker–Döring equations: in that case, the (normalized) sizes of the particles range in the set of positive integers and a particle can only modify its size by gaining or shedding a particle with unit size [2]. Another example are min-driven coagulation equations: given a positive integer k, at each step of the process, a particle with the smallest size ℓ is chosen and broken into k daughter particles with size ℓ/k, which are then pasted to other particles chosen at random in the population with equal probability [7]. Another model may be described as follows: at each step, an integer $k \geq 1$ is chosen randomly according to some probability p_k and one particle with the smallest size ℓ and k other particles are chosen at random and merged into a single particle [4,9,18].

In this paper, we focus on the min-driven coagulation equation introduced in [7] with $k = 1$ (that is, there is no break-up of the particle of minimal size) but relax the assumption of deposition with equal probability. More specifically, the coalescence mechanism we are interested in is the following: consider an initial configuration $X=(X_i)_{i\ge 1}$ of particles, X_i denoting the number of particles of size $i\ge 1$, and define the minimal size ℓ_X of X as the smallest integer $i\ge 1$ for which $X_i>0$ (that is, $X_{\ell_X}>0$ and $X_i=0$ for $i\in\{1,\dots,\ell_X-1\}$ if $\ell_X>1$). We pick a particle of size ℓ_X, choose at random another particle of size $j\ge\ell_X$ according to a certain law, and merge the two particles to form a particle of size ℓ_X+j. The system of particles thus jumps from the state X to the state $Y=(Y_i)_{i\ge 1}$ given by $Y_k=X_k$ if $k\notin\{\ell_X,j,\ell_X+j\}$ and

$$\begin{aligned} &Y_{\ell_X}=X_{\ell_X}-1, \qquad Y_j=X_j-1, \qquad && Y_{\ell_X+j}=X_{\ell_X+j}+1 \quad && \text{if } j>\ell_X,\\ &Y_{\ell_X}=X_{\ell_X}-2, && Y_{2\ell_X}=X_{2\ell_X}+1 && \text{if } j=\ell_X. \end{aligned}$$

Observe that no matter is lost during this event. It remains to specify the probability for this jump to take place: instead of assuming it to be uniform and independent of the sizes of the particles involved in the coalescence event as in [7], we consider the more general case where the jump from the state X to the state Y occurs at a rate $K(\ell_X,j)$, the coagulation kernel $K(i,j)$ being a positive function defined for $1\le i\le j$.

A more precise description of the stochastic process is to be found in the next section, where a renormalized version of this process is also introduced. We will show that, as the number of initial particles becomes large, the renormalized process converges to a deterministic limit which solves a countably infinite system of ordinary differential equations (Theorem 1.3). The convergence holds true provided the coagulation kernel $K(i,j)$ does not increase too fast as $i,j\to\infty$, a typical example being

$$K(i,j)=\phi(i), \quad 1\le i\le j, \text{ for some positive and non-decreasing function } \phi. \tag{1.1}$$

Well-posedness of the system solved by the deterministic limit is also investigated (Theorem 1.1) and reveals an interesting phenomenon, namely the possibility that the minimal size becomes infinite in finite time according to the growth of K (Theorem 1.4). Such a property also shows up for the stochastic min-driven coagulation process in a suitable sense (Theorem 1.5). It is worth pointing out that coagulation kernels K of the form (1.1) play a special role here.

1.1. *The stochastic min-driven coagulation process*

We now describe more precisely the stochastic min-driven coagulation process to be studied in this paper. It is somehow reminiscent of the Marcus–Lushnikov process [16,17] (which is related to the Smoluchowski coagulation equation). As in this process, two particles are chosen at random according to a certain law and merged but there is here an additional constraint; namely, one of the particles involved in the coalescence event has to be of minimal size among all particles in the system. To be more precise, we fix some positive integer N and an initial condition $X_0^N=(X_{i,0}^N)_{i\ge 1}\in\ell^1_{\mathbb{N}}$ such that

$$\sum_{i=1}^{\infty} iX_{i,0}^N=N, \tag{1.2}$$

where $X_{i,0}^N$ is the number of particles of size $i\ge 1$ and $\ell^1_{\mathbb{N}}$ denotes the space of summable non-negative and integer-valued sequences

$$\ell^1_{\mathbb{N}}:=\bigl\{X_0=(X_{i,0})_{i\ge 1}\in\ell^1\bigl(\mathbb{N}\setminus\{0\}\bigr):\ X_{i,0}\in\mathbb{N} \text{ for all } i\ge 1\bigr\}. \tag{1.3}$$

We next consider a time-dependent random variable $X^N(t)=(X_i^N(t))_{i\ge 1}$ which encodes the state of the process at time t starting from the configuration X_0^N, its ith-component $X_i^N(t)$ standing for the number of particles of size $i\ge 1$ at time $t\ge 0$. We assume that $X^N(0)=X_0^N$, so that N is equal to the total mass initially present in the system. The process $(X^N(t))_{t\ge 0}$ evolves then as a Markov process with the following transition rules: if, at a time t, the process is in the state $X^N(t)=X=(X_i)_{i\ge 1}$ with minimal size $\ell_X\ge 1$ (that is, $X_{\ell_X}>0$ and $X_i=0$ for $1\le i\le\ell_X-1$ if $\ell_X>1$), then each particle present, of size $j\ge\ell_X$ say, coagulates at rate $K(\ell_X,j)$ with another particle chosen (randomly) from the set of particles of minimal size ℓ_X. Mathematically, this means that the process jumps from the state $X^N(t) = X$ to a state of the form

$$Y = (0, \dots, 0, X_{\ell_X} - 1, X_{\ell_X+1}, \dots, X_j - 1, \dots, X_{\ell_X+j} + 1, \dots) \quad \text{with rate } K(\ell_X, j) X_j$$

for some $j > \ell_X$ or to the state

$$Z = (0, \dots, 0, X_{\ell_X} - 2, X_{\ell_X+1}, \dots, X_{2\ell_X} + 1, \dots) \quad \text{with rate } K(\ell_X, \ell_X)(X_{\ell_X} - 1).$$

Equivalently, this means that the process waits an exponential time of parameter

$$\lambda_X := \left(\sum_{j=\ell_X}^{\infty} K(\ell_X, j) X_j \right) - K(\ell_X, \ell_X)$$

and then jumps to the state Y with probability $K(\ell_X, j)X_j/\lambda_X$ for $j > \ell_X$ and to the state Z with probability $K(\ell_X, \ell_X)(X_{\ell_X} - 1)/\lambda_X$. Observe that, as X_{ℓ_X} could be equal to 1 or 2, there might be no particle of size ℓ_X after this jump and the minimal size thus increases. In addition, we obviously have

$$\sum_{i=1}^{\infty} i Y_i = \sum_{i=1}^{\infty} i Z_i = \sum_{i=1}^{\infty} i X_i,$$

so that the total mass contained in the system of particles does not change during the jumps. Consequently,

$$\sum_{i=1}^{\infty} i X_i^N(t) = \sum_{i=1}^{\infty} i X_{i,0}^N = N \quad \text{for all } t \geq 0. \tag{1.4}$$

As already mentioned, one aim of this paper is to prove that, under some assumptions on the coagulation kernel K and the initial data $(X_0^N)_{N\geq 1}$, a suitably renormalized version of the stochastic process converges to a deterministic limit as N tends to infinity. More precisely, we introduce $\tilde{X}^N := X^N/N$ and, for further use, list some properties of this process. By the above construction, the generator $\mathcal{L}^N = (\mathcal{L}_k^N)_{k\geq 1}$ of this renormalized process has the form

$$\begin{aligned}
\big(\mathcal{L}_k^N f\big)(\xi) = N \Bigg(\sum_{j=\ell_\xi}^{\infty} K(\ell_\xi, j)\xi_j \bigg[f_k\bigg(\xi + \frac{\mathbf{e}_{\ell_\xi+j}}{N} - \frac{\mathbf{e}_{\ell_\xi}}{N} - \frac{\mathbf{e}_j}{N} \bigg) - f_k(\xi) \bigg] \Bigg) \\
- K(\ell_\xi, \ell_\xi) \bigg[f_k\bigg(\xi + \frac{\mathbf{e}_{2\ell_\xi}}{N} - 2\frac{\mathbf{e}_{\ell_\xi}}{N} \bigg) - f_k(\xi) \bigg],
\end{aligned} \tag{1.5}$$

where $f = (f_k)_{k\geq 1} : \ell^1(\mathbb{N} \setminus \{0\}) \to \ell^1(\mathbb{N} \setminus \{0\})$ and $(\mathbf{e}_i)_{i\geq 1}$ denotes the canonical basis of $\ell^1(\mathbb{N} \setminus \{0\})$. Moreover, the quadratic variation $\mathcal{Q}^N = (\mathcal{Q}_k^N)_{k\geq 1}$ of the martingale

$$f\big(\tilde{X}^N(t)\big) - \int_0^t \big(\mathcal{L}^N f\big)\big(\tilde{X}^N(s)\big)\,\mathrm{d}s$$

is

$$\begin{aligned}
\big(\mathcal{Q}_k^N f\big)(\xi) = N \Bigg(\sum_{j=\ell_\xi}^{\infty} K(\ell_\xi, j)\xi_j \bigg[f_k\bigg(\xi + \frac{\mathbf{e}_{\ell_\xi+j}}{N} - \frac{\mathbf{e}_{\ell_\xi}}{N} - \frac{\mathbf{e}_j}{N} \bigg) - f_k(\xi) \bigg]^2 \Bigg) \\
- K(\ell_\xi, \ell_\xi) \bigg[f_k\bigg(\xi + \frac{\mathbf{e}_{2\ell_\xi}}{N} - \frac{2\mathbf{e}_{\ell_\xi}}{N} \bigg) - f_k(\xi) \bigg]^2.
\end{aligned} \tag{1.6}$$

Let $\tilde{\beta}(\xi)$ be the drift of the process $\tilde{X}^N$ when it is in state ξ, so that

$$\tilde{\beta}(\xi) := \sum_{\xi' \neq \xi} q\big(\xi, \xi'\big)\big(\xi' - \xi\big),$$

where $q(\xi, \xi')$ is the jump rate from ξ to ξ'. Taking $f = id$ in (1.5) leads to the following formula for the drift

$$\begin{cases} \tilde{\beta}_j(\xi) := 0 & \text{if } 1 \leq j \leq \ell_\xi - 1, \\ \tilde{\beta}_{\ell_\xi}(\xi) := -\sum_{j=\ell_\xi+1}^{\infty} K(\ell_\xi, j)\xi_j - 2K(\ell_\xi, \ell_\xi)\xi_{\ell_\xi} + \frac{2}{N}K(\ell_\xi, \ell_\xi), & \\ \tilde{\beta}_j(\xi) := K(\ell_\xi, j - \ell_\xi)\xi_{j-\ell_\xi} - K(\ell_\xi, j)\xi_j & \text{if } j \geq \ell_\xi + 1,\ j \neq 2\ell_\xi, \\ \tilde{\beta}_{2\ell_\xi}(\xi) := K(\ell_\xi, \ell_\xi)\big(\xi_{\ell_\xi} - \frac{1}{N}\big) - K(\ell_\xi, 2\ell_\xi)\xi_{2\ell_\xi}. & \end{cases} \tag{1.7}$$

Here and below, we set $K(i, j) = 0$ for $i > j \geq 1$. We also define

$$\tilde{\alpha}(\xi) := \sum_{\xi' \neq \xi} q\big(\xi, \xi'\big)\big\|\xi' - \xi\big\|_2^2 = \sum_{j=1}^{\infty}\sum_{\xi' \neq \xi} q(\xi, \xi')\big|\xi'_j - \xi_j\big|^2. \tag{1.8}$$

Then

$$\tilde{\alpha}(\xi) = \sum_{j=1}^{\infty} \tilde{\alpha}_j(\xi),$$

where $\tilde{\alpha}_j$ is obtained by taking $f(\xi) = \xi_j \mathbf{e}_j$ in (1.6), so that

$$\begin{cases} \tilde{\alpha}_j(\xi) := 0 & \text{if } 1 \leq j \leq \ell_\xi - 1, \\ \tilde{\alpha}_{\ell_\xi}(\xi) := \frac{1}{N}\sum_{j=\ell_\xi+1}^{\infty} K(\ell_\xi, j)\xi_j + \frac{4}{N}K(\ell_\xi, \ell_\xi)\xi_{\ell_\xi} - \frac{4}{N^2}K(\ell_\xi, \ell_\xi), & \\ \tilde{\alpha}_j(\xi) := \frac{1}{N}K(\ell_\xi, j - \ell_\xi)\xi_{j-\ell_\xi} + \frac{1}{N}K(\ell_\xi, j)\xi_j & \text{if } j \geq \ell_\xi + 1,\ j \neq 2\ell_\xi, \\ \tilde{\alpha}_{2\ell_\xi}(\xi) := \frac{1}{N}K(\ell_\xi, \ell_\xi)\big(\xi_{\ell_\xi} - \frac{1}{N}\big) + \frac{1}{N}K(\ell_\xi, 2\ell_\xi)\xi_{2\ell_\xi}. & \end{cases} \tag{1.9}$$

1.2. *Main results*

For $p \in [1, \infty)$, let ℓ^p be the Banach space of p-summable real-valued sequences

$$\ell^p := \left\{ x = (x_i)_{i \geq 1} \colon \|x\|_p := \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p} < \infty \right\}.$$

We next define the space $\mathcal{X}_{1,1}$ of real-valued sequences with finite first moment by

$$\mathcal{X}_{1,1} := \left\{ x = (x_i)_{i \geq 1} \colon \|x\|_{1,1} := \sum_{i=1}^{\infty} i|x_i| < \infty \right\}, \tag{1.10}$$

which is a Banach space for the norm $\|\cdot\|_{1,1}$, and its positive cone

$$\mathcal{X}_{1,1}^+ := \big\{ x = (x_i)_{i \geq 1} \in \mathcal{X}_{1,1} \colon x_i \geq 0 \text{ for } i \geq 1 \big\}.$$

For $m \geq 2$, let $\mathcal{X}_{1,m}$ be the subspace of $\mathcal{X}_{1,1}$ of sequences having their $m - 1$ first components equal to zero, namely

$$\mathcal{X}_{1,m} := \big\{ x = (x_i)_{i \geq 1} \in \mathcal{X}_{1,1} \colon x_i = 0 \text{ for } i \in \{1, \dots, m - 1\} \big\}, \tag{1.11}$$

and $\mathcal{X}_{1,m}^+ := \mathcal{X}_{1,m} \cap \mathcal{X}_{1,1}^+$.

We assume that there is $\kappa > 0$ such that

$$0 \le K(i,j) \le \kappa ij, \quad 1 \le i \le j, \quad \text{and} \quad \delta_i := \inf_{j \ge i} \{K(i,j)\} > 0 \quad \text{for } i \ge 1. \tag{1.12}$$

Next, for $i \ge 1$, we define the function $b^{(i)} = (b_j^{(i)})_{j \ge 1}$ on $\mathcal{X}_{1,1}$ by

$$\begin{cases} b_j^{(i)}(x) := 0 & \text{if } 1 \le j \le i-1, \\ b_i^{(i)}(x) := -2K(i,i)x_i - \sum_{j=i+1}^{\infty} K(i,j)x_j, \\ b_j^{(i)}(x) := K(i,j-i)x_{j-i} - K(i,j)x_j & \text{if } j \ge i+1, \end{cases} \tag{1.13}$$

recalling that we have set $K(i,j) = 0$ for $i > j \ge 1$. Let us point out here that $b^{(i)}(x)$ is closely related to the drift $\tilde{\beta}(x)$ defined by (1.7) for $x \in \mathcal{X}_{1,i}$.

Consider an initial condition $x_0 = (x_{i,0})_{i \ge 1}$ such that

$$x_0 \in \mathcal{X}_{1,1}^+ \quad \text{with } x_{1,0} > 0 \text{ and } \|x_0\|_{1,1} = 1. \tag{1.14}$$

Theorem 1.1. *Assume that the coagulation kernel K and the initial condition x_0 satisfy* (1.12) *and* (1.14), *respectively. There is a unique pair of functions (ℓ, x) having the following properties*:

(i) *There is an increasing sequence of times $(t_i)_{i \ge 0}$ with $t_0 = 0$ such that*

$$\ell(t) := i \quad \text{for } t \in [t_{i-1}, t_i) \text{ and } i \ge 1.$$

We define

$$t_\infty := \sup_{i \ge 0} t_i = \lim_{i \to \infty} t_i \in (0, \infty]. \tag{1.15}$$

(ii) $x = (x_i)_{i \ge 1} \in \mathcal{C}([0, t_\infty); \mathcal{X}_{1,1})$ *satisfies* $x(0) = x_0$,

$$x(t) \in \mathcal{X}_{1,\ell(t)}^+ \setminus \mathcal{X}_{1,\ell(t)+1} \quad \text{for } t \in [0, t_\infty), \tag{1.16}$$

and solves

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) = b^{(\ell(t))}\big(x(t)\big) \quad \text{for } t \in [0, t_\infty) \setminus \{t_i \colon i \ge 0\}. \tag{1.17}$$

In addition,

$$x_j(t) > 0 \quad \text{for } t \in (t_{i-1}, t_i] \text{ and } j \ge i+1 \tag{1.18}$$

and

$$\|x(t)\|_{1,1} = \|x_0\|_{1,1} = 1 \quad \text{for } t \in [0, t_\infty). \tag{1.19}$$

In other words, for each $i \ge 1$, $x(t) \in \mathcal{X}_{1,i}^+$ and $x_i(t) > 0$ for $t \in [t_{i-1}, t_i)$ and $\mathrm{d}x(t)/\mathrm{d}t = b^{(i)}(x(t))$ for $t \in (t_{i-1}, t_i)$. Given $t \in [0, t_\infty)$, Theorem 1.1 asserts that $x(t) \in \mathcal{X}_{1,\ell(t)}^+$ with $x_{\ell(t)}(t) > 0$, so that $\ell(t)$ is the minimal size of the particles at time t.

Remark 1.2. *The assumption $\|x_0\|_{1,1} = 1$ is actually not restrictive*: *indeed, given $\bar{x}_0 \in \mathcal{X}_{1,1}^+$ such that $\bar{x}_{1,0} > 0$, the initial condition $x_0 = \bar{x}_0 / \|\bar{x}_0\|_{1,1}$ satisfies* (1.14). *If x denotes the corresponding solution to* (1.17) *with minimal size ℓ and $\bar{x} := \|\bar{x}_0\|_{1,1} x$, it is straightforward to check that the pair $(\ell, \bar{x})$ satisfies all the requirements of Theorem* 1.1 *except* (1.19) *which has to be replaced by $\|\bar{x}(t)\|_{1,1} = \|\bar{x}_0\|_{1,1}$ for $t \in [0, t_\infty)$.*

We now turn to the connection between the deterministic and stochastic models and establish the following convergence result.

Theorem 1.3. *Let K and x_0 be a coagulation kernel and a deterministic initial condition satisfying* (1.12) *and* (1.14), *respectively. Consider a sequence $(X_0^N)_{N\geq 1}$ of stochastic initial configurations in $\ell^1_{\mathbb{N}}$ satisfying* (1.2) *which are close to x_0 in the following sense*:

$$\mathbb{P}\left(\left\|\frac{1}{N}X_0^N - x_0\right\|_1 > \frac{1}{N^{1/4}}\right) \leq \frac{1}{N^{1/4}}. \tag{1.20}$$

Assume further that, for any $i \geq 0$, there is $\kappa_i > 0$ such that

$$K(i,j) \leq \kappa_i, \quad j \geq i, \quad \text{and} \quad \kappa_\infty := \sup\left\{\frac{\kappa_i}{i}\right\} < \infty. \tag{1.21}$$

Let x be the corresponding solution to (1.17) *with maximal existence time t_∞ defined by* (1.15) *and, for $N \geq 1$, let X^N be the Markov process starting from X_0^N defined in Section* 1.1. *Then for all $t \in (0, t_\infty)$ there exist constants $C(t), D(t) > 0$ such that for N large enough*:

$$\mathbb{P}\left(\sup_{0\leq s\leq t}\left\|\frac{1}{N}X^N(s) - x(s)\right\|_1 \geq \frac{D(t)}{N^{1/4}}\right) \leq \frac{C(t)}{N^{1/4}}.$$

We next turn to the lifespan of the deterministic and stochastic min-driven coagulation models and investigate the possible values of t_∞ as well as the behaviour of the time T^{X_0} after which the stochastic min-driven coagulation process X starting from $X_0 \in \ell^1_{\mathbb{N}}$ ($\ell^1_{\mathbb{N}}$ being defined in (1.3)) no longer evolves, that is,

$$T^{X_0} := \inf\{t \geq 0: \|X(t)\|_1 = 1\}. \tag{1.22}$$

We first establish conditions on the growth of the coagulation kernel K which ensure that t_∞ is finite or infinite. Note that, in the former case, this means that the minimal size ℓ blows up in finite time.

Theorem 1.4. *Consider an initial condition x_0 satisfying* (1.14) *and let x be the corresponding solution to the min-driven coagulation equations given in Theorem* 1.1 *defined on $[0, t_\infty)$, t_∞ being defined in* (1.15):

(i) *If $K(i,j) \leq \ln(i+1)/(4A_0)$ for $1 \leq i \leq j$ and some $A_0 > 0$ then $t_\infty = \infty$.*
(ii) *If $K(i,j) \geq a_0(\ln(i+1))^{1+\alpha}$ for $1 \leq i \leq j$ and some $a_0 > 0$ and $\alpha > 0$, then $t_\infty < \infty$.*

A more precise result is available for the stochastic min-driven coagulation process under a stronger structural assumption on the coagulation kernel.

Theorem 1.5. *Assume that the coagulation kernel K is of the form*

$$K(i,j) = \phi(i), \quad 1 \leq i \leq j, \text{ where } \phi \text{ is a positive increasing function.} \tag{1.23}$$

Then

$$\sup_{X_0\in\ell^1_{\mathbb{N}}} \mathbb{E}\left(T^{X_0}\right) < \infty \quad \text{if and only if} \quad \sum_{i=1}^{\infty}\frac{1}{i\phi(i)} < \infty,$$

the space $\ell^1_{\mathbb{N}}$ being defined in (1.3).

More precisely, when the series $\sum 1/(i\phi(i))$ diverges, we have

$$T^{n\mathbf{e}_1} \underset{n\to\infty}{\longrightarrow} \infty \quad \text{in probability,}$$

where $n\mathbf{e}_1$ denotes the initial configuration with n particles of size 1.

The above two results provide conditions on the coagulation kernel K which guarantee that, in a finite time, some mass escapes to infinity, or forms a giant particle, of the order of the system. This is the behaviour known as *gelation* for the Smoluchowski coagulation equation and the Marcus–Lushnikov process, and is known to occur when the coagulation kernel K satisfies $K(i,j) \geq c(ij)^{\lambda/2}$ for some $\lambda \in (1,2]$ [8,10]. We observe that the growth required on the coagulation kernel is much weaker for the min-driven coagulation models. In fact the behaviour we have shown is more extreme than gelation, in that all the mass goes to infinity or joins the giant particle. A similar phenomenon has been called *complete gelation* in the context of the Marcus–Lushnikov process, and is known to occur instantaneously, as $N \to \infty$, whenever $K(i,j) \geq ij(\log(i+1)\log(j+1))^\alpha$ and $\alpha > 1$ [11].

2. The deterministic min-driven coagulation equation

In this section, we investigate the well-posedness of the min-driven coagulation equation (1.17). It is clearly an infinite system of ordinary differential equations which is linear on the time intervals where the minimal size ℓ is constant. We will thus first study the well-posedness for this reduced system, assuming initially that the coefficients are bounded, in order to be able to apply the Cauchy–Lipschitz theorem and relaxing this assumption afterwards by a compactness method. We also pay attention to the vanishing time of the first component which was initially positive. The proof of Theorem 1.1 is then performed by an induction argument.

2.1. *An auxiliary infinite system of differential equations*

Consider $i \geq 1$ and a sequence $(a_j)_{j\geq 1}$ of real numbers satisfying

$$a_j = 0, \quad j < i, \quad \text{and} \quad 0 < a_j \leq Aj, \quad j \geq i, \tag{2.1}$$

for some $A > 0$. We define the function $F = (F_j)_{j\geq 1}$ on $\mathcal{X}_{1,1}$ by

$$\begin{cases} F_j(y) := 0 & \text{if } 1 \leq j \leq i-1, \\ F_i(y) := -a_i y_i - \sum_{j=i}^{\infty} a_j y_j, & \\ F_j(y) := a_{j-i} y_{j-i} - a_j y_j & \text{if } j \geq i+1 \end{cases} \tag{2.2}$$

for $y \in \mathcal{X}_{1,1}$. Note that (2.1) ensures that $F(y) \in \ell^1$ for $y \in \mathcal{X}_{1,1}$ and that $F(y) \in \mathcal{X}_{1,i}$.

Proposition 2.1. *Consider a sequence $(a_j)_{j\geq 1}$ satisfying (2.1) and an initial condition $y_0 = (y_{j,0})_{j\geq 1} \in \mathcal{X}_{1,i}$. There is a unique solution $y \in \mathcal{C}([0,\infty); \mathcal{X}_{1,i})$ to the Cauchy problem*

$$\frac{\mathrm{d}y}{\mathrm{d}t} = F(y), \qquad y(0) = y_0. \tag{2.3}$$

Moreover, for each $t > 0$, y and $\mathrm{d}y/\mathrm{d}t$ belong to $L^\infty(0,t;\mathcal{X}_{1,i})$ and $L^\infty(0,t;\ell^1)$, respectively, and

$$\sum_{j=i}^{\infty} j y_j(t) = \sum_{j=i}^{\infty} j y_{j,0}. \tag{2.4}$$

We first consider the case of a bounded sequence $(a_j)_{j\geq 1}$.

Lemma 2.2. *Consider a sequence $(a_j)_{j\geq 1}$ satisfying*

$$a_j = 0, \quad j < i, \quad \textit{and} \quad 0 < a_j \leq A_0, \quad j \geq i, \tag{2.5}$$

for some $A_0 > 0$ and an initial condition $y_0 = (y_{j,0})_{j\geq 1} \in \mathcal{X}_{1,i}$. Then there is a unique solution $y \in \mathcal{C}([0,\infty); \mathcal{X}_{1,i})$ to the Cauchy problem (2.3) and

$$\sum_{j=i}^{\infty} j y_j(t) = \sum_{j=i}^{\infty} j y_{j,0}, \quad t \geq 0. \tag{2.6}$$

Proof. It readily follows from (2.2) and (2.5) that, given $y \in \mathcal{X}_{1,i}$ and $\hat{y} \in \mathcal{X}_{1,i}$, we have

$$\left\| F(y) - F(\hat{y}) \right\|_{1,1} \le 4A_0 \|y - \hat{y}\|_{1,1}, \tag{2.7}$$

while the first $i-1$ components of $F(y)$ vanish. Therefore, F is a Lipschitz continuous map from $\mathcal{X}_{1,i}$ to $\mathcal{X}_{1,i}$ and the Cauchy–Lipschitz theorem guarantees the existence and uniqueness of a solution $y \in \mathcal{C}([0,\infty); \mathcal{X}_{1,i})$ to (2.3).

Next, let $(g_j)_{j\ge1}$ be a sequence of real numbers satisfying $0 \le g_j \le Gj$ for $j \ge 1$ and some $G > 0$. We deduce from (2.3), (2.5) and the summability properties of y that

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_{j=i}^{\infty} g_j y_j(t) = \sum_{j=i}^{\infty} (g_{i+j} - g_i - g_j) a_j y_j(t), \quad t \ge 0. \tag{2.8}$$

In particular, the choice $g_j = j$, $j \ge 1$, gives (2.6). □

Proof of Proposition 2.1. For $m \ge 1$ and $j \ge 1$, we put $a_j^m := a_j \wedge m$. Since the sequence $(a_j^m)_{j\ge1}$ is bounded, it follows from Lemma 2.2 that there is a unique solution $y^m = (y_j^m)_{j\ge1} \in \mathcal{C}([0,\infty); \mathcal{X}_{1,i})$ to the Cauchy problem

$$\frac{\mathrm{d}y_i^m}{\mathrm{d}t} = -a_i^m y_i^m - \sum_{j=i}^{\infty} a_j^m y_j^m, \tag{2.9}$$

$$\frac{\mathrm{d}y_j^m}{\mathrm{d}t} = a_{j-i}^m y_{j-i}^m - a_j^m y_j^m, \quad j \ge i+1, \tag{2.10}$$

with initial condition $y^m(0) = y_0$. Introducing $\sigma_j^m := \mathrm{sign}(y_j^m)$, we infer from (2.1), (2.9) and (2.10) that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \|y^m\|_{1,1} &= \sum_{j=i}^{\infty} j \sigma_j^m \frac{\mathrm{d}y_j^m}{\mathrm{d}t} \\
&= -i a_i^m |y_i^m| - \sum_{j=i}^{\infty} i a_j^m \sigma_i^m y_j^m + \sum_{j=2i}^{\infty} j a_{j-i}^m \sigma_j^m y_{j-i}^m - \sum_{j=i+1}^{\infty} j a_j^m |y_j^m| \\
&= \sum_{j=i}^{\infty} \big((i+j)\sigma_{i+j}^m \sigma_j^m - i \sigma_i^m \sigma_j^m - j \big) a_j^m |y_j^m| \\
&\le 2i \sum_{j=i}^{\infty} a_j^m |y_j^m| \le 2Ai \|y^m\|_{1,1},
\end{aligned}$$

hence

$$\left\| y^m(t) \right\|_{1,1} \le \|y_0\|_{1,1} e^{2Ait}, \quad t \ge 0. \tag{2.11}$$

It next readily follows from (2.1), (2.9) and (2.10) that

$$\left| \frac{\mathrm{d}y_i^m}{\mathrm{d}t} \right| \le Ai |y_i^m| + A \|y^m\|_{1,1},$$

$$\left| \frac{\mathrm{d}y_j^m}{\mathrm{d}t} \right| \le A(j-i) |y_{j-i}^m| + Aj |y_j^m|, \quad j \ge i+1,$$

and thus

$$\sum_{j=i}^{\infty} \left| \frac{\mathrm{d}y_j^m}{\mathrm{d}t}(t) \right| \le 3A \left\| y^m(t) \right\|_{1,1} \le 3A \|y_0\|_{1,1} e^{2Ait}, \quad t \ge 0, \tag{2.12}$$

by (2.11).

Now, for all $j \geq 1$ and $T > 0$, the sequence of functions $(y_j^m)_{N\geq 1}$ is bounded in $W^{1,\infty}(0,T)$ by (2.11) and (2.12) and thus relatively compact in $\mathcal{C}([0,T])$ by the Arzelà–Ascoli theorem. Consequently, there is a subsequence $(m_k)_{k\geq 1}$, $m_k \to \infty$, and there is a sequence of functions $y = (y_j)_{j\geq 1}$ such that

$$\lim_{k\to\infty} \sup_{t\in[0,T]} \left| y_j^{m_k}(t) - y_j(t) \right| = 0 \quad \text{for } j \geq 1 \text{ and } T > 0. \tag{2.13}$$

If $j \geq i+1$, it is straightforward to deduce from (2.10) and (2.13) that y_j actually belongs to $\mathcal{C}^1([0,\infty))$ and solves

$$\frac{dy_j}{dt} = a_{j-i} y_{j-i} - a_j y_j, \quad y_j(0) = y_{j,0}. \tag{2.14}$$

In addition, (2.11) and (2.13) imply that $y(t) \in \mathcal{X}_{1,i}$ for all $t \geq 0$ and satisfies

$$\left\| y(t) \right\|_{1,1} \leq \|y_0\|_{1,1} e^{2Ait}, \quad t \geq 0. \tag{2.15}$$

Passing to the limit in (2.9) is more difficult because of the infinite series on the right. For that purpose, we need an additional estimate to control the tail of the series which we derive now: we first recall that, since $y_0 \in \mathcal{X}_{1,1}$, a refined version of de la Vallée–Poussin's theorem ensures that there is a non-negative and non-decreasing convex function $\zeta \in \mathcal{C}^\infty([0,\infty))$ such that $\zeta(0) = 0$, ζ' is a concave function,

$$\lim_{r\to\infty} \frac{\zeta(r)}{r} = \infty \quad \text{and} \quad \sum_{j=i}^{\infty} \zeta(j) |y_{j,0}| < \infty, \tag{2.16}$$

see [6,14]. We infer from (2.1), (2.9), (2.10) and the properties of ζ that

$$\begin{aligned}
\frac{d}{dt} \sum_{j=i}^{\infty} \zeta(j) \left| y_j^m \right| &= \sum_{j=i}^{\infty} \left(\zeta(i+j) \operatorname{sign}\left(y_{i+j}^m\right) \operatorname{sign}\left(y_j^m\right) - \zeta(i) \operatorname{sign}\left(y_i^m\right) \operatorname{sign}\left(y_j^m\right) - \zeta(j) \right) a_j^m \left| y_j^m \right| \\
&\leq \sum_{j=i}^{\infty} \left(\zeta(i+j) + \zeta(i) - \zeta(j) \right) a_j^m \left| y_j^m \right| \\
&\leq \sum_{j=i}^{\infty} \left(\int_0^j \int_0^i \zeta''(r+s)\, ds\, dr + 2\zeta(i) \right) a_j^m \left| y_j^m \right| \\
&\leq \sum_{j=i}^{\infty} \left(\int_0^j i\zeta''(r)\, dr + 2\zeta(i) \right) a_j^m \left| y_j^m \right| \\
&\leq \sum_{j=i}^{\infty} \left(i\zeta'(j) + 2\zeta(i) \right) a_j^m \left| y_j^m \right| \\
&\leq 2A\zeta(i) \left\| y^m \right\|_{1,1} + Ai \sum_{j=i}^{\infty} j\zeta'(j) \left| y_j^m \right|.
\end{aligned}$$

By the concavity of ζ', we have $j\zeta'(j) \leq 2\zeta(j)$ for $j \geq 1$, see Lemma A.1 in [12]. Inserting this estimate in the previous inequality and using (2.11), we end up with

$$\frac{d}{dt} \sum_{j=i}^{\infty} \zeta(j) \left| y_j^m(t) \right| \leq 2Ai \sum_{j=i}^{\infty} \zeta(j) \left| y_j^m(t) \right| + 2A\zeta(i) \|y_0\|_{1,1} e^{2Ait}, \quad t \geq 0,$$

and thus

$$\sum_{j=i}^{\infty} \zeta(j)\left|y_j^m(t)\right| \le \left(\sum_{j=i}^{\infty} \zeta(j)|y_{j,0}| + 2A\zeta(i)\|y_0\|_{1,1} t\right) \mathrm{e}^{2Ait}, \quad t \ge 0, \tag{2.17}$$

the right-hand side of (2.17) being finite by (2.16). It first follows from (2.13) and (2.17) by Fatou's lemma that

$$\sum_{j=i}^{\infty} \zeta(j)\left|y_j(t)\right| \le \left(\sum_{j=i}^{\infty} \zeta(j)|y_{j,0}| + 2A\zeta(i)\|y_0\|_{1,1} t\right) \mathrm{e}^{2Ait}, \quad t \ge 0. \tag{2.18}$$

Notice next that, thanks to the superlinearity (2.16) of ζ, the estimates (2.17) and (2.18) provide us with a control of the tail of the series $\sum j y_j^m$ and $\sum j y_j$ which does not depend on m. More precisely, we infer from (2.17), (2.18) and the convexity of ζ that, for $T > 0$, $t \in [0, T]$, and $J \ge 2i$,

$$\begin{aligned}
\left\|\left(y^{m_k} - y\right)(t)\right\|_{1,1} &\le \sum_{j=i}^{J-1} j\left|\left(y_j^{m_k} - y_j\right)(t)\right| + \sum_{j=J}^{\infty} j\left(\left|y_j^{m_k}(t)\right| + \left|y_j(t)\right|\right) \\
&\le \sum_{j=i}^{J-1} j\left|\left(y_j^{m_k} - y_j\right)(t)\right| + \frac{J}{\zeta(J)} \sum_{j=J}^{\infty} \zeta(j)\left(\left|y_j^{m_k}(t)\right| + \left|y_j(t)\right|\right) \\
&\le \sum_{j=i}^{J-1} j\left|\left(y_j^{m_k} - y_j\right)(t)\right| + \frac{2J}{\zeta(J)} \left(\sum_{j=i}^{\infty} \zeta(j)|y_{j,0}| + 2A\zeta(i)\|y_0\|_{1,1} T\right) \mathrm{e}^{2AiT}.
\end{aligned}$$

By (2.13), we may pass to the limit as $k \to \infty$ in the preceding inequality to deduce that

$$\limsup_{k\to\infty} \sup_{t\in[0,T]} \left\|\left(y^{m_k} - y\right)(t)\right\|_{1,1} \le \frac{2J}{\zeta(J)} \left(\sum_{j=i}^{\infty} \zeta(j)|y_{j,0}| + 2A\zeta(i)\|y_0\|_{1,1} T\right) \mathrm{e}^{2AiT}.$$

We next use (2.16) to let $J \to \infty$ and conclude that

$$\lim_{k\to\infty} \sup_{t\in[0,T]} \left\|\left(y^{m_k} - y\right)(t)\right\|_{1,1} = 0. \tag{2.19}$$

Recalling (2.1), it is straightforward to deduce from (2.19) that

$$\lim_{k\to\infty} \sup_{t\in[0,T]} \left|\sum_{j=i}^{\infty} a_j^{m_k} y_j^{m_k}(t) - \sum_{j=i}^{\infty} a_j y_j(t)\right| = 0$$

for all $T > 0$, from which we conclude that y_i belongs to $\mathcal{C}^1([0, \infty))$ and solves

$$\frac{\mathrm{d}y_i}{\mathrm{d}t} = -a_i y_i - \sum_{j=i}^{\infty} a_j y_j, \qquad y_i(0) = y_{i,0}. \tag{2.20}$$

Another consequence of (2.19) is that $y \in \mathcal{C}([0, \infty); \mathcal{X}_{1,i})$ and is thus locally bounded in $\mathcal{X}_{1,1}$. This property in turn provides the boundedness of $\mathrm{d}y/\mathrm{d}t$ in ℓ^1, the proof being similar to that of (2.12). We finally use once more (2.19) to deduce from (2.6) (satisfied by y^{m_k} thanks to Lemma 2.2) that (2.4) holds true. We have thus established the existence part of Proposition 2.1.

As for uniqueness, if y and $\hat{y}$ are two solutions to the Cauchy problem (2.3), a computation similar to that leading to (2.11) gives $\|y(t) - \hat{y}(t)\|_{1,1} \le \|y(0) - \hat{y}(0)\|_{1,1} \mathrm{e}^{2Ait} = 0$ for $t \ge 0$. Consequently, $y = \hat{y}$ and the uniqueness assertion of Proposition 2.1 is proved. $\square$

Remark 2.3. *In fact, the derivation of* (2.17) *is formal as the series* $\sum \zeta(j) y_j^m$ *is not known to converge a priori* (*recall that* $\zeta(j)$ *is superlinear by* (2.16)). *It can be justified rigorously by using classical truncation arguments. More specifically, for* $R \geq 1$, *define* $\zeta_R(r) = \zeta(r)$ *for* $r \in [0, R]$ *and* $\zeta_R(r) = \zeta(R) + \zeta'(R)(r - R)$ *for* $r \geq R$. *Then* ζ_R *enjoys the same properties as* ζ *and the sequence* $(\zeta_R(j))_{j\geq 1}$ *grows linearly with respect to* j. *We can then use* (2.8) *to perform a similar computation to the one above leading to* (2.17) *and obtain a bound on* $\sum \zeta_R(j) y_j^m$ *which depends neither on* R *nor on* m. *The desired result then follows by letting* $R \to \infty$ *with the help of Fatou's lemma.*

We now turn to specific properties of solutions to (2.3) when $y_0 \in \mathcal{X}_{1,i}^+$.

Proposition 2.4. *Consider a sequence* $(a_j)_{j\geq 1}$ *satisfying* (2.1), *an initial condition* $y_0 = (y_{j,0})_{j\geq 1} \in \mathcal{X}_{1,i}$ *such that*

$$y_0 \in \mathcal{X}_{1,i}^+ \quad \text{and} \quad y_{i,0} > 0, \tag{2.21}$$

and let y *be the corresponding solution to the Cauchy problem* (2.3). *There exist* $t_* \in (0, \infty]$ *and* $t_{*,1} \in [t_*, \infty]$ *such that*

$$y_i(t) > 0 \quad \text{for } t \in [0, t_*) \text{ and } y_i(t_*) = 0, \tag{2.22}$$

$$y_{ki}(t) > 0 \quad \text{for } t \in (0, t_*) \text{ and } k \geq 2, \tag{2.23}$$

$$y_j(t) \geq 0 \quad \text{for } t \in [0, t_*) \text{ and } j \geq i+1, \tag{2.24}$$

$$y_j(t) > 0 \quad \text{for } t \in [0, t_*) \text{ if } j \geq i+1 \text{ and } y_{j,0} > 0, \tag{2.25}$$

$$\frac{\mathrm{d}y_i}{\mathrm{d}t}(t) < 0 \quad \text{for } t \in [0, t_{*,1}) \tag{2.26}$$

and

$$\left\| y(t) \right\|_{1,1} = \|y_0\|_{1,1} \quad \text{for } t \in [0, t_*). \tag{2.27}$$

If $t_* < \infty$, *then* $t_{*,1} > t_*$ *and the properties* (2.23)–(2.25) *and* (2.27) *hold true also for* $t = t_*$.

Proof. We define

$$t_* := \sup\left\{t > 0\colon\ y_i(s) > 0 \text{ for } s \in [0, t)\right\},$$

and first notice that $t_* > 0$ due to the continuity of y_i and the positivity (2.21) of $y_{i,0}$. Clearly, y_i satisfies (2.22).

Consider next $j \in \{i+1, \ldots, 2i-1\}$ (if this set is non-empty). Since $y(t) \in \mathcal{X}_{1,i}$ for $t \geq 0$, it follows from (2.3) that, for $t \in [0, t_*)$, $\mathrm{d}y_j(t)/\mathrm{d}t = -a_j y_j(t)$ and thus $y_j(t) = y_{j,0}\mathrm{e}^{-a_j t} \geq 0$. We next deduce from (2.3) that, for $t \in [0, t_*)$, $\mathrm{d}y_{2i}(t)/\mathrm{d}t = a_i y_i(t) - a_{2i} y_{2i}(t) \geq -a_{2i} y_{2i}(t)$, whence $y_{2i}(t) \geq y_{2i,0}\mathrm{e}^{-a_{2i} t} \geq 0$. We next argue in a similar way to prove by induction that $y_j(t) \geq 0$ for $t \in [0, t_*)$ so that y satisfies (2.24).

We now improve the positivity properties of y and prove (2.23) and (2.25). Consider first $j \geq i+1$ for which $y_{j,0} > 0$. By (2.3) and (2.24), we have $\mathrm{d}y_j(t)/\mathrm{d}t = a_{j-i} y_{j-i}(t) - a_j y_j(t) \geq -a_j y_j(t)$ for $t \in [0, t_*)$, whence $y_j(t) \geq y_{j,0}\mathrm{e}^{-a_j t} > 0$ and (2.25). To prove (2.23), we argue by contradiction and assume that there are $k \geq 2$ and $t_0 \in (0, t_*)$ (or $t_0 \in (0, t_*]$ if $t_* < \infty$) such that $y_{ki}(t_0) = 0$. We infer from (2.3) and the variation of constants formula that

$$0 = y_{ki}(t_0) = \mathrm{e}^{-a_{ki} t_0} y_{ki,0} + a_{(k-1)i} \int_0^{t_0} \mathrm{e}^{-a_{ki}(t_0 - s)} y_{(k-1)i}(s)\, \mathrm{d}s.$$

The non-negativity of $y_{ki,0}$ and $y_{(k-1)i}$ and the continuity of $y_{(k-1)i}$ then imply that $y_{ki,0} = 0$ and $y_{(k-1)i}(t) = 0$ for $t \in [0, t_0]$. At this point, either $k = 2$ and we have a contradiction with (2.22), or $k > 2$ and we proceed by induction to show that $y_{li}(t) = 0$ for $t \in [0, t_0]$ and $l \in \{1, \ldots, k\}$, again leading us to a contradiction with (2.22).

The property (2.26) now follows from (2.1) and (2.23): indeed, by (2.3) we have

$$\frac{\mathrm{d}y_i}{\mathrm{d}t}(t) = -a_i y_i(t) - \sum_{j=i}^{\infty} a_j y_j(t) \leq -a_{2i} y_{2i}(t) < 0$$

for $t \in [0, t_*)$ (and also for $t = t_*$ if $t_* < \infty$) so that

$$t_{*,1} := \sup\left\{t > 0\colon \frac{\mathrm{d}y_i}{\mathrm{d}t}(s) < 0 \text{ for } s \in [0, t)\right\} \geq t_*,$$

and $t_{*,1} > t_*$ if $t_* < \infty$.

Finally, since $y(t)$ belongs to $\mathcal{X}_{1,i}^+$ for $t \in [0, t_*)$, (2.27) readily follows from (2.4). □

We next study of the finiteness of the time t_* defined in Proposition 2.4.

Proposition 2.5. *Consider a sequence $(a_j)_{j\geq 1}$ satisfying (2.1), an initial condition $y_0 = (y_{j,0})_{j\geq 1} \in \mathcal{X}_{1,i}$ satisfying (2.21) and let y be the corresponding solution to the Cauchy problem (2.3). Assume further that there is $\delta_0 > 0$ such that*

$$0 < \delta_0 \leq a_j, \quad j \geq i. \tag{2.28}$$

If $t_* \in (0, \infty]$ denotes the time introduced in Proposition 2.4, then $t_* \in (0, \infty)$.

Proof. For $t \geq 0$, we put

$$M_0(t) := \sum_{j=i}^{\infty} y_j(t) \quad \text{and} \quad M_{-1}(t) := \sum_{j=i}^{\infty} \frac{y_j(t)}{j}.$$

By (2.22), $M_0(t) > 0$ for $t \in [0, t_*)$ and it follows from (2.8) that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{M_{-1}}{M_0}\right) &= \frac{1}{M_0}\sum_{j=i}^{\infty}\left(\frac{1}{i+j} - \frac{1}{i} - \frac{1}{j}\right)a_j y_j + \frac{M_{-1}}{M_0^2}\sum_{j=i}^{\infty} a_j y_j \\
&= \frac{1}{M_0}\sum_{j=i}^{\infty}\left(\frac{1}{i+j} - \frac{1}{j} + \frac{M_{-1}}{M_0} - \frac{1}{i}\right)a_j y_j.
\end{aligned}$$

Observing that

$$\frac{1}{i+j} \leq \frac{1}{j} \quad \text{and} \quad \frac{M_{-1}}{M_0} \leq \frac{1}{i},$$

we infer from (2.28) that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{M_{-1}}{M_0}\right) &\leq \frac{\delta_0}{M_0}\sum_{j=i}^{\infty}\left(\frac{1}{i+j} - \frac{1}{j} + \frac{M_{-1}}{M_0} - \frac{1}{i}\right)y_j \\
&\leq \frac{\delta_0}{M_0}\left(\sum_{j=i}^{\infty}\left(\frac{1}{i+j} - \frac{1}{i}\right)y_j - M_{-1} + \frac{M_{-1}}{M_0}M_0\right) \\
&\leq -\frac{\delta_0}{M_0}\sum_{j=i}^{\infty}\frac{j}{i(i+j)}y_j \leq -\frac{\delta_0}{2iM_0}\sum_{j=i}^{\infty} y_j \leq -\frac{\delta_0}{2i}.
\end{aligned}$$

Consequently, we have

$$0 \leq \frac{M_{-1}}{M_0}(t) \leq \frac{M_{-1}}{M_0}(0) - \frac{\delta_0}{2i}t$$

for $t \in [0, t_*)$ which implies that $t_* \leq (2iM_{-1}(0))/(\delta_0 M_0(0)) \leq 2/\delta_0$ and is thus finite. □

2.2. *Proof of Theorem* 1.1

The construction of the functions (ℓ, x) is performed by induction on the minimal size, noticing that x solves an infinite system of ordinary differential equations similar to (2.3) on each time interval where ℓ is constant.

Proof of Theorem 1.1.
Step 1: By (1.12), the sequence $(K(1,j))_{j\ge 1}$ satisfies the assumptions (2.1) (with $A=\kappa$) and (2.28) (with $\delta_0=\delta_1$) while x_0 satisfies (2.21) with $i=1$. According to Propositions 2.1, 2.4 and 2.5, there is a unique solution $x^{(1)} \in \mathcal{C}([0,\infty);\mathcal{X}_{1,1})$ to the Cauchy problem

$$\frac{\mathrm{d}x^{(1)}}{\mathrm{d}t} = b^{(1)}\big(x^{(1)}\big), \qquad x^{(1)}(0) = x_0,$$

and there is $t_1 \in (0,\infty)$ such that

$$\begin{aligned}
&x_1^{(1)}(t) > 0 \quad \text{for } t \in [0,t_1) \text{ and } x_1^{(1)}(t_1) = 0,\\
&x_j^{(1)}(t) > 0 \quad \text{for } t \in (0,t_1] \text{ and } j \ge 2,\\
&\big\|x^{(1)}(t)\big\|_{1,1} = \|x_0\|_{1,1} \quad \text{for } t \in [0,t_1].
\end{aligned}$$

We then put

$$\ell(t) := 1 \quad \text{and} \quad x(t) := x^{(1)}(t) \quad \text{for } t \in [0,t_1).$$

Clearly, x satisfies (1.16), (1.17) and (1.19) for $i=1$.
Step 2: Assume now that we have constructed (ℓ, x) up to some time t_i for some $i \ge 1$. On the one hand, by (1.12), the sequence $(K(i+1,j))_{j\ge i+1}$ satisfies the assumptions (2.1) (with $A=\kappa(i+1)$) and (2.28) (with $\delta_0=\delta_{i+1}$). On the other hand, the sequence $x(t_i)$ belongs to $\mathcal{X}_{1,i+1}^+$ with $x_j(t_i) > 0$ for $j \ge i+1$ by (1.18). We are then in a position to apply Propositions 2.1, 2.4 and 2.5 and conclude that there is a unique solution $x^{(i+1)} \in \mathcal{C}([t_i,\infty);\mathcal{X}_{1,i+1})$ to the Cauchy problem

$$\frac{\mathrm{d}x^{(i+1)}}{\mathrm{d}t} = b^{(i+1)}\big(x^{(i+1)}\big), \quad x^{(i+1)}(t_i) = x(t_i),$$

and there is $t_{i+1} \in (0,\infty)$ such that

$$\begin{aligned}
&x_{i+1}^{(i+1)}(t) > 0 \quad \text{for } t \in [t_i,t_{i+1}) \text{ and } x_{i+1}^{(i+1)}(t_{i+1}) = 0,\\
&x_j^{(i+1)}(t) > 0 \quad \text{for } t \in (t_i,t_{i+1}] \text{ and } j \ge i+2,\\
&\big\|x^{(i+1)}(t)\big\|_{1,1} = \big\|x(t_i)\big\|_{1,1} \quad \text{for } t \in [t_i,t_{i+1}].
\end{aligned}$$

We then put

$$\ell(t) := i+1 \quad \text{and} \quad x(t) := x^{(i+1)}(t) \quad \text{for } t \in [t_i,t_{i+1}).$$

It is then easy to check that $x \in \mathcal{C}([0,t_{i+1};\mathcal{X}_{1,1})$ and satisfies (1.16)–(1.19) for $j \in \{1,\dots,i+1\}$. This completes the inductive step and the proof of the existence part of Theorem 1.1.
Step 3: If (ℓ, x) and $(\hat{\ell}, \hat{x})$ both satisfy the properties listed in Theorem 1.1, we deduce from Proposition 2.1 that $x(t) = \hat{x}(t)$ for $t \in [0, t_1 \wedge \hat{t}_1]$. In particular, x_1 and $\hat{x}_1$ vanish at the same time $t_1 \wedge \hat{t}_1$ which implies that $t_1 = \hat{t}_1$. We next argue by induction to conclude that $\ell = \hat{\ell}$ and $x = \hat{x}$. $\square$

3. Convergence of the stochastic process

In this section, we study the stochastic process introduced in Section 1.1 and prove Theorem 1.3. The proof is performed along the lines of the general scheme developed in [5] with the following main differences: the deterministic system of ordinary differential equations (1.17) considered herein has its solutions in an infinite-dimensional vector space and changes when the minimal size ℓ jumps.

Let K be a coagulation kernel satisfying (1.21). We fix an initial condition x_0 satisfying (1.14) and let x be the corresponding solution to (1.17). By (1.19) and (1.21), we may argue as in the proof of Proposition 2.1 to show that, for $i \geq 1$,

$$\left\| \frac{\mathrm{d}x}{\mathrm{d}t}(t) \right\|_1 \leq 3\kappa_i, \quad t \in [t_{i-1}, t_i]. \tag{3.1}$$

Consider a sequence of random initial data $(X_0^N)_{N\geq 1}$ in $\ell^1_{\mathbb{N}}$ satisfying (1.2) and (1.20). For each $N \geq 1$, X^N denotes the Markov process described in Section 1.1 starting from X_0^N and $\tilde{X}^N := X^N/N$ its renormalized version. To prove Theorem 1.3, we need to introduce some specific times relative to the extinction of some sizes of particle. Let $T_0^N = 0$ and define

$$T_i^N := \inf\{t > T_{i-1}^N : X_i^N(t) = 0\}, \qquad \sigma_i^N := T_i^N - T_{i-1}^N, \quad i \geq 1. \tag{3.2}$$

We also put $s_i := t_i - t_{i-1}$ for $i \geq 1$, the times $(t_i)_{i\geq 0}$ being defined in Theorem 1.1.

We begin by proving the following proposition.

Proposition 3.1. *For all $I \geq 0$, there exist positive constants $C_0(I)$, $C_0(I)'$, and an integer $N_0(I)$ such that*

$$\mathbb{P}\left(\sup_{0\leq t\leq T_I^N} \|\tilde{X}^N(t) - x(t)\|_1 > \frac{C_0(I)}{N^{1/4}} \right) \leq \frac{C_0(I)'}{N^{1/4}} \quad \text{for } N \geq N_0(I).$$

Two steps are needed to prove Proposition 3.1: we first consider $i \geq 1$ and work on the interval $[T_{i-1}^N, T_i^N]$, showing that the behaviour at any time $t \in (T_{i-1}^N, T_i^N]$ depends only on the behaviour at the "initial" time T_{i-1}^N (Proposition 3.2). We then argue by induction on i to prove a "global" convergence result (Proposition 3.3).

Proposition 3.2. *For all $i \geq 1$ and $\gamma > 0$, there exist positive constants $C_1(\gamma, i)$, $C_1(i)'$, $\bar{s}_i \in (s_i, s_i + 1)$, η_i, and an integer $N_1(\gamma, i)$ such that*

$$x_i^{(i)}(t_{i-1} + \bar{s}_i) < 0, \qquad \frac{\mathrm{d}x_i^{(i)}}{\mathrm{d}t}(t_{i-1} + s) \leq -\eta_i < 0 \quad \text{for } s \in [0, \bar{s}_i], \tag{3.3}$$

$$\mathbb{P}\left(\sup_{0\leq s\leq \sigma_i^N} \|\tilde{X}^N(T_{i-1}^N + s) - x^{(i)}(t_{i-1} + s)\|_1 > \frac{C_1(\gamma, i)}{N^{1/4}} \right) \leq \frac{C_1(i)'}{N^{1/4}} + \mathbb{P}(\Omega_{i,\gamma}^c),$$

$$\mathbb{P}(\sigma_i^N > \bar{s}_i) \leq \frac{C_1(i)'}{N^{1/4}} + \mathbb{P}(\Omega_{i,\gamma}^c)$$

for $N \geq N_1(\gamma, i)$, where

$$\Omega_{i,\gamma} := \left\{ \|\tilde{X}^N(T_{i-1}^N) - x(t_{i-1})\|_1 \leq \frac{\gamma}{N^{1/4}} \right\},$$

and $x^{(i)} : [t_{i-1}, \infty) \to \mathcal{X}_{1,1}$ denotes the solution to the differential equation

$$\frac{\mathrm{d}x^{(i)}}{\mathrm{d}t}(t) = b^{(i)}(x^{(i)}(t)) \quad \text{for } t \geq t_{i-1}, x^{(i)}(t_{i-1}) = x(t_{i-1}). \tag{3.4}$$

Proof. Fix $i \geq 1$ and set $\tilde{x} := x^{(i)}$ to simplify the notation. Recall that $x(t) = x^{(i)}(t)$ for $t \in [t_{i-1}, t_i]$. By Section 1.1, we have for $0 \leq s \leq \sigma_i^N$,

$$\tilde{x}(t_{i-1}+s) = x(t_{i-1}) + \int_0^s b^{(i)}\big(\tilde{x}(t_{i-1}+t)\big)\,\mathrm{d}t,$$

$$\tilde{X}^N\big(T_{i-1}^N+s\big) = \tilde{X}^N(T_{i-1}) + \int_0^s \tilde{\beta}\big(\tilde{X}^N\big(T_{i-1}^N+t\big)\big)\,\mathrm{d}t + M_s^N,$$

where $(M_s^N)_{s\geq 0}$ is a $\mathbb{F}_s^{(i)}$-martingale, $\mathbb{F}_s^{(i)} := \sigma(X_{T_{i-1}^N+t}\colon\ t \in [0,s])$, and $\tilde{\beta}$ is the drift of the process $\tilde{X}^N$ defined in (1.7). Subtracting the above two identities, we obtain

$$\begin{aligned}
&\tilde{X}^N\big(T_{i-1}^N+s\big) - \tilde{x}(t_{i-1}+s)\\
&\quad= \tilde{X}^N\big(T_{i-1}^N\big) - x(t_{i-1}) + \int_0^s \big[\tilde{\beta}\big(\tilde{X}^N\big(T_{i-1}^N+t\big)\big) - b^{(i)}\big(\tilde{X}^N\big(T_{i-1}^N+t\big)\big)\big]\,\mathrm{d}t\\
&\qquad+ \int_0^s \big[b^{(i)}\big(\tilde{X}^N\big(T_{i-1}^N+t\big)\big) - b^{(i)}\big(\tilde{x}(t_{i-1}+t)\big)\big]\,\mathrm{d}t + M_s^N.
\end{aligned} \tag{3.5}$$

We now aim to use the representation formula (3.5) to estimate $\|\tilde{X}^N(T_{i-1}^N+s) - \tilde{x}(t_{i-1}+s)\|_1$ for $s \in [0, \sigma_i^N]$. This requires in particular to estimate the martingale term M_s^N in ℓ^1. However, a classical way to estimate M_s^N is to use Doob's inequality which gives an L^2-bound not suitable for our purposes. To remedy this difficulty, we only use (3.5) for the first d components of $\tilde{X}^N(T_{i-1}^N+s) - \tilde{x}(t_{i-1}+s)$, the integer d being suitably chosen, and control the tail of the series by the first moment. More precisely, given $d \geq 1$, we introduce the projections p_d and q_d defined in ℓ^1 by $p_d(y) := (y_1, \ldots, y_d, 0, \ldots)$ and $q_d(y) = y - p_d(y)$, $y \in \ell^1$. Clearly,

$$\big\|p_d(y)\big\|_1 \leq \sqrt{d}\big\|p_d(y)\big\|_2, \quad y \in \ell^1, \tag{3.6}$$

and

$$\big\|q_d(y)\big\|_1 \leq \frac{\|y\|_{1,1}}{d}, \quad y \in \mathcal{X}_{1,1}. \tag{3.7}$$

By (3.7) and the boundedness of the first moment of $\tilde{X}^N$ and $\tilde{x}$ (see (1.4), (1.19) and Lemma 2.2), we have for $s \in [0, \sigma_i^N]$

$$\begin{aligned}
&\big\|\tilde{X}^N\big(T_{i-1}^N+s\big) - \tilde{x}(t_{i-1}+s)\big\|_1\\
&\quad\leq \big\|p_d\big(\tilde{X}^N\big(T_{i-1}^N+s\big) - \tilde{x}(t_{i-1}+s)\big)\big\|_1 + \big\|q_d\big(\tilde{X}^N\big(T_{i-1}^N+s\big)\big)\big\|_1 + \big\|q_d\big(\tilde{x}(t_{i-1}+s)\big)\big\|_1\\
&\quad\leq \big\|p_d\big(\tilde{X}^N\big(T_{i-1}^N+s\big) - \tilde{x}(t_{i-1}+s)\big)\big\|_1 + \frac{\|\tilde{X}^N(T_{i-1}^N+s)\|_{1,1}}{d} + \frac{\|\tilde{x}(t_{i-1}+s)\|_{1,1}}{d}\\
&\quad\leq \big\|p_d\big(\tilde{X}^N\big(T_{i-1}^N+s\big) - \tilde{x}(t_{i-1}+s)\big)\big\|_1 + \frac{(1+\|x(t_{i-1})\|_{1,1}\mathrm{e}^{4\kappa_i s})}{d}\\
&\quad\leq \big\|p_d\big(\tilde{X}^N\big(T_{i-1}^N+s\big) - \tilde{x}(t_{i-1}+s)\big)\big\|_1 + \frac{(1+\|x_0\|_{1,1}\mathrm{e}^{4\kappa_i s})}{d}.
\end{aligned} \tag{3.8}$$

Since $\tilde{\beta}_j - b_j^{(i)} = 0$ for all $j \geq 1$ except for $j \in \{i, 2i\}$ for which $\tilde{\beta}_i - b_i^{(i)} = 2K(i,i)/N$ and $\tilde{\beta}_{2i} - b_{2i}^{(i)} = -K(i,i)/N$ we have

$$\big\|\tilde{\beta}(y) - b^{(i)}(y)\big\|_1 \leq \frac{3K(i,i)}{N} \leq \frac{3\kappa_i}{N}, \quad y \in \mathcal{X}_{1,1}, \tag{3.9}$$

by (1.21). Observing next that $b^{(i)}$ is Lipschitz continuous in ℓ^1 with Lipschitz constant $3\kappa_i$, we infer from (3.5), (3.6) and (3.9) that

$$\begin{aligned}&\|p_d(\tilde{X}^N(T_{i-1}^N+s)-\tilde{x}(t_{i-1}+s))\|_1\\&\quad\le\|p_d(\tilde{X}^N(T_{i-1})-\tilde{x}(t_{i-1}))\|_1+\frac{3\kappa_i s}{N}+3\kappa_i\int_0^s\|\tilde{X}^N(T_{i-1}^N+t)-\tilde{x}(t_{i-1}+t)\|_1\,\mathrm{d}t+\sqrt{d}\|p_d(M_s^N)\|_2.\end{aligned}$$

Combining the above inequality with (3.8) gives

$$\begin{aligned}\|\tilde{X}^N(T_{i-1}^N+s)-\tilde{x}(t_{i-1}+s)\|_1&\le\|\tilde{X}^N(T_{i-1})-\tilde{x}(t_{i-1})\|_1+\frac{3\kappa_i s}{N}+\frac{(1+\|x_0\|_{1,1}\mathrm{e}^{4\kappa_i s})}{d}\\&\quad+3\kappa_i\int_0^s\|\tilde{X}^N(T_{i-1}^N+t)-\tilde{x}(t_{i-1}+t)\|_1\,\mathrm{d}t+\sqrt{d}\|M_s^N\|_2.\end{aligned}\tag{3.10}$$

At this point, we fix $\bar{s}_i\in(s_i,s_i+1)$ and $\eta_i>0$ such that $\tilde{x}_i(t_{i-1}+\bar{s}_i)<0$ and $\mathrm{d}\tilde{x}_i/\mathrm{d}t(t_{i-1}+s)<-\eta_i$ for $s\in[0,\bar{s}_i]$ (such a pair $(\bar{s}_i,\eta_i)$ exists as $\tilde{x}_i(t_i)=x_i(t_{i-1}+s_i)=0$ and $\mathrm{d}\tilde{x}_i/\mathrm{d}t<0$ in $[t_{i-1},t_i]$ by (2.26)). Let $\gamma>0$ and introduce

$$\Omega_i':=\left\{\sup_{s\in[0,\bar{s}_i\wedge\sigma_i^N]}\|M_s^N\|_2\le\frac{1}{N^{3/8}}\right\}.$$

Choosing an integer $d\in(N^{1/4},2N^{1/4})$, we deduce from (3.10) that, in $\Omega_{i,\gamma}\cap\Omega_i'$, we have for $s\in[0,\bar{s}_i\wedge\sigma_i^N]$

$$\begin{aligned}&\|\tilde{X}^N(T_{i-1}^N+s)-\tilde{x}(t_{i-1}+s)\|_1\\&\quad\le\frac{\gamma}{N^{1/4}}+\frac{3\kappa_i s}{N}+\frac{(1+\|x_0\|_{1,1}\mathrm{e}^{4\kappa_i s})}{N^{1/4}}+3\kappa_i\int_0^s\|\tilde{X}^N(T_{i-1}^N+t)-\tilde{x}(t_{i-1}+t)\|_1\,\mathrm{d}t+\frac{\sqrt{2}}{N^{1/4}}\\&\quad\le\frac{\gamma+C_2}{N^{1/4}}\mathrm{e}^{4\kappa_i s}+3\kappa_i\int_0^s\|\tilde{X}^N(T_{i-1}^N+t)-\tilde{x}(t_{i-1}+t)\|_1\,\mathrm{d}t\end{aligned}$$

for some positive constant C_2. After integration, we end up with

$$\sup_{s\in[0,\bar{s}_i\wedge\sigma_i^N]}\|\tilde{X}^N(T_{i-1}^N+s)-\tilde{x}(t_{i-1}+s)\|_1\le5\frac{\gamma+C_2}{N^{1/4}}\mathrm{e}^{4\kappa_i\bar{s}_i}\le5\frac{\gamma+C_2}{N^{1/4}}\mathrm{e}^{4\kappa_i(1+s_i)}.\tag{3.11}$$

In particular, in $\{\sigma_i^N>\bar{s}_i\}\cap\Omega_{i,\gamma}\cap\Omega_i'$, we have

$$0\le\tilde{X}_i^N(T_{i-1}^N+\bar{s}_i)\le\tilde{x}_i(t_{i-1}+\bar{s}_i)+5\frac{\gamma+C_2}{N^{1/4}}\mathrm{e}^{4\kappa_i(1+s_i)}<0$$

for N large enough. Consequently, there exists $N_1(\gamma,i)$ such that

$$\Omega_{i,\gamma}\cap\Omega_i'\subset\{\sigma_i^N\le\bar{s}_i\}\quad\text{for } N\ge N_1(\gamma,i).$$

Recalling (3.11), we have thus established that, for $N\ge N_1(\gamma,i)$,

$$\begin{aligned}\mathbb{P}\left(\sup_{s\in[0,\bar{s}_i\wedge\sigma_i^N]}\|\tilde{X}^N(T_{i-1}^N+s)-\tilde{x}(t_{i-1}+s)\|_1\ge\frac{C_1(\gamma,i)}{N^{1/4}}\right)&\le\mathbb{P}\big((\Omega_{i,\gamma}\cap\Omega_i')^c\big)\\&\le\mathbb{P}(\Omega_{i,\gamma}^c)+\mathbb{P}(\Omega_i'^c)\end{aligned}\tag{3.12}$$

and

$$\mathbb{P}(\sigma_i^N>\bar{s}_i)\le\mathbb{P}\big((\Omega_{i,\gamma}\cap\Omega_i')^c\big)\le\mathbb{P}(\Omega_{i,\gamma}^c)+\mathbb{P}(\Omega_i'^c),\tag{3.13}$$

with $C_1(\gamma,i) := 5(\gamma + C_2)e^{4\kappa_i(1+s_i)}$.

To complete the proof, it remains to bound $\mathbb{P}(\Omega_i'^c)$. By Doob's inequality, we have:

$$\mathbb{E}\Big(\sup_{s\in[0,\bar{s}_i\wedge\sigma_i^N]}\|M_s^N\|_2^2\Big) \le 4\mathbb{E}\big(\|M_{\bar{s}_i\wedge\sigma_i^N}^N\|_2^2\big) \le 4\mathbb{E}\bigg(\int_0^{\bar{s}_i\wedge\sigma_i^N}\tilde{\alpha}\big(\tilde{X}^N\big(T_{i-1}^N+t\big)\big)\,\mathrm{d}t\bigg),$$

where $\tilde{\alpha}$ is defined by (1.8). According to Section 1.1 and (1.21), it is easy to show that, if $y\in\mathcal{X}_{1,i}$, we have $\tilde{\alpha}(y)\le 5\kappa_i\|y\|_1/N$. Since $X^N(s)\in\mathcal{X}_{1,i}$ for $s\in[T_{i-1}^N,T_i^N]$ and $\bar{s}_i<s_i+1$, we conclude that

$$\mathbb{E}\Big(\sup_{s\in[0,\bar{s}_i\wedge\sigma_i^N]}\|M_s^N\|_2^2\Big) \le \frac{C_3(i)}{N}.$$

Therefore, observing that

$$\mathbb{P}\big(\Omega_i'^c\big) = \mathbb{P}\bigg(\sup_{s\in[0,\bar{s}_i\wedge\sigma_i^N]}\|M_s^N\|_2^2 > \frac{1}{N^{3/4}}\bigg),$$

Markov's inequality yields

$$\mathbb{P}\big(\Omega_i'^c\big) \le N^{3/4}\mathbb{E}\Big(\sup_{s\in[0,\bar{s}_i\wedge\sigma_i^N]}\|M_s^N\|_2^2\Big) \le \frac{C_3(i)}{N^{1/4}}.$$

Proposition 3.2 then readily follows from (3.12), (3.13) and the above bound with $C_1(i)' := C_3(i)$. □

Proposition 3.3. *For all $i\ge 1$, there exist positive constants a_i, b_i and an integer $N_2(i)$ such that*

$$\mathbb{P}\bigg(\big\|\tilde{X}^N\big(T_{i-1}^N\big) - x(t_{i-1})\big\|_1 > \frac{b_i}{N^{1/4}}\bigg) \le \frac{a_i}{N^{1/4}}\quad \textit{for all } N\ge N_2(i). \tag{3.14}$$

Proof. We argue by induction on $i\ge 1$ and first note that (3.14) holds true for $i=1$ with $a_1=b_1=1$ by (1.20).

Assume next that (3.14) holds true for some $i\ge 1$. Setting $\tilde{x} := x^{(i)}$, the function $x^{(i)}$ being defined in Proposition 3.2, we have

$$\big\|\tilde{X}^N\big(T_i^N\big) - x(t_i)\big\|_1 \le \big\|\tilde{X}^N\big(T_i^N\big) - \tilde{x}\big(t_{i-1}+\sigma_i^N\big)\big\|_1 + \big\|\tilde{x}\big(t_{i-1}+\sigma_i^N\big) - \tilde{x}(t_i)\big\|_1. \tag{3.15}$$

On the one hand, it follows from (3.14) for i and Proposition 3.2 with $\gamma = b_i$ that we have

$$\begin{aligned}\mathbb{P}\bigg(\big\|\tilde{X}^N\big(T_i^N\big) - \tilde{x}\big(t_{i-1}+\sigma_i^N\big)\big\|_1 > \frac{C_1(b_i,i)}{N^{1/4}}\bigg) &\le \frac{C_1(i)'}{N^{1/4}} + \mathbb{P}\bigg(\big\|\tilde{X}^N\big(T_{i-1}^N\big) - \tilde{x}(t_{i-1})\big\|_1 > \frac{b_i}{N^{1/4}}\bigg)\\ &\le \frac{C_1(i)'+a_i}{N^{1/4}}\end{aligned} \tag{3.16}$$

and

$$\mathbb{P}\big(\sigma_i^N > \bar{s}_i\big) \le \frac{C_1(i)'+a_i}{N^{1/4}} \tag{3.17}$$

for $N\ge N_1(b_i,i)+N_2(i)$, the constant $\bar{s}_i$ being defined in (3.3).

On the other hand, if $|\sigma_i^N - s_i| > C_1(b_i,i)/(\eta_i N^{1/4})$, we have either $\sigma_i^N > \bar{s}_i$ or $\sigma_i^N \le \bar{s}_i$ and we deduce from (3.3) that

$$\big|\tilde{x}_i\big(t_{i-1}+\sigma_i^N\big)\big| = \big|\tilde{x}_i\big(t_{i-1}+\sigma_i^N\big) - \tilde{x}_i(t_{i-1}+s_i)\big| = \bigg|\int_{\sigma_i^N}^{s_i}\frac{\mathrm{d}\tilde{x}_i}{\mathrm{d}t}(t)\,\mathrm{d}t\bigg| \ge \eta_i\big|\sigma_i^N - s_i\big| > \frac{C_1(b_i,i)}{N^{1/4}},$$

so that

$$\left\{\left|\sigma_i^N - s_i\right| > \frac{C_1(b_i,i)}{\eta_i N^{1/4}}\right\} \subset \left\{\sigma_i^N > \bar{s}_i\right\} \cup \left\{\left|\tilde{X}_i^N\left(T_i^N\right) - \tilde{x}_i\left(t_{i-1}+\sigma_i^N\right)\right| > \frac{C_1(b_i,i)}{N^{1/4}}\right\}$$

since $\tilde{X}_i^N(T_i^N) = 0$. We then infer from (3.16), (3.17) and the above inclusion that, for $N \geq N_1(b_i,i) + N_2(i)$,

$$\mathbb{P}\left(\left|\sigma_i^N - s_i\right| > \frac{C_1(b_i,i)}{\eta_i N^{1/4}}\right) \leq 2\frac{(C_1(i)' + a_i)}{N^{1/4}}. \tag{3.18}$$

This estimate now allows us to handle the second term in the right-hand side of (3.15). Indeed, by Proposition 2.1, if $\sigma_i^N \leq \bar{s}_i$,

$$\left\|\tilde{x}\left(t_{i-1}+\sigma_i^N\right) - \tilde{x}(t_i)\right\|_1 \leq \left|\sigma_i^N - s_i\right| \sup_{t\in[t_{i-1},t_{i-1}+\bar{s}_i]} \left\|\frac{\mathrm{d}\tilde{x}}{\mathrm{d}t}(t)\right\|_1 \leq C_4(i)\left|\sigma_i^N - s_i\right|,$$

and it follows from (3.17) and (3.18) that, for $N \geq N_1(b_i,i) + N_2(i)$,

$$\begin{aligned}\mathbb{P}\left(\left\|\tilde{x}\left(t_{i-1}+\sigma_i^N\right) - \tilde{x}(t_i)\right\|_1 > \frac{C_1(b_i,i)C_4(i)}{\eta_i N^{1/4}}\right) &\leq \mathbb{P}\left(\sigma_i^N > \bar{s}_i\right) + \mathbb{P}\left(\left|\sigma_i^N - s_i\right| > \frac{C_1(b_i,i)}{\eta_i N^{1/4}}\right) \\ &\leq 3\frac{(C_1(i)' + a_i)}{N^{1/4}}.\end{aligned} \tag{3.19}$$

Setting

$$a_{i+1} := 4\left(a_i + C_1'(i)\right), \qquad b_{i+1} := 2\frac{(1+C_4(i))C_1(b_i,i)}{\eta_i}, \qquad N_2(i+1) := N_1(b_i,i) + N_2(i), \tag{3.20}$$

we infer from (3.15), (3.16) and (3.19) that, for $N \geq N_2(i+1)$,

$$\begin{aligned}\mathbb{P}\left(\left\|\tilde{X}^N\left(T_i^N\right) - x(t_i)\right\|_1 > \frac{b_{i+1}}{N^{1/4}}\right) &\leq \mathbb{P}\left(\left\|\tilde{X}^N\left(T_i^N\right) - \tilde{x}\left(t_{i-1}+\sigma_i^N\right)\right\|_1 > \frac{C_1(b_i,i)}{N^{1/4}}\right) \\ &\quad + \mathbb{P}\left(\left\|\tilde{x}\left(t_{i-1}+\sigma_i^N\right) - \tilde{x}(t_i)\right\|_1 > \frac{C_1(b_i,i)C_4(i)}{\eta_i N^{1/4}}\right) \\ &\leq \frac{a_{i+1}}{N^{1/4}},\end{aligned}$$

which completes the proof. □

Corollary 3.4. *For all $i \geq 1$, there are positive constants A_i, B_i and an integer $N_3(i)$ such that*

$$\mathbb{P}\left(\left|T_i^N - t_i\right| > \frac{B_i}{N^{1/4}}\right) \leq \frac{A_i}{N^{1/4}} \quad \textit{for } N \geq N_3(i).$$

Proof. Recalling (3.18) and (3.20), we have

$$\mathbb{P}\left(\left|\sigma_i^N - s_i\right| > \frac{b_{i+1}}{N^{1/4}}\right) \leq \frac{a_{i+1}}{N^{1/4}} \quad \text{for } N \geq N_2(i+1)$$

and $i \geq 1$. Fix $i \geq 1$ and put

$$N_3(i) := \max_{1\leq j\leq i} N_2(j+1), \qquad A_i := \sum_{j=1}^{i} a_{j+1}, \qquad B_i := \sum_{j=1}^{i} b_{j+1}.$$

As

$$T_i^N - t_i = \sum_{j=1}^{i} \left(\sigma_j^N - s_j\right),$$

we have

$$\mathbb{P}\left(\left|T_i^N - t_i\right| > \frac{B_i}{N^{1/4}}\right) \le \sum_{j=1}^{i} \mathbb{P}\left(\left|\sigma_j^N - s_j\right| > \frac{b_{j+1}}{N^{1/4}}\right) \le \sum_{j=1}^{i} \frac{a_{j+1}}{N^{1/4}} = \frac{A_i}{N^{1/4}}$$

as claimed. □

We are now able to prove Proposition 3.1.

Proof of Proposition 3.1. For $I \ge 1$, consider

$$\Lambda_I := \bigcap_{i=1}^{I} \left\{ \sup_{0 \le s \le \sigma_i^N} \left\| \tilde{X}^N\left(T_{i-1}^N + s\right) - x^{(i)}(t_{i-1} + s) \right\|_1 \le \frac{C_1(b_i, i)}{N^{1/4}} \text{ and } \left|T_i^N - t_i\right| \le \frac{B_i}{N^{1/4}} \right\},$$

and

$$N_4(i) := \max_{1 \le i \le I} \max\left\{N_1(b_i, i), N_2(i), N_3(i)\right\}.$$

According to Propositions 3.2, 3.3 and Corollary 3.4, we have for $N \ge N_4(i)$

$$\begin{aligned}
\mathbb{P}\left(\Lambda_I^c\right) &\le \sum_{i=1}^{I} \mathbb{P}\left(\sup_{s \in [0, \sigma_i^N]} \left\| \tilde{X}^N\left(T_{i-1}^N + s\right) - x^{(i)}(t_{i-1} + s) \right\|_1 > \frac{C_1(b_i, i)}{N^{1/4}} \right) + \sum_{i=1}^{I} \mathbb{P}\left(\left|T_i^N - t_i\right| > \frac{B_i}{N^{1/4}} \right) \\
&\le \sum_{i=1}^{I} \left(\mathbb{P}\left(\left\| \tilde{X}^N\left(T_{i-1}^N\right) - x^{(i)}(t_{i-1}) \right\|_1 > \frac{b_i}{N^{1/4}} \right) + \frac{C_1(i)'}{N^{1/4}} \right) + \sum_{i=1}^{I} \frac{A_i}{N^{1/4}} \\
&\le \sum_{i=1}^{I} \frac{a_i + C_1(i)' + A_i}{N^{1/4}},
\end{aligned}$$

$$\mathbb{P}\left(\Lambda_I^c\right) \le \frac{C_5(I)}{N^{1/4}}. \tag{3.21}$$

Consider now $t \ge 0$. In $\Lambda_I \cap \{T_I^N \ge t\}$, there are $i \in \{1, \dots, I-1\}$, and $s \in [0, \sigma_i^N)$ such that $t = T_{i-1}^N + s$ and

$$T_{i-1}^N + s \le T_{i-1}^N + \sigma_i^N = T_i^N - t_i + t_i \le t - I + \frac{B_i}{N^{1/4}} \le \vartheta_I := \min\left\{1 + t_I, \frac{t_I + t_\infty}{2}\right\}, \tag{3.22}$$

$$t_{i-1} + s \le t_{i-1} + \sigma_i^N = t_{i-1} - T_{i-1}^N + T_i^N - t_i + t_i \le t_I + \frac{2B_i}{N^{1/4}} \le \vartheta_I \tag{3.23}$$

for $N \ge N_5(I)$ large enough. Consequently, recalling that $x^{(i)}$ is defined in Proposition 3.2, it follows from (3.1) that, in $\Lambda_I \cap \{T_I^N \ge t\}$

$$\begin{aligned}
\left\| \tilde{X}^N(t) - x(t) \right\|_1 \le{}& \left\| \tilde{X}^N\left(T_{i-1}^N + s\right) - x^{(i)}(t_{i-1} + s) \right\|_1 + \left\| x^{(i)}(t_{i-1} + s) - x(t_{i-1} + s) \right\|_1 \\
&+ \left\| x(t_{i-1} + s) - x\left(T_{i-1}^N + s\right) \right\|_1
\end{aligned}$$

$$\leq \frac{C_1(b_i,i)}{N^{1/4}} + \left\| x^{(i)}(t_{i-1}+s) - x(t_{i-1}+s) \right\|_1 + \left| T_{i-1}^N - t_{i-1} \right| \sup_{t\in[0,\vartheta_I]} \left\| \frac{\mathrm{d}x}{\mathrm{d}t}(t) \right\|_1$$

$$\leq \frac{C_6(I)}{N^{1/4}} + \left\| x^{(i)}(t_{i-1}+s) - x(t_{i-1}+s) \right\|_1 \tag{3.24}$$

for $N \geq N_5(I)$.

Now, since $0 \leq s < \sigma_i^N$ in $\Lambda_I \cap \{T_I^N \geq t\}$, we have the following dichotomy:

(a) either $s \leq s_i$ and $x^{(i)}(t_{i-1}+s) = x(t_{i-1}+s)$,

(b) or $s_i < s < \sigma_i^N$ and, for $N \geq N_5(I)$, we infer from Proposition 2.1, (3.1), (3.23) and the identity $x^{(i)}(t_i) = x(t_i)$ that

$$\begin{aligned}
\left\| x^{(i)}(t_{i-1}+s) - x(t_{i-1}+s) \right\|_1 &\leq \left\| x^{(i)}(t_{i-1}+s) - x^{(i)}(t_i) \right\|_1 + \left\| x(t_i) - x(t_{i-1}+s) \right\|_1 \\
&\leq |s - s_i| \left(\sup_{t\in[0,\vartheta_I]} \left\| \frac{\mathrm{d}x^{(i)}}{\mathrm{d}t}(t) \right\|_1 + \sup_{t\in[0,\vartheta_I]} \left\| \frac{\mathrm{d}x}{\mathrm{d}t}(t) \right\|_1 \right) \\
&\leq C_7(I) \left| \sigma_i^N - s_i \right| \\
&\leq C_7(I) \left(\left| T_i^N - t_i \right| + \left| T_{i-1}^N - t_{i-1} \right| \right) \\
&\leq \frac{C_8(I)}{N^{1/4}}.
\end{aligned}$$

Combining (3.24) and the above analysis, we conclude that, in $\Lambda_I \cap \{T_I^N \geq t\}$,

$$\left\| \tilde{X}^N(t) - x(t) \right\|_1 \leq \frac{C_9(I)}{N^{1/4}}$$

for $N \geq N_5(I)$ and thus

$$\Lambda_I \subset \left\{ \sup_{0\leq t\leq T_I^N} \left\| \tilde{X}^N(t) - x(t) \right\|_1 \leq \frac{C_9(I)}{N^{1/4}} \right\}.$$

Proposition 3.1 then follows from (3.21) and the above set inclusion. □

Proof of Theorem 1.3. Let $t \in (0, t_\infty)$. There exists $I \geq 1$ such that $t < t_I$. Clearly,

$$\left\{ \sup_{0\leq s\leq t} \left\| \tilde{X}^N(s) - x(s) \right\|_1 > \frac{C_0(I)}{N^{1/4}} \right\} \subset \left\{ \sup_{0\leq s\leq T_I^N} \left\| \tilde{X}^N(s) - x(s) \right\|_1 > \frac{C_0(I)}{N^{1/4}} \right\} \cup \left\{ t_I > T_I^N \right\},$$

the constant $C_0(I)$ being defined in Proposition 3.1. Theorem 1.3 then follows from Proposition 3.1 and Corollary 3.4. □

4. Deterministic maximal existence time

4.1. *Global existence*

Proof of Theorem 1.4(i). Recall that we assume that there exists $A_0 > 0$ such that for all $1 \leq i \leq j$,

$$K(i,j) \leq \frac{\ln(i+1)}{4A_0}.$$

For $t \in [0, t_\infty)$ and $i \geq 1$, we define

$$\phi_i := \frac{\ln(i+1)}{4A_0} \quad \text{and} \quad M_0(t) := \sum_{j=1}^{\infty} x_j(t).$$

For $i \geq 1$ and $t \in (t_{i-1}, t_i)$, we infer from the upper bound on K and (2.8) that

$$0 = \frac{\mathrm{d}M_0}{\mathrm{d}t}(t) + \sum_{j=i}^{\infty} K(i,j)x_j(t) \leq \frac{\mathrm{d}M_0}{\mathrm{d}t}(t) + \phi_i M_0(t).$$

Integrating with respect to time and using the time continuity of x in $\mathcal{X}_{1,1}$ gives

$$M_0(t_i)\mathrm{e}^{\phi_i t_i} \geq M_0(t_{i-1})\mathrm{e}^{\phi_i t_{i-1}} = M_0(t_{i-1})\mathrm{e}^{\phi_{i-1} t_{i-1}}\mathrm{e}^{(\phi_i - \phi_{i-1})t_{i-1}}.$$

Arguing by induction, we conclude that

$$M_0(t_i)\mathrm{e}^{\phi_i t_i} \geq M_0(0)\prod_{j=1}^{i-1}\mathrm{e}^{(\phi_{j+1}-\phi_j)t_j}, \quad i \geq 2.$$

By (1.19) we have

$$M_0(t_i) \leq \frac{1}{i}\sum_{j=i}^{\infty} jx_j(t_i) = \frac{1}{i}, \quad i \geq 2.$$

Combining the above two estimates gives

$$\frac{1}{i}\mathrm{e}^{\phi_i t_i} \geq M_0(0)\prod_{j=1}^{i-1}\mathrm{e}^{(\phi_{j+1}-\phi_j)t_j},$$

$$\phi_i t_i \geq \ln i + \sum_{j=1}^{i-1}(\phi_{j+1}-\phi_j)t_j + \ln\big(M_0(0)\big), \quad i \geq 2,$$

$$t_i \geq 4A_0\frac{\ln i}{\ln(i+1)} + \frac{1}{\ln(i+1)}\sum_{j=1}^{i-1}\ln\left(\frac{j+2}{j+1}\right)t_j + \frac{4A_0}{\ln(i+1)}\ln\big(M_0(0)\big). \tag{4.1}$$

In particular, for $I \geq 2$ and $i > I$, we infer from (4.1) and the monotonicity of $(t_j)_{j\geq 1}$ that

$$\begin{aligned} t_i &\geq 4A_0\frac{\ln i}{\ln(i+1)} + \frac{1}{\ln(i+1)}\sum_{j=I}^{i-1}\ln\left(\frac{j+2}{j+1}\right)t_I + \frac{1}{\ln(i+1)}\sum_{j=1}^{I-1}\ln\left(\frac{j+2}{j+1}\right)t_1 + \frac{4A_0}{\ln(i+1)}\ln\big(M_0(0)\big) \\ &\geq 4A_0\frac{\ln i}{\ln(i+1)} + \frac{\ln(i+1)-\ln(I+1)}{\ln(i+1)}t_I + \frac{\ln(I+1)-\ln 2}{\ln(i+1)}t_1 + \frac{4A_0}{\ln(i+1)}\ln\big(M_0(0)\big). \end{aligned}$$

Assume now for contradiction that $t_\infty < \infty$. We may let $i \to \infty$ in the previous inequality to conclude that $t_\infty \geq 4A_0 + t_I$ for all $I \geq 2$. Letting $I \to \infty$ then implies that $t_\infty \geq 4A_0 + t_\infty$ and a contradiction. Therefore, $t_\infty = \infty$. □

4.2. *Finite time blow-up of the minimal size*

We actually establish a stronger version of the second assertion of Theorem 1.4.

Proposition 4.1. *Consider a coagulation kernel K and an initial condition x_0 satisfying* (1.12) *and* (1.14), *respectively. Let x be the corresponding solution to the min-driven coagulation equations given in Theorem* 1.1 *defined on* $[0, t_\infty)$, t_∞ *being defined in* (1.15). *Assume further that there exist a non-decreasing sequence $(\phi_j)_{j\ge1}$ of non-negative real numbers, a non-increasing sequence $(\psi_j)_{j\ge1}$ of non-negative real numbers, and $\varepsilon > 0$ such that*

$$K(i,j) \ge \phi_i \quad \text{and} \quad \phi_i(\psi_i - \psi_{i+j}) \ge \varepsilon \quad \text{for } j \ge i \ge 1. \tag{4.2}$$

Then $t_\infty < \infty$.

Proof. For $t \in [0, t_\infty)$, define

$$M_0(t) := \sum_{j=1}^{\infty} x_j(t) \quad \text{and} \quad M_\psi(t) := \sum_{j=1}^{\infty} \psi_j x_j(t).$$

Given $i \ge 1$ and $t \in (t_{i-1}, t_i)$, it follows from (1.17) and (2.8) that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{M_\psi}{M_0}\right) &= \frac{1}{M_0}\sum_{j=i}^{\infty}(\psi_{i+j} - \psi_i - \psi_j)K(i,j)x_j + \frac{M_\psi}{M_0^2}\sum_{j=i}^{\infty}K(i,j)x_j \\
&= \frac{1}{M_0}\sum_{j=i}^{\infty}\left(\psi_{i+j} - \psi_j + \frac{M_\psi}{M_0} - \psi_i\right)K(i,j)x_j.
\end{aligned}$$

By the monotonicity of $(\psi_j)_{j\ge1}$, we have

$$\psi_{i+j} \le \psi_j \quad \text{and} \quad \frac{M_\psi}{M_0} \le \psi_i, \quad j \ge i,$$

so that (4.2) entails that

$$\left(\psi_{i+j} - \psi_j + \frac{M_\psi}{M_0} - \psi_i\right)K(i,j) \le \left(\psi_{i+j} - \psi_j + \frac{M_\psi}{M_0} - \psi_i\right)\phi_i, \quad j \ge i.$$

Then,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{M_\psi}{M_0}\right) &\le \frac{\phi_i}{M_0}\sum_{j=i}^{\infty}\left(\psi_{i+j} - \psi_j + \frac{M_\psi}{M_0} - \psi_i\right)x_j \\
&\le \frac{\phi_i}{M_0}\left(\sum_{j=i}^{\infty}\psi_{i+j}x_j - M_\psi + \frac{M_\psi}{M_0}M_0 - \psi_i M_0\right) \\
&\le \frac{1}{M_0}\sum_{j=i}^{\infty}\phi_i(\psi_{i+j} - \psi_i)x_j \\
&\le -\varepsilon,
\end{aligned}$$

and so,

$$\left(\frac{M_\psi}{M_0}\right)(t_i) + \varepsilon(t_i - t_{i-1}) \le \left(\frac{M_\psi}{M_0}\right)(t_{i-1}).$$

Summing the above inequality with respect to i gives

$$\varepsilon t_\infty \le \lim_{i\to\infty}\left(\frac{M_\psi}{M_0}\right)(t_i) + \varepsilon t_\infty \le M_\psi(0)/M_0(0) < \infty$$

and completes the proof. □

Let us now give some examples of sequences $(\phi_j)_{j\geq 1}$ which satisfy (4.2):

- if $\phi_j = j^\alpha$ for $j \geq 1$ and some $\alpha > 0$, then (4.2) is satisfied with $\psi_j = j^{-\alpha}$, $j \geq 1$, and $\varepsilon = (1 - 2^{-\alpha})$.
- if $\phi_j = (\ln(j+1))^{1+\alpha}$ for $j \geq 1$ and some $\alpha > 0$, then (4.2) is satisfied with $\psi_j = (\ln(j+1))^{-\alpha}$, $j \geq 1$, and $\varepsilon = \alpha 2^{-1-\alpha}\ln(3/2)$.

In particular, Theorem 1.4(ii) follows by combining the second example above with Proposition 4.1.

5. Finite or infinite stochastic time of the last coalescence event

In this section, we study the boundedness or unboundedness of the expectation of the last coalescence time T^{X_0} defined in (1.22) with respect to the initial condition $X_0 \in \ell^1_{\mathbb{N}}$, the space $\ell^1_{\mathbb{N}}$ being defined in (1.3). We focus on the class of coagulation kernels K having the special structure (1.23), namely,

$$K(i,j) = \phi(i), \quad 1 \leq i \leq j \text{ for some positive increasing function } \phi.$$

To this end, we prove some specific properties of the stochastic min-driven coagulation process for this type of kernel. In fact, a crucial argument in the analysis is that this structure allows us to compare the evolution of the process from an arbitrary initial configuration with that starting from monodisperse initial data (that is, initial data of the form $n\mathbf{e}_i$ for $n \geq 1$ and $i \geq 1$, $(\mathbf{e}_i)_{i\geq 1}$ being the canonical basis of ℓ^1 defined in Section 1.1).

Before going on, we introduce some notation. If $Z \in \ell^1_{\mathbb{N}}$ with $\|Z\|_1 = n$, the vector $(S_1(Z), \ldots, S_n(Z)) \in \mathbb{N}^n$ denotes the collection of the sizes of the particles encoded by Z sorted in increasing order, that is,

$$S_m(Z) := 1 \quad \text{if } 1 \leq m \leq Z_1, \qquad S_m(Z) := s \quad \text{if } 1 + \sum_{j=1}^{s-1} Z_j \leq m \leq \sum_{j=1}^{s} Z_j \text{ and } 2 \leq s \leq n. \tag{5.1}$$

Next, given an initial condition $X_0 \in \ell^1_{\mathbb{N}}$ with $n := \|X_0\|_1$, let X be the stochastic min-driven coagulation process starting from X_0 in Section 1.1 and recall that T^{X_0} is defined by

$$T^{X_0} = \inf\{t \geq 0\colon\ \|X(t)\|_1 = 1\}.$$

For $i \geq 1$, we also introduce the time

$$T_i^{X_0} := \inf\{t > 0\colon\ X_1(t) = \cdots = X_i(t) = 0\}, \tag{5.2}$$

when particles of size smaller or equal than i have disappeared (note that the time T_i^N defined in (3.2) in Section 3 corresponds to $T_i^{X_0^N}$ with the notation introduced in (5.2)). In addition, since X_0 contains n particles, the stochastic process X undergoes $n-1$ coalescence events between $t = 0$ and T^{X_0} and we define $L(m)$ to be the minimal size of X after the $(m-1)$th coalescence event and before the mth coalescence event, $1 \leq m \leq n-1$. Before the latter event, the rate of coagulation is $(n-m)\phi(L(m))$ since K satisfies $K(i,j) = \phi(i) \wedge \phi(j)$. Consequently,

$$T^{X_0} = \sum_{m=1}^{n-1} \frac{\varepsilon_m}{(n-m)\phi(L(m))}, \tag{5.3}$$

where $(\varepsilon_m)_{1\leq m\leq n-1}$ is a sequence of i.i.d. random variables with law $exp(1)$.

The first step towards the proof of Theorem 1.5 is a monotonicity property.

Lemma 5.1. *Let X_0 and Y_0 be two initial conditions in $\ell^1_{\mathbb{N}}$ such that $\|X_0\|_1 = \|Y_0\|_1$ and*

$$S_m(Y_0) \leq S_m(X_0) \quad \textit{for all } 1 \leq m \leq \|X_0\|_1. \tag{5.4}$$

Then, we can construct the stochastic min-driven coagulation processes starting from X_0 and Y_0 on the same probability space such that $T_i^{X_0} \le T_i^{Y_0}$ for all $i \ge 1$ and $T^{X_0} \le T^{Y_0}$. In particular, for all initial data $X_0 \in \ell^1_{\mathbb{N}}$,

$$T_1^{X_0} \le T_1^{\|X_0\|_1 \mathbf{e}_1} \quad \text{and} \quad T^{X_0} \le T^{\|X_0\|_1 \mathbf{e}_1}.$$

Proof. Let X and Y denote the stochastic min-driven coagulation processes starting from X_0 and Y_0, respectively, and define $n := \|X_0\|_1 = \|Y_0\|_1$. Between $t = 0$ and T^{X_0}, the process X reaches n different states $\{\hat{X}(j)\colon 0 \le j \le n-1\}$ with $\hat{X}(0) = X_0$ and $\|\hat{X}(j)\|_1 = n - j$. In other words, $\hat{X}(j)$ is the state of X after the jth coalescence event and actually equals $X(\theta_j)$, θ_j being the time at which the jth coalescence event occurs. Analogously, between $t = 0$ and T^{Y_0}, the process Y reaches n different states $\{\hat{Y}(j)\colon 0 \le j \le n-1\}$ with $\hat{Y}(0) = Y_0$ and $\|\hat{Y}(j)\|_1 = n - j$.

We first prove by induction that we can construct the processes X and Y on the same probability space such that

$$S_m\big(\hat{Y}(j)\big) \le S_m\big(\hat{X}(j)\big), \quad 1 \le m \le n-j, 0 \le j \le n-1. \tag{5.5}$$

By (5.4), this inequality is clearly satisfied for $j = 0$. Assume now that (5.5) holds true for some $j \in \{0, \dots, n-2\}$ and set

$$S_m^{X,j} := S_m\big(\hat{X}(j)\big) \quad \text{and} \quad S_m^{Y,j} := S_m\big(\hat{Y}(j)\big), \quad 1 \le m \le n-j.$$

Since the coagulation kernel K is of the form (1.23), we may couple the two processes X and Y in such a way that $\hat{X}(j+1)$ is obtained by coalescing the particles of sizes $S_1^{X,j}$ and $S_k^{X,j}$ and $\hat{Y}(j+1)$ by coalescing the particles of sizes $S_1^{Y,j}$ and $S_k^{Y,j}$ with the same index k chosen in $\{2, \dots, n-i\}$ with uniform law. Thus,

$$\begin{aligned}
\big\{S_m\big(\hat{X}(j+1)\big)\colon 1 \le m \le n-j-1\big\} &= \big\{S_2^{X,j}, \dots, S_{k-1}^{X,j}, S_{k+1}^{X,j}, \dots, S_{n-j}^{X,j}\big\} \cup \big\{S_1^{X,j} + S_k^{X,j}\big\},\\
\big\{S_m\big(\hat{Y}(j+1)\big)\colon 1 \le m \le n-j-1\big\} &= \big\{S_2^{Y,j}, \dots, S_{k-1}^{Y,j}, S_{k+1}^{Y,j}, \dots, S_{n-j}^{Y,j}\big\} \cup \big\{S_1^{Y,j} + S_k^{Y,j}\big\}.
\end{aligned}$$

At this stage, the inequality (5.5) is not obvious as the reordering of the sizes can be different in $\hat{X}(j+1)$ and $\hat{Y}(j+1)$. The situation can be represented as follows:

$$\begin{aligned}
&S_1^{Y,j} \le \cdots \le S_{k-1}^{Y,j} \le \cdots \le S_1^{Y,j} + S_k^{Y,j} \le \cdots \le \cdots \le \cdots \le S_{n-i}^{Y,j},\\
&S_1^{X,j} \le \cdots \le S_{k-1}^{X,j} \le \cdots \le \cdots \le \cdots \le S_1^{X,j} + S_k^{X,j} \le \cdots \le S_{n-i}^{X,j}.
\end{aligned}$$

Nevertheless, we observe that

$$S_m\big(\hat{Y}(j+1)\big) = \begin{cases} S_{m+1}^{Y,j} & \text{for } 1 \le m \le k-2,\\ \max\big\{\min\big\{S_{m+2}^{Y,j}, S_1^{Y,j} + S_k^{Y,j}\big\}, S_{m+1}^{Y,j}\big\} & \text{for } m \ge k-1, \end{cases}$$

and

$$S_m\big(\hat{X}(j+1)\big) = \begin{cases} S_{m+1}^{X,j} & \text{for } 1 \le m \le k-2,\\ \max\big\{\min\big\{S_{m+2}^{X,j}, S_1^{X,j} + S_k^{X,j}\big\}, S_{m+1}^{X,j}\big\} & \text{for } m \ge k-1, \end{cases}$$

from which (5.5) for $j+1$ readily follows thanks to (5.5) for j.

We next claim that the random number of coalescence events needed to exhaust the particles of size $i \ge 1$ is smaller for X than for Y, that is,

$$n_i^{X_0} \le n_i^{Y_0}, \quad i \ge 1, \tag{5.6}$$

where

$$\begin{aligned}
n_i^{X_0} &:= \inf\big\{j \in \{0, \dots, n-1\}\colon S_1\big(\hat{X}(j)\big) \ge i+1\big\},\\
n_i^{Y_0} &:= \inf\big\{j \in \{0, \dots, n-1\}\colon S_1\big(\hat{Y}(j)\big) \ge i+1\big\}.
\end{aligned}$$

Indeed, we have $S_1(\hat{Y}(j)) \le S_1(\hat{X}(j)) \le i$ for $1 \le j \le n_i^{X_0} - 1$ by (5.5).

We can now prove the lemma. For $i \ge 1$, we have

$$T_i^{X_0} = \sum_{j=1}^{n_i^{X_0}} \frac{\varepsilon_j}{(n-j)\phi(S_1(\hat{X}(j-1)))} \quad \text{and} \quad T_i^{Y_0} = \sum_{j=1}^{n_i^{Y_0}} \frac{\varepsilon_j}{(n-j)\phi(S_1(\hat{Y}(j-1)))},$$

where $(\varepsilon_k)_{k\ge 1}$ is a sequence of i.i.d. random variables with law $exp(1)$. Concerning T^{X_0} and T^{Y_0}, we have

$$T^{X_0} = \sum_{j=1}^{n-1} \frac{\varepsilon_j}{(n-j)\phi(S_1(\hat{X}(j-1)))} \quad \text{and} \quad T^{Y_0} = \sum_{j=1}^{n-1} \frac{\varepsilon_j}{(n-j)\phi(S_1(\hat{Y}(j-1)))}.$$

The desired result then follows by (5.5), (5.6) and the monotonicity of ϕ. □

We next prove that the expectation of the time $T_1^{X_0}$ after which all particles of size 1 have disappeared is bounded independently of the initial condition X_0 (as soon as $X_0 \ne \mathbf{e}_1$). According to Lemma 5.1, it will be sufficient to prove such a bound for monodisperse initial data of the form $n\mathbf{e}_1$, $n \ge 2$.

Lemma 5.2. *There exists $C > 0$ such that, for any initial condition $X_0 \in \ell^1_{\mathbb{N}}$ with $X_0 \ne \mathbf{e}_1$,*

$$\mathbb{E}\big(T_1^{X_0}\big) \le C,$$

the time $T_1^{X_0}$ being defined in (5.2).

Proof. Let $n := \|X_0\|_1$ be the initial number of particles. If $n = 1$ and $X_0 \ne \mathbf{e}_1$, then $T_1^{X_0} = 0$. So, we assume that $n \ge 2$. By Lemma 5.1, we have the stochastic domination $T_1^{X_0} \le T_1^{n\mathbf{e}_1}$, so that

$$\mathbb{E}\big(T_1^{X_0}\big) \le \mathbb{E}\big(T_1^{n\mathbf{e}_1}\big), \tag{5.7}$$

and it suffices to obtain an upper bound on $(T_1^{n\mathbf{e}_1})$ which does not depend on $n \ge 2$.

We consider the solution x to the deterministic min-driven coagulation equation (1.17) with monodisperse initial condition $x_0 = (x_{i,0})_{i\ge 1}$ given by $x_{1,0} = 1$ and $x_{i,0} = 0$ for $i \ge 2$. It follows from Corollary 3.4 that

$$\mathbb{P}\left(\big|T_1^{n\mathbf{e}_1} - t_1\big| > \frac{B_1}{n^{1/4}}\right) \le \frac{A_1}{n^{1/4}}, \quad n \ge N_3(1),$$

from which we deduce that there is $C > 0$ such that

$$\mathbb{P}\big(T_1^{n\mathbf{e}_1} > B_1 + t_1\big) \le \frac{C}{n^{1/4}}, \quad n \ge 2. \tag{5.8}$$

Introducing the (random) number of coalescence events n_1 performed between $t = 0$ and $T_1^{n\mathbf{e}_1}$, we have

$$T_1^{n\mathbf{e}_1} = \sum_{m=1}^{n_1} \frac{\varepsilon_m}{(n-m)\phi(1)},$$

where $(\varepsilon_m)_{1\le m\le n-1}$ is a sequence of i.i.d. random variables with law $exp(1)$. Obviously, $n_1 \le n-1$ which gives the bound

$$T_1^{n\mathbf{e}_1} \le \frac{1}{\phi(1)} \sum_{m=1}^{n-1} \frac{\varepsilon_m}{m}.$$

Since $\mathbb{E}(\varepsilon_m) = 1$ and $\mathbb{E}(\varepsilon_m^2) = 2$ for $1 \le m \le n$, we deduce from (5.8), the Hölder inequality, and the above estimate that

$$\begin{aligned}
\mathbb{E}\big(T_1^{n\mathbf{e}_1}\big) &= \mathbb{E}\big(T_1^{n\mathbf{e}_1}\mathbb{1}_{[0,B_1+t_1]}\big(T_1^{n\mathbf{e}_1}\big)\big) + \mathbb{E}\big(T_1^{n\mathbf{e}_1}\mathbb{1}_{(B_1+t_1,\infty)}\big(T_1^{n\mathbf{e}_1}\big)\big) \\
&\le B_1 + t_1 + \frac{1}{\phi(1)}\sum_{m=1}^{n-1}\frac{1}{m}\mathbb{E}\big(\varepsilon_m \mathbb{1}_{(B_1+t_1,\infty)}\big(T_1^{n\mathbf{e}_1}\big)\big) \\
&\le B_1 + t_1 + \frac{1}{\phi(1)}\sum_{m=1}^{n-1}\frac{1}{m}\mathbb{E}\big(\varepsilon_m^2\big)^{1/2}\mathbb{P}\big(T_1^{n\mathbf{e}_1} > B_1 + t_1\big)^{1/2} \\
&\le B_1 + t_1 + \frac{C}{\phi(1)n^{1/8}}\sum_{m=1}^{n-1}\frac{1}{m} \\
&\le B_1 + t_1 + C\frac{\ln n}{n^{1/8}}.
\end{aligned}$$

Since B_1 and t_1 do not depend on n (actually one has $t_1 = 1/\phi(1)$), we have established the upper bound from which Lemma 5.2 follows by (5.7). □

The next step is to establish a connection between the early stages of the dynamics of the processes starting from monodisperse initial data.

Lemma 5.3. *For $n \ge 2$ and $i \ge 1$ we have*

$$T_i^{n\mathbf{e}_i} \overset{law}{=} \frac{\phi(1)}{\phi(i)} T_1^{n\mathbf{e}_1}.$$

Proof. As in the proof of Lemma 5.1, a coupling can be done between the processes starting from $n\mathbf{e}_1$ and $n\mathbf{e}_i$ so that

$$T_1^{n\mathbf{e}_1} = \sum_{m=1}^{n_1}\frac{\varepsilon_m}{(n-m)\phi(1)} \quad \text{and} \quad T_i^{n\mathbf{e}_i} = \sum_{m=1}^{n_1}\frac{\varepsilon_m}{(n-m)\phi(i)}$$

with the same random number of coalescence events n_1 and sequence $(\varepsilon_m)_{1\le m\le n-1}$ of i.i.d. random variables with law *exp*(1) for both processes. □

Proof of Theorem 1.5. Assume first that

$$\sum_{i=1}^{\infty}\frac{1}{i\phi(i)} < \infty.$$

By Lemma 5.1, we just have to show that $\mathbb{E}(T^{n\mathbf{e}_1})$ is bounded independently of $n \ge 1$.

To this end, we fix $n \ge 1$. Let us first notice that, if $n = 1$, then $T^{n\mathbf{e}_1} = 0$. Assume now that $n \ge 2$ and for $i \ge 1$, let X be the stochastic min-driven coagulation process starting from $n\mathbf{e}_i$. Clearly, $T_j^{n\mathbf{e}_i} = 0$ for $1 \le j \le i-1$ and we define the (random) number $n_* := \|X(T_i^{n\mathbf{e}_i})\|_1$ of particles in the system at time $T_i^{n\mathbf{e}_i}$ and $Y := X(T_i^{n\mathbf{e}_i})$. Notice that $Y_j = X_j(T_i^{n\mathbf{e}_i}) = 0$ for $1 \le j \le 2i-1$ and the conservation of mass warrants that $n_* \le n/2$ as

$$2in_* = 2i\big\|X\big(T_i^{n\mathbf{e}_i}\big)\big\|_1 \le \big\|X\big(T_i^{n\mathbf{e}_i}\big)\big\|_{1,1} = \|n\mathbf{e}_i\|_{1,1} = ni.$$

Moreover, the properties of Y and Lemma 5.1 yield the stochastic domination $T^Y \le T^{n_*\mathbf{e}_{2i}}$. Since

$$T^{n\mathbf{e}_i} \overset{\text{law}}{=} T_i^{n\mathbf{e}_i} + T^Y,$$

where, conditionally on Y, $T_i^{n\mathbf{e}_i}$ and T^Y are independent, it follows from Lemma 5.3 that

$$T^{n\mathbf{e}_i} \le \frac{\phi(1)}{\phi(i)} T_1^{n\mathbf{e}_1} + T^{n_*\mathbf{e}_{2i}}. \tag{5.9}$$

Let us now prove by induction on n that the property

$$\mathcal{P}(n)\colon\ \mathbb{E}\big(T^{m\mathbf{e}_{2^i}}\big) \le C \sum_{j=i}^{\infty} \frac{\phi(1)}{\phi(2^j)} \quad \text{for all } i \ge 0 \text{ and } 0 \le m \le n,$$

holds true for all $n \ge 0$, where C is the constant appearing in Lemma 5.2.

It is clear for $n = 0$. Consider $n \ge 1$ and assume $\mathcal{P}(n-1)$. For $i \ge 0$, it follows from (5.9) and $\mathcal{P}(n-1)$ that there is $n_* \le n/2$ such that

$$\begin{aligned}
\mathbb{E}\big(T^{n\mathbf{e}_{2^i}}\big) &\le \frac{\phi(1)}{\phi(2^i)} \mathbb{E}\big(T_1^{n\mathbf{e}_1}\big) + \mathbb{E}\big(T^{n_*\mathbf{e}_{2^{i+1}}}\big) \\
&\le \frac{\phi(1)}{\phi(2^i)} \mathbb{E}\big(T_1^{n\mathbf{e}_1}\big) + \sum_{m=1}^{n/2} \mathbb{P}\big(n^* = m\big) \mathbb{E}\big(T^{m\mathbf{e}_{2^{i+1}}}\big) \\
&\le \frac{\phi(1)}{\phi(2^i)} \mathbb{E}\big(T_1^{n\mathbf{e}_1}\big) + \sup_{1 \le m \le n/2} \mathbb{E}\big(T^{m\mathbf{e}_{2^{i+1}}}\big) \\
&\le \frac{\phi(1)}{\phi(2^i)} \mathbb{E}\big(T_1^{n\mathbf{e}_1}\big) + C \sum_{j=i+1}^{\infty} \frac{\phi(1)}{\phi(2^j)} \quad \text{(by induction hypothesis)} \\
&\le C \sum_{j=i}^{\infty} \frac{\phi(1)}{\phi(2^j)},
\end{aligned}$$

which proves $\mathcal{P}(n)$.

We then infer from property $\mathcal{P}(n)$ for $i = 0$ that

$$\mathbb{E}\big(T^{n\mathbf{e}_1}\big) \le C\phi(1) \sum_{i=0}^{\infty} \frac{1}{\phi(2^i)} < \infty,$$

the convergence of the series $\sum 1/\phi(2^i)$ being ensured by that of $\sum 1/(i\phi(i))$ and the monotonicity of ϕ.

To prove the converse part of Theorem 1.5, we assume that

$$\sum_{i=1}^{\infty} \frac{1}{i\phi(i)} = \infty,$$

and show that, for each constant $R > 0$, there exists a configuration X_0 such that $\mathbb{E}(T^{X_0}) \ge R$. More precisely, we will prove that

$$\lim_{n\to\infty} \mathbb{P}\big(T^{n\mathbf{e}_1} \le R\big) = 0 \quad \text{for all } R > 0, \tag{5.10}$$

which clearly implies that

$$\lim_{n\to\infty} \mathbb{E}\big(T^{n\mathbf{e}_1}\big) = \infty. \tag{5.11}$$

Indeed, let $n \geq 2$. By (5.3), we have

$$T^{n\mathbf{e}_1} = \sum_{m=1}^{n-1} \frac{\varepsilon_m}{(n-m)\phi(L(m))},$$

where $(\varepsilon_m)_{1\leq m\leq n-1}$ is a sequence of i.i.d. random variables with law $exp(1)$. The sequence $(L(m))_{1\leq m\leq n-1}$ is random but let us notice the bound

$$L(m) \leq \frac{n}{n-m+1} \leq \frac{n}{n-m}, \quad 1 \leq m \leq n-1,$$

which follows from the conservation of mass since there remain $n-m+1$ particles in the system before the mth coalescence event. Therefore, by the monotonicity of ϕ, we have the stochastic domination

$$T^{n\mathbf{e}_1} \geq \Lambda_n := \sum_{m=1}^{n-1} \frac{\varepsilon_{n-m}}{m\phi(n/m)}.$$

In particular,

$$\mathbb{P}\big(T^{n\mathbf{e}_1} \leq R\big) \leq \mathbb{P}(\Lambda_n \leq R) \quad \text{for } R > 0. \tag{5.12}$$

We next infer from the divergence of the series $\sum 1/(i\phi(i))$ that

$$\mathbb{P}\Big(\lim_{n\to\infty} \Lambda_n = \infty\Big) = 1,$$

see Theorem 2.3.2 in [19], for instance. Combining the above property with (5.12) implies that $\mathbb{P}(T^{n\mathbf{e}_1} \leq R) \longrightarrow 0$ as $n \longrightarrow \infty$ for all $R > 0$. In other words, $T^{n\mathbf{e}_1} \longrightarrow \infty$ in probability. $\square$

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